1. Suppose that $X$ is a random variable with mean $E[X] = 3$ and variance $\text{Var}(X) = 10$. Further, let $M_X(t)$ denote the moment generating function of $X$.

(a) Find $\frac{d}{dt} M_X(t)|_{t=0}$.

**Solution:** $\frac{d}{dt} M_X(t)|_{t=0} = E[X] = 3$. ♦

(b) Find $\frac{d^2}{dt^2} M_X(t)|_{t=0}$.

**Solution:** $\frac{d^2}{dt^2} M_X(t)|_{t=0} = E[X^2] = \text{Var}(X) + (E[X])^2 = 19$. ♦

2. If $X$ has m.g.f. $M_X(t) = 0.8e^t + 0.2$, name the distribution of $X$ (with parameter value(s)).

**Solution:** Bern(0.8) or Bin(1, 0.8). ♦

3. Suppose $X$ has p.d.f. $f(x) = 3x^2$, $0 \leq x \leq 1$. What is the p.d.f. of $Y = \sqrt{X}$?

**Solution:** The c.d.f. of $Y$ is

$$G(y) = \Pr(Y \leq y) = \Pr(\sqrt{X} \leq y)$$

$$= \Pr(X \leq y^2) = \int_0^{y^2} 3x^2 \, dx$$

$$= y^6.$$

Thus, the required p.d.f. is $g(y) = 6y^5$, $0 \leq y \leq 1$. ♦

4. Suppose that $X$ is a nice, happy Exp(3) random variable. Find the distribution of the “nasty” random variable $1 - e^{-3X}$.

**Solution:** Note that the c.d.f. of $X$ is $F(x) = 1 - e^{-3x}$. Then the Inverse Transform Theorem immediately implies that $F(X) = 1 - e^{-3X}$ is Unif(0,1). ♦
5. Consider the joint p.m.f. of \((X, Y)\) for which \(f(0, 0) = 0.1\), \(f(0, 1) = 0.3\), \(f(0, 2) = 0.2\), \(f(2, 0) = 0.2\), \(f(2, 1) = 0.2\), and \(f(2, 2) = 0\).

(a) Find \(\Pr(Y = 1)\).

**Solution:** First of all, the joint p.m.f. can be written as follows:

<table>
<thead>
<tr>
<th>(f(x, y))</th>
<th>(x = 0)</th>
<th>(x = 1)</th>
<th>(f_Y(y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y = 0)</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>(y = 1)</td>
<td>0.3</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>(y = 2)</td>
<td>0.2</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>(f_X(x))</td>
<td>0.6</td>
<td>0.4</td>
<td>1</td>
</tr>
</tbody>
</table>

Then \(\Pr(Y = 1) = f_Y(1) = 0.5\). ◊

(b) Are \(X\) and \(Y\) independent?

**Solution:** No. For example, \(f(1, 2) = 0 \neq f_X(1)f_Y(2) = 0.08\). ◊

6. TRUE or FALSE? If \(X\) and \(Y\) are independent, then the conditional p.d.f. \(f(y|x) = f_X(x)\) for all \(x\) and \(y\).

**Solution:** FALSE. It should be \(f(x|y) = f_X(x)\) for all \(x\) and \(y\). ◊

7. Suppose that \(X\) and \(Y\) have joint p.d.f. \(f(x, y) = 3x\), \(0 < y < x < c\).

(a) Find \(c\).

**Solution:** Set

\[
1 = \int_{\infty}^{\infty} \int_{\infty}^{\infty} f(x, y) \, dy \, dx = \int_{0}^{c} \int_{0}^{x} 3x \, dy \, dx = c^3,
\]

so to obtain \(c = 1\). ◊

(b) Find \(E[X]\).

**Solution:** We have

\[
f_X(x) = \int_{\infty}^{\infty} f(x, y) \, dy = \int_{0}^{x} 3x \, dy = 3x^2, \quad 0 < x < 1.
\]
Thus,
\[
E[X] = \int_{-\infty}^{\infty} f_X(x) \, dx = \int_{0}^{1} x \, 3x^2 \, dx = 3/4. \quad \diamond
\]

(c) Find \( \text{Cov}(X, Y) \).

**Solution:** Similarly,
\[
f_Y(y) = \int_{y}^{1} 3x \, dx = \frac{3}{2} (1 - y^2), \quad 0 < y < 1,
\]
so that
\[
E[Y] = \int_{0}^{1} y \frac{3}{2} (1 - y^2) \, dy = 3/8.
\]
In addition,
\[
E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x, y) \, dy \, dx = \int_{0}^{1} \int_{0}^{x} xy \, 3x \, dy \, dx = 3/10.
\]
Thus, we have
\[
\text{Cov}(X, Y) = E[XY] - E[X] E[Y] = \frac{3}{10} - \frac{9}{32} = \frac{3}{100} = 0.01875. \quad \diamond
\]

(d) Find the conditional p.d.f. \( f(x|y) \).

**Solution:** From previous results, we have
\[
f(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{\frac{3x}{2} (1 - y^2)}{\frac{3}{2} (1 - y^2)} = \frac{2x}{1 - y^2}, \quad 0 < y < x < 1. \quad \diamond
\]

(e) Find \( E[X|Y = y] \).

**Solution:** By definition, we have
\[
E[X|Y] = \int_{-\infty}^{\infty} xf(x|y) \, dx
= \int_{y}^{1} \frac{2x}{1 - y^2} \, dx
= \frac{2(1 - y^3)}{3(1 - y^2)} = \frac{2(1 + y + y^2)}{3(1 + y)}. \quad 0 < y < 1, \quad \diamond
\]
(f) Find \(E[EX|Y]]\).

**Solution:** By LOTUS and previous results, we have

\[
E[E[X|Y]] = \int_{-\infty}^{\infty} E[X|y]f_Y(y)
dy = \int_{0}^{1} \frac{2(1-y^3)}{3(1-y^2)} \frac{3}{2} (1-y^2)
dy = \int_{0}^{1} (1-y^3)
dy = \frac{3}{4}. \quad \diamondsuit
\]

Of course, this isn’t a surprise since the double expectation theorem says that \(E[E[X|Y]] = E[X]\). (You could’ve done the problem this way in about one second.) \(\diamondsuit\)

8. TRUE or FALSE? \(E[\ell n(Z)] = E[E[\ell n(Z)|Y]]\).

**Solution:** TRUE (by double expectation). \(\diamondsuit\)

9. Suppose that the joint p.d.f. of \(X\) and \(Y\) is \(f(x, y) = cxy/(1+x+y)\), for \(0 < x < 1\) and \(0 < y < 1\) and some appropriate constant \(c\). Are \(X\) and \(Y\) independent?

**Solution:** No. You can’t factor \(f(x, y)\). \(\diamondsuit\)

10. Suppose that \(E[X] = 3, E[Y] = 4, \Var(X) = 8, \Var(Y) = 16,\) and \(\Cov(X, Y) = -4\).

   (a) Find \(\Var(X - Y)\).

   **Solution:** \(\Var(X - Y) = \Var(X) + \Var(Y) - 2\Cov(X, Y) = 32. \quad \diamondsuit\)

   (b) Find \(\Corr(X, Y)\).

   **Solution:** \(\Corr(X, Y) = \frac{\Cov(X, Y)}{\sqrt{\Var(X)\Var(Y)}} = -1/\sqrt{8} = -0.354. \quad \diamondsuit\)
11. If $X$ and $Y$ are independent $\text{Bern}(p)$ random variables, find $E[(X + Y)^2]$.

**Solution:** Since $X$ and $Y$ are independent, and all $\text{Bern}(p)$ moments are equal to $p$, we have

\[
\]

12. Suppose that the questions on a test are i.i.d. in the sense that you will be able to answer any question correctly with probability 0.8. What is the expected number of questions that you will do until you make your second incorrect answer?

**Solution:** Let $W = \text{number of questions until second error}$. Then $W \sim \text{Neg-Bin}(2,0.2)$. (You can also write $W$ as the sum of two geometric random variables.) Then $E[W] = r/p = 2/0.2 = 10$.

13. I am a discrete distribution with the memoryless property. Name me.

**Solution:** Geometric($p$).

14. I am the sum of 5 i.i.d. Exp($\lambda = 1/3$) random variables. Name me (including parameter(s)).

**Solution:** Erlang$_{k=5}(\lambda = 1/3)$.

15. TRUE or FALSE? A Poisson counting process never has simultaneous arrivals.

**Solution:** TRUE.

16. I’m a continuous distribution used to model stock prices. Name me.

**Solution:** lognormal.
17. The number of defects on a length of wire is a Poisson process with rate \( \lambda = 1.5/\text{meter} \). Find the probability that we will find exactly 1 defect on a certain 2-meter section of the wire.

**Solution:** Let \( X \) denote the number of defects on a 2-meter section. Then \( X \sim \text{Pois}(3) \), and \( \Pr(X = 1) = e^{-3}3^1/1! = 0.1494 \). ♦

18. Customers arrive at a store according to a Poisson process at the rate of 3/hour. Name the distribution (with parameter(s)) of the time between the 2nd and 3rd arrivals.

**Solution:** \( \text{Exp}(\lambda = 3) \). ♦

19. What kind of random variables does the Box-Muller method generate?

**Solution:** \( \text{Nor}(0,1) \). ♦

20. Suppose that \( Z \) is a standard normal random variable, having c.d.f. \( \Phi(z) \). What is the value of \( \Phi^{-1}(0.95) \)? (You may want to look at the following table.)

<table>
<thead>
<tr>
<th>( z )</th>
<th>( \Pr(Z \leq z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>0.8159</td>
</tr>
<tr>
<td>0.95</td>
<td>0.8289</td>
</tr>
<tr>
<td>1</td>
<td>0.8413</td>
</tr>
<tr>
<td>1.28</td>
<td>0.9000</td>
</tr>
<tr>
<td>1.5</td>
<td>0.9332</td>
</tr>
<tr>
<td>1.645</td>
<td>0.9500</td>
</tr>
<tr>
<td>1.96</td>
<td>0.9750</td>
</tr>
<tr>
<td>2</td>
<td>0.9773</td>
</tr>
</tbody>
</table>

**Solution:** \( \Phi^{-1}(0.95) \) is that value of \( z \) such that \( \Pr(Z \leq z) = 0.95 \). The table shows that the desired value is \( z = 1.645 \). ♦

21. Suppose that \( Z \sim \text{Nor}(0,1) \). Find \( \Pr(-1.5 \leq Z \leq 1.5) \).
Solution: From the table, we have $\Pr(-1.5 \leq Z \leq 1.5) = 2\Phi(1.5) - 1 = 2(0.9332) - 1 = 0.8664$. ◊

22. Suppose that the diameter of a ball bearing is $D \sim \text{Nor}(10, 4)$, where the units are in millimeters. Find the probability that a particular ball bearing will be greater than 12 mm.

Solution: $\Pr(D > 12) = \Pr(Z > \frac{12 - 10}{\sqrt{4}}) = \Pr(Z > 1) = 0.1587$. ◊

23. The amount of time to perform Job A is $\text{Nor}(10, 2)$, and the amount of time to perform Job B is $\text{Nor}(6, 0.5)$. Assuming that the times $A$ and $B$ are independent, find $\Pr(A > 2B)$, i.e., the probability that $A$ takes more than twice as long as $B$.

Solution: Since $A$ and $B$ are independent and normal, then any linear combination of $A$ and $B$ is normal. Moreover,


and

$$\text{Var}(A - 2B) = \text{Var}(A) + 4\text{Var}(B) = 4.$$ 

Putting this all together, we have $A - 2B \sim \text{Nor}(-2, 4)$.

Then $\Pr(A > 2B) = \Pr(A - 2B > 0) = \Pr(\text{Nor}(-2, 4) > 0) = \Pr(Z > \frac{0 - (-2)}{\sqrt{4}}) = \Pr(Z > 1) = 0.1587$. ◊

24. Consider i.i.d. dice tosses $X_1, X_2, \ldots, X_{25}$, and let the sample mean $\bar{X} \equiv \sum_{i=1}^{25} X_i / 25$. What is the variance of $\bar{X}$?

Solution: First of all,

$$\text{Var}(X_i) = E[X_i^2] - (E[X_i])^2 = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6} - (3.5)^2 = \frac{35}{12} = 2.9167.$$ 

Then $\text{Var}(\bar{X}) = \text{Var}(X_i) / 25 = 7/60 = 0.1167$. ◊
25. If $X_1, X_2, \ldots, X_{25}$ are i.i.d. $\text{Nor}(-5, 10)$, what’s the distribution (with parameter(s)) of the sample mean $\bar{X}$?

**Solution:** $\bar{X} \sim \text{Nor}(-5, 0.4)$

26. Suppose that $X_1, X_2, \ldots, X_{1200}$ are i.i.d. $\text{Pois}(3)$. Find the approximate value of $\Pr(\bar{X} \geq 2.9)$, where $\bar{X}$ is the sample mean.

**Solution:** Note that $E[\bar{X}] = \lambda = 3$ and $\text{Var}(\bar{X}) = \lambda/n = 3/1200 = 1/400$. Then by the CLT,

$$
\Pr(\bar{X} \geq 2.9) = \Pr \left( \frac{\bar{X} - 3.0}{\sqrt{1/400}} \geq \frac{2.9 - 3.0}{\sqrt{1/400}} \right)
\approx \Pr(Z \geq -2)
= 1 - \Phi(-2) = \Phi(2) = 0.9773.
$$

27. What is our most important theorem? (Pick one of the choices below.)

(a) The central lemon theorem.
(b) The sentry limit theorem.
(c) The central limit theorem.
(d) The central limit serum.
(e) All of the above.

**Solution:** (c).