NAME →

ISyE 6739 — Test 1 Solutions — Summer 2017

This is a take-home test. But please limit the total work time to less than about 3 hours.

1. Suppose that
   \[ P(\text{It rains today} \mid \text{It’s cold outside}) = 0.9 \]
   and
   \[ P(\text{It rains today and it’s cold outside}) = 0.4. \]

   Find the probability that it’s cold outside.

   Hint: See https://www.youtube.com/watch?v=NNTSk8EsoAM.

   **Solution:** By definition,
   \[
   P(\text{Rain} \mid \text{Cold}) = \frac{P(\text{Rain and Cold})}{P(\text{Cold})},
   \]
   so that
   \[
   P(\text{Cold}) = \frac{P(\text{Rain and Cold})}{P(\text{Rain} \mid \text{Cold})} = \frac{4}{9}. \quad \Box
   \]

2. 1960s British Rock Question: The probability that a random person likes The Beatles \((B)\) is 80%, the probability that a person likes The Zombies \((Z)\) is 60%.

   (a) YES or NO? Can \(B\) and \(Z\) be independent?

   **Solution:** In order for \(B\) and \(Z\) to be independent, we’ll need
   \[ P(B \cap Z) = P(B)P(Z) = 0.48, \]
   which is possible. So the answer is YES. \(\Box\)

   (b) YES or NO? Can the complements \(\bar{B}\) and \(\bar{Z}\) be disjoint?

   **Solution:** In order for \(\bar{B}\) and \(\bar{Z}\) to be disjoint, we’ll need
   \[ P(\bar{B} \cup \bar{Z}) = P(\bar{B}) + P(\bar{Z}) = 0.6, \]
   which is possible. For example, it would work with \(P(B \cap Z) = 0.4\). In any case, the answer is YES. \(\Box\)
(c) Suppose the probability that a person likes The Beatles or The Zombies (or both) is 0.9. What’s the probability that a person likes both groups?

Solution: \[ P(B \cap Z) = P(B) + P(Z) - P(B \cup Z) = 0.8 + 0.6 - 0.9 = 0.5. \]

3. If \( P(A) = 0.7 \), \( P(B) = 0.5 \), and \( P(C) = 0.2 \), and \( A, B, \) and \( C \) are independent, find the probability that exactly two of \( A \), \( B \), and \( C \) occur.

Solution: We want
\[
P(AB\bar{C}) + P(A\bar{B}C) + P(\bar{A}BC)
= P(A)P(B)P(\bar{C}) + P(A)P(\bar{B})P(C) + P(\bar{A})P(B)P(C)
= (0.7)(0.5)(0.8) + (0.7)(0.5)(0.2) + (0.3)(0.5)(0.2) = 0.38.
\]

4. Consider a box of 10 sox — 4 blue and 6 red. Pick three sox without replacement. What’s the probability that the third sock is red?

Solution: Given that you have no knowledge of how the first two picks went,
\[ P(\text{third red}) = P(\text{first red}) = \frac{6}{10}. \]

5. What is 0! (zero factorial)?

Solution: 1. \[ \square \]

6. Toss a die 5 times. What’s the probability that you’ll never see a “3” or a “4”?

Solution: Since the events are independent, the answer is
\[
[P(\text{no 3 or 4 on a given toss})]^5 = \left(\frac{2}{3}\right)^5 = \frac{32}{243}.
\]

7. Suppose that \( A \) and \( B \) are independent events. Which of the following statements are true (may be more than one correct answer)?

(a) \( P(A \cap B) = P(A)P(B) \)
(b) $P(B|A) = P(B)$
(c) $P(A|\bar{B}) = P(A)$
(d) $P(\bar{A}|B) = 1 - P(A)$
(e) $P(A \cup B) = P(A) + P(B)$

Solution:

(a) TRUE, by definition of independence.
(b) TRUE. $P(B|A) = P(A \cap B)/P(A) = P(A)P(B)/P(A) = P(B)$.
(c) TRUE. Similar to (b).
(d) TRUE. $P(\bar{A}|B) = 1 - P(A|B) = 1 - P(A)$.
(e) FALSE. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B)$. □

8. Suppose that exactly 5% of the people in Atlanta drool. In addition, it is known that the probability that a drooler likes UGA is 95% while the probability that a non-drooler likes UGA is 20%. Now, let’s suppose that we observe a person cheering for UGA. What is the probability that the person is a drooler?

Solution: $D$ : Drooler, $N$ : Non-Drooler, $L$ : Like UGA. By Bayes Theorem,

$$P(D|L) = \frac{P(L|D)P(D)}{P(L|D)P(D) + P(L|N)P(N)} = \frac{(0.95)(0.05)}{(0.95)(0.05) + (0.20)(0.95)} = \frac{1}{5}. \quad \square$$

9. Suppose there are 4 people in the room. Assuming that all birthdays are independent and all 365 days are equally likely, what’s the probability that there are no matches among the 4 people? (You don’t have to simplify.)

Solution:

$$P(\text{No matches among the 4 people}) = \frac{P_{365,4}}{365^4} = \frac{(364)(363)(362)}{365^3} = 0.9836. \quad \square$$
10. My friends have 4 kids. I know that the oldest is a girl. What’s the probability that my friends have at least 3 girls?

**Solution:** We have

\[
P(\geq 3 \text{ girls}|\text{XXXG}) = P(\geq 2 \text{ girls out of first 3}) = P(BGG \text{ or } GBG \text{ or } GGB \text{ or } GGG) = \frac{1}{2}.\]

\[
\square
\]

11. Suppose I toss 6 dice. What’s the probability that I’ll see exactly three pairs? (An example of such an outcome is 3,5,5,3,1,1. A four-of-a-kind, etc., doesn’t count.)

**Solution:**

The # of ways to choose numbers for three pairs is \(\binom{6}{3}\).

The # of ways to place the first pair is \(\binom{6}{2}\).

The # of ways to place the second pair is \(\binom{4}{2}\).

The # of ways to place the third pair is \(\binom{2}{2}\).

Thus, \(\frac{\binom{6}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{6^6} = 0.03858.\) \(\square\)

12. Pick 8 cards from a standard deck. What’s the probability that I’ll see a flush of 8 cards? (That is, all 8 cards come from the same suit.)

**Solution:**

The # of ways to choose suits is 4.

The # of ways to choose 8 cards out of 13 is \(\binom{13}{8}\).

Thus, \(\frac{\binom{13}{8}}{\binom{52}{8}} = 0.00000684.\) \(\square\)

13. What is the loneliest number?

**Solution:** One is the loneliest number (though two can be as bad as one). See http://www.youtube.com/watch?v=UiiKcd7yPLdU for a proof. \(\square\)
14. Suppose that $X$ is a discrete random variable that can equal 1 with probability $p$ and 5 with probability $1 - p$. If $E[X] = 2$, find $p$.

**Solution:** $E[X] = 1(p) + 5(1 - p) = 2$. Thus, $p = 0.75$. □

15. Suppose $X$ has p.d.f. $f(x) = 3x^2$ for $0 < x < 1$.

(a) Find $P(X > 0.6 | X > 0.5)$.

**Solution:** The c.d.f. is $F(x) = x^3$ for $0 < x < 1$. Thus,

\[
P(X > 0.6 | X > 0.5) = \frac{P(X > 0.6, X > 0.5)}{P(X > 0.5)} = \frac{P(X > 0.6)}{P(X > 0.5)} = \frac{1 - F(0.6)}{1 - F(0.5)} = \frac{1 - 0.216}{1 - 0.125} = 0.896.
\]

(b) Find $E[3X - 2]$.

**Solution:** We have

\[
E[X] = \int_R x f(x) \, dx = \int_0^1 x(3x^2) \, dx = \frac{3}{4},
\]


(c) Find $E[1/X]$.

**Solution:** By LOTUS, we have

\[
E[1/X] = \int_R (1/x) f(x) \, dx = \int_0^1 (1/x)(3x^2) \, dx = \frac{3}{2}.
\]
16. TRUE or FALSE? \( \text{Var}(Y - a) < 0 \) for some sufficiently large constant \( a \).

**Solution:** FALSE. To show this, use the fact that variances are invariant to shifts and then the definition: \( \text{Var}(Y - a) = \text{Var}(Y) = E\left[(Y - E[Y])^2\right] \geq 0. \)

17. TRUE or FALSE? If \( X \) is continuous, then its p.d.f. is the derivative of its c.d.f.

**Solution:** TRUE (by the Fundamental Theorem of Calculus).

18. Suppose \( X \sim \text{Exponential}(\lambda) \), i.e., its p.d.f. is \( f(x) = \lambda e^{-\lambda x} \) for \( x \geq 0 \).

(a) What’s \( E[X] \)? (You’ve already seen this in class notes and/or practice tests.)

**Solution:** By hook or by crook (e.g., integration by parts + l’Hôpital’s Rule), we have

\[
E[X] = \int_{\mathbb{R}} x f(x) \, dx = \int_{0}^{\infty} x \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda}.
\]

(b) Find \( E[e^{tX}] \).

**Solution:** By LOTUS and calculus,

\[
E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x) \, dx = \int_{0}^{\infty} e^{-tx} \lambda e^{-\lambda x} \, dx = \frac{\lambda}{\lambda - t}, \quad \text{for } t < \lambda.
\]

You may (or you will soon) recognize this as the moment generating function of the exponential distribution.

(c) Calculate \( \frac{d}{dt}E[e^{tX}] \bigg|_{t=0} \). (That is, take the derivative and then evaluate it at \( t = 0 \).)

**Solution:** We have

\[
\frac{d}{dt}E[e^{tX}] \bigg|_{t=0} = \frac{d}{dt} \left( \frac{\lambda}{\lambda - t} \right) \bigg|_{t=0} = \frac{\lambda}{(\lambda - t)^2} \bigg|_{t=0} = \frac{1}{\lambda}.
\]

(Of course, this is the mean of the exponential, so the m.g.f. has provided us without another way to find that quantity.)
19. **PROVE or DISPROVE:** If \( X \) is a continuous random variable that is always positive, then \( \mathbb{E}[X^n] = \int_0^\infty nx^{n-1} \mathbb{P}(X > x) \, dx \).

**Solution:** This is TRUE, because

\[
\int_0^\infty nx^{n-1} \mathbb{P}(X > x) \, dx = \int_0^\infty \int_x^\infty f(t) \, dt \, dx \\
= \int_0^\infty \int_0^t nx^{n-1} f(t) \, dx \, dt \quad \text{(interchange order of integration)} \\
= \int_0^\infty f(t) \int_0^t nx^{n-1} \, dx \, dt \\
= \int_0^\infty f(t)t^n \, dt \\
= \mathbb{E}[X^n]. \quad \text{(by LOTUS).} \, \square
\]

20. If \( X = -3 \) with probability 0.6 and \( X = 0 \) with probability 0.4, find \( \text{Var}(-3X + 2) \).

**Solution:** First of all, note that

\[
\mathbb{E}[X] = \sum_x xf(x) = -3(0.6) + 0(0.4) = -1.8
\]

and

\[
\mathbb{E}[X^2] = \sum_x x^2 f(x) = (-3)^2(0.6) + 0^2(0.4) = 5.4,
\]

so that \( \text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 2.16 \).

Thus, \( \text{Var}(-3X + 2) = (-3)^2 \text{Var}(X) = 9(2.16) = 19.44. \, \square \)

21. Suppose that \( X \) is a continuous random variable with p.d.f.

\[
f(x) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1 - x)^{b-1}, \quad 0 < x < 1,
\]

where \( a \) and \( b \) are positive constants, and \( \Gamma(a) = \int_0^\infty t^{a-1}e^{-t} \, dt \) is the Gamma function (that you may remember from Calculus class). Find \( \mathbb{E}[X] \).
Solution: Applying a little calculus elbow grease, we have

\[ E[X] = \int_{\mathbb{R}} xf(x) \, dx \]
\[
= \int_0^1 x \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1 - x)^{b-1} \, dx \\
= \int_0^1 \Gamma(a + b) \frac{\Gamma(a)}{\Gamma(a)\Gamma(b)} x^a(1 - x)^{b-1} \, dx \\
= \frac{\Gamma(a + b)}{\Gamma(a)} \frac{\Gamma(a + 1)}{\Gamma(a + 1 + b)} \int_0^1 \frac{\Gamma((a + 1) + b)}{\Gamma(a + 1)\Gamma(b)} x^{(a+1)-1}(1 - x)^{b-1} \, dx \\
= \frac{\Gamma(a + b)}{\Gamma(a)} \frac{\Gamma(a + 1)}{\Gamma(a + 1 + b)} \\
\text{(the integral equals 1 since it’s the original p.d.f. but with } a + 1 \text{ in place of } a) \\
= \frac{\Gamma(a + b) a\Gamma(a)}{\Gamma(a) (a + b)\Gamma(a + b)} \\
= \frac{a}{a + b}. \]

In fact, this thing is a beta distribution with parameters \(a\) and \(b\), and the result is well known. □