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ISyE 6739 — Test 1 Solutions — Summer 2016

(revised 6/27/17)

This is a take-home test. But please limit the total work time to less than about 3 hours.

1. TRUE or FALSE? If $A_1, A_2, \ldots$ are disjoint events, then $\mathsf{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathsf{P}(A_i)$. Give an example to support your claim.

   Solution: TRUE. (This is actually one of the probability axioms.) ♢

   For example, let's flip a coin until we obtain our first H. Let A_i be the event that we obtain our first H on the ith toss, e.g., $A_4 = \{TTTH\}$. Obviously, the A_i's are disjoint. Moreover,

$$\begin{aligned}
\mathsf{P}(\cup_{i=1}^{\infty} A_i) &= \mathsf{P}(\text{eventually we toss } H) \\
&= \mathsf{P}(\{H\} \cup \{TH\} \cup \{TTH\} \cup \{TTTH\} \cdots) \\
&= \mathsf{P}(H) + \mathsf{P}(TH) + \mathsf{P}(TTH) + \cdots \\
&= \sum_{i=1}^{\infty} \mathsf{P}(A_i) \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \\
&= 1. \quad \diamondsuit
\end{aligned}$$

2. Something is fishy! The probability that a random person likes sushi (S) is 70%; the probability that a person wears glasses (G) is 40%.

   (a) YES or NO? Can S and G be disjoint?

   Solution: In order for S and G to be disjoint, we'll need $\mathsf{P}(S \cup G) = \mathsf{P}(S) + \mathsf{P}(G) = 1.1$, which is impossible. So the answer is NO. ♢

   (b) YES or NO? Can $\bar{S}$ and G be disjoint?

   Solution: In order for $\bar{S}$ and G to be disjoint, we'll need $\mathsf{P}(\bar{S} \cup G) = \mathsf{P}(\bar{S}) + \mathsf{P}(G) = 0.7$, which is possible. So the answer is YES. ♢

(c) YES or NO? Can S and G be independent?

Solution: In order for S and G to be independent, we'll need $\mathsf{P}(S \cap G) = \mathsf{P}(S)\mathsf{P}(G) = 0.28$, which is possible. So the answer is YES. $\diamondsuit$

(d) YES or NO? Can S and $\bar{G}$ be independent?

Solution: In order for S and $\bar{G}$ to be independent, we'll need $\mathsf{P}(S \cap \bar{G}) = \mathsf{P}(S)\mathsf{P}(\bar{G}) = 0.42$, which is possible. So the answer is YES. $\diamondsuit$

(e) Suppose the probability that a person likes sushi *and* wears glasses is 30%. What's the probability that a random person likes sushi but doesn't wear glasses?

Solution: $\mathsf{P}(S \cap \bar{G}) = \mathsf{P}(S) - \mathsf{P}(S \cap G) = 0.7 - 0.3 = 0.4.$ $\diamondsuit$

3. TRUE or FALSE? If A and B are disjoint, then A and B are independent.

   Solution: FALSE. (Almost the opposite, actually. If you know that A and B are disjoint, then if A occurs, you have tremendous information about B; so they can't be independent.) $\diamondsuit$

4. TRUE or FALSE? If $\mathsf{P}(A \cap B \cap C) = \mathsf{P}(A)\mathsf{P}(B)\mathsf{P}(C)$, then A, B, and C are independent. Give an example to support your claim.

   Solution: FALSE. (They also have to be *pairwise* independent.) $\diamondsuit$

5. Consider a box containing 6 red sox and 3 blue sox. Suppose I select 3 sox from the box. What's the probability that the *second* sock is blue?

   Solution: Since we don't know what happened with the first or third sox, the desired probability is $3/9 = 1/3$. $\diamondsuit$

6. My friends have 4 kids. I know that at least 2 of these kids are girls. What's the probability that all 4 are girls?

   Solution: Similar to an example in class,

   $$\mathsf{P}(\text{GGGG}|\geq \text{two girls}) \;=\; \frac{\mathsf{P}(\text{GGGG})}{\mathsf{P}(\geq \text{two girls})} \;=\; \frac{1/16}{11/16} \;=\; \frac{1}{11}. \quad \diamondsuit$$

   (If you don't see how I got the 11/16, just list out all of the 16 B/G possibilities.)

7. Flip a fair coin 5 times. What's the probability that you'll get at least two heads in a row during your sequence of 5 flips?

   Solution: There are 32 possibilities: HHHHH, HHHHT, . . . , TTTTT. 19 of these have "HH" appear. (Write all 32 out and you will see.) So the desired probability is 19/32. $\diamondsuit$

8. I made name tags for 6 people; but unfortunately, I handed them out randomly — oops! Find the probability that none of the 6 people get their correct name tag.

   Solution: This is a straightforward application of the envelope problem. The probability of no matches is

   $$1-\left[1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!}-\frac{1}{6!}\right] \;\approx\; \frac{1}{e} \;\approx\; 0.368. \quad \diamondsuit$$

9. Suppose I toss 6 dice. What's the probability that I'll see one pair and one 4-of-a-kind, e.g., 4,5,5,4,5,5?

   Solution:

   - The # ways to toss 6 dice is 6^6.
   - The # ways to pick the two ranks (one for the pair, one for the 4-of-a-kind, where order matters) is $P_{6,2} = 6(5) = 30$.
   - The # ways to place the pair (and hence the 4-of-a-kind) is $\binom{6}{2} = 15$.

   Thus, the desired probability is $\frac{450}{6^6} = 0.00965$. $\diamondsuit$

10. Pick 6 cards from a standard deck. What's the probability that I'll see one pair and one 4-of-a-kind? (Example: $K\clubsuit, 10\diamondsuit, 10\heartsuit, K\heartsuit, K\spadesuit, K\diamondsuit$.)

Solution:

- The # ways to pick the two ranks (one for the pair, one for the 4-of-a-kind, where order matters) is $P_{13,2} = 13(12) = 156$.
- The # ways to choose suits for the pair is $\binom{4}{2} = 6$.
- The # ways to choose suits for the 4-of-a-kind is $\binom{4}{4} = 1$.

Thus, the desired probability is

$$\frac{156(6)}{\binom{52}{6}} = 0.0000460. \quad \diamondsuit$$

11. Suppose that 40% of the people in a population are males. In addition, it is known that on any given day, the probability that a male drinks wine is 0.8, and the probability that a female drinks wine is 0.6. Further, all days are em independent. Now, let's suppose that we observe a person enjoying wine *on Monday and Tuesday*. What is the probability that the person is a male?

Solution: WW : drinks wine two days on a row, M : male, F : female. By Bayes Theorem,

$$\begin{aligned} \mathsf{P}(M|WW) &= \frac{\mathsf{P}(WW|M)\mathsf{P}(M)}{\mathsf{P}(WW|M)\mathsf{P}(M) + \mathsf{P}(WW|F)\mathsf{P}(F)} \\ &= \frac{(0.64)(0.4)}{(0.64)(0.4) + (0.36)(0.6)} \\ &= 0.542. \quad \diamondsuit \end{aligned}$$

12. Suppose X is discrete with the following probability mass function:

x	-1	0	1	2
$\mathsf{P}(X = x)$	0.4	0.3	0.2	0.1

(a) Find $\mathsf{P}(0.5 \le X \le 2)$.

Solution: $\mathsf{P}(X = 1 \text{ or } 2) = 0.3. \quad \diamondsuit$

(b) Find $\mathsf{P}(0 \le X \le 1 \,|\, 0.5 \le X \le 2)$.

Solution:

$$\frac{\mathsf{P}((X=0 \text{ or } 1) \cap (X=1 \text{ or } 2))}{\mathsf{P}(X=1 \text{ or } 2)} = \frac{\mathsf{P}(X=1)}{\mathsf{P}(X=1 \text{ or } 2)} = 2/3. \quad \diamondsuit$$

(c) Find $\mathsf{E}[X]$.

Solution: $-1(0.4) + 0(0.3) + 1(0.2) + 2(0.1) = 0. \quad \diamondsuit$

(d) Find $\mathsf{E}[e^X]$.

Solution: By LOTUS, we have

$$\mathsf{E}[e^X] \;=\; \sum_x e^x f(x) \;=\; e^{-1}(0.4) + e^0(0.3) + e^1(0.2) + e^2(0.1) \;=\; 1.730. \quad \diamondsuit$$

13. Suppose that X is a continuous random variable with p.d.f. $f(x) = x^2$ for $-1 < x < c$.

(a) Find c.

Solution:

$$1 \;=\; \int_{-1}^{c} x^2\,dx \;=\; \frac{x^3}{3}\Big|_{-1}^{c} \;=\; \frac{c^3+1}{3},$$

which implies that $c = 2^{1/3}$. $\diamondsuit$

(b) Find the c.d.f. $F(x)$.

Solution:

$$F(x) \;=\; \int_{-1}^{x} t^2\,dt \;=\; \frac{x^3+1}{3}, \quad -1 < x < 2^{1/3}. \quad \diamondsuit$$

(c) Find $\mathsf{E}[X]$.

Solution:

$$\mathsf{E}[X] \;=\; \int_{-1}^{2^{1/3}} x^3\,dx \;=\; \frac{2^{4/3}-1}{4} \;=\; 0.3800. \quad \diamondsuit$$

(d) Find $\mathsf{Var}(X)$.

Solution:

$$\mathsf{E}[X^2] = \int_{-1}^{2^{1/3}} x^4\,dx = \frac{2^{5/3}+1}{5} = 0.8350.$$

Thus, $\mathsf{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2 = 0.6906$. ♢

14. PROVE or DISPROVE: If X is a continuous random variable that is always positive, then $\mathsf{E}[X] = \int_0^\infty \mathsf{P}(X > x)\,dx$.

Solution: This is TRUE, because

$$\begin{aligned}\int_0^\infty \mathsf{P}(X > x)\,dx &= \int_0^\infty \int_x^\infty f(t)\,dt\,dx \\ &= \int_0^\infty \int_0^t f(t)\,dx\,dt \quad \text{(interchange order of integration)} \\ &= \int_0^\infty f(t) \int_0^t dx\,dt \\ &= \int_0^\infty f(t)t\,dt \\ &= \mathsf{E}[X]. \quad \diamondsuit\end{aligned}$$

15. If $\mathsf{E}[X] = -3$ and $\mathsf{E}[X^2] = 11$, find $\mathsf{Var}(-2X - 7)$.

Solution: $\mathsf{Var}(-2X-7) = (-2)^2\mathsf{Var}(X) = 4\{\mathsf{E}[X^2] - (\mathsf{E}[X])^2\} = 4(11-9) = 8$. ♢

16. PROVE or DISPROVE: $\mathsf{E}[X^2] \ge (\mathsf{E}[X])^2$.

Solution: This is TRUE, since $0 \le \mathsf{Var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2$. ♢

17. Suppose that X is a continuous random variable with p.d.f.

$$f(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, \quad x > 0,$$

where α and β are positive constants, and $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt$ is the Gamma function (that you may remember from Calculus class). Find $\mathsf{E}[1/X]$.

Solution:

$$\begin{aligned}
\mathsf{E}[1/X] &= \int_0^\infty \frac{\beta^\alpha x^{\alpha-2}e^{-\beta x}}{\Gamma(\alpha)}\,dx \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\infty \beta^\alpha \left(\frac{t}{\beta}\right)^{\alpha-2} e^{-t}\,\frac{dt}{\beta} \quad \text{(take } t = \beta x\text{)} \\
&= \frac{\beta}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-2}e^{-t}\,dt \\
&= \frac{\beta\Gamma(\alpha-1)}{\Gamma(\alpha)} \quad \text{(for } \alpha > 2\text{)}.
\end{aligned}$$

You could probably stop at the above step, but it turns out that a nice property of the gamma function is that $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$. Thus, $\mathsf{E}[1/X] = \beta/(\alpha-1)$. $\diamondsuit$

18. Let X denote the lifetime of a light bulb, and suppose it has p.d.f. $f(x) = \lambda e^{-\lambda x}$, for $x > 0$, where λ is a positive constant.

    (a) Find $\mathsf{E}[X]$.

    Solution: Using integration by parts and then L'Hôspital's Rule, we have $\mathsf{E}[X] = \int_0^\infty \lambda x e^{-\lambda x}\,dx = 1/\lambda$. $\diamondsuit$

    (Of course, if you had recognized that $X \sim \text{Exp}(\lambda)$, then you're done immediately.)

    (b) Find the *median* of X, i.e., the point m such that $\mathsf{P}(X \le m) = \mathsf{P}(X > m) = 0.5$.

    Solution: The c.d.f. of the Exp(λ) is $F(x) = \int_0^x f(t)\,dt = 1 - e^{-\lambda x}$.

    Setting $0.5 = F(m) = 1 - e^{-\lambda m}$, we have $m = \frac{1}{\lambda}\ell\text{n}(2)$. $\diamondsuit$

    (c) Prove that $\mathsf{P}(X > s+t | X > s) = \mathsf{P}(X > t)$ for positive constants s and t.

Solution: This is called the *memoryless property* of the exponential distribution.

$$\begin{aligned}\mathsf{P}(X > s+t | X > s) &= \frac{\mathsf{P}(X > s+t \cap X > s)}{\mathsf{P}(X > s)} = \frac{\mathsf{P}(X > s+t)}{\mathsf{P}(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathsf{P}(X > t). \quad \diamondsuit\end{aligned}$$

Remarks: Thus, if the exponential survives at least s time units, then the probability that it survives at least an additional t units is the same as the unconditional probability that it survives at least t — it's just like the product is new! Moreover, it turns out that the exponential is the only continuous distribution with this property. We'll talk more about the exponential later in the course.

(d) Find $\mathsf{P}(X > \mathsf{E}[X])$, i.e., the probability that X survives longer than its mean life.

Solution:

$$\mathsf{P}(X > \mathsf{E}[X]) = 1 - F(\mathsf{E}[X]) = \exp(-\lambda \mathsf{E}[X]) = e^{-1} = 0.3679. \quad \diamondsuit$$

(e) Now find $\mathsf{P}(X > 2\mathsf{E}[X] \,|\, X > \mathsf{E}[X])$, i.e., the probability that X survives another mean life given than it made it past its mean life to begin with.

Solution: By the memoryless property and the previous problem,

$$\mathsf{P}(X > 2\mathsf{E}[X] \,|\, X > \mathsf{E}[X]) = \mathsf{P}(X > \mathsf{E}[X]) = e^{-1} = 0.3679. \quad \diamondsuit$$

(f) Find what is known as the *hazard rate*, $h(x) \equiv f(x)/\mathsf{P}(X > x)$, which is regarded as the instantaneous rate of death.

Solution:

$$h(x) = \frac{f(x)}{\mathsf{P}(X > x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda. \quad \diamondsuit$$

Remark: Cool — this is a constant hazard rate. The exponential is the only continuous distribution with this property.

19. Suppose that X has p.d.f. $f(x) = \frac{1}{\sqrt{2\pi}} \exp\{-x^2/2\}$, for all real x, i.e., the *standard normal* distribution. Show me how to use Excel (or any other reasonable software package) to find $\mathsf{P}(-1 < X < 1)$, $\mathsf{P}(-2 < X < 2)$, and $\mathsf{P}(-3 < X < 3)$.

Solution: if you use Excel, then

$$\mathsf{P}(-1 < X < 1) \;=\; \mathsf{NORM.S.DIST}(1,1) - \mathsf{NORM.S.DIST}(-1,1) \;=\; 0.6827.$$

Similarly,

$$\mathsf{P}(-2 < X < 2) \;=\; \mathsf{NORM.S.DIST}(2,1) - \mathsf{NORM.S.DIST}(-2,1) \;=\; 0.9545,$$

and

$$\mathsf{P}(-3 < X < 3) \;=\; \mathsf{NORM.S.DIST}(3,1) - \mathsf{NORM.S.DIST}(-3,1) \;=\; 0.9973. \quad \diamondsuit$$

20. I've had a little too much to drink and I'm having some trouble walking around. In fact, I tend to step to the left with probability 0.5, and to the right w.p. 0.5. All steps are independent. Suppose I take 6 steps. What's the probability that I'll be within 1 step (to the left or right) of my starting place?

Solution: Suppose I start at point 0, and let's say that I'm walking on the integers — a step to the right increases my position by 1, and a step to the left decreases my position by one. So if I take 4 steps to the right (and 2 to the left), I end up at position 2.

Let $X \sim \text{Binomial}(6, 1/2)$ denote the number of steps that I take to the right. Thus, $X = x$ means that I take x steps to the right and $6 - x$ to the left, so that I end up at position $2x - 6$. This means that the only possible positions after 6 steps are $-6, -4, -2, 0, 2, 4, 6$.

With all of this in mind, we have

$$\begin{aligned}
&\mathsf{P}(\text{I'll be within 1 step of position 0}) \\
&\quad = \mathsf{P}(\text{I'm at position 0 after the 6 steps}) \\
&\quad = \mathsf{P}(\text{I took 3 steps in each direction}) \\
&\quad = \binom{n}{k} p^k q^{n-k} \quad \text{(number of steps right is Bin(6,1/2))} \\
&\quad = \binom{6}{3} (1/2)^3 (1/2)^3 \;=\; 0.3125. \quad \diamondsuit
\end{aligned}$$