ISyE 6739 — Summer 2017

Homework #7 (Modules 3.3–3.4) — Solutions

1. (Hines et al., 7-26. CLT.) 100 small bolts are packed in a box. Each weighs an average of 1 ounce, with a standard deviation of 0.1 ounce. Find the probability that a box weighs more than 102 ounces.

   Solution: Let X_i be the weight of the ith bolt and let $Y = \sum_{i=1}^{100} X_i$ be the weight of the box. Note that $\mathsf{E}(X_i) = 1$, $\mathsf{Var}(X_i) = 0.01$, $i = 1, 2, \ldots, 100$.

   Assuming that the X_i's are independent, we use the central limit theorem to approximate the distribution of $Y \sim \text{Nor}(100, 1)$. Then

$$\mathsf{Pr}(Y > 102) \;=\; \mathsf{Pr}\Big(Z > \frac{102 - 100}{1}\Big) \;=\; 1 - \Phi(2) \;=\; 0.02275. \quad \diamondsuit$$

2. (Hines et al., 7–29(a). CLT.) A production process produces items, of which 8% are defective. A random sample of 200 items is selected every day and the number of defective items X is counted. Using the normal approximation to the binomial, find $\mathsf{Pr}(X \le 16)$.

   Solution: $p = 0.08$, $n = 200$, $np = 16$, $\sqrt{npq} = 3.84$. Let's incorporate the "continuity correction," and then the CLT:

$$\begin{aligned}
\mathsf{Pr}(X \le 16) &= \mathsf{Pr}(X \le 16.5) \\
&\approx \mathsf{Pr}\Big(Z \le \frac{16.5 - np}{\sqrt{npq}}\Big) \quad (\text{where } Z \sim \text{Nor}(0,1)) \\
&= \mathsf{Pr}\Big(Z \le \frac{16.5 - 16}{3.84}\Big) \\
&= \Phi(0.13) \;=\; 0.55172. \quad \diamondsuit
\end{aligned}$$

3. (Hines et al., 7–37. lognormal.) The random variable $Y = \ell\text{n}(X)$ has a $\text{Nor}(50, 25)$ distribution. Find the mean, variance, mode, and median of X.

   Solution: I got these answers by directly plugging into the equations from the book. For example, in general, $\mathsf{E}[X] = \exp(\mu + \sigma^2/2) = e^{62.5}$. And similarly, $\mathsf{Var}(X) = e^{125}(e^{25} - 1)$, $\text{median}(X) = e^{50}$, $\text{mode}(X) = e^{25}$. $\quad \diamondsuit$

4. Computer Exercises — Random Variate Generation

(a) Let's start out with something easy — the Uniform(0,1) distribution. To generate a Uniform(0,1) random variable in Excel, you simply type `= RAND()`. Copy an entire column of 100 of these guys and make a histogram. If things don't look particularly uniform, try the same exercise for 1000 observations. By the way, you can use the `<F9>` key to get an independent realization of your experiment.

(b) It's very easy to generate an Exponential(1) random variable in Excel. Just use

$$= -\texttt{LN(RAND())}$$

(This result uses the inverse transform method from Module 2.6.) Generate 1000 or so of these guys and make a nice histogram.

(c) In Excel, you can generate a Normal(0,1) random variable using

$$= \texttt{NORMINV(RAND(), 0, 1)} \qquad \text{(inverse transform method)}$$

or

$$= \texttt{SQRT}(-2 * \texttt{LN(RAND())}) * \texttt{COS}(2 * \texttt{PI()} * \texttt{RAND()}) \qquad \text{(Box–Muller method)}$$

Generate a bunch of normals using one of the above equations and make a histogram.

(d) Triangular distribution. Generate two columns of Uniform(0,1)'s. In the third column, add up the respective entries from the previous two columns, e.g., `C1 = A1 + B1`, etc. Make a histogram of the third column. Guess what you get?

Solution: You get a triangular p.d.f. Surprise! ♢

(e) Normal distribution from the Central Limit Theorem. Generate *twelve* columns of Uniform(0,1)'s. In the 13th column, add up the respective entries from the previous 12 columns. Make a histogram of the 13th column. Guess what you get this time?

Solution: You get what looks like a normal p.d.f. The CLT works! ♢

(f) Cauchy distribution. It turns out that you can generate a Cauchy random variable as the ratio of two i.i.d. Nor(0,1)'s. Make a histogram and comment. Does the CLT work for this distribution?

Solution: You get a mess that has extreme values. If you zoom in towards $x = 0$, it looks vaguely normal — but the tails are way too fat to actually be normal. If you try to apply the CLT, it fails — in fact, you get another Cauchy. The reason for the CLT failure is that the variance of the Cauchy is infinite, thus violating one of the CLT assumptions. $\diamondsuit$