3. Bivariate Random Variables

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Lesson 3.1 — Introduction

In this introductory lesson, we'll cover . . .

- What we mean by bivariate (or joint) random variables.
- The discrete case.
- The continuous case.
- Bivariate cdf's.

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Example: Choose a person at random. Look at their height and weight (X, Y). Obviously, X and Y will be related somehow.

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It's easy to see how this generalizes the 1-dimensional pdf, $f(x)$.

Example: Choose a point (X, Y) at random in the interior of the circle inscribed in the unit square, e.g., $C \equiv (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \leq \frac{1}{4}$.

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$$f(x,y) = \begin{cases} 4/\pi & \text{if } (x,y) \in C \\ 0 & \text{otherwise.} \quad \Box \end{cases}$$

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Moral: Be careful with limits! ☐

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$F(x,y)$ is continuous from the right in both x and y.

Example: Suppose

$$F(x,y) \;=\; \begin{cases} 1-e^{-x}-e^{-y}+e^{-(x+y)} & \text{if } x \geq 0,\, y \geq 0 \\ 0 & \text{if } x < 0 \text{ or } y < 0. \end{cases}$$

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$Y = 40$	0.01	0.07	0.12	0.2
$Y = 60$	0.29	0.03	0.48	0.8
$P(X = x)$	0.3	0.1	0.6	1

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Remark: Hmmm…. Compared to the last example, this has the *same marginals* but *different joint* distribution! That's because the joint distribution contains *much more information* than just the marginals.

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Note that $2/(1-x^4)$ is a *constant* with respect to y, and we can check to see that $f(y|x)$ is a legit conditional pdf:

$$\int_{\mathbb{R}} f(y|x)\,dy \;=\; \int_{x^2}^{1}\frac{2y}{1-x^4}\,dy \;=\; 1. \quad \Box$$

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$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx = \int_y^1 2\,dx = 2(1-y),\ 0 < y < 1. \quad \square$$

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Definition: X and Y are **independent** RVs if, for all x and y,

$$\boxed{f(x,y) \;=\; f_X(x)f_Y(y).}$$

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Similarly, X and Y independent implies $f(x|y) = f_X(x)$.

Example (discrete): $f(x,y) = P(X = x, Y = y)$.

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$f(x,y)$	$X = 1$	$X = 2$	$f_Y(y)$
$Y = 2$	0.12	0.28	0.4
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X and Y are independent since $f(x,y) = f_X(x)f_Y(y), \forall x, y$. $\square$

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Theorem: X and Y are independent iff $f(x, y) = a(x)b(y), \forall x, y$, for some functions $a(x)$ and $b(y)$ (not necessarily pdf's).

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Thus, X and Y are independent (as above). $\square$

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Now that we can figure out if X and Y are independent, what can we do with that knowledge?

Lesson 3.5 — Consequences of Independence

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Definition/Theorem (**two-dimensional Unconscious Statistician**):
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$$\boxed{\mathrm{E}[h(X,Y)] \;=\; \begin{cases} \sum_x \sum_y h(x,y) f(x,y) & \text{discrete} \\ \int_{\mathbb{R}} \int_{\mathbb{R}} h(x,y) f(x,y)\, dx\, dy & \text{continuous.} \end{cases}}$$

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Theorem: *Whether or not* X and Y are independent,

$$\boxed{\mathrm{E}[X+Y] \;=\; \mathrm{E}[X] + \mathrm{E}[Y].}$$

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One can generalize this result to more than two random variables.

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Corollary: If $X_1, X_2, \ldots, X_n$ are RVs, then

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Proof: Induction. □

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Remark: The above theorem is *not* necessarily true if X and Y are *dependent*. See the upcoming discussion on covariance.

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\end{aligned}$$

Remark: The assumption of independence really is important here. If X and Y aren't independent, then the result might not hold!

Can generalize. . .

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Corollary: If $X_1, X_2, \ldots, X_n$ are *independent* RVs, then

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Can generalize…

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Corollary: If $X_1, X_2, \ldots, X_n$ are *independent* RVs, then

$$\text{Var}\left(\sum_{i=1}^{n} a_i X_i + b\right) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i).$$

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So the mean of $\bar{X}$ is the same as the mean of X_i. $\square$

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So the mean of $\bar{X}$ is the same as the mean of X_i, but the *variance decreases*! This makes $\bar{X}$ a great *estimator* for μ (which is usually unknown in practice); the result is referred to as the **Law of Large Numbers**. Stay tuned.

Lesson 3.7 — Conditional Expectation

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The Next Few Lessons:

- Conditional expectation — definition and examples.
- “Double” expectation — a very cool theorem.
- Honors Class: First-step analysis.
- Honors Class: Random sums of random variables.
- Honors Class: The standard conditioning argument and its applications.

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Note that $\mathrm{E}[Y|X = x]$ is a function of x.

Discrete Example:

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$f(x,y)$	$X=0$	$X=3$	$f_Y(y)$
$Y=2$	0.11	**0.34**	0.45
$Y=5$	0.00	**0.05**	0.05
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This compares to the unconditional expectation $\mathrm{E}[Y] = 6.15$. So information that $X = 3$ pushes the conditional expected value of Y down to 5.05. □

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So, e.g., $\mathrm{E}[Y|X=0.5] = \frac{2}{3}\cdot\frac{63}{64}/\frac{15}{16} = 0.70.$ $\square$

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Believe it or not, sometimes it's easier to calculate $\mathrm{E}[Y]$ indirectly by using our double expectation trick.

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&= \mathrm{E}[Y]. \quad \Box
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By previous examples, we know that

$$\begin{aligned} f_X(x) &= \frac{21}{8}x^2(1-x^4), \quad \text{if } -1 \le x \le 1 \\ f_Y(y) &= \frac{7}{2}y^{5/2}, \quad \text{if } 0 \le y \le 1 \end{aligned}$$

Old Example: Suppose $f(x,y) = \frac{21}{4}x^2y$, if $x^2 \le y \le 1$.

Find $\mathrm{E}[Y]$ *two ways*.

By previous examples, we know that

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$$\mathrm{E}[Y] \;=\; \int_{\mathbb{R}} y f_Y(y)\,dy \;=\; \int_0^1 \frac{7}{2} y^{7/2}\,dy \;=\; \frac{7}{9}.$$

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$$\begin{aligned}
\mathrm{E}[Y] &= \mathrm{E}[\mathrm{E}(Y|X)] \\
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&= \int_{-1}^{1} \left(\frac{2}{3} \cdot \frac{1-x^6}{1-x^4}\right)\left(\frac{21}{8} x^2 (1-x^4)\right) dx
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Notice that both answers are the same (good)! ☐

Lesson 3.9 — Honors Class: First-Step Analysis

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Solving, we get $\text{E}[Y] = 1/p$ (which is the correct answer)! □

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In any case, it's obvious that A and B are iid Geom$(p = 1/2)$, so by the previous example, $\mathrm{E}[Y] = \mathrm{E}[A] + \mathrm{E}[B] = (1/p) + (1/p) = 4$. $\Box$

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This example didn’t involve first-step analysis (besides using the expected value of a geometric RV). But the next related example will....

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&= 1.5 + 0.75\,\mathrm{E}[Y].
\end{aligned}$$

Solving, we obtain $\mathrm{E}[Y] = 6$, which is perhaps surprising given the result from the previous example. $\square$

Lesson 3.10 — Honors Class: Random Sums of Random Variables

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Remark: You have to be very careful here. In particular, note that $\mathrm{E}\left[\sum_{i=1}^{N} X_i\right] \neq N\mathrm{E}[X_1]$, since the LHS is a number and the RHS is random.

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$$\mathrm{E}\left(\sum_{i=1}^{N} X_i\right) \;=\; \mathrm{E}\left[\mathrm{E}\left(\sum_{i=1}^{N} X_i \,\middle|\, N\right)\right]$$

Proof (cf. Ross): By double expectation,

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Theorem: Under the same conditions as before,

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Proof: See, for instance, Ross. ☐

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$$Y \;=\; 1_A \;\equiv\; \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathrm{E}[Y] \;=\; \sum_y y f_Y(y) \;=\; P(Y=1) \;=\; P(A).$$

Similarly, for any RV X, we have

$$\mathrm{E}[Y|X=x] \;=\; \sum_y y f(y|x) \;=\; P(Y=1|X=x) \;=\; P(A|X=x).$$

Lesson 3.11 — Honors Class: Standard Conditioning Argument

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These results suggest an alternative way of calculating $P(A)$....

Theorem: If X is a continuous RV (similar result if X is discrete), then

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Proof:

$$\begin{aligned} P(A) &= \mathrm{E}[Y] \quad \text{(where we take } Y = 1_A\text{)} \\ &= \mathrm{E}[\mathrm{E}(Y|X)] \quad \text{(double expectation)} \end{aligned}$$

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&= \mathrm{E}[\mathrm{E}(Y|X)] \quad \text{(double expectation)} \\
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Remark: We call this the **"standard conditioning argument."**

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Remark: We call this the **"standard conditioning argument."** Yes, it looks complicated. But sometimes you need to take a step backward to go two steps forward!

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Remark: Think of X as the time until the next male driver shows up at a parking lot (at rate α / hour) and Y as the time for the next female driver (at rate β / hour). Then $P(Y \le X) = \beta/(\alpha + \beta)$ is the intuitively reasonable probability that the next driver to arrive will be female. $\Box$

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This turns out to mean that $Z \sim$ **Gamma**$(2, \lambda)$, aka **Erlang**$_2(\lambda)$. □

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Example/Theorem: Suppose X and Y are two independent integer-valued RVs with pmf's $f_X(x)$ and $f_Y(y)$. Then the pmf of $Z = X + Y$ is

$$\boxed{f_Z(z) \;=\; P(Z=z) \;=\; \sum_{x=-\infty}^{\infty} f_X(x) f_Y(z-x).}$$

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Thus, $Z \sim \text{Bin}(2, p)$, a fond blast from the past! $\square$

Lesson 3.12 — Covariance and Correlation

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Definition: The **covariance** between X and Y is

$$\mathrm{Cov}(X,Y) \equiv \sigma_{XY} \equiv \mathrm{E}[(X-\mathrm{E}[X])(Y-\mathrm{E}[Y])].$$

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Remark: $\text{Cov}(X,X) = \text{E}[(X-\text{E}[X])^2] = \text{Var}(X)$.

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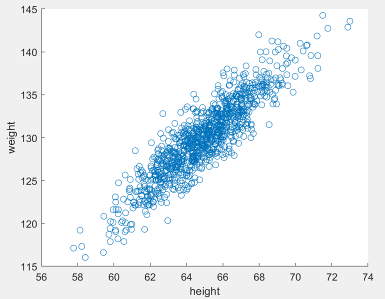

If X and Y have negative covariance, then X and Y move “in opposite directions.” Think snowfall and temperature.

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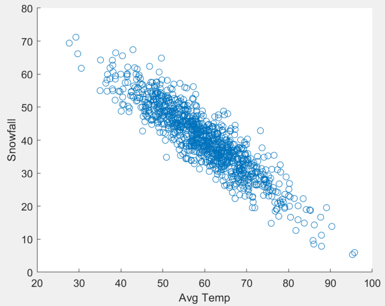

If X and Y are *independent*, then of course they have no association with each other. In fact, we'll prove below that independence implies that the covariance is 0 (but not the other way around).

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Example: IBM stock price vs. temperature on Mars are independent — at least that's what they want you to believe!

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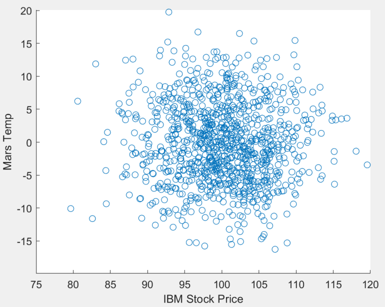

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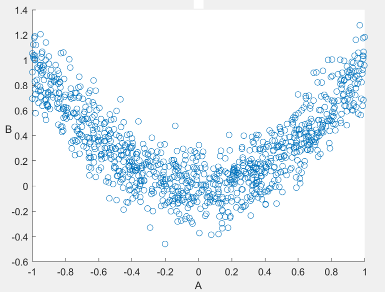

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Example of a zero correlation relationship with causality! We've seen that it's possible for two dependent RVs to be uncorrelated. □

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These items can be often be established via mathematical analysis, statistical analysis of appropriate data, or consultation with appropriate experts.

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$Y=40$	0.0	0.2	0.2	0.4
$Y=50$	0.1	0.1	0.0	0.2
$Y=60$	0.4	0.0	0.0	0.4
$f_X(x)$	0.5	0.3	0.2	1

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We'll spare the details, but here are the relevant calculations. . .

$$\mathrm{E}[X] \;=\; \sum_x x f_X(x) \;=\; 2.7,$$

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Lesson 3.15 — Some Useful Covariance / Correlation Theorems

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Theorem: $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)$, *whether or not* X and Y are independent.

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Theorem:

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) \;=\; \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2\sum\sum\nolimits_{i<j} \operatorname{Cov}(X_i, X_j).$$

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Corollary: If all X_i's are *independent*, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) \;=\; \sum_{i=1}^{n} \mathrm{Var}(X_i).$$

Theorem: $\text{Cov}(aX, bY + c) = ab\,\text{Cov}(X, Y)$.

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Proof: Put the above two results together. ☐

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$$\mathrm{E}[X^k] \;=\; \frac{d^k}{dt^k} M_X(t) \Big|_{t=0}, \quad k = 1, 2, \ldots.$$

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Old Theorem (identifying distributions): *In this class,* each distribution has a unique mgf.

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which implies that Y has the same distribution as $3X - 2$, where $X \sim \text{Bin}(15, 0.75)$. $\square$

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This is the mgf of the $\text{Bin}(\sum_{i=1}^{k} n_i, p)$, so we're done. $\square$

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Goal: Now let's give a *general result* on the distribution of functions of *two* random variables, the proof of which is beyond the scope of our class.

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Corollary: If X and Y are *independent*, then the joint pdf of V and W is

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Remark: Although the notation is nasty, the application isn't really so bad.

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This is the Gamma$(2,\lambda)$ pdf, which matches our answer from earlier in the current module. □

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Note that you have to be careful about the limits of v and w, but this thing really does double integrate to 1! □

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which is a little weird-looking and unexpected to me (it's flat for $w \le 1$, and then decreases to 0 pretty quickly for $w > 1$). $\Box$

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This is a **Triangle(0,1,2)** pdf. Can you see why? Is there an intuitive explanation for this pdf? $\square$

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