Some Almost-sure Convergence Properties Useful in Sequential Analysis

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Abstract: Kim and Nelson propose sequential procedures for selecting the simulated system with the largest steady-state mean from a set of alternatives that yield stationary output processes. Each procedure uses a triangular continuation region so that sampling stops when the relevant test statistic first reaches the region's boundary. In applying the generalized continuous mapping theorem to prove the asymptotic validity of these procedures as the indifference-zone parameter tends to zero, we are given (i) a sequence of functions on the unit interval (which are right-continuous with left-hand limits) converging to a realization of a certain Brownian motion process with drift; and (ii) a sequence of triangular continuation regions corresponding to the functions in sequence (i) and converging to the triangular continuation region for the Brownian motion process. From each function in sequence (i) and its corresponding continuation region in sequence (ii), we obtain the associated boundary-hitting point; and we prove that the resulting sequence of such points converges almost surely to the boundary-hitting point for the Brownian motion process. We also discuss the application of this result to a statistical process-control scheme for autocorrelated data and to other selection procedures for steady-state simulation experiments.

Keywords: Brownian motion with drift; Crossing problems; Sequential ranking-and-selection procedures; Steady-state computer simulation.

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1. INTRODUCTION

This article is a follow-up to the recent work of Kim and Nelson (2005) on sequential selection procedures for steady-state simulation experiments. The main result of Kim and Nelson (2005) is that their selection procedures $\mathcal{KN}+$ and $\mathcal{KN}{++}$ work asymptotically in the sense that those procedures deliver at least the prespecified probability of correct selection in all configurations of interest as the indifference-zone parameter

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δ goes to zero; and in such configurations, the largest mean response is at least the amount δ larger than all the other mean responses. In this article we provide a complete justification for the way that Kim and Nelson (2005) apply the generalized continuous mapping theorem in the proofs of their Theorems 1 and 2. We are given a sequence in $D[0, 1]$, the space of functions on [0, 1] that are right-continuous and have left-hand limits; see Section 14 of Billingsley (1968). We also have a sequence of triangular continuation regions that corresponds to the given sequence of functions and that converges to the triangular continuation region for the Brownian motion process. For each function and its corresponding triangular continuation region, we observe the associated point at which the function first reaches ("hits") the boundary of its continuation region; and we show that the resulting sequence of such points converges almost surely to the boundary-hitting point for the Brownian motion process and its triangular continuation region.

The results developed in this note can also be applied to the analysis of statistical process-control (SPC) schemes for autocorrelated data such as those proposed by Kim et al. (2005). In the design of an SPC scheme, an essential performance measure is the average run length (ARL), which is the expected number of measurements sampled until an out-of-control alarm is raised. An alarm occurs when a monitoring statistic exits its control limits. To determine the control limits for a monitoring scheme, first we determine the target value ARL_0, which represents the ARL before an out-of-control alarm is (incorrectly) raised when the process is actually in control. Then we set the control limits such that the actual in-control ARL is equal to the target value ARL_0. Determining the control limits for independent and identically distributed (i.i.d.) normal data is relatively easy, and this can be done analytically for some SPC charts such as the Shewhart chart.

Determining SPC control limits for autocorrelated data is more difficult. In this case the control limits are often determined under the assumption that the underlying process follows a specific probabilistic model—such as an AR(1) or ARMA(1) process—or by trial-and-error using simulation. Kim et al. (2005) develop a CUSUM-based sequential SPC procedure for autocorrelated data that has the following advantages. (i) It is model-free, meaning that it does not require any assumptions about the distribution of the underlying process. (ii) It uses raw measurements rather than batch means of those measurements so as to avoid delays in raising legitimate out-of-control alarms, where such delays might be caused by the use of a large batch size (see Runger and Willemain 1995). (iii) Its control limits are determined analytically rather than by simulation.

Kim et al. (2005) use the same type of triangular continuation regions as in Kim and Nelson (2005); and to justify their sequential SPC procedure, Kim et al. (2005) first show that an appropriately standardized continuous-time version of their CUSUM monitoring statistic converges weakly to a Brownian motion process as a parameter δ, which is similar to the indifference-zone parameter, tends to zero (so that the sample size tends to infinity). Then they use Proposition 3.2 of this note to show that given realizations of the CUSUM monitoring statistic for progressively increasing sample sizes with its successive continuous-time versions converging to a realization of the Brownian motion process as $\delta \to 0$, the corresponding sequence of boundary-hitting points for the continuous-time version of the CUSUM monitoring statistic converges with probability 1 to the boundary-hitting point for the Brownian motion process, when considering the distribution of probability over all possible realizations of the Brownian motion process. This development allows Kim et al. (2005) to compute approximate control limits for monitoring an autocorrelated process that will yield the target value ARL_0 when that process is in control.

Although in this note we consider only the triangular continuation regions of Kim and Nelson (2005) and Kim et al. (2005), our approach can accommodate other types of continuation regions that arise in other sequential-analysis procedures. For example, the selection procedures of Batur and Kim (2004) use parabolic continuation regions, but these procedures are limited to experiments in which i.i.d. normal data are generated by each alternative system to be compared. If the selection procedures of Batur and Kim are to be extended to steady-state simulation experiments (where data are generally neither i.i.d. nor normally distributed), then to justify the asymptotic validity of the extended selection procedures, we must formulate an appropriately standardized continuous-time version of the target output process that has the following properties: (i) it converges weakly to a Brownian motion process as the indifference-zone parameter tends to zero; and (ii) it has the same kind of almost-sure convergence property detailed above for the simulation selection procedures $\mathcal{KN}+$ and $\mathcal{KN}++$ of Kim and Nelson (2005) and for the SPC scheme of Kim et al. (2005). By a straightforward modification to handle continuation regions that are parabolic rather than triangular, Propositions 3.1 and 3.2 of this note can be used to provide a rigorous justification of the proposed extension of the selection procedures of Batur and Kim (2004) to apply to steady-state simulation experiments.

This note is organized as follows. In Section 2 we define the notation used throughout the discussion. Formal statements and proofs of the relevant almost-sure convergence properties are contained in Section 3 and Appendix A, respectively. Kim, Nelson, and Wilson (2005) provide a more detailed version of the development summarized here.

2. SETUP AND NOTATION

First we outline the setup of Kim and Nelson (2005) for formulating sequential selection procedures for steady-state simulation. We consider a continuous-time test statistic $\mathcal{G}(t, r)$ (for $t \in [0, 1]$ and $r = 1, 2, \ldots$) that belongs to $D[0, 1]$ with $\mathcal{G}(0, r) = 0$. Interest centers on the point at which $\mathcal{G}(\cdot, r)$ first reaches the boundary of a triangular continuation region whose upper and lower boundaries are defined, respectively, by the lines $U(t) = \mathcal{A} - \mathcal{B}t$ and $L(t) = -\mathcal{A} + \mathcal{B}t$ for $t \in [0, 1]$, where $\mathcal{A} > 0$, $\mathcal{B} > 0$, and $\mathcal{A}/\mathcal{B} \leq 1$.

Definition 2.1. For $Y \in D[0, 1]$, let $T_Y = T_Y(U) \equiv \inf\{t : |Y(t)| \geq U(t)\}$; and define the function $s : Y \in D[0, 1] \to s(Y) \in \mathbb{R}$ by $s(Y) \equiv Y\big(T_Y\big)$. Sometimes we write $T_Y(U)$ rather than T_Y to emphasize the dependence on U of the time T_Y at which the process Y first reaches the boundary defined by U.

Given a monotonically decreasing sequence $\big\{\delta_n : n = 1, 2, \ldots\big\}$ of positive numbers tending to zero, we define quantities $\mathcal{A}\big(\delta_n\big)$ and $\mathcal{B}\big(\delta_n\big)$ that satisfy $\lim_{n\to\infty} \mathcal{A}\big(\delta_n\big) = \mathcal{A}$ and $\lim_{n\to\infty} \mathcal{B}\big(\delta_n\big) = \mathcal{B}$ and that specify triangular continuation regions whose upper and lower boundaries are given, respectively, by the lines $U_n(t) = \mathcal{A}\big(\delta_n\big) - \mathcal{B}\big(\delta_n\big)t$ and $L_n(t) = -\mathcal{A}\big(\delta_n\big) + \mathcal{B}\big(\delta_n\big)t$ for $t \in [0, 1]$ and $n = 1, 2, \ldots$.

Definition 2.2. For $Y \in D[0, 1]$, let $T_Y\big(U_n\big) \equiv \inf\big\{t : |Y(t)| \geq U_n(t)\big\}$; and define the function $s_n : Y \in D[0, 1] \to s_n(Y) \in \mathbb{R}$ by $s_n(Y) \equiv Y\big[T_Y\big(U_n\big)\big]$.

Definition 2.3. (Billingsley 1968) Let Λ denote the class of strictly increasing, continuous mappings of $[0, 1]$ onto itself such that for every $\lambda \in \Lambda$, we have $\lambda(0) = 0$ and $\lambda(1) = 1$. If $X, Y \in D[0, 1]$, then the Skorohod metric $\rho(X, Y)$ is the infimum of those positive ω for which there exists $\lambda \in \Lambda$ such that $\sup_{t\in[0,1]} |\lambda(t) - t| \leq \omega$ and $\sup_{t\in[0,1]} |X(t) - Y[\lambda(t)]| \leq \omega$.

Definition 2.4. Let $\mathbb{D}_s \equiv \Big\{x \in D[0, 1] :$ for some sequence $\big\{x_n\big\} \subset D[0, 1]$ with $\lim_{n\to\infty} \rho\big(x_n, x\big) = 0$, the sequence $\big\{s_n\big(x_n\big)\big\}$ does not converge to $s(x)\Big\}$.

Definition 2.5. Let $\mathcal{W}_\Delta(\cdot)$ denote a Brownian motion process on $[0, \infty)$ with drift parameter $\Delta > 0$ so that $\mathrm{E}\big[\mathcal{W}_\Delta(t)\big] = \Delta t$ and $\mathrm{Var}\big[\mathcal{W}_\Delta(t)\big] = t$ for all $t \geq 0$.

Remark 2.1. In proving Theorems 1 and 2 of Kim and Nelson (2005), the authors show that under broadly applicable assumptions on the underlying simulation-generated output process, the corresponding standardized continuous-time process $\mathcal{G}(\cdot, r)$ has asymptotic behavior described by the functional central limit theorem (FCLT) $\mathcal{G}(\cdot, r) \Longrightarrow \mathcal{W}_\Delta(\cdot)$ as $r \to \infty$, where $\Longrightarrow$ denotes weak convergence. A key step in their proofs is to use the FCLT to show that $s_r[\mathcal{G}(\cdot, r)] \Longrightarrow s\big[\mathcal{W}_\Delta(\cdot)\big]$ as $r \to \infty$. To invoke the generalized continuous mapping theorem—that is, Theorem 5.5 of Billingsley (1968)—as justification for the desired conclusion, we will prove that the event $\mathbb{D}_s$ has Wiener measure zero. In Remark 3.1 and Proposition 3.2 of the next section, we particularize the development to the specific result required to complete the proofs of Theorems 1 and 2 of Kim and Nelson (2005).

3. ALMOST-SURE CONVERGENCE PROPERTIES

Proposition 3.1. *If* $s(\cdot)$ *and* $s_n(\cdot)$ *are given by Definitions 2.1 and 2.2, respectively, and the event* $\mathbb{D}_s$ *is given by Definition 2.4, then* $\Pr\{\mathcal{W}_\Delta \in D[0,1] - \mathbb{D}_s\} = 1$.

Appendix A contains the lengthy proof of Proposition 3.1, but the approach is summarized as follows. Just after the time $T_{\mathcal{W}_\Delta} = T_{\mathcal{W}_\Delta}(U)$ that $\mathcal{W}_\Delta(\cdot)$ first hits the boundary U (say), $\mathcal{W}_\Delta(\cdot)$ crosses that boundary infinitely often as characterized by the local version of the law of the iterated logarithm (see Equations (A.1) and (A.2) below). Thus if $x_n(\cdot)$ and $U_n(\cdot)$ are sufficiently close to $\mathcal{W}_\Delta(\cdot)$ and $U(\cdot)$, respectively, in the Skorohod metric on $D[0,1]$, then the time $T_{x_n} = T_{x_n}(U_n)$ that $x_n(\cdot)$ first hits the boundary U_n must also be close to $T_{\mathcal{W}_\Delta} = T_{\mathcal{W}_\Delta}(U)$ (see Equations (A.3), (A.4), and (A.6) below). But if $x_n(\cdot)$ and $\mathcal{W}_\Delta(\cdot)$ are close to each other in $D[0,1]$, then the function values $x_n(T_{x_n})$ and $\mathcal{W}_\Delta(T_{\mathcal{W}_\Delta})$ at their respective boundary-hitting times T_{x_n} and $T_{\mathcal{W}_\Delta}$ must also be close to each other (see Equation (A.5) below). This is precisely the behavior sought for almost all realizations of $\mathcal{W}_\Delta$—that is, with probability 1.

Remark 3.1. In the proofs of their Theorems 1 and 2, Kim and Nelson (2005) consider two systems, indexed by k and i, whose respective steady-state mean responses μ_k and μ_i satisfy $\mu_k \geq \mu_i + \delta_n$. Let X_{kj} and X_{ij} respectively denote the jth observations sampled from systems k and i for $j = 1, 2, \ldots, r$. The difference process $\{Z_{ki}(j) = X_{kj} - X_{ij} : j = 1, 2, \ldots, r\}$ has steady-state mean $\mu_k - \mu_i$ and variance parameter $v_{ik}^2 \equiv \lim_{r\to\infty} r\mathrm{Var}[\bar{Z}_{ki}(r)]$, where $\bar{Z}_{ki}(r)$ is the sample mean of $\{Z_{ki}(j) : j = 1, 2, \ldots, r\}$. Corresponding to the indifference-zone parameter δ_n, Kim and Nelson (2005) formulate a maximum sample size $N_{ik}(\delta_n)$ to be taken from both systems such that

$$\lim_{n\to\infty} \frac{(N_{ik}(\delta_n) + 1)\delta_n}{v_{ik}\sqrt{N_{ik}(\delta_n) + 1}} = \Delta, \tag{3.1}$$

the drift parameter specified in Definition 2.5. Thus in their setup for using the FCLT stated in Remark 2.1, Kim and Nelson (2005) replace $\mathcal{G}(t, r)$ with the following specific expression,

$$\mathcal{G}_{ki}\big(t, \delta_n\big) = \frac{\Big\{\sum_{j=1}^{\lfloor (N_{ik}(\delta_n)+1)t \rfloor} Z_{ki}(j)\Big\} - (N_{ik}(\delta_n)+1)(\mu_k - \mu_i)t}{v_{ik}\sqrt{N_{ik}(\delta_n)+1}} + \frac{(N_{ik}(\delta_n)+1)\delta_n t}{v_{ik}\sqrt{N_{ik}(\delta_n)+1}} \tag{3.2}$$

for $t \in [0, 1]$ and $n = 1, 2, \ldots$; and they prove that

$$\mathcal{G}_{ki}\big(\cdot, \delta_n\big) \underset{n\to\infty}{\Longrightarrow} \mathcal{W}_{\Delta}(\cdot). \tag{3.3}$$

From (3.1)–(3.3) we prove the asymptotic result required for Theorems 1 and 2 of Kim and Nelson (2005).

Proposition 3.2. *If difference process* $\big\{Z_{ki}(j) = X_{kj} - X_{ij} : j = 1, 2, \ldots\big\}$ *has variance parameter* $v_{ik}^2 \in (0, \infty)$ *and if (3.1), (3.2), and (3.3) hold, then* $s_n\big[\mathcal{G}_{ki}\big(\cdot, \delta_n\big)\big] \underset{n\to\infty}{\Longrightarrow} s\big[\mathcal{W}_{\Delta}(\cdot)\big]$.

Proof. The result follows from (3.3), Proposition 3.1, and the generalized continuous mapping theorem. ∎

APPENDIX A: PROOF OF PROPOSITION 3.1

We will show that $\Pr\big\{\mathcal{W}_{\Delta} \in D[0, 1] - \mathbb{D}_s \,\big|\, \mathcal{W}_{\Delta}\big(T_{\mathcal{W}_{\Delta}}\big) = \mathcal{A} - \mathcal{B}T_{\mathcal{W}_{\Delta}}\big\} = 1$; a parallel analysis establishes a similar result conditional on $\big\{\mathcal{W}_{\Delta}\big(T_{\mathcal{W}_{\Delta}}\big) = -\mathcal{A} + \mathcal{B}T_{\mathcal{W}_{\Delta}}\big\}$ so that the desired conclusion follows. Conditional on the event $\big\{\mathcal{W}_{\Delta}\big(T_{\mathcal{W}_{\Delta}}\big) = \mathcal{A} - \mathcal{B}T_{\mathcal{W}_{\Delta}}\big\}$, we exploit three key properties of $\mathcal{W}_{\Delta}(\cdot)$ that each hold almost surely: (i) the continuity of sample paths of $\mathcal{W}_{\Delta}(\cdot)$ on [0, 1]; (ii) the local version of the law of the iterated logarithm for Brownian motion; and (iii) the inequality $0 < T_{\mathcal{W}_{\Delta}} < 1$, where $T_{\mathcal{W}_{\Delta}}$ is a stopping time for $\mathcal{W}_{\Delta}$ (see Fabian 1974). Because $\mathcal{W}_{\Delta}$ has independent increments, it follows from (ii) and (iii) that

$$\text{With probability 1, we have } \limsup_{u\to 0+} \frac{\big[\mathcal{W}_{\Delta}\big(T_{\mathcal{W}_{\Delta}} + u\big) - \Delta u\big] - \mathcal{W}_{\Delta}\big(T_{\mathcal{W}_{\Delta}}\big)}{\sqrt{2u \ln[\ln(1/u)]}} = 1. \tag{A.1}$$

Thus we may restrict attention to an event $\mathcal{H} \subset D[0, 1]$ such that $\mathcal{H} \subset \big\{\mathcal{W}_{\Delta} \in D[0, 1] : \text{(i)– (iii) hold}\big\}$ and $\Pr\big\{\mathcal{W}_{\Delta} \in \mathcal{H} \,\big|\, \mathcal{W}_{\Delta}\big(T_{\mathcal{W}_{\Delta}}\big) = \mathcal{A} - \mathcal{B}T_{\mathcal{W}_{\Delta}}\big\} = 1$.

Choose $\mathcal{W}_{\Delta} \in \mathcal{H}$ arbitrarily. If $\big\{x_n\big\} \subset D[0, 1]$ converges to $\mathcal{W}_{\Delta}$, then we will prove that $\big\{s_n\big(x_n\big)\big\}$ converges to $s\big(\mathcal{W}_{\Delta}\big)$ so that the conclusion of Proposition 3.1 follows immediately. In terms of the coefficients $\mathcal{A}_n^{\pm} \equiv \mathcal{A} \pm \big|\mathcal{A} - \mathcal{A}\big(\delta_n\big)\big|$ and $\mathcal{B}_n^{\pm} \equiv \mathcal{B} \mp \big|\mathcal{B} - \mathcal{B}\big(\delta_n\big)\big|$ for $n = 1, 2, \ldots$, we define the boundaries of an enclosing envelope, $U_n^{\pm}(t) \equiv \mathcal{A}_n^{\pm} - \mathcal{B}_n^{\pm} t$ for all $t \geq 0$ and for $n = 1, 2, \ldots$. We may assume that $\big|\mathcal{A} - \mathcal{A}\big(\delta_n\big)\big| < \mathcal{A}$ and $\big|\mathcal{B} - \mathcal{B}\big(\delta_n\big)\big| < \mathcal{B}$ so that $\mathcal{A}_n^{\pm} > 0$ and $\mathcal{B}_n^{\pm} > 0$ for $n = 1, 2, \ldots$. With this setup, we see that $U_n^{-}(t) \leq U_n(t),\ U(t) \leq U_n^{+}(t)$ for all $t \geq 0$ and for $n = 1, 2, \ldots$.

For any sequence $\{x_n\}$ converging to $\mathcal{W}_\Delta$ in $D[0,1]$, Definition 2.3 implies there exists $\lambda_n \in \Lambda$ such that $\sup_{t\in[0,1]} |\lambda_n(t) - t| \le \rho(x_n, \mathcal{W}_\Delta) + n^{-1}$ and $\sup_{t\in[0,1]} |x_n(t) - \mathcal{W}_\Delta[\lambda_n(t)]| \le \rho(x_n, \mathcal{W}_\Delta) + n^{-1}$ for $n = 1, 2, \ldots$. Taking $g_n \equiv \sup_{t\in[0,1]} |\mathcal{W}_\Delta(t) - \mathcal{W}_\Delta[\lambda_n(t)]|$ for $n = 1, 2, \ldots$, we see from the uniform continuity of $\mathcal{W}_\Delta(t)$ on $[0,1]$ and the definition of the $\{g_n\}$ that $\lim_{n\to\infty} g_n = 0$. Moreover, if we take $\varepsilon_n \equiv 3n^{-1} + 3\sup\{\rho(x_\ell, \mathcal{W}_\Delta) + g_\ell : \ell = n, n+1, \ldots\}$ for $n = 1, 2, \ldots$, then we see that $\{\varepsilon_n\}$ is a monotonically decreasing sequence of positive numbers with limit zero.

We take $T_{\mathcal{W}_\Delta \pm \varepsilon_n}(U_n^{\pm})$ as in Definitions 2.1 and 2.2. Because $\varepsilon_n > 0$ for $n = 1, 2, \ldots$, we have $T_{\mathcal{W}_\Delta + \varepsilon_n}(U_n^-) \le T_{\mathcal{W}_\Delta}(U_n^-) \le T_{\mathcal{W}_\Delta}(U_n)$, $T_{\mathcal{W}_\Delta}(U) \le T_{\mathcal{W}_\Delta}(U_n^+) \le T_{\mathcal{W}_\Delta - \varepsilon_n}(U_n^+)$ for $n = 1, 2, \ldots$; and thus we have $t_* \equiv \limsup_{n\to\infty} T_{\mathcal{W}_\Delta + \varepsilon_n}(U_n^-) \le T_{\mathcal{W}_\Delta}(U)$ and $t^* \equiv \liminf_{n\to\infty} T_{\mathcal{W}_\Delta - \varepsilon_n}(U_n^+) \ge T_{\mathcal{W}_\Delta}(U)$.

Next we must prove that $\lim_{n\to\infty} T_{\mathcal{W}_\Delta + \varepsilon_n}(U_n^-) = t_* = T_{\mathcal{W}_\Delta}(U) = t^* = \lim_{n\to\infty} T_{\mathcal{W}_\Delta - \varepsilon_n}(U_n^+)$. First we show that $t^* = T_{\mathcal{W}_\Delta}(U)$. In view of (A.1), for any $\xi \in (0,1)$, there exists a monotonically decreasing sequence $\{t_k : k = 1, 2, \ldots\} \subset (T_{\mathcal{W}_\Delta}, 1)$ such that $\lim_{k\to\infty} t_k = T_{\mathcal{W}_\Delta}$; and for $k = 1, 2, \ldots$, we have

$$\mathcal{W}_\Delta(t_k) - \Delta(t_k - T_{\mathcal{W}_\Delta}) - \mathcal{W}_\Delta(T_{\mathcal{W}_\Delta}) > \vartheta_k \equiv \xi\sqrt{2(t_k - T_{\mathcal{W}_\Delta})\ln\{\ln[1/(t_k - T_{\mathcal{W}_\Delta})]\}}. \tag{A.2}$$

Notice that $\vartheta_k > 0$ for $k = 1, 2, \ldots$, and $\lim_{k\to\infty} \vartheta_k = 0$. Pick k arbitrarily. Since $t_k > T_{\mathcal{W}_\Delta}$, we have $\mathcal{W}_\Delta(T_{\mathcal{W}_\Delta}) = \mathcal{A} - \mathcal{B}T_{\mathcal{W}_\Delta} > \mathcal{A} - \mathcal{B}t_k = U(t_k)$; and thus by the basic properties of the $\{\mathcal{A}_n^{\pm}\}$, $\{U_n(\cdot)\}$, and $\{U_n^{\pm}(\cdot)\}$, there exists a positive integer N_1 such that $U(t_k) \le U_n^+(t_k) < \mathcal{W}_\Delta(T_{\mathcal{W}_\Delta})$ and $\varepsilon_n < \vartheta_k$ for every $n \ge N_1$. If $n \ge N_1$, then $\mathcal{W}_\Delta(t_k) - \varepsilon_n > [\mathcal{W}_\Delta(T_{\mathcal{W}_\Delta}) + \Delta(t_k - T_{\mathcal{W}_\Delta}) + \vartheta_k] - \varepsilon_n > \mathcal{W}_\Delta(T_{\mathcal{W}_\Delta}) > U_n^+(t_k)$; and from the definition of $T_{\mathcal{W}_\Delta - \varepsilon_n}(U_n^+)$, it follows immediately that $T_{\mathcal{W}_\Delta - \varepsilon_n}(U_n^+) \le t_k$ for every $n \ge N_1$. Thus we see that $T_{\mathcal{W}_\Delta}(U) \le t^* = \liminf_{n\to\infty} T_{\mathcal{W}_\Delta - \varepsilon_n}(U_n^+) \le \limsup_{n\to\infty} T_{\mathcal{W}_\Delta - \varepsilon_n}(U_n^+) \le t_k$. Because k is arbitrary and $\lim_{k\to\infty} t_k = T_{\mathcal{W}_\Delta} = T_{\mathcal{W}_\Delta}(U)$, we have

$$\lim_{n\to\infty} T_{\mathcal{W}_\Delta - \varepsilon_n}(U_n^+) = t^* = T_{\mathcal{W}_\Delta}(U). \tag{A.3}$$

See Equations (43)–(50) of Kim, Nelson, and Wilson (2005) for a simple proof of the complementary result

$$\lim_{n\to\infty} T_{\mathcal{W}_\Delta + \varepsilon_n}(U_n^-) = t_* = T_{\mathcal{W}_\Delta}(U). \tag{A.4}$$

From the definition of ε_n we have $\rho(x_n, \mathcal{W}_\Delta) < \varepsilon_n/2$ and $g_n < \varepsilon_n/2$ for $n = 1, 2, \ldots$. This latter result, the triangle inequality, Definition 2.3, and the definition of g_n ensure that for every $t \in [0,1]$, we have $|x_n(t) - \mathcal{W}_\Delta(t)| \le |x_n(t) - \mathcal{W}_\Delta[\lambda_n(t)]| + |\mathcal{W}_\Delta[\lambda_n(t)] - \mathcal{W}_\Delta(t)| < \varepsilon_n/2 + \varepsilon_n/2 = \varepsilon_n$ for $n = 1, 2, \ldots$, so that we have

$$\mathcal{W}_{\vartriangle}(t) - \varepsilon_n < x_n(t), \mathcal{W}_{\vartriangle}(t) < \mathcal{W}_{\vartriangle}(t) + \varepsilon_n \text{ for all } t \in [0, 1] \text{ and for } n = 1, 2, \ldots . \tag{A.5}$$

From (A.5) together with the definitions of $T_{\mathcal{W}_{\vartriangle}+\varepsilon_n}\big(U_n^-\big)$, $T_{\mathcal{W}_{\vartriangle}-\varepsilon_n}\big(U_n^+\big)$, $T_{x_n}\big(U_n^{\pm}\big)$ and $T_{x_n}\big(U_n\big)$, we have

$$T_{\mathcal{W}_{\vartriangle}+\varepsilon_n}\big(U_n^-\big) \le T_{x_n}\big(U_n^-\big) \le T_{x_n}\big(U_n\big) \le T_{x_n}\big(U_n^+\big) \le T_{\mathcal{W}_{\vartriangle}-\varepsilon_n}\big(U_n^+\big) \text{ for } n = 1, 2, \ldots . \tag{A.6}$$

Finally from (A.3), (A.4), and (A.6), we have $\lim_{n\to\infty} T_{x_n}\big(U_n\big) = T_{\mathcal{W}_{\vartriangle}}(U)$. Taking $t = T_{x_n}\big(U_n\big)$ in (A.5), we have $\mathcal{W}_{\vartriangle}\big[T_{x_n}(U_n)\big] - \varepsilon_n \le x_n\big[T_{x_n}(U_n)\big] = s_n\big(x_n\big) \le \mathcal{W}_{\vartriangle}\big[T_{x_n}(U_n)\big] + \varepsilon_n$ for $n = 1, 2, \ldots$; thus by the continuity of $\mathcal{W}_{\vartriangle}(\cdot)$ we have $\lim_{n\to\infty} s_n\big(x_n\big) = \lim_{n\to\infty} \mathcal{W}_{\vartriangle}\big[T_{x_n}(U_n)\big] = \mathcal{W}_{\vartriangle}\big[T_{\mathcal{W}_{\vartriangle}}(U)\big] = s(\mathcal{W}_{\vartriangle})$. ∎

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