Capital Asset Pricing Model

1 Introduction

In this handout we develop a model that can be used to determine how an investor can choose an optimal asset portfolio in this sense:

- the investor will earn the highest possible expected return given the level of volatility the investor is willing to accept or, equivalently,
- the investor’s portfolio will have the lowest level of volatility given the level of expected return the investor requires.

The techniques used are called mean-variance optimization and the underlying theory is called the Capital Asset Pricing Model (CAPM). Under the assumptions of CAPM, it is possible to determine the expected “risk-adjusted” return of any asset/security, which incorporates the security’s expected return, volatility and its correlation with the “market portfolio.”

2 Mean-Variance Efficient Frontier

2.1 Setup

We consider a market with \( n \) risky assets, \( i = 1, 2, \ldots, n \) and a risk-free asset labeled 0. An investor wishes to invest \( B \) dollars in this market. Let

- \( B_i, i = 0, 1, 2, \ldots, n \), denote the allocation of the budget to asset \( i \) so that \( \sum_{i=0}^{n} B_i = B \),
- \( x_i := B_i/B \) denote the portfolio weight of asset \( i \), namely, the fraction of the investor’s budget allocated to asset \( i \),
- \( R_i \) denote the random one-period return on asset \( i, i = 1, 2, \ldots, n \), and let
- \( r_f \) denote the risk-free return.

Example 1 Consider a portfolio of 200 shares of firm A worth $30/share and 100 shares of firm B worth $40/share. The total value of the portfolio is

\[
200 \times (30) + 100(40) = $10,000. \tag{1}
\]

The respective portfolio weights are

\[
x_A = \frac{200 \times (30)}{10,000} = 60\%, \quad x_B = \frac{100 \times (40)}{10,000} = 40\%. \tag{2}
\]
2.2 Portfolio return

The investor’s one-period return on his/her portfolio is given by

\[
\text{One-period return} = \frac{\sum_{i=0}^{n} B_i R_i}{B} = \sum_{i=0}^{n} \frac{B_i}{B} R_i = \sum_{i=0}^{n} x_i R_i. \tag{3}
\]

We shall think of \(B\) as fixed and hereafter identify a portfolio of the \(n + 1\) assets with a vector

\[
x = (x_0, x_1, x_2, \ldots, x_n) \text{ such that } \sum_{i=0}^{n} x_i = 1. \tag{4}
\]

The portfolio’s random return will be denoted by

\[
R_P = R(x) := \sum_{i=1}^{n} x_i R_i. \tag{5}
\]

We shall use the symbols ‘\(x\)’ or \(P\) to refer to a portfolio.

**Remark 1** When \(x_i < 0\) the holder of the portfolio is short-selling asset \(i\). In these notes we permit *unlimited* short-selling. In practice, however, there are limits to the magnitude of short-selling. If short-selling is not permitted, i.e., each \(x_i\) is constrained to be non-negative, then the solution approach outlined in these notes does not directly apply, though the problem can be easily solved with commercial software.

**Example 2** You bought the portfolio of Example 1. Suppose firm A’s share price goes up to $36 and firm B’s share price falls to $38. What is the new value of the portfolio? What return did it earn? After the price change, what are the new portfolio weights? The new value of the portfolio is

\[
200 \times \$36 + 100 \times \$38 = \$11,000 \tag{6}
\]

for a gain of $1,000 or 10% return on investment. A’s return was \(36/30 - 1 = 20\%\) and B’s return was \(38/40 - 1 = -5\%\). Since the initial portfolio weights were \(x_A = 60\%\) and \(x_B = 40\%\), we can also compute the portfolio’s return as

\[
R_P = x_A R_A + x_B R_B = 0.60(20\%) + 0.40(-5\%) = 10\%. \tag{7}
\]

The new portfolio weights are

\[
x_A = \frac{200 \times \$36}{\$11,000} = 65.45\%, \quad x_B = \frac{100 \times \$38}{\$11,000} = 34.55\%. \tag{8}
\]
2.3 Expected portfolio return

The expected return of a portfolio \( P \) is given by

\[
\mu(x) := E[R_P] = E[\sum_i x_i R_i] = \sum_i E[x_i R_i] = \sum_i x_i E[R_i].
\]  

(9)

**Example 3** You invest $1000 in stock A, $3000 in stock B. You expect a return of 10% for stock A and 18% in stock B. What is your portfolio’s expected return? Your portfolio weights are \( x_A = \frac{1000}{4000} = 25\% \) and \( x_B = \frac{3000}{4000} = 75\% \). Therefore,

\[
E[R_P] = 0.25(10\%) + 0.75(18\%) = 16\%.
\]  

(10)

2.4 Portfolio optimization via expected utility

The investor’s end-of-period wealth is \( B(1 + R(x)) \). We assume the investor is risk-averse and that his/her utility function \( U(\cdot) \) is sufficiently differentiable and strictly concave. The investor’s optimal portfolio \( x^* \) should maximize his/her expected utility of end-of-period wealth, as follows:

Investor’s problem: \( \max_{x: \sum_i x_i = 1} E[U(1 + R(x))] \),

(11)

where, for convenience, we have normalized the budget to one.

The investor’s problem is a non-trivial one. One difficulty is to obtain the distribution of \( R(x) \), which depends on \( x \). In lieu of solving the investor’s problem directly, we shall approximate the expected utility via a second-order Taylor series expansion about mean end-of-period wealth given by \( 1 + E[R(x)] := 1 + \mu(x) \).

For each realization of the random variable \( R(x) \), and ignoring higher-order terms,

\[
U(1 + R(x)) \approx U(1 + \mu(x)) + [(1 + R(x)) - (1 + \mu(x))]U'(1 + \mu(x)) + \frac{1}{2}[(1 + R(x)) - (1 + \mu(x))]^2 U''(1 + \mu(x)).
\]  

(12)

Taking expectations of both sides of (12),

\[
E[U(1 + R(x))] \approx E[U(1 + \mu(x))] + U'(1 + \mu(x))E[R(x) - \mu(x)] + \frac{1}{2} U''(1 + \mu(x))E[(R(x) - \mu(x))^2].
\]  

(13)

Using the fact that

\[
E[R(x) - \mu(x)] = E[R(x)] - \mu(x) = 0,
\]  

(14)

\[
E[R(x) - \mu(x)]^2 = Var[R(x)],
\]  

(15)

we have

\[
E[U(1 + R(x))] \approx U(1 + \mu(x)) + \frac{1}{2} U''(1 + \mu(x)) Var[R(x)].
\]  

(16)
Assuming that the approximation of expected utility given in (16) is reasonably accurate, maximizing the left-hand side of (16) is (essentially) equivalent to maximizing the right-hand side of (16).

Now fix the value for mean return $\mu(x)$ say to $\bar{R}_0$; that is, consider only those portfolios $x$ for which $\mu(x) = \bar{R}_0$. In words, the investor is fixing his/her desired expected portfolio return. Since the sign of $U''(\cdot)$ is negative, and since $U(1 + \mu(x))$ and $U''(1 + \mu(x))$ are constants, it follows from the right-hand side of (16) that the investor’s optimal portfolio must

- must minimize the variance of the portfolio’s return subject to achieving the required expected return or, equivalently,
- maximize the portfolio’s expected return subject to fixing the portfolio’s variance to an acceptable level.

Such a portfolio is called mean-variance efficient. The set of all such mean-variance efficient portfolios is called the mean-variance efficient frontier.

3 Portfolio Variance

In this section we only consider the risky assets $i = 1, 2, \ldots, n$. Keep in mind each $R_i$ is a random variable.

3.1 Basic definitions and properties

We recall some basic definitions and properties from statistics. Let

\[ \bar{R}_i := E[R_i] \] \hspace{1cm} (17)
\[ \text{Var}[R_i] := E[(R_i - \bar{R}_i)^2] = E[R_i^2] - (\bar{R}_i)^2 \] \hspace{1cm} (18)
\[ \sigma_i := \sqrt{\text{Var}[R_i]} \] \hspace{1cm} (19)

denote, respectively, the mean, variance and standard deviation of $R_i$.

Recall that the covariance between two random variables $X$ and $Y$ is defined as

\[ \text{Cov}(X,Y) = \sigma_{XY} := E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y], \] \hspace{1cm} (20)

and the correlation between two random variables $X$ and $Y$ is defined as

\[ \text{Corr}(X,Y) = \rho_{XY} := \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} \] \hspace{1cm} (21)
Recall that a correlation value must always be a number between -1 and 1. Note that covariance can be determined from correlation via

\[ \sigma_{XY} = \text{Cov}(X, Y) = \text{Corr}(X, Y)\sigma_X\sigma_Y = \rho_{XY}\sigma_X\sigma_Y. \]  

(22)

It follows directly from (20) that

• variance may be expressed in terms of covariance:

\[ \text{Var}(X) = \text{Cov}(X, X), \]  

(23)

• covariance is symmetric:

\[ \text{Cov}(X, Y) = \text{Cov}(Y, X), \]  

(24)

• covariance is bilinear:

\[ \text{Cov}\left(\sum_{i=1}^{n} x_i, Y\right) = \sum_{i=1}^{n} \text{Cov}(X_i, Y), \]  

(25)

\[ \text{Cov}(\alpha X, Y) = \text{Cov}(X, \alpha Y) = \alpha \text{Cov}(X, Y), \quad \alpha \text{ a real number}. \]  

(26)

### 3.2 Variance formula

In what follows, we let

\[ \sigma_{ij} := \text{Cov}(R_i, R_j) = \sigma_i\sigma_j\text{Corr}(R_i, R_j) = \sigma_i\sigma_j\rho_{ij} \]  

(27)

and let \( \Sigma \) denote the \( n \) by \( n \) matrix whose \((i,j)\)th entry is given by \( \sigma_{ij} \). The matrix \( \Sigma \) is called the covariance matrix. Due to (24), \( \sigma_{ij} = \sigma_{ji} \), which implies that the covariance matrix is symmetric, namely, \( \Sigma = \Sigma^T \) (here, the symbol ‘T’ denotes transpose).

Using (23) and repeated application of (25) and (26), the portfolio’s (return) variance is

\[ \text{Var}[R(x)] = \text{Var}\left[\sum_{i=1}^{n} x_i R_i\right] \]  

(28)

\[ = \text{Cov}\left(\sum_{i=1}^{n} x_i R_i, \sum_{j=1}^{n} x_j R_j\right) \]  

(29)

\[ = \sum_{i} x_i \text{Cov}(R_i, \sum_{j} x_j R_j) \]  

(30)

\[ = \sum_{i} x_i \left(\sum_{j} x_j \text{Cov}(R_i, R_j)\right) \]  

(31)
\[
\begin{align*}
&= x^T \Sigma x \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j.
\end{align*}
\]

### 3.3 Volatility of a two-asset portfolio

When \( n = 2 \) the portfolio’s variance is

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_{ij} x_i x_j = \sigma_1^2 x_1^2 + \sigma_{12} x_1 x_2 + \sigma_{21} x_2 x_1 + \sigma_2^2 x_2^2
\]

\[
= \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\sigma_{12} x_1 x_2
\]

\[
= \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\sigma_1\sigma_2\rho_{12} x_1 x_2
\]

Here, we have used the fact that the covariances between the two assets, namely, \( \sigma_{12} \) and \( \sigma_{21} \), are equal.

**Remark 2** In matrix notation, the portfolio’s variance may be represented as

\[
\begin{align*}
\text{Var}[R(x)] &= x^T \Sigma x \\
&= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 x_1 + \sigma_{12} x_2 \\ \sigma_{21} x_1 + \sigma_2^2 x_2 \end{pmatrix} \\
&= (x_1\sigma_1^2 x_1 + x_1\sigma_{12} x_2) + (x_2\sigma_{21} x_1 + \sigma_2^2 x_2) \\
&= \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\sigma_{12} x_1 x_2.
\end{align*}
\]

**Example 4** Stock returns will be more highly correlated when they are similarly affected by the same economic events. This is why stocks in the same industry tend to have highly correlated returns than stocks in somewhat different industries. Table 1 (Table 11.3, p. 336 in Corporate Finance by Berk and DeMarzo) provides some examples.

What is the covariance between the returns for Microsoft and Dell?

\[
\text{Cov}(R_M, R_D) = \sigma_m \sigma_D \rho_{MD} = (0.37)(0.50)(0.62) = 0.1147.
\]

What is the standard deviation of a portfolio with equal amounts invested in these two stocks?

\[
\text{Var}(R_P) = \text{Var}(0.5R_m + 0.5R_D)
\]

\[
= \sigma_m^2 x_M^2 + \sigma_D^2 x_D^2 + 2\sigma_{MD} x_M x_D
\]

\[
= (0.37)^2(0.5)^2 + (0.5)^2(0.5)^2 + 2(0.1147) = 0.1541
\]

\[
\sigma_P = \sqrt{0.1541} = 39.26%.
\]
Table 1: Historical Annual Volatilities and Correlations for Selected Stocks (based on monthly returns, 1996-2008).

<table>
<thead>
<tr>
<th></th>
<th>Microsoft</th>
<th>Dell</th>
<th>Alaska</th>
<th>Southwest Airlines</th>
<th>Ford Motor</th>
<th>General Motors</th>
<th>General Mills</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility (StDev)</td>
<td>37%</td>
<td>50%</td>
<td>38%</td>
<td>31%</td>
<td>42%</td>
<td>41%</td>
<td>18%</td>
</tr>
<tr>
<td>Correlation with:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Microsoft</td>
<td>1.00</td>
<td>0.62</td>
<td>0.25</td>
<td>0.23</td>
<td>0.26</td>
<td>0.23</td>
<td>0.10</td>
</tr>
<tr>
<td>Dell</td>
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<td>1.00</td>
<td>0.19</td>
<td>0.21</td>
<td>0.31</td>
<td>0.28</td>
<td>0.07</td>
</tr>
<tr>
<td>Alaska Air</td>
<td>0.25</td>
<td>0.19</td>
<td>1.00</td>
<td>0.30</td>
<td>0.16</td>
<td>0.13</td>
<td>0.11</td>
</tr>
<tr>
<td>Southwest Airlines</td>
<td>0.23</td>
<td>0.21</td>
<td>0.30</td>
<td>1.00</td>
<td>0.25</td>
<td>0.22</td>
<td>0.20</td>
</tr>
<tr>
<td>Ford Motor</td>
<td>0.26</td>
<td>0.31</td>
<td>0.16</td>
<td>0.25</td>
<td>1.00</td>
<td>0.62</td>
<td>0.07</td>
</tr>
<tr>
<td>General Motors</td>
<td>0.23</td>
<td>0.28</td>
<td>0.13</td>
<td>0.22</td>
<td>0.62</td>
<td>1.00</td>
<td>0.02</td>
</tr>
<tr>
<td>General Mills</td>
<td>0.10</td>
<td>0.07</td>
<td>0.11</td>
<td>0.20</td>
<td>0.07</td>
<td>0.02</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Example 5 Consider a portfolio of Intel and Coca-Cola stocks. Data shown in Table 2. Let $x$ denote the weight on Intel and $1 - x$ denote the weight on Coca-Cola. As $x$ varies from 0 to 1, the portfolio’s mean return $E[R(x)]$ and standard deviation $StDev(R(x)) = \sqrt{Var(R(x))}$ will vary. We can plot the points $\{(StDev(R(x)), E[R(x)])\}$ in the “(x,y) space”. This collection of points traces out a curve. The shape of the curve critically depends on the correlation $\rho$ between these two stocks. In fact, we can trace out a collection of such curves, one for each possible value of $\rho \in [-1, 1]$.

What is the shape of the curve when $\rho = +1$? $\rho = -1$? How about values for $\rho$ in between? The expected return

$$E[R(x)] = 0.26x + 0.06(1 - x) = 0.06 + 0.20x$$

(48)

is linear in $x$ and does not depend on $\rho$. As for the variance:

$$\begin{align*}
\text{General } \rho : \quad Var(R(x)) &= (0.5)^2 x^2 + (0.25)^2 (1-x)^2 + 2x(1-x)(0.5)(0.25)\rho \\
\rho = +1 : \quad Var(R(x)) &= (0.5)^2 x^2 + (0.25)^2 (1-x)^2 + 2x(1-x)(0.5)(0.25) \\
&= (0.5x + 0.25(1-x))^2
\end{align*}$$

(49)
\[
\text{StDev}(R(x)) = (0.25 + 0.25x)^2 \quad (52)
\]
\[
\rho = -1: \quad \text{Var}(R(x)) = (0.5)^2x^2 + (0.25)^2(1-x)^2 - 2x(1-x)(0.5)(0.25) \quad (53)
\]
\[
= (0.5x - 0.25(1-x))^2 \quad (54)
\]
\[
= (0.75x - 0.25)^2 \quad (55)
\]
\[
\text{StDev}(R(x)) = |0.75x - 0.25|, \quad x \in [0,1]. \quad (56)
\]

Note how the variance is a quadratic function of \( x \). Note further than when \( \rho = +1 \), the \text{StDev} is linear in \( x \), whereas when \( \rho = -1 \), the \text{StDev} is “piecewise linear” and will actually be zero when \( x = 1/3 \! \! \! \! . \)

**Example 6** Consider the data of Example 5. Assume \( \rho = 0 \). Suppose you have $20,000 in cash to invest. You decide to short sell $10,000 worth of Coco-Cola stock and invest the proceeds from your short sale plus your $20,000 in Intel. What is the expected return and volatility of your portfolio?

Your short sale is a negative investment of $10,000 in Coca-Cola stock. You have invested $30,000 in Intel. Your total investment is still $20,000. Your portfolio weights are 1.5 or 150% in Intel and -0.5 or -50% in Coca-Cola. Thus, the expected portfolio return is

\[
1.5(0.26) + -0.5(0.06) = 36% \quad (58)
\]

and its variance is

\[
0.50^2(1.5)^2 + 0.25^2(-0.5)^2 = 0.578125 \quad (59)
\]

for a StDev of 76%. By short-selling you have dramatically increased the expected return but you have also significantly increased the volatility.

### 3.4 Minimum variance portfolio of two stocks

By definition, the **minimum variance portfolio** achieves the lowest variance, regardless of expected return. For two stocks, such as examined in Example 5, it is easy to solve for the minimum variance portfolio. (We shall see that it is also easy to solve for the minimum variance portfolio for the general case.) From the two-stock portfolio variance formula (36) with \( x_2 = 1 - x_1 \), the variance is

\[
\text{Var}(x_1R_1 + (1-x_1)R_2) = \sigma_1^2x_1^2 + \sigma_2^2(1-x_1)^2 + 2x_1(1-x_1)\sigma_{12}. \quad (60)
\]

Since there are no constraints on the values for \( x \), we can take the derivative with respect to \( x \) and set equal to zero:

\[
0 := \frac{d}{dx_1}\text{Var}(R(x)) = 2\sigma_1^2x_1 - 2(1-x_1)\sigma_2^2 - 2\sigma_{12}(1-2x_1) \quad (61)
\]

\[
= 2(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12})x_1 - (\sigma_2^2 - \sigma_{12}). \quad (62)
\]
Thus,

\[
x_1^* = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \quad (63)
\]

\[
x_2^* = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \quad (64)
\]

**Remark 3** When the stock returns are uncorrelated, the covariance \(\sigma_{12}\) equals zero. Then, the optimal portfolio is

\[
(x_1^*, x_2^*) = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right) \quad (65)
\]

**Example 7** Let’s return to the data of Example 5. Assume the correlation coefficient is 0.20. Then the covariance \(\sigma_{12} = (0.50)(0.25)(0.2) = 0.025\). To minimize the variance, the optimal weight for the Intel stock is

\[
\frac{(0.25)^2 - 0.025}{(0.50)^2 + (0.25)^2 - 2(0.025)} = \frac{0.0375}{0.2625} = 1/7, \quad (66)
\]

and

\[
\begin{align*}
Var(R(x^*)) &= (0.50)^2(1/7)^2 + (0.25)^2(6/7)^2 + 2(1/7)(6/7)(0.025) = 0.05714 \quad (67) \\
StDev(R(x^*)) &= \sqrt{0.05714} = 23.9\% \quad (68)
\end{align*}
\]

Note how the minimum volatility is less than the volatility of either stock!

### 3.5 Volatility of a large portfolio: the benefits of diversification

Let \(AvgVar\) denote the average variance of the individual stocks, and let \(AvgCov\) denote the average covariance between the stocks. Imagine a world where the variance of each stock was constant and equal to \(AvgVar\) and the covariance between stocks was constant and equal to \(AvgCov\). Now consider an equally-weighted portfolio of these stocks in which \(x_i = 1/n\) for each \(i = 1, 2, \ldots, n\). What is the variance of this equally-weighted portfolio? The covariance matrix \(\Sigma\) has \(n^2\) entries, \(n\) along the diagonal and \(n^2 - n = n(n - 1)\) off-diagonal entries. The entries along the diagonal correspond to each stock’s variance which equals \(AvgVar\), whereas the off-diagonal entries correspond to the covariances which each equal \(AvgCov\). Therefore, in this special setting

\[
Var(R_P) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \quad (69)
\]
\[
\begin{align*}
\sigma^2_P &= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \\
&= \frac{1}{n^2} \left\{ n \cdot \text{AvgVar} + n(n-1) \cdot \text{AvgCov} \right\} \\
&= \frac{\text{AvgVar}}{n} + \left( 1 - \frac{1}{n} \right) \cdot \text{AvgCov}.
\end{align*}
\]

We can see that \( \text{Var}(R_P) \rightarrow \text{AvgCov} \) as \( n \rightarrow \infty \). The limiting portfolio standard deviation \( \text{StDev}(R_P) = \sqrt{\text{AvgCov}} \).

The historical volatility (standard deviation) of the return of a large stock is about 40\% and the typical correlation between the returns of large stocks is about 28\%. On average, then,

\[
\begin{align*}
\text{AvgVar} &= (0.40)^2 = 0.16, \\
\text{AvgCov} &= (0.40)(0.40)(0.28) = 0.0448, \\
\text{StDev}(R_P) &= \sqrt{0.16/n + 0.0448(1 - 1/n)} \longrightarrow 21.17\%.
\end{align*}
\]

Note that when \( n = 20 \) the \( \text{StDev}(R_P) = 22.49\% \), which is very close to the limiting volatility.

What about a portfolio with arbitrary weights? We have:

\[
\begin{align*}
\sigma_P^2 &= \text{Var}(R_P) = \text{Cov}(R_P, R_P) \\
&= \text{Cov} \left( \sum_{i} x_i R_i, R_P \right) \\
&= \sum_{i} x_i \text{Cov}(R_i, R_P) \\
&= \sum_{i} x_i \sigma_i \sigma_p \text{Corr}(R_i, R_P).
\end{align*}
\]

Dividing both sides of this equation by \( \sigma_P \) yields this very important decomposition of the volatility of a portfolio:

\[
\text{StDev}(R_P) = \sum_{i} \frac{x_i}{\text{Amount of i held}} \cdot \frac{\sigma_i}{\text{Total risk of i}} \cdot \frac{\text{Corr}(R_i, R_P)}{\text{Fraction of i’s risk common to P}}
\]

**Remark 4** Assuming that not all stocks have a perfect positive correlation of +1, we can conclude that when the weights are all positive, the volatility of the portfolio will be lower than the weighted average volatility given by \( \sum_{i} x_i \sigma_i \). Note how the expected return of a portfolio \( \sum_{i} x_i E[R_i] \) is the weighted average of the individual returns. So, you can eliminate some of the volatility by diversifying.
3.6 Minimum variance portfolio of risky assets

The core optimization problem is formulated as follows:

\[ \mathcal{P} : \min \ Var[R(x)] = \frac{1}{2} x^T \Sigma x \quad (81) \]

subject to:

\[ x_1 + x_2 + \cdots + x_n = 1 \quad (82) \]

In vector form, equation (82) may be expressed as

\[ e^T x = 1 \quad (83) \]

where \( e^T := (1,1,\ldots,1) \) is the \( n \)-vector whose components are all ones.

**Remark 5** The objective function is a (convex) quadratic function of the vector \( x \) and the one constraint is linear in \( x \). So Problem \( \mathcal{P} \) is called a quadratic programming problem, a special case of convex optimization problems. Quadratic programming problems are easy to solve using commercial software.

Define

\[ L(x, \lambda) := \frac{1}{2} x^T \Sigma x - \lambda(e^T x - 1). \quad (84) \]

Here, \( \lambda \) is a real number. The function \( L(\cdot) \) is called the Lagrangian for this problem.

**Example 8** When \( n = 2 \) the core optimization problem is given by:

\[ \mathcal{P} : \quad \min \left\{ \frac{1}{2} (\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\sigma_{12} x_1 x_2) \right\} \quad (85) \]

subject to:

\[ x_1 + x_2 = 1. \quad (86) \]

The associated Lagrangian is given by:

\[ L(x_1, x_2, \lambda) := \frac{1}{2} \left\{ \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\sigma_{12} x_1 x_2 \right\} - \lambda(x_1 + x_2 - 1). \quad (87) \]

Now consider the following unconstrained problem:

\[ \mathcal{L}(\lambda) : \quad V(\lambda) := \min_x L(x, \lambda). \quad (88) \]

Let \( x(\lambda) \) denote an optimal solution to problem \( \mathcal{L}(\lambda) \). The following is a fundamental theorem of convex analysis:
Theorem 1

(1) If $x^*$ is an optimal solution to $\mathcal{P}$, then there exists a $\lambda$ such that $x^* = x(\lambda)$.

(2) If $x(\lambda)$ satisfies the portfolio constraint (82), then $x(\lambda)$ is an optimal solution to $\mathcal{P}$.

Remark 6 The variable $\lambda$ is called a dual variable. As in linear programming, there is one dual variable for each constraint. Note that there is only one dual variable here regardless of the number of $x$ variables. It turns out that primal problem $\mathcal{P}$ has a dual problem given by

$$D : \max_{\lambda} \mathcal{V}(\lambda),$$

such that the optimal values of the primal and dual problems are equal. The dual problem is a concave optimization problem. Essentially, this duality extends the well-known duality of linear programming.

We can use Theorem 1 to obtain a closed-form solution to our original problem $\mathcal{P}$. Here is the “game plan.”

**Step 1.** Let $x^*$ be an optimal solution to $\mathcal{P}$ and pick a $\lambda$ such that $x^* = x(\lambda)$. (Theorem 1(a) guarantees that this is possible.) Since $x(\lambda)$ is an optimal solution to (88), an unconstrained problem, we know from calculus that all $n$ partial derivatives of $L(\cdot, \lambda)$ evaluated at $x(\lambda)$ must equal zero. These $n$ equations in $n$ unknowns may be solved to determine $x(\lambda)$.

**Step 2.** It remains to pin down the value for $\lambda$, which requires one additional equation. Theorem 1(b) provides it, namely, the portfolio constraint (83).

We illustrate this game plan in the special case that $n = 2$. When presented in matrix form, the equations we derive all hold true for the general case.

First, the partial derivatives must vanish, i.e.,

$$\frac{\partial L}{\partial x_1} = \sigma_1^2 x_1 + \sigma_{12} x_2 - \lambda = 0. \quad (90)$$

$$\frac{\partial L}{\partial x_2} = \sigma_2^2 x_2 + \sigma_{12} x_1 - \lambda = 0. \quad (91)$$

(To simplify notation we suppress the dependence of $x$ on $\lambda$.) In matrix form equations (90)-(91) may be represented as

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (92)$$

or equivalently as

$$\Sigma x = \lambda e. \quad (93)$$
These equations — keep in mind that there are \( n \) of them! — are called the **first-order optimality conditions**. Now by multiplying both sides of (93) by \( \Sigma^{-1} \) we solve for the optimal \( x = x(\lambda) \) as:

\[
x(\lambda) = \begin{pmatrix} x_1(\lambda) \\ x_2(\lambda) \end{pmatrix} = \lambda \begin{pmatrix} \Sigma^{-1}e \end{pmatrix}.
\]  

(94)

The vector \( x(\lambda) \) is **proportional** to the vector \( \Sigma^{-1}e \), with \( \lambda \) being the proportionality constant. Since we know there is some \( \lambda \) for which \( x^* = x(\lambda) \), the optimal solution to \( P \) is also proportional to the vector \( \Sigma^{-1}e \). Since the \( \sum_i x_i^* = 1 \), the proportionality constant is simply the reciprocal of the sum of the components of \( \Sigma^{-1}e \), i.e., to arrive at the optimal solution \( x^* \) from \( \Sigma^{-1}e \) all one has to do is simply “normalize” the latter vector by its sum. For example, suppose the vector \( \Sigma^{-1}e = (10, 20, 30) \). The sum is 60 and so the unique vector that is proportional to \( (10, 20, 30) \) and which sums to one is the vector \( (1/6, 1/3, 1/2) \). Finally, we observe that the multiplication of \( \Sigma^{-1} \) by the vector \( e \) simply computes the vector of row sums of the matrix \( \Sigma^{-1} \).

**Example 9** Let’s return to the data of Examples 5 and 7. Recall that we found that the minimum variance portfolio was \( (1/7, 6/7) \). In this example,

\[
\Sigma = \begin{pmatrix} 0.2500 & 0.0250 \\ 0.0250 & 0.0625 \end{pmatrix},
\]  

(95)

and

\[
\Sigma^{-1} = \frac{1}{(0.25)(0.0625) - (0.025)^2} \begin{pmatrix} 0.0625 & -0.0250 \\ -0.0250 & 0.2500 \end{pmatrix},
\]  

(96)

\[
= \frac{1}{0.015} \begin{pmatrix} 0.0625 & -0.0250 \\ -0.0250 & 0.2500 \end{pmatrix},
\]  

(97)

\[
= \begin{pmatrix} 4.16 & -1.6 \\ -1.6 & 16.6 \end{pmatrix}.
\]  

(98)

The vector of row sums associated with \( \Sigma^{-1} \) is \( (2.5, 15) \). The sum of these two numbers 17.5. Thus, the optimal portfolio vector is \( (2.5/17.5, 15/17.5) = (1/7, 6/7) \)!  

**Remark 7** It is not difficult to show that

\[
\lambda = \frac{1}{e^T \Sigma^{-1} e}.
\]  

(99)

In words, the value of \( \lambda \) can be obtained by merely summing up the values for all elements of the inverse of the covariance matrix, and taking the reciprocal. With a little more work, you can show that \( \lambda \) actually equals the minimum variance! Returning to the example above, the sum of the elements of \( \Sigma^{-1} \) equals 17.5. The minimum variance equals 1/17.5 and the minimum standard deviation equals \( \sqrt{1/17.5} = 23.9\% \), as before!
Remark 8 If you are just interested in the minimum variance portfolio, you do not need to calculate the determinant here as it is just a proportionality constant. (You have to normalize the vector $\Sigma^{-1}e$ anyway, so the constant $1/0.015$ is unnecessary.)

Example 10 Consider three risky assets whose covariance matrix $\Sigma$ is

$$
\Sigma = \begin{pmatrix}
216 & 70 & -324 \\
70 & 25 & -150 \\
-324 & -150 & 1596 \\
\end{pmatrix}.
$$

(100)

(Here, we measured returns in percents.) The inverse of the covariance matrix is

$$
\Sigma^{-1} = \begin{pmatrix}
0.1480 & -0.5367 & -0.0204 \\
-0.5367 & 2.0388 & 0.0827 \\
-0.0204 & 0.0827 & 0.0043 \\
\end{pmatrix}.
$$

(101)

Thus,

$$
\Sigma^{-1}e = \begin{pmatrix}
-0.4081 \\
1.5971 \\
0.0661 \\
\end{pmatrix} \\
$$

(102)

$$
e^T \Sigma^{-1}e = -0.4081 + 1.5971 + 0.0661 = 1.2379.
$$

(103)

The sum of all entries of $\Sigma^{-1}$ is 1.2379 and this value equals the reciprocal of the minimum variance. Dividing the vector $(-0.4081, 1.5971, 0.0661)$ by 1.2379 yields $(-0.3297, 1.2756, 0.0534)$, and this is the minimum variance portfolio. Its standard deviation is $\sqrt{1/1.2379} = 80.78\%$.

4 Computing the Tangent Portfolio

In this section we assume the market includes a risk-free security that returns $r_f$ in all states. The tangent (or market) portfolio solves the following optimization problem:

$$(TP) : \text{Max} \left\{ \frac{\bar{R}^T x - r_f}{\sqrt{x^T \Sigma x}} : e^T x = 1 \right\},$$

(104)

where $r_f$ denotes the risk-free rate. The objective function is called the Sharpe ratio.

It can be shown that all mean-variance efficient portfolios must be a combination of the risk-free security and the tangent portfolio. That is, each investor should allocate a portion of their budget to the risk-free security and the remaining portion to the tangent portfolio. The sub-allocations to the individual risky securities are determined by the weights $x^*$ that define the tangent portfolio (to be determined). A very risk-averse investor allocates very little to the
tangent portfolio, whereas an investor with a large appetite for risk may hold a negative portion
in the risk-free security (i.e. borrow at the risk-free rate) to invest more than the budget into
the tangent portfolio.

We now turn to explaining how to calculate the tangent portfolio. Define

\[ \hat{R} := \bar{R} - r_f, \tag{105} \]

which is the vector of excess returns obtained by simply subtracting the risk-free rate from each
of the asset’s expected returns. Problem \((TP)\) may be expressed as:

\[ (TP): \quad \text{Max} \left\{ \frac{\hat{R}^T x}{\sqrt{x^T \Sigma x}} : e^T x = 1 \right\}, \tag{106} \]

Note that if one doubles the vector \(x\) the objective function in \((TP)\) remains the same. (The
objective function is said to be “homogeneous of degree 0”.) Thus, we can ignore the constraint
that \(e^T x = 1\), because at the end we can perform the necessary normalization. (This assumes
that the sum of the \(x_i\) is positive.)

Now imagine we knew the expected return \(\bar{R}_0\) of the tangent portfolio. Let \(\hat{R}_0 := \bar{R}_0 - r_f\)
denote the excess expected return of the tangent portfolio. The tangent portfolio would then
have to satisfy the following mean-variance optimization problem:

\[ \text{Min} \left\{ \frac{1}{2} x^T \Sigma x : \hat{R}^T x = \hat{R}_0 \right\}. \tag{107} \]

Inspect this optimization problem closely. It is the same as our original optimization problem \(P\) except
that the constraint now has a vector \(\hat{R}\) that replaces the vector \(e\) and a value \(\hat{R}_0\) that
replaces the 1. From a structural standpoint, however, nothing has fundamentally changed,
other than the “names to protect the innocent.”

Let \(\mu\) denote the dual variable for this problem. The tangent portfolio necessarily satisfies
the first-order optimality conditions

\[ \Sigma x^* = \mu \hat{R}. \tag{108} \]

Consequently, the tangent portfolio must be proportional to \(\Sigma^{-1} \hat{R}\), and thus it may be found
in the analogous way we found the minimum variance portfolio.

**Example 11** Consider again our three asset data of the previous example. Suppose the ex-
pected returns are \((\bar{R}_1, \bar{R}_2, \bar{R}_3) = (18\%, 10\%, 8\%)\), and suppose the risk-free rate is 3\%.
The excess expected returns are \((18\% - 3\%, 10\% - 3\%, 8\% - 3\%) = (15\%, 7\%, 5\%)\). The tangent
portfolio is proportional to

\[ \Sigma^{-1} \begin{pmatrix} 15 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} -1.6389 \\ 6.6346 \\ 0.2944 \end{pmatrix}. \tag{109} \]
Thus the tangent portfolio vector is \((-0.3098, 1.2542, 0.0557)\). (Divide the right-hand side of (109) by 5.2901 = \(-1.6389 + 6.6346 + 0.2944\).) Its expected return is 7.4112 and its variance equals 0.8338.

Remark 9 You can verify that the variance of the tangent portfolio equals \(\mu \hat{R}_0\) and \(\mu = (e^T \Sigma^{-1} \hat{R})^{-1}\). In the example, \(\hat{R}_0 = 7.4112 - 3 = 4.112\) and \(\mu = 1/5.2901\) and so the variance equals \(4.112/5.2901 = 0.8338\).

5 Security Market Line (SML)

The first-order optimality conditions (108) imply a relationship between a security’s expected return and a measure of its risk. This relationship is called the Security Market Line (SML) and is the most widely used approach to determine an appropriate expected return for a security (or asset). The SML can also be used to decide whether an asset should be added to an existing portfolio and, if so, how to determine its weight in the new portfolio.

5.1 Derivation of SML from the first-order optimality conditions

Let
\[ x_M = (x_{M1}, x_{M2}, \ldots, x_{Mn}) \]
denote the tangent portfolio (of risky assets), and let
\[ \hat{R}_M = \sum_i x_{iM} \hat{R}_i = \hat{R}^T x_M \]
denote the expected return on the tangent portfolio. We will derive the SML from the first-order optimality conditions (108).

We begin by noting that the \(i^{th}\) equation of (108) in non-matrix notation is simply
\[ \sum_j \sigma_{ij} x_{Mj} = \mu(\hat{R}_i - r_f). \] (110)

Since \(\sigma_{ij} = \text{Cov}(R_i, R_j)\), it follows from the properties of covariance that
\[ \mu(\hat{R}_i - r_f) = \sum_j \text{Cov}(R_i, R_j)x_{Mj} = \text{Cov}(R_i, \sum_j R_jx_{Mj}) = \text{Cov}(R_i, R_M). \] (111)

Consequently,
\[ \hat{R}_i = r_f + \frac{1}{\mu} \text{Cov}(R_i, R_M). \] (112)
It remains to pin down the value of $\mu$. Substitute $x_M$ into (108) and multiply both sides by $x_M$ to obtain that

$$x_M^T \Sigma x_M = Var(R_M) = \mu(x_M^T \hat{R}). \quad (113)$$

Now we use the fact that

$$x_M^T \hat{R} = x_M^T (\bar{R} - r_f e) = x_M^T \bar{R} - r_f (x_M^T e) = \bar{R}_M - r_f \quad (114)$$

to derive that

$$Var(R_M) = \mu(\bar{R}_M - r_f) \Rightarrow \frac{1}{\mu} = \frac{\bar{R}_M - r_f}{Var(R_M)}. \quad (115)$$

From (112) and (115), we conclude that

$$\bar{R}_i = r_f + \frac{Cov(R_i, R_M)}{Var(R_M)} (\bar{R}_M - r_f). \quad (116)$$

Define the parameter

$$\beta_i := \frac{Cov(R_i, R_M)}{Var(R_M)} = \frac{\rho_{iM} \sigma_i \sigma_M}{\sigma^2_M} = \frac{\rho_{iM} \sigma_i}{\sigma_M}. \quad (117)$$

It is called the security’s beta. Using the parameter $\beta_i$, we can rewrite (116) as

$$\bar{R}_i = r_f + \beta_i (\bar{R}_M - r_f) \quad \text{Security Market Line (SML)} \quad (118)$$

The term $\bar{R}_M - r_f$ is called the Market Premium. So, the Security Market Line states that

Security’s expected return = the risk-free rate + (security’s beta) * (Market Premium), \quad (119)

or, equivalently,

$$\hat{R}_i = \beta_i \hat{R}_M \quad \text{(security’s beta)} \ast \hat{R}_M \quad (120)$$

In the portfolio context it is the security’s $\beta$ that is considered an appropriate measure of a security’s “risk.”

**Remark 10** Observe that there is a linear relationship between a security’s expected return and its risk (as measured by its $\beta$). This relationship is what economists call an equilibrium relationship in the following sense: If a security’s expected return was higher (lower) than what is predicted by the SML, then there would be more (less) demand for it, thereby raising (lowering) its price and ultimately lowering (raising) its expected return to place it back on the SML.

**Example 12** Based on the past 50 years, the market premium is approximately 5%. With a risk-free rate of 2% the SML becomes $\bar{R} = 0.02 + 0.05 \ast \beta$. 

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• If $\beta = 0$, then $\bar{R} = 0.02$. So when a security is uncorrelated with the market, it only “deserves” an expected return equal to the risk-free rate. This is because you can diversify away any of its firm-specific risk with a large portfolio.

• If $\beta = 1$, then $\bar{R} = \bar{R}_M$. So when a security’s $\beta$ equals one, then its expected return should equal the market portfolio’s expected return.

• Suppose $\beta < 0$, i.e., a security is negatively correlated with the market portfolio? The SML tells us its expected rate of return should be lower than the risk-free rate! The reason for this is that this security provides an additional benefit by lowering the standard deviation of the market portfolio due to its negative correlation. This security acts like a “hedge.” Investors will accept a lower expected rate of return for such a security because it provides a benefit for lowering overall portfolio risk.

Example 13  Stock A has a beta of 0.50 and Stock B has a beta of 1.25. Suppose $r_f = 4\%$ and $\bar{R}_M = 10\%$. What is the expected return of an equally weighted portfolio of these two stocks? Applying the SML, we have:

\[
\bar{R}_A = r_f + \beta_A \bar{R}_M - r_f = 4\% + 0.50(10\% - 4\%) = 7\%, \quad (121)
\]
\[
\bar{R}_B = r_f + \beta_B \bar{R}_M - r_f = 4\% + 1.25(10\% - 4\%) = 11.5\% \quad (122)
\]

Expected return of an equally weighted portfolio is $0.50(7\%) + 0.50(11.5\%) = 9.25\%$.

What is the beta of a portfolio $P$?

\[
\beta_P = \frac{Cov(R_P, R_M)}{Var(R_M)} = \frac{Cov(\sum_i x_i R_i, R_M)}{Var(R_M)} = \sum_i x_i \frac{Cov(R_i, R_M)}{Var(R_M)} = \sum_i x_i \beta_i. \quad (123)
\]

We can see that the beta of a portfolio is the weighted average of the beta of the assets in the portfolio.

Example 14  Consider the data of the previous example. The portfolio’s beta is $0.50(0.50) + 0.50(1.25) = 0.875$. Applying the SML again, we have that

\[
\bar{R}_P = r_f + \beta_P (\bar{R}_M - r_f) = 4\% + (0.875)(10\% - 4\%) = 9.25\%.
\]

5.2  Estimation of $\beta$ via regression

One can estimate $\beta$ via linear regression. Its value corresponds to the slope of the best-fitting line in the plot of the security’s excess return versus the market’s excess return. Linear regression models the excess return of a security as the sum of three components:

\[
\hat{R}_i = \alpha_i + \beta_i \hat{R}_M + \varepsilon_i. \quad (124)
\]
• $\alpha_i$ is the constant or intercept term of the regression, also called the stock’s alpha.

• $\beta_i \hat{R}_M$ represents the sensitivity of the stock to market risk.

• $\varepsilon_i$ is the error (or residual) term. The average error is assumed to be zero.

Taking the expectations of both sides of (124), we have

$$E[\hat{R}_i] = \beta_i E[\hat{R}_M] + \alpha_i. \quad (125)$$

The stock’s alpha, $\alpha_i$, measures the historical performance of the security relative to the expected return predicted by the SML. It is a risk-adjusted measure of a stock’s historical performance. According to the SML, $\alpha_i$ should not be significantly different from zero.

**Example 15** Suppose you estimate that stock A has a volatility of 26% and a beta of 1.45, whereas stock B has a volatility of 37% and a beta of 0.79. Which stock has more total risk? Which stock has more market risk? Total risk is measured by volatility; therefore stock B has higher total risk. Market risk is measured by beta; therefore stock A has higher market risk.

Suppose the risk-free rate is 3% and you estimate the market’s expected return as 8%. Which firm has a higher cost of equity capital? According to the SML:

$$E[R_A] = r_f + \beta_A(E[R_M] - r_f) = 3\% + 1.45(8\% - 3\%) = 10.25\%. \quad (126)$$

$$E[R_B] = r_f + \beta_B(E[R_M] - r_f) = 3\% + 0.79(8\% - 3\%) = 6.95\%. \quad (127)$$

Market risk cannot be diversified. It is the market risk that determines the cost of capital. We conclude that firm A has the higher cost of equity capital even though it is less volatile.

### 5.3 Derivation of SML from incremental analysis

Consider this question: You are given a portfolio $P$ and an asset $i$ not in portfolio $P$. Will adding asset $i$ improve your portfolio? According to the theory, adding asset $i$ (either with a positive or negative weight) to $P$ will be beneficial if by doing so it will increase the (new) Sharpe ratio.

**Example 16** Suppose portfolio $P$’s expected return is 10%, its volatility is 20% and the risk-free rate is 2%. Suppose further that a particular mix of asset $i$ and $P$ yields a portfolio $P'$ with an expected return of 9.5% and a volatility of 15%. Is asset $i$ beneficial? The Sharpe ratio of $P$ is $(0.10-0.02)/0.20 = 0.40$ and the Sharpe ratio for $P'$ is $(0.095-0.02)/0.15 = 0.50$.

Here is what we can do. For every dollar invested in $P'$ we can borrow $1/3$ at the risk-free rate and invest the proceeds in $P'$. The revised portfolio $P''$ has a weight of $4/3$ on $P$ and a
weight of $-1/3$ on the risk-free asset. The volatility of $P''$ is $4/3$ the volatility of $P' = 4/3(0.15) = 20\%$ and its expected return is $(4/3)(9.5) + (-1/3)2 = 12\%$. By adding asset $i$ and borrowing, we have constructed a new portfolio $P''$ whose volatility is the same as the original portfolio $P$, yet has an expected return that is $2\%$ higher. Clearly, asset $i$ is beneficial.

Consider modifying portfolio $P$, as follows: for every dollar invested in $P$ we will either (a) borrow $x > 0$ dollars at the risk-free rate and invest the proceeds in asset $i$, or (b) short $-x > 0$ dollars of asset $i$ and invest the proceeds at the risk-free rate. The revised portfolio $P_x$ will still have a weight of 1 on $P$, but will now have a weight of $x$ on asset $i$ and a weight of $-x$ on the risk-free asset. The expected return and variance of $P_x$ are

\[
E[R_{P_x}] = \hat{R}_P + x(\hat{R}_i - r_f) = \hat{R}_P + x\hat{R}_i
\]

\[
Var[R_{P_x}] = \sigma_P^2(1)^2 + \sigma_i^2x^2 + 2(1)(x)\sigma_{iP} = \sigma_P^2 + 2\sigma_{iP}x + \sigma_i^2x^2,
\]

and so the Sharpe ratio $SR(x)$ associated with $P_x$ is

\[
SR(x) = \frac{\hat{R}_P + \hat{R}_ix}{\sqrt{\sigma_P^2 + 2\sigma_{iP}x + \sigma_i^2x^2}} := \frac{\mathcal{R}(x)}{\sqrt{\mathcal{Var}(x)}}.
\]

With this notation, note that $SR(0)$ equals the Sharpe ratio for $P$, $\mathcal{R}(0)$ equals $\hat{R}_P$ and $\sqrt{\mathcal{Var}(0)}$ equals the volatility of $P$.

Since the value of $x$ can be positive or negative, the Sharpe ratio for $P_x$ will be higher than that for $P$ if the derivative of $SR(x)$ at $x = 0$ is not zero. For example, if $SR'(0) > 0$, then $SR(x) > SR(0)$ for sufficiently small values of $x > 0$; or if $SR'(0) < 0$, then $SR(x) > SR(0)$ for sufficiently small values of $x < 0$. So, if the Sharpe ratio cannot be increased with asset $i$, then it must necessarily be the case that $SR'(0) = 0$. Conversely, the condition $SR'(0) = 0$ is also sufficient, namely, if it holds, then the Sharpe ratio cannot be increased with asset $i$.

Now for the calculus. Recall that

\[
\frac{d}{dx}(f(x)/g(x)) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2},
\]

and so

\[
SR'(x) = \frac{\mathcal{R}'(x)\sqrt{\mathcal{Var}(x)} - \mathcal{R}(x)\frac{\mathcal{Var}(x)}{2\sqrt{\mathcal{Var}(x)}}}{\mathcal{Var}(x)} = \frac{\hat{R}_i\sigma_P - \hat{R}_P\frac{2\sigma_{iP}}{\sigma_P}}{\sigma_P^2},
\]

Since $\sigma_P^2 > 0$,

\[
SR'(0) \neq 0 \iff \hat{R}_i\sigma_P \neq \frac{\hat{R}_P\sigma_{iP}}{\sigma_P} \iff \hat{R}_i \neq \frac{\sigma_{iP}}{\sigma_P}\hat{R}_P = \beta_i \hat{R}_P,
\]
from which we may conclude that a necessary and sufficient condition for not being able to improve the Sharpe ratio of $P$ with asset $i$ is that

$$R_i = r_f + \beta^P_i (\bar{R}_P - r_f) := \text{required return for asset } i. \quad (135)$$

We can think of the right-hand side of (135) as the required return for asset $i$ to compensate for the risk it will contribute to portfolio $P$. If $\bar{R}_i$ is higher than its required return, then adding asset $i$ to $P$ with some positive weight will increase the Sharpe ratio; if $\bar{R}_i$ is lower than its required return, then adding asset $i$ to $P$ with some negative weight will increase the Sharpe ratio.

**Remark 11** The derivation above will apply if asset $i$ already belongs to the portfolio $P$ with some non-zero weight. If asset $i$’s expected return does not equal its required return, this means that the existing weight $x_i$ in portfolio $P$ can be adjusted (up or down) to improve the Sharpe ratio. When portfolio $P$ is mean-variance efficient, its Sharpe ratio cannot be improved, which implies that existing portfolio weights cannot be adjusted without lowering the Sharpe ratio.

**Example 17** You are currently invested in a broad-based fund with an expected return of 15% and a volatility of 20%. Your broker suggests that you add a real estate fund to your portfolio. It has an expected return of 9%, a volatility of 35% and a correlation of 0.10 with your existing fund. Will adding the real estate fund improve your portfolio? The risk-free rate is 3%.

Let $P$ denote the portfolio consisting of the existing fund and let $i$ denote the real estate fund. Here,

$$\beta^P_i = \rho_{iP} \sigma_i / \sigma_P = 0.10(0.35)/(0.20) = 0.175, \quad (136)$$

and so

$$r_f + \beta^P_i (\bar{R}_P - r_f) = 3% + 0.175(15% - 3%) = 5.1%. \quad (137)$$

As 5.1% < 9%, it will certainly be advantageous to invest some amount in the real estate fund to improve your Sharpe ratio of (0.15-0.03)/0.20 = 0.60.

**Example 18** Consider the data for the previous example. What is the optimal choice of $x$? Here, $\sigma_{iP} = (0.35)(0.20)(0.1) = 0.007$, and so the Sharpe ratio (130) is

$$SR(x) = \frac{0.12 + 0.06x}{\sqrt{0.04 + 0.014x + 0.1225x^2}}. \quad (138)$$

You can try different examples for $x$ in increments of 0.01 and choose the best. For example, $SR(0.10) = 0.6103$. It turns out that $x = 0.11$ is the best and yields a Sharpe ratio of 0.6104. So, if you had $10,000 in your fund, you can borrow $1,100 at the risk-free rate and invest in the real estate fund. If you did not want to borrow, you would have to sell a portion of your existing broad-based fund. In this case, the portfolio weights are $1/(1 + x)$ and $x/(1 + x)$ on the broad-based and real-estate funds, respectively. The amount invested in the broad-based fund will equal $10,000/(1 + x) = \$9,009$, so you will sell $991 of the existing fund and buy this amount of the real estate fund.
Remark 12 It is possible to determine the value of $x$ in our example without having to enumerate all possibilities. As you add asset $i$ to the portfolio $P$, the $\beta_i^{P^x}$ will change, since asset $i$ now belongs to the new portfolio. We should stop adding asset $i$ precisely when $SR'(x) = 0$, which will happen exactly when $\hat{R}_i = \beta_i^{P^x} \hat{R}_{P^x}$. It is important to keep in mind here that both $\beta_i^{P^x}$ and $\hat{R}_{P^x}$ depend on the value of $x$.

Recall that the return of portfolio $P^x$ is $R_P + x(R_i - r_f)$, and so

$$\hat{R}_{P^x} = \hat{R}_P + \hat{R}_i x;$$

moreover,

$$\beta_i^{P^x} = \frac{\text{Cov}(R_i, R_P + x(R_i - r_f))}{\text{Var}(R_P + x(R_i - r_f))} = \frac{\sigma_{iP} + \sigma_i^2 x}{\sigma_P^2 + 2\sigma_{iP} x + \sigma_i^2 x^2}.\quad (140)$$

Consequently,

$$SR'(x) = 0 \iff \hat{R}_i = \frac{\sigma_{iP} + \sigma_i^2 x}{\sigma_P^2 + 2\sigma_{iP} x + \sigma_i^2 x^2}(\hat{R}_P + \hat{R}_i x).\quad (141)$$

An inspection of (141) shows that the quadratic term in $x$ will cancel, thereby leaving just a linear expression that can easily be solved. After a little algebra, we have

$$x^* = \frac{\sigma_P^2 \hat{R}_i - \sigma_{iP} \hat{R}_P}{\sigma_i^2 R_P - \sigma_{iP} R_i}.\quad (142)$$

Returning to the previous example, $\sigma_P^2 = (0.20)^2 = 0.04$, $\sigma_i^2 = (0.35)^2 = 0.1225$, $\sigma_{iP} = (0.20)(0.35)(0.10) = 0.007$, $R_P = 0.15 - 0.03 = 0.12$ and $\hat{R}_i = 0.09 - 0.03 = 0.06$, and so

$$x^* = \frac{(0.04)(0.06) - (0.007)(0.12)}{(0.1225)(0.12) - (0.007)(0.06)} = 0.109244 = 11\%.\quad (143)$$

6 CAPM Assumptions and Implications

There are three assumptions underlying the CAPM:

- Investors can buy or sell all securities at competitive market prices (without transaction costs) and can borrow and lend at the risk-free rate.
- Investors hold only efficient portfolios of traded assets. Such portfolios yield the maximum expected return for a given level of volatility.
- Investors have the same (homogeneous) expectations regarding the expected returns, volatilities and correlations of the securities.
These assumptions imply the following conclusions. First, each investor will identify the same portfolio of risky assets that has the highest Sharpe ratio. This is the tangent portfolio of risky assets. The only difference between each investor is the amount of money invested in the risk-free asset, which depends on their respective appetites for risk. For example, one investor may choose to invest 50% of his wealth in the risk-free rate and the remaining 50% in the tangent portfolio. The actual weight of this investor’s portfolio in risky asset \( i \) is simply \( 0.50x_i^* \). Another investor may choose to invest only 10% of her wealth in the risk-free asset; the actual weight of this investor’s portfolio in risky asset \( i \) is \( 0.90x_i^* \). The sum of all investor’s portfolios must equal the portfolio of all risky securities in the market, which is commonly referred to as the market portfolio. (It is often proxied by the S&P 500.) Therefore, the tangent portfolio of risky securities equals the market portfolio. This statement can be viewed as saying that “supply equals demand.” As we alluded to before, prices in the market adjust so that all securities are held in the tangent (market) portfolio in just the right amounts determined by the SML.

In sum, according to the assumptions of CAPM, all mean-variance efficient portfolios must lie on the Capital Market Line (CML) defined by:

\[
\bar{R}_P = r_f + \left( \frac{\bar{R}_M - r_f}{\sigma_M} \right) \sigma_P.
\]  

That is, a portfolio \( P \) is efficient if and only if it has the same Sharpe ratio as the market’s, which is just another way of saying that all mean-variance efficient portfolios must be some combination of the risk-free asset and the market (tangent) portfolio.
7 Summary of key definitions, notation and formulae

Risk-free rate = \( r_f \)

Random return on asset i = \( R_i \)

Expected return on asset i = \( E[R_i] = \bar{R}_i \)

Expected excess return on asset i = \( \bar{R}_i - r_f = \hat{R}_i \)

Standard deviation of return on asset i = \( \sigma_i \)

Covariance between returns on assets i and j = \( \sigma_{ij} \)

Covariance matrix = \( \Sigma \)

Correlation between returns on assets i and j = \( \rho_{ij} = \sigma_{ij}/\sigma_i \sigma_j \)

Portfolio weight on asset i = \( x_i \)

Portfolio random return = \( R_P = R(x) = \sum_i x_i R_i \)

Portfolio expected return = \( \sum_i x_i \bar{R}_i \)

Portfolio expected excess return = \( \sum_i x_i \hat{R}_i \)

Portfolio variance \( Var(R_P) = \sum_i \sum_j \sigma_{ij}^2 x_i x_j \)

Portfolio volatility \( StDev(R_P) = \sqrt{Var(R_P)} \)

Portfolio variance for two assets = \( \sigma_i^2 x_i^2 + \sigma_j^2 x_j^2 + 2 x_i x_j \sigma_i \sigma_j \rho_{ij} \)

Minimum variance portfolio of two assets = \( (x_1^*, x_2^*) = \left( -\frac{\sigma_{12}^2}{\sigma_1^2 + \sigma_2^2 - 2 \sigma_1 \sigma_2}, \frac{\sigma_{12}^2}{\sigma_1^2 + \sigma_2^2 - 2 \sigma_1 \sigma_2} \right) \)

Variance of a large portfolio = \( \text{AvgVar} + \left( 1 - \frac{1}{n} \right) \text{AvgCov} \)

Security i’s contribution to volatility of P = \( x_i \lambda \sum_j \sigma_{ij} \rho_{ij} \)

Security beta \( \beta_i = \frac{\text{Cov}(R_i, R_M)}{\text{Var}(R_M)} = \frac{\rho_{iM} \sigma_i \sigma_M}{\sigma_M^2} = \frac{\rho_{ij} \sigma_i}{\sigma_M} \)

Security market line (SML): \( \bar{R}_i = r_f + \beta_i (\bar{R}_M - r_f) = \text{Required return on asset i} \)

Capital market line (CML): \( R_P = r_f + \left( \frac{\bar{R}_M - r_f}{\sigma_M} \right) \sigma_P \)