Chapter 5

Interest Rates
Chapter Outline

5.1 Interest Rate Quotes and Adjustments
5.2 Application: Discount Rates and Loans
5.3 The Determinants of Interest Rates
5.4 Risk and Taxes
5.5 The Opportunity Cost of Capital
5.1 Interest Rate Quotes and Adjustments

• The Effective Annual Rate
  – Indicates the total amount of interest that will be earned at the end of one year
  – Considers the effect of compounding
    • Also referred to as the effective annual yield (EAY) or annual percentage yield (APY)
5.1 Interest Rate Quotes and Adjustments (cont'd)

• Adjusting the Discount Rate to Different Time Periods
  – Earning a 5% return annually is not the same as earning 2.5% every six months.

• General Equation for Discount Rate Period Conversion

  Equivalent $n$-Period Discount Rate = $(1 + r)^n - 1$

  • $(1.05)^{0.5} - 1 = 1.0247 - 1 = .0247 = 2.47\%$
    – Note: $n = 0.5$ since we are solving for the six month (or 1/2 year) rate
Textbook Example 5.1

Valuing Monthly Cash Flows

Problem
Suppose your bank account pays interest monthly with the interest rate quoted as an effective annual rate (EAR) of 6%. What amount of interest will you earn each month? If you have no money in the bank today, how much will you need to save at the end of each month to accumulate $100,000 in 10 years?
Textbook Example 5.1 (cont'd)

Solution
From Eq. 5.1, a 6% EAR is equivalent to earning \((1.06)^{1/12} - 1 = 0.4868\%\) per month. We can write the timeline for our savings plan using monthly periods as follows:

<table>
<thead>
<tr>
<th>Month</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\ldots</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow:</td>
<td></td>
<td>C</td>
<td>C</td>
<td>\ldots</td>
<td>C</td>
</tr>
</tbody>
</table>

That is, we can view the savings plan as a monthly annuity with \(10 \times 12 = 120\) monthly payments. We can calculate the total amount saved as the future value of this annuity, using Eq. 4.9:

\[
FV(\text{annuity}) = C \times \frac{1}{r} \left[ (1 + r)^n - 1 \right]
\]

We can solve for the monthly payment \(C\) using the equivalent monthly interest rate \(r = 0.4868\%\), and \(n = 120\) months:

\[
C = \frac{FV(\text{annuity})}{\frac{1}{r} \left[ (1 + r)^n - 1 \right]} = \frac{100,000}{\frac{1}{0.004868} \left[ (1.004868)^{120} - 1 \right]} = 615.47 \text{ per month}
\]

We can also compute this result using the annuity spreadsheet:

<table>
<thead>
<tr>
<th>NPER</th>
<th>RATE</th>
<th>PV</th>
<th>PMT</th>
<th>FV</th>
<th>Excel Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given</td>
<td>120</td>
<td>0.4868%</td>
<td>0</td>
<td>100,000</td>
<td></td>
</tr>
<tr>
<td>Solve for PMT</td>
<td></td>
<td></td>
<td>(-615.47)</td>
<td></td>
<td>=PMT(0.004868,120,0,100000)</td>
</tr>
</tbody>
</table>

Thus, if we save $615.47 per month and we earn interest monthly at an effective annual rate of 6%, we will have $100,000 in 10 years.
Alternative Example 5.1

• Problem
  – Suppose an investment pays interest quarterly with the interest rate quoted as an effective annual rate (EAR) of 9%.
    • What amount of interest will you earn each quarter?
  – If you have no money in the bank today, how much will you need to save at the end of each quarter to accumulate $25,000 in 5 years?
Alternative Example 5.1

• Solution

– From Equation 5.1, a 9% EAR is approximately equivalent to earning $(1.09)^{1/4} - 1 = 2.1778\%$ per quarter.

– To determine the amount to save each quarter to reach the goal of $25,000 in five years, we must determine the quarterly payment, $C$:

$$C = \frac{FV(Annuity)}{\frac{1}{r} \left[ (1 + r)^n - 1 \right]} = \frac{\$25,000}{\frac{1}{.021778} \left[ 1.021778^{20} - 1 \right]} = \$1,010.82 \text{ per quarter}$$
Annual Percentage Rates

• The **annual percentage rate (APR)**, indicates the amount of simple interest earned in one year.
  
  – **Simple interest** is the amount of interest earned *without* the effect of compounding.
  
  – The APR is typically less than the effective annual rate (EAR).
Annual Percentage Rates (cont'd)

- **The APR itself cannot be used as a discount rate.**
  - The APR with $k$ compounding periods is a way of quoting the actual interest earned each compounding period:

  \[
  \text{Interest Rate per Compounding Period} = \frac{\text{APR}}{k \text{ periods / year}}
  \]
Annual Percentage Rates (cont'd)

• Converting an APR to an EAR

\[ 1 + \text{EAR} = \left( 1 + \frac{\text{APR}}{k} \right)^k \]

- The EAR increases with the frequency of compounding.
  - Continuous compounding is compounding every instant.
Annual Percentage Rates (cont'd)

Table 5.1  Effective Annual Rates for a 6% APR with Different Compounding Periods

<table>
<thead>
<tr>
<th>Compounding Interval</th>
<th>Effective Annual Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual</td>
<td>$(1 + 0.06/1)^1 - 1 = 6%$</td>
</tr>
<tr>
<td>Semiannual</td>
<td>$(1 + 0.06/2)^2 - 1 = 6.09%$</td>
</tr>
<tr>
<td>Monthly</td>
<td>$(1 + 0.06/12)^{12} - 1 = 6.1678%$</td>
</tr>
<tr>
<td>Daily</td>
<td>$(1 + 0.06/365)^{365} - 1 = 6.1831%$</td>
</tr>
</tbody>
</table>

- A 6% APR with continuous compounding results in an EAR of approximately 6.1837\%. 
Textbook Example 5.2

Converting the APR to a Discount Rate

Problem
Your firm is purchasing a new telephone system, which will last for four years. You can purchase the system for an upfront cost of $150,000, or you can lease the system from the manufacturer for $4000 paid at the end of each month. Your firm can borrow at an interest rate of 5% APR with semiannual compounding. Should you purchase the system outright or pay $4000 per month?
Textbook Example 5.2 (cont'd)

Solution
The cost of leasing the system is a 48-month annuity of $4000 per month:

<table>
<thead>
<tr>
<th>Month:</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payment:</td>
<td>$4000</td>
<td>$4000</td>
<td>...</td>
<td></td>
<td>$4000</td>
</tr>
</tbody>
</table>

We can compute the present value of the lease cash flows using the annuity formula, but first we need to compute the discount rate that corresponds to a period length of one month. To do so, we convert the borrowing cost of 5% APR with semiannual compounding to a monthly discount rate. Using Eq. 5.2, the APR corresponds to a six-month discount rate of 5%/2 = 2.5%. To convert a six-month discount rate into a one-month discount rate, we compound the six-month rate by 1/6 using Eq. 5.1:

\[(1.025)^{1/6} - 1 = 0.4124\% \text{ per month}\]

(Alternatively, we could first use Eq. 5.3 to convert the APR to an EAR: \(1 + EAR = (1 + 0.05/2)^2 = 1.050625\). Then we can convert the EAR to a monthly rate using Eq. 5.1: \((1.050625)^{1/12} - 1 = 0.4124\% \text{ per month}\).)

Given this discount rate, we can use the annuity formula (Eq. 4.8) to compute the present value of the 48 monthly payments:

\[PV = 4000 \times \frac{1}{0.004124} \left(1 - \frac{1}{1.004124^{48}}\right) = 173,867\]

We can also use the annuity spreadsheet:

<table>
<thead>
<tr>
<th>NPER</th>
<th>RATE</th>
<th>PV</th>
<th>PMT</th>
<th>FV</th>
<th>Excel Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given</td>
<td>48</td>
<td>0.4124%</td>
<td>$-4000$</td>
<td>0</td>
<td>=PV(0.004124,48,−4000,0)</td>
</tr>
<tr>
<td>Solve for PV</td>
<td>173,867</td>
<td>173,867</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus, paying $4000 per month for 48 months is equivalent to paying a present value of $173,867 today. This cost is $173,867 − $150,000 = $23,867 higher than the cost of purchasing the system, so it is better to pay $150,000 for the system rather than lease it. We can interpret this result as meaning that at a 5% APR with semiannual compounding, by promising to repay $4000 per month, your firm can borrow $173,867 today. With this loan it could purchase the phone system and have an additional $23,867 to use for other purposes.
Alternative Example 5.2

• Problem

- A firm is considering purchasing or leasing a luxury automobile for the CEO. The vehicle is expected to last 3 years. You can buy the car for $65,000 up front, or you can lease it for $1,800 per month for 36 months. The firm can borrow at an interest rate of 8% APR with quarterly compounding. Should you purchase the system outright or pay $1,800 per month?
Alternative Example 5.2 (cont’d)

• Solution

– The first step is to compute the discount rate that corresponds to monthly compounding. To convert an 8% rate compounded quarterly to a monthly discount rate, compound the quarterly rate using Equations 5.3 and 5.1:

\[
(1 + \frac{0.08}{4})^4 = 1.082432 \rightarrow 1.082432^{\frac{1}{12}} - 1 = 0.66227\% \text{ per month}
\]
Alternative Example 5.2 (cont’d)

• Solution
  
  – Given a monthly discount rate of 0.66227%, the present value of the 36 monthly payments can be computed:

  \[ PV = \frac{1,800 \times \frac{1}{0.0066227} \left(1 - \frac{1}{1.0066227^{36}}\right)}{1} = 57,486 \]

  – Paying $1,800 per month for 36 months is equivalent to paying $57,486 today. This is $65,000 - $57,486 = $7,514 lower than the cost of purchasing the system, so it is better to lease the vehicle rather than buy it.
5.2 Application: Discount Rates and Loans

• Computing Loan Payments
  – Payments are made at a set interval, typically monthly.
  – Each payment made includes the interest on the loan plus some part of the loan balance.
  – All payments are equal and the loan is fully repaid with the final payment.
5.2 Application: Discount Rates and Loans (cont'd)

• Computing Loan Payments

- Consider a $30,000 car loan with 60 equal monthly payments, computed using a 6.75% APR with monthly compounding.

• 6.75% APR with monthly compounding corresponds to a one-month discount rate of 6.75% / 12 = 0.5625%.

\[
C = \frac{P}{\frac{1}{r} \left(1 - \frac{1}{(1 + r)^N}\right)} = \frac{1}{0.005625} \left(\frac{30,000}{1 - \frac{1}{(1 + 0.005625)^{60}}}\right) = \$590.50
\]
Computing the Outstanding Loan Balance

Problem
Two years ago your firm took out a 30-year amortizing loan to purchase a small office building. The loan has a 4.80% APR with monthly payments of $2623.33. How much do you owe on the loan today? How much interest did the firm pay on the loan in the past year?
Solution
After 2 years, the loan has 28 years, or 336 months, remaining:

\[
\begin{array}{cccccc}
0 & 1 & 2 & \ldots & 336 \\
\$-2623.33 & \$-2623.33 & \$-2623.33 & \ldots & \$-2623.33 \\
\end{array}
\]

The remaining balance on the loan is the present value of these remaining payments, using the loan rate of \(4.8\%/12 = 0.4\%\) per month:

\[
\text{Balance after 2 years} = \$2623.33 \times \frac{1}{0.004} \left( 1 - \frac{1}{1.004^{336}} \right) = \$484,332
\]

During the past year, your firm made total payments of \(\$2623.33 \times 12 = \$31,480\) on the loan. To determine the amount that was interest, it is easiest to first determine the amount that was used to repay the principal. Your loan balance one year ago, with 29 years (348 months) remaining, was

\[
\text{Balance after 1 year} = \$2623.33 \times \frac{1}{0.004} \left( 1 - \frac{1}{1.004^{348}} \right) = \$492,354
\]

Therefore, the balance declined by \(\$492,354 - \$484,332 = \$8022\) in the past year. Of the total payments made, \$8022 was used to repay the principal and the remaining \(\$31,480 - \$8022 = \$23,458\) was used to pay interest.
5.3 The Determinants of Interest Rates

• Inflation and Real Versus Nominal Rates

  – **Nominal Interest Rate:** The rates quoted by financial institutions and used for discounting or compounding cash flows

  – **Real Interest Rate:** The rate of growth of your purchasing power, after adjusting for inflation
5.3 The Determinants of Interest Rates (cont'd)

Growth in Purchasing Power = $1 + r_r = \frac{1 + r}{1 + i} = \frac{\text{Growth of Money}}{\text{Growth of Prices}}$

- The Real Interest Rate
  
  $r_r = \frac{r - i}{1 + i} \approx r - i$
Textbook Example 5.4

Calculating the Real Interest Rate

**Problem**
At the start of 2005, one-year U.S. government bond rates were about 2.8%, while the rate of inflation that year was 3.4%. At the start of 2008, one-year interest rates were about 3.2%, and inflation that year was about 0.1%. What was the real interest rate in 2005 and 2008?
Solution

Using Eq. 5.5, the real interest rate in 2005 was \((2.8\% - 3.4\%)/(1.034) = -0.58\%\). In 2008, the real interest rate was \((3.2\% - 0.1\%)/(1.001) = 3.10\%\). Note that the real interest rate was negative in 2005, indicating that interest rates were insufficient to keep up with inflation: Investors in U.S. government bonds were able to buy less at the end of the year than they could have purchased at the start of the year. On the other hand, there was hardly any inflation in 2008, and so the real interest rate earned was only slightly below the nominal interest rate.
Alternative Example 5.4

• Problem

– In the year 2008, the average 1-year Treasury Constant Maturity rate was about 1.82% and the rate of inflation was about 0.28%.

– What was the real interest rate in 2008?
Alternative Example 5.4

• Solution

– Using Equation 5.5, the real interest rate in 2008 was:

• \[(1.82\% - 0.28\%) \div (1.0028) = 1.54\%\]

  – Which is (approximately) equal to the difference between the nominal rate and inflation: \(1.82\% - 0.28\% = 1.54\%\)
Investment and Interest Rate Policy

• An increase in interest rates will typically reduce the NPV of an investment.

  – Consider an investment that requires an initial investment of $10 million and generates a cash flow of $3 million per year for four years. If the interest rate is 5%, the investment has an NPV of:

\[
NPV = -10 + \frac{3}{1.05} + \frac{3}{1.05^2} + \frac{3}{1.05^3} + \frac{3}{1.05^4} = $0.638 \text{ million}
\]
Investment and Interest Rate Policy (cont'd)

– If the interest rate rises to 9%, the NPV becomes negative and the investment is no longer profitable:

\[ NPV = -10 + \frac{3}{1.09} + \frac{3}{1.09^2} + \frac{3}{1.09^3} + \frac{3}{1.09^4} = $0.281 \text{ million} \]
Monetary Policy, Deflation, and the 2008 Financial Crisis

• When the 2008 financial crisis struck, the Federal Reserve responded by cutting its short-term interest rate target to 0%.

• While this use of monetary policy is generally quite effective, because consumer prices were falling in late 2008, the inflation rate was negative, and so even with a 0% nominal interest rate the real interest rate remained positive.
The Yield Curve and Discount Rates

- **Term Structure:** The relationship between the investment term and the interest rate

- **Yield Curve:** A graph of the term structure
Figure 5.2 Term Structure of Risk-Free U.S. Interest Rates, November 2006, 2007, and 2008

<table>
<thead>
<tr>
<th>Term (years)</th>
<th>Date Oct-06</th>
<th>Oct-07</th>
<th>Oct-08</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5.15%</td>
<td>3.20%</td>
<td>0.44%</td>
</tr>
<tr>
<td>1</td>
<td>5.02%</td>
<td>3.15%</td>
<td>0.60%</td>
</tr>
<tr>
<td>2</td>
<td>4.83%</td>
<td>3.14%</td>
<td>0.96%</td>
</tr>
<tr>
<td>3</td>
<td>4.71%</td>
<td>3.20%</td>
<td>1.35%</td>
</tr>
<tr>
<td>4</td>
<td>4.64%</td>
<td>3.32%</td>
<td>1.75%</td>
</tr>
<tr>
<td>5</td>
<td>4.62%</td>
<td>3.47%</td>
<td>2.13%</td>
</tr>
<tr>
<td>6</td>
<td>4.62%</td>
<td>3.63%</td>
<td>2.49%</td>
</tr>
<tr>
<td>7</td>
<td>4.65%</td>
<td>3.78%</td>
<td>2.81%</td>
</tr>
<tr>
<td>8</td>
<td>4.68%</td>
<td>3.93%</td>
<td>3.09%</td>
</tr>
<tr>
<td>9</td>
<td>4.71%</td>
<td>4.06%</td>
<td>3.32%</td>
</tr>
<tr>
<td>10</td>
<td>4.75%</td>
<td>4.17%</td>
<td>3.51%</td>
</tr>
<tr>
<td>15</td>
<td>4.87%</td>
<td>4.44%</td>
<td>3.90%</td>
</tr>
<tr>
<td>20</td>
<td>4.88%</td>
<td>4.45%</td>
<td>3.84%</td>
</tr>
</tbody>
</table>
The Yield Curve and Discount Rates (cont'd)

- The term structure can be used to compute the present and future values of a risk-free cash flow over different investment horizons.

\[
PV = \frac{C_n}{(1 + r_n)^n}
\]

- Present Value of a Cash Flow Stream Using a Term Structure of Discount Rates

\[
PV = \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2} + \ldots + \frac{C_N}{(1 + r_N)^N} = \sum_{n=1}^{N} \frac{C_N}{(1 + r_n)^n}
\]
Textbook Example 5.5

Using the Term Structure to Compute Present Values

Problem
Compute the present value of a risk-free five-year annuity of $1000 per year, given the yield curve for November 2008 in Figure 5.2.
Textbook Example 5.5 (cont'd)

**Solution**
To compute the present value, we discount each cash flow by the corresponding interest rate:

\[
PV = \frac{1000}{1.0091} + \frac{1000}{1.0098^2} + \frac{1000}{1.0126^3} + \frac{1000}{1.0169^4} + \frac{1000}{1.0201^5} = \$4775.25
\]

Note that we cannot use the annuity formula here because the discount rates differ for each cash flow.
Alternative Example 5.5

• Problem

– Compute the present value of a risk-free three-year annuity of $500 per year, given the following yield curve:

<table>
<thead>
<tr>
<th>Term (Years)</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.261%</td>
</tr>
<tr>
<td>2</td>
<td>0.723%</td>
</tr>
<tr>
<td>3</td>
<td>1.244%</td>
</tr>
</tbody>
</table>

November-09
Alternative Example 5.5

- **Solution**

  Each cash flow must be discounted by the corresponding interest rate:

  \[ PV = \frac{\$500}{1.00261} + \frac{\$500}{1.00723^2} + \frac{\$500}{1.01244^3} \]

  \[ PV = \$498.70 + \$492.85 + 481.79 = \$1,473.34 \]
The Yield Curve and the Economy

• Interest Determination
  – The Federal Reserve determines very short-term interest rates through its influence on the federal funds rate, which is the rate at which banks can borrow cash reserves on an overnight basis.
  – All other interest rates on the yield curve are set in the market and are adjusted until the supply of lending matches the demand for borrowing at each loan term.
The Yield Curve and the Economy

• Interest Rate Expectations
  – The shape of the yield curve is influenced by interest rate expectations.
  • An inverted yield curve indicates that interest rates are expected to decline in the future.
    – Because interest rates tend to fall in response to an economic slowdown, an inverted yield curve is often interpreted as a negative forecast for economic growth.

  » Each of the last six recessions in the United States was preceded by a period in which the yield curve was inverted.

  » The yield curve tends to be sharply increasing as the economy comes out of a recession and interest rates are expected to rise.
5.4 Risk and Taxes

- Risk and Interest Rates
  - U.S. Treasury securities are considered “risk-free.” All other borrowers have some risk of default, so investors require a higher rate of return.
Figure 5.4 Interest Rates on Five-Year Loans for Various Borrowers, March 2009

- U.S. Government (Treasury Notes): 2.0%
- Wal-Mart Stores: 3.1%
- Coca-Cola: 3.7%
- Walt Disney: 4.3%
- Safeway: 5.4%
- FedEx: 6.0%
- GE Capital: 10.0%
Textbook Example 5.7

Discounting Risky Cash Flows

Problem
Suppose the U.S. government owes your firm $1000, to be paid in five years. Based on the interest rates in Figure 5.4, what is the present value of this cash flow? Suppose instead FedEx owes your firm $1000. Estimate the present value in this case.
**Solution**

Assuming we can regard the government’s obligation as risk free (there is no chance you won’t be paid), then we discount the cash flow using the risk-free Treasury interest rate of 2%:

\[ PV = \frac{1000}{(1.02)^5} = 905.73 \]

The obligation from FedEx is not risk-free. There is no guarantee that FedEx will not have financial difficulties and fail to pay the $1000. Because the risk of this obligation is likely to be comparable to the five-year loan quoted in Figure 5.4, the 6% interest rate of the loan is a more appropriate discount rate to use to compute the present value in this case:

\[ PV = \frac{1000}{(1.06)^5} = 747.26 \]

Note the substantially lower present value than the government debt in this case, due to the risk of default.
After-Tax Interest Rates

- Taxes reduce the amount of interest an investor can keep, and we refer to this reduced amount as the after-tax interest rate.

\[ r - (\tau \times r) = r \left(1 + \tau\right) \]
Textbook Example 5.8

Comparing After-Tax Interest Rates

Problem
Suppose you have a credit card with a 14% APR with monthly compounding, a bank savings account paying 5% EAR, and a home equity loan with a 7% APR with monthly compounding. Your income tax rate is 40%. The interest on the savings account is taxable, and the interest on the home equity loan is tax deductible. What is the effective after-tax interest rate of each instrument, expressed as an EAR? Suppose you are purchasing a new car and are offered a car loan with a 4.8% APR and monthly compounding (which is not tax deductible). Should you take the car loan?
**Solution**

Because taxes are typically paid annually, we first convert each interest rate to an EAR to determine the actual amount of interest earned or paid during the year. The savings account has a 5% EAR. Using Eq. 5.3, the EAR of the credit card is \((1 + 0.14/12)^{12} - 1 = 14.93\%\), and the EAR of the home equity loan is \((1 + 0.07/12)^{12} - 1 = 7.23\%\).

Next, we compute the after-tax interest rate for each. Because the credit card interest is not tax deductible, its after-tax interest rate is the same as its pre-tax interest rate, 14.93%. The after-tax interest rate on the home equity loan, which is tax deductible, is \(7.23\% \times (1 - 0.40) = 4.34\%\). The after-tax interest rate that we will earn on the savings account is \(5\% \times (1 - 0.40) = 3\%\).

Now consider the car loan. Its EAR is \((1 + 0.048/12)^{12} - 1 = 4.91\%\). It is not tax deductible, so this rate is also its after-tax interest rate. Therefore, the car loan is not our cheapest source of funds. It would be best to use savings, which has an opportunity cost of foregone after-tax interest of 3%. If we don’t have sufficient savings, we should use the home equity loan, which has an after-tax cost of 4.34%. And we should certainly not borrow using the credit card!
5.5 The Opportunity Cost of Capital

- **Opportunity Cost of Capital**: The best available expected return offered in the market on an investment of comparable risk and term to the cash flow being discounted
  - Also referred to as *Cost of Capital*
Continuous Rates and Cash Flows

• Discount Rate of a Continuously Compounded APR
  – Some investments compound more frequently than daily.
  • As we move from daily to hourly to compounding every second, we approach the limit of continuous compounding, in which we compound every instant.

The EAR for a Continuously Compounded APR

\[(1 + EAR) = e^{APR}\]
Continuous Rates and Cash Flows (cont’d)

• Discount Rate of a Continuously Compounded APR
  - Alternatively, if we know the EAR and want to find the corresponding continuously compounded APR, the formula is:

  **The Continuously Compounded APR for an EAR**

  \[ APR = \ln(1 + EAR) \]

  Continuously compounded rates are not often used in practice.
Continuous Rates and Cash Flows (cont’d)

• Continuously Arriving Cash Flows
  – Consider the cash flows of an online book retailer. Suppose the firm forecasts cash flows of $10 million per year. The $10 million will be received throughout each year, not at year-end, that is, the $10 million is paid continuously throughout the year.
  – We can compute the present value of cash flows that arrive continuously using a version of the growing perpetuity formula.
Continuous Rates and Cash Flows (cont’d)

• Continuously Arriving Cash Flows
  – If cash flows arrive, starting immediately, at an initial rate of $C$ per year, and if the cash flows grow at rate $g$ per year, then given a discount rate of $r$ per year, the present value of the cash flows is:

  **Present Value of a Continuously Growing Perpetuity**

  \[
  PV = \frac{C}{r_{cc} - g_{cc}}
  \]

  where $r_{cc} = \ln(1+r)$ and $g_{cc} = \ln(1+g)$ are the discount and growth rates expressed as continuously APRs, respectively.
Continuous Rates and Cash Flows (cont’d)

- Continuously Arriving Cash Flows
  - The present value of a continuously growing perpetuity can be approximated by:

\[
PV = \frac{C}{r_{cc} - g_{cc}} \approx \frac{\bar{C}_1}{r - g} \times (1 + r)^{1/2}
\]

where \(\bar{C}_1\) is the total cash received during the first year.
Textbook Example 5A.1

Valuing Projects with Continuous Cash Flows

Problem
Your firm is considering buying an oil rig. The rig will initially produce oil at a rate of 30 million barrels per year. You have a long-term contract that allows you to sell the oil at a profit of $1.25 per barrel. If the rate of oil production from the rig declines by 3% over the year and the discount rate is 10% per year (EAR), how much would you be willing to pay for the rig?
Textbook Example 5A.1

Solution
According to the estimates, the rig will generate profits at an initial rate of (30 million barrels per year) × ($1.25/barrel) = $37.5 million per year. The 10% discount rate is equivalent to a continuously compounded APR of $r_{cc} = \ln(1 + 0.10) = 9.531\%$; similarly, the growth rate has an APR of $g_{cc} = \ln(1 - 0.03) = -3.046\%$. From Eq. 5A.3, the present value of the profits from the rig is

$$PV\text{ (profits)} = \frac{37.5}{r_{cc} - g_{cc}} = \frac{37.5}{0.09531 + 0.03046} = $298.16\text{ million}$$

Alternatively, we can closely approximate the present value as follows. The initial profit rate of the rig is $37.5$ million per year. By the end of the year, the profit rate will have declined by 3% to $37.5 \times (1 - 0.03) = $36.375 million per year. Therefore, the average profit rate during the year is approximately $(37.5 + 36.375)/2 = $36.938$ million. Valuing the cash flows as though they occur at the middle of each year, we have

$$PV\text{ (profits)} = \left[\frac{36.938}{(r - g)}\right] \times (1 + r)^{1/2}$$

$$= \left[\frac{36.938}{0.10 + 0.03}\right] \times (1.10)^{1/2} = $298.01\text{ million}$$

Note that both methods produce very similar results.