Mean–Variance Efficient Frontier

1 Introduction

We consider a market with $n$ risky assets. An investor wishes to invest $B$ dollars in this market. Let $B_i$, $i = 1, 2, \ldots, n$, denote the allocation of the budget to asset $i$ so that $\sum_{i=1}^{n} B_i = B$ and let $r_i$ denote the random one-period return on asset $i$, $i = 1, 2, \ldots, n$. The investor’s one-period return on his portfolio $(B_1, B_2, \ldots, B_n)$ is given by

\[
\text{One-period return } = \frac{\sum_i B_i r_i}{B} := \sum_i x_i r_i, \tag{1}\]

where we let $x_i := B_i/B$ denote the weight of asset $i$ in the portfolio. We shall think of $B$ as fixed and hereafter identify a portfolio of the $n$ assets with a vector $x = (x_1, x_2, \ldots, x_n)$ such that $\sum_{i=1}^{n} x_i = 1$. The portfolio’s random return will be denoted by

\[
r(x) := \sum_{i=1}^{n} x_i r_i. \tag{2}\]

Remark 1 When $x_i < 0$ the holder of the portfolio is short-selling asset $i$. In these notes we permit unlimited short-selling. In practice, however, there are limits to the magnitude of short-selling. If short-selling is not permitted, i.e., each $x_i$ is constrained to be non-negative, then the solution approach outlined in these notes does not directly apply, though the problem can be easily solved with commercial software.

Let $W_0$ denote the investor’s initial wealth and let $R(x) := 1 + r(x)$ denote the one-period total random return. The investor’s end-of-period wealth is $W_0 R(x)$. We assume the investor is risk-averse and that his utility function $U(\cdot)$ is sufficiently differentiable and strictly concave. The investor’s optimal portfolio $x^*$ should maximize his expected utility of end-of-period wealth, as follows:

\[
\text{Investor’s problem: } \max_x \left\{ E[U(R(x))] : \sum_{i=1}^{n} x_i = 1 \right\}, \tag{3}\]

where, for convenience, we have normalized the initial wealth $W_0$ to one.

The investor’s problem is a non-trivial one. One difficulty is to obtain the distribution of $R(x)$, which depends on $x$. In lieu of solving the investor’s problem directly, we shall approximate the expected utility via a second-order Taylor series expansion about $\mu(x) := E[R(x)]$. For each realization of the random variable $R(x)$, and ignoring higher-order terms,

\[
U(R(x)) \approx U(\mu(x)) + [R(x) - \mu(x)]U'(\mu(x)) + \frac{1}{2} [R(x) - \mu(x)]^2 U''(\mu(x)). \tag{4}\]
Taking expectations of both sides of (4), we have
\[
E[U(R(x))] \approx E[U(\mu(x))] + U'(\mu(x))E[R(x) - \mu(x)] + \frac{1}{2}U''(\mu(x))E[(R(x) - \mu(x))^2] \quad (5)
\]
\[
= U(\mu(x)) + \frac{1}{2}U''(\mu(x))Var[R(x)],
\]
where the second line follows since \(U(\mu(x))\) is a constant and \(E[R(x)] = \mu(x)\) and \(E[(R(x) - \mu(x))^2] = Var[R(x)]\). We assume that the approximation of expected utility given in (4) is accurate. Consequently, maximizing the left-hand side of (4) is essentially equivalent to maximizing the right-hand side of (4).

Now fix the value for \(\mu(x)\) say to \(\bar{r}_0\); that is, consider only those portfolios \(x\) for which \(\mu(x) = \bar{r}_0\). In words, the investor is fixing his desired expected portfolio return. Since the sign of \(U''(\cdot)\) is negative, and since \(U(\mu(x))\) and \(U''(\mu(x))\) are constants, it immediately follows from the right-hand side of (4) that the investor’s optimal portfolio (subject to the expected return constraint) must minimize the variance of the portfolio’s return. Such a portfolio is called mean-variance efficient. As the level of \(\bar{r}_0\) is varied a collection of mean-variance efficient points will be generated that trace out what is termed the mean-variance efficient frontier. Using results from convex analysis, calculating the mean-variance efficient frontier turns out to be easily implementable.

2 Portfolio Mean and Variance

We recall some basic definitions/properties from statistics. Keep in mind each \(r_i\) is a random variable.

\[
\bar{r}_i := E[r_i] \quad (7)
\]
\[
\sigma_i^2 := Var[r_i] = E(r_i - \bar{r}_i)^2 = E[r_i^2] - (\bar{r}_i)^2 \quad (8)
\]
denote, respectively, the mean and variance of \(r_i\). For two random variables \(X\) and \(Y\),
\[
\]
It follows directly from (9) that the variance may be expressed in terms of covariance, i.e.,
\[
Var(X) = COV(X,X), \quad (10)
\]
and that covariance is symmetric, i.e.,
\[
COV(X,Y) = COV(Y,X). \quad (11)
\]
In addition, covariance has the following properties:
\[
COV\left(\sum_{i=1}^{n} X_i, Y\right) = \sum_{i=1}^{n} COV(X_i,Y), \quad (12)
\]
\[
COV(\alpha X, Y) = COV(X,\alpha Y) = \alpha COV(X,Y), \quad \alpha \text{ a real number.} \quad (13)
\]
In what follows, we let
\[ \sigma_{ij} := \text{COV}(r_i, r_j) \]  \hspace{1cm} (14)
and let \( \Sigma \) denote the \( n \) by \( n \) matrix whose \((i,j)\)th entry is given by \( \sigma_{ij} \). The matrix \( \Sigma \) is called the covariance matrix. Due to (11), \( \sigma_{ij} = \sigma_{ji} \), which implies that the covariance matrix is symmetric, namely, \( \Sigma = \Sigma^T \) (here, the symbol ‘T’ denotes transpose).

The portfolio’s mean return or “portfolio’s return” is given by
\[ E[r(x)] = E\left[ \sum_{i=1}^n r_i x_i \right] = \sum_{i=1}^n \bar{r}_i x_i. \]  \hspace{1cm} (15)

Using (10) and repeated application of (12) and (13), the portfolio’s variance is given by
\[ \text{Var}[r(x)] = \text{Var}\left[ \sum_{i=1}^n r_i x_i \right] \]  \hspace{1cm} (16)
\[ = \text{COV}\left( \sum_{i=1}^n r_i x_i, \sum_{j=1}^n r_j x_j \right) \]  \hspace{1cm} (17)
\[ = \sum_{i=1}^n \sum_{j=1}^n \text{COV}(r_i, r_j)x_i x_j \]  \hspace{1cm} (18)
\[ = \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_i x_j. \]  \hspace{1cm} (19)

**Example 1** When \( n = 2 \) the portfolio’s variance is given by
\[ \sum_{i=1}^2 \sum_{j=1}^2 \sigma_{ij} x_i x_j = \sigma_1^2 x_1^2 + \sigma_{12} x_1 x_2 + \sigma_{21} x_2 x_1 + \sigma_2^2 x_2^2 \]  \hspace{1cm} (20)
\[ = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\sigma_{12} x_1 x_2. \]  \hspace{1cm} (21)

In matrix notation, the portfolio’s variance may be represented as
\[ \text{Var}[r(x)] = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \]  \hspace{1cm} (22)
\[ = x^T \Sigma x. \]  \hspace{1cm} (23)

**Remark 2** It may be verified that the representation
\[ \text{Var}[r(x)] = x^T \Sigma x \]  \hspace{1cm} (24)
holds true for \( n > 2 \).
3 The Mean-Variance Optimization Problem

The core optimization problem is formulated as follows:

\[ P(\vec{r}_0) : \quad \text{Min} \quad \text{Var}[r(x)] = \frac{1}{2} x^T \Sigma x \]  

subject to:

\[ x_1 + x_2 + \cdots + x_n = 1 \]  

\[ \vec{r}_1 x_1 + \vec{r}_2 x_2 + \cdots + \vec{r}_n x_n = \vec{r}_0. \]  

In vector form, equations (26) and (27) may be expressed as

\[ e^T x = 1 \]  

\[ \vec{r}^T x = \vec{r}_0, \]  

where \( e^T := (1, 1, \ldots, 1) \) is the \( n \)-vector whose components are all ones and \( \vec{r}^T := (\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n) \).

Remark 3 A convex optimization problem is defined by a convex objective function subject to constraints defined by convex functions. (Linear programming is a special, well-known case.) The constraints in \( P(\vec{r}_0) \) are linear, which are convex. The objective function in \( P(\vec{r}_0) \) is convex because the variance cannot be negative. The objective function is a quadratic function of the vector \( x \) and so Problem \( P(\vec{r}_0) \) is called a quadratic programming problem. Quadratic programming problems are easy to solve using commercial software.

Define

\[ L(x, \lambda, \mu) := \frac{1}{2} x^T \Sigma x - \lambda (e^T x - 1) - \mu (\vec{r}^T x - \vec{r}_0). \]  

Here, \( \lambda \) and \( \mu \) are real numbers. The function \( L(\cdot) \) is called the Lagrangian for this problem.

Example 2 When \( n = 2 \) the core optimization problem is given by:

\[ P(\vec{r}_0) : \quad \text{Min} \quad \frac{1}{2} \{ \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2 \sigma_{12} x_1 x_2 \} \]  

subject to:

\[ x_1 + x_2 = 1 \]  

\[ \vec{r}_1 x_1 + \vec{r}_2 x_2 = \vec{r}_0. \]  

The associated Lagrangian is given by:

\[ L(x_1, x_2, \lambda, \mu) := \frac{1}{2} \{ \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2 \sigma_{12} x_1 x_2 \} - \lambda (x_1 + x_2 - 1) - \mu (\vec{r}_1 x_1 + \vec{r}_2 x_2 - \vec{r}_0). \]
Now consider the following unconstrained problem:

\[ L(\bar{r}_0) : \quad V(\lambda, \mu) := \min_x L(x, \lambda, \mu). \]  

(35)

Let \( x(\lambda, \mu) \) denote an optimal solution to problem \( L(\bar{r}_0) \). The following is a fundamental theorem of convex analysis:

**Theorem 1**

1. If \( x^* \) is an optimal solution to \( P(\bar{r}_0) \), then values for \( \lambda \) and \( \mu \) exist for which \( x^* = x(\lambda, \mu) \).
2. If \( x(\lambda, \mu) \) satisfies (26) and (27), then \( x(\lambda, \mu) \) is an optimal solution to \( P(\bar{r}_0) \).

**Remark 4** The variables \( \lambda \) and \( \mu \) are called the dual variables. As in linear programming, there is one dual variable for every constraint. Note that there are only two dual variables regardless of the number of \( x \) variables. It turns out that primal problem \( P(\bar{r}_0) \) has a dual problem given by

\[ V^*(\bar{r}_0) := \max_{\lambda, \mu} V(\lambda, \mu), \]  

(36)

such that \( V^* = \text{Var}[R(x^*)] \), where \( x^* \) is an optimal solution to problem \( P(\bar{r}_0) \). The dual problem is a concave optimization problem. Essentially, this duality extends the well-known duality of linear programming.

4 Optimality Conditions

We can use Theorem 1 to obtain a closed-form solution to our original problem \( P(\bar{r}_0) \). Here is the “game plan”.

**Step 1.** Let \( x^* \) be an optimal solution to \( P(\bar{r}_0) \) and suppose we were given values for \( \lambda \) and \( \mu \) such that \( x^* = x(\lambda, \mu) \). (Theorem 1(a) guarantees that this is possible.) Since \( x(\lambda, \mu) \) is an optimal solution to (35), which is an unconstrained problem, we know from calculus that all \( n \) partial derivatives of \( L(\cdot) \) evaluated at \( x(\lambda, \mu) \) must vanish. These \( n \) equations in \( n \) unknowns may be solved to determine \( x(\lambda, \mu) \).

**Step 2.** It remains to pin down values for \( \lambda \) and \( \mu \), which requires two additional equations. Theorem 1(b) provides them: the portfolio constraint (28) and the portfolio return constraint (29).

We illustrate this game plan in the special case that \( n = 2 \). When presented in matrix form, the equations we derive all hold true for the general case.
First, the partial derivatives must vanish, i.e.,
\[ \frac{\partial L}{\partial x_1} = \sigma_1^2 x_1 + \sigma_{12} x_2 - \lambda - \mu \bar{r}_1 = 0. \]  
\[ (37) \]
\[ \frac{\partial L}{\partial x_2} = \sigma_2^2 x_2 + \sigma_{12} x_1 - \lambda - \mu \bar{r}_2 = 0. \]  
\[ (38) \]
(To simplify notation we suppress the dependence of \( x \) on \( \lambda \) and \( \mu \).) In matrix form equations (37)-(38) may be represented as
\[ \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & \bar{r}_1 \\ \bar{r}_2 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \]  
\[ (39) \]
or equivalently as
\[ \Sigma x = \lambda e + \mu \bar{r}. \]  
\[ (40) \]
Now by multiplying both sides of (40) by \( \Sigma^{-1} \) we solve for the optimal \( x = x(\lambda, \mu) \) as:
\[ x(\lambda, \mu) = \begin{pmatrix} x_1(\lambda, \mu) \\ x_2(\lambda, \mu) \end{pmatrix} = \begin{pmatrix} \Sigma^{-1} e & \Sigma^{-1} \bar{r} \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}. \]  
\[ (41) \]
Since \( x(\lambda, \mu) \) must solve the two feasibility constraints (28) and (29), we have:
\[ \begin{pmatrix} 1 \\ \bar{r}_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \bar{r}_1 & \bar{r}_2 \end{pmatrix} \begin{pmatrix} x_1(\lambda, \mu) \\ x_2(\lambda, \mu) \end{pmatrix} = \begin{pmatrix} e^T \\ \bar{r}^T \end{pmatrix} \begin{pmatrix} x_1(\lambda, \mu) \\ x_2(\lambda, \mu) \end{pmatrix} \]  
\[ = \begin{pmatrix} e^T \Sigma^{-1} e & e^T \Sigma^{-1} \bar{r} \\ \bar{r}^T \Sigma^{-1} e & \bar{r}^T \Sigma^{-1} \bar{r} \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}. \]  
\[ (42) \]
\[ (43) \]
\[ (44) \]
It follows from equation (44) that
\[ \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} e^T \Sigma^{-1} e & e^T \Sigma^{-1} \bar{r} \\ \bar{r}^T \Sigma^{-1} e & \bar{r}^T \Sigma^{-1} \bar{r} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \bar{r}_0 \end{pmatrix}. \]  
\[ (45) \]
It may be verified that equations (40), (41), (44) and (45) all hold for \( n > 2 \) risky assets.

**Remark 5** If \( x \) were a single variable, then the Lagrangian in (30) would be more commonly written as
\[ L(x, \lambda, \mu) = \left( \frac{1}{2} \Sigma \right) x^2 - \lambda (ex - 1) + \mu (\bar{r}x - \bar{r}_0), \]

since \( \Sigma, e \) and \( \bar{r} \) would be all constants. The derivative of \( L(\cdot) \) with respect to the variable \( x \) would be easily computed as \( \Sigma x - \lambda e - \mu \bar{r} \), which when set equal to zero yields the equation \( \Sigma x = \lambda e + \mu \bar{r} \) and \( x = \Sigma^{-1}(\lambda e + \mu \bar{r}) \). The analogous operations when \( x \) is a vector are given in (40) and (41). Setting the collection of partial derivatives of a quadratic function of several variables to zero yields a system of linear equations.
5 Algorithm to Compute Mean-Variance Efficient Points

Here is how you could solve $P(\bar{r}_0)$ step by step:

- **STEP 1.** Compute the $n \times 2$ matrix $B := \begin{pmatrix} \Sigma^{-1}e & \Sigma^{-1}\bar{r} \end{pmatrix}$.

- **STEP 2.** Compute the $2 \times 2$ matrix $C := \begin{pmatrix} e^T \Sigma^{-1}e & e^T \Sigma^{-1}\bar{r} \\ \bar{r}^T \Sigma^{-1}e & \bar{r}^T \Sigma^{-1}\bar{r} \end{pmatrix} := \begin{pmatrix} C_{ee} & C_{e\bar{r}} \\ C_{e\bar{r}} & C_{\bar{r}\bar{r}} \end{pmatrix}$.

- **STEP 3.** Compute the optimal values for $\lambda$ and $\mu$ as $\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = C^{-1} \begin{pmatrix} 1 \\ \bar{r}_0 \end{pmatrix}$.

- **STEP 4.** Compute the optimal solution $x(\bar{r}_0)$ as $B \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = (BC^{-1}) \begin{pmatrix} 1 \\ \bar{r}_0 \end{pmatrix}$.

- **STEP 5.** Compute the minimum variance $Var(\bar{r}_0)$ as $x(\bar{r}_0)^T \Sigma x(\bar{r}_0)$.

**Remark 6** STEP 3 is unnecessary for the purpose of solving the problem, but it does give the values for the dual variables.

**Remark 7** The computation in STEP 5 can be simplified. In what follows we write $x$ in lieu of $x(\bar{r}_0)$. Using (40),

$$
Var(\bar{r}_0) = x^T \Sigma x = x^T [\lambda e + \mu \bar{r}] = \lambda (x^T e) + \mu (x^T \bar{r}) = \lambda + \mu \bar{r}_0
$$

(46)

(47)

Thus, the minimum variance is a quadratic function of $\bar{r}_0$.

6 Example

Consider 3 risky assets whose expected returns are $\bar{r}_1 = 18$, $\bar{r}_2 = 10$ and $\bar{r}_3 = 8$. The covariance matrix $\Sigma$ is

$$
\begin{pmatrix}
216 & 70 & -324 \\
70 & 25 & -150 \\
-324 & -150 & 1596
\end{pmatrix}.
$$

(49)
The inverse of the covariance matrix is

\[
\begin{pmatrix}
0.1480 & -0.5367 & -0.0204 \\
-0.5367 & 2.0388 & 0.0827 \\
-0.0204 & 0.0827 & 0.0043
\end{pmatrix}
\]  

(50)

You should verify that

\[
\mathbf{B} = \begin{pmatrix}
-0.4092 & -2.8673 \\
1.5847 & 11.3878 \\
0.0665 & 0.4932
\end{pmatrix}
\]  

(51)

\[
\mathbf{C} = \begin{pmatrix}
1.2420 & 9.0136 \\
9.0136 & 66.2109
\end{pmatrix}
\]  

(52)

\[
\mathbf{C}^{-1} = \begin{pmatrix}
66.9280 & -9.112 \\
-9.112 & 1.2555
\end{pmatrix}
\]  

(53)

\[
\mathbf{BC}^{-1} = \begin{pmatrix}
-1.2608 & 0.1283 \\
2.3039 & -0.1417 \\
-0.0431 & 0.0133
\end{pmatrix}
\]  

(54)

Thus the optimal portfolio vector is

\[
x(\bar{r}_0) = \begin{pmatrix}
-1.2608 & 0.1283 \\
2.3039 & -0.1417 \\
-0.0431 & 0.0133
\end{pmatrix}
\begin{pmatrix}
1 \\
\bar{r}_0
\end{pmatrix} = \begin{pmatrix}
-1.2608 + 0.1283\bar{r}_0 \\
2.3039 - 0.1417\bar{r}_0 \\
-0.0431 + 0.0133\bar{r}_0
\end{pmatrix}
\]  

(55)

and the minimum variance is

\[
\text{Var}(\bar{r}_0) = \left(1 \quad \bar{r}_0\right) \mathbf{C}^{-1} \left(\begin{array}{c} 1 \\ \bar{r}_0 \end{array}\right) = 66.9280 - 18.2224\bar{r}_0 + 1.2555\bar{r}_0^2.
\]  

(56)

Note that \(x(\bar{r}_0)\) does indeed satisfy the feasibility constraints (26) and (27).

7 Minimum Variance Portfolio

It is easy to find the mean-variance efficient portfolio that minimizes the portfolio variance, otherwise called the minimum variance portfolio. There are two approaches.

**Approach 1: Minimize the Variance Formula.** Here, one simply minimizes \(\text{Var}(\bar{r}_0)\) given in (56), which is achieved by taking its derivative and setting it equal to zero. For the example in the previous section,

\[
0 = \frac{dV(\bar{r}_0)}{d\bar{r}_0} = -18.2224 + 2.5111\bar{r}_0,
\]  

(57)
which means that the portfolio that achieves the minimum variance has an expected mean return equal to 7.257. Once the value for \( \bar{r}_0 \) for the minimum variance portfolio is known, plug its value into (55) to obtain the minimum variance portfolio \( x^* = (-0.3297, 1.2756, 0.0534) \), and then plug this value into (56) to obtain the minimum variance of 0.8078.

**Approach 2: Solve the Problem Directly.** The problem of finding the minimum variance portfolio is the same as our original problem \( P(\bar{r}_0) \) except that the expected return constraint (27) is not there. Eliminating a constraint is equivalent to setting its dual variable \( \mu \) to zero. An inspection of (41) shows that the minimum variance portfolio must be proportional to the vector \( \Sigma^{-1}e \). To obtain the minimum variance portfolio, compute the vector \( \Sigma^{-1}e \) and then “normalize” this vector so that the sum of its components is one. (This is achieved by merely dividing each of this vector’s components by the sum of all the components.) That is,

\[
\text{minimum variance portfolio} = \frac{1}{e^T \Sigma^{-1}e} \Sigma^{-1}e. \tag{58}
\]

One may further verify that the minimum variance is given by

\[
\text{minimum variance} = \left( e^T \Sigma^{-1}e \right)^{-1}. \tag{59}
\]

In our example, \( \Sigma^{-1}e = (-0.4081, 1.5971, 0.0661)^T \). The sum of the components of this vector is 1.2379 (which equals the reciprocal of the minimum variance). Dividing \((-0.4081, 1.5971, 0.0661)\) by 1.2379 yields \((-0.3297, 1.2756, 0.0534)\), as in Approach 1.

**Remark 8** Two observations yields “computational savings”. First, the minimum variance portfolio is proportional to the vector \( \Sigma^{-1}e \), which is given in the first column of the matrix \( \mathcal{B} \). Second, in words \( e^T \Sigma^{-1}e \) merely represents the sum of all elements of the matrix \( \Sigma^{-1} \).

### 8 Computing the Tangent (Market) Portfolio

In this section we assume the market includes a risk-free security that returns \( r_f \) in all states. The tangent or market portfolio solves the following optimization problem:

\[
(TP) : \quad \max \left\{ \frac{r^T x - r_f}{\sqrt{x^T \Sigma x}} : e^T x = 1 \right\}, \tag{60}
\]

where \( r_f \) denotes the risk-free rate. The objective function is called the *Sharpe ratio*.

It can be shown that *all mean-variance efficient portfolios must be a combination of the risk-free security and the tangent portfolio*. That is, each investor should allocate a portion of their budget to the risk-free security and the remaining portion to the tangent portfolio. The sub-allocations to the individual risky securities are determined by the weights \( x^* \) that define
the tangent portfolio (to be determined). A very risk-averse investor allocates very little to
the tangent portfolio, whereas an investor with a large appetite for risk may hold a negative
portion in the risk-free security (i.e. borrow from the money market) to invest more than his
budget into the tangent portfolio.

We now turn to explaining how to calculate the tangent portfolio. Define
\[ \hat{\mathbf{r}} := \bar{\mathbf{r}} - r_{f} \mathbf{e}, \] (61)
which is obtained by simply subtracting the risk-free rate from each of the asset’s expected
returns. Problem (TP) may be expressed as:
\[ (TP) : \quad \max \left\{ \frac{\hat{\mathbf{r}}^T \mathbf{x}}{\sqrt{\mathbf{x}^T \Sigma \mathbf{x}}} : \mathbf{e}^T \mathbf{x} = 1 \right\}, \] (62)
Note that if one doubles the vector \( \mathbf{x} \) the objective function in (TP) remains the same. (The
objective function is said to be “homogeneous of degree 0”.) Thus, we can ignore the constraint
that \( \mathbf{e}^T \mathbf{x} = 1 \), because at the end we can perform the necessary normalization.

Now imagine we knew the expected return \( \bar{\mathbf{r}} \) of the tangent portfolio. The tangent portfolio
would then have to satisfy the following mean-variance optimization problem:
\[ \min \left\{ \mathbf{x}^T \Sigma \mathbf{x} : \hat{\mathbf{r}}^T \mathbf{x} = \bar{\mathbf{r}} - r_{f} := \hat{\mathbf{r}} \right\}. \] (63)
Inspect this optimization problem closely. It is the same as our original optimization problem
\( P(\bar{\mathbf{r}}) \ except that the first feasibility constraint (26) is now removed. This is equivalent to
setting its dual variable \( \lambda \) equal to zero. That is, the tangent portfolio necessarily satisfies the
first-order optimality conditions
\[ \Sigma \mathbf{x}^* = \mu \hat{\mathbf{r}}. \] (64)
Consequently, the tangent portfolio must be proportional to \( \Sigma^{-1} \hat{\mathbf{r}} \), and thus may be found in
the analogous way we found the minimum variance portfolio via Approach 1.

Remark 9 Note that
\[ \Sigma^{-1} \hat{\mathbf{r}} = \Sigma^{-1}(\bar{\mathbf{r}} - r_{f} \mathbf{e}) = (\Sigma^{-1} \bar{\mathbf{r}}) - r_{f} (\Sigma^{-1} \mathbf{e}) \] (65)
\[ = \begin{pmatrix} \Sigma^{-1} \mathbf{e} & \Sigma^{-1} \bar{\mathbf{r}} \end{pmatrix} \begin{pmatrix} -r_{f} \\ 1 \end{pmatrix} = \begin{pmatrix} -r_{f} \\ 1 \end{pmatrix}. \] (66)

Example 3 Suppose the risk-free rate is 3 in our example. The tangent portfolio is proportional to
\[ \Sigma^{-1} \begin{pmatrix} 15 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} -1.6389 \\ 6.6346 \\ 0.2944 \end{pmatrix}. \] (67)
Thus the tangent portfolio vector is \( (-0.3098, 1.2542, 0.0557) \). (Divide the right-hand side of
(67) by 5.2901 = -1.6389 + 6.6346 + 0.2944.) It’s expected return is 7.4112 and its variance
equals 0.8376 from (56). You will get the same answer if you use (66).
9 Security Market Line

Let \( x_M = (x_{M1}, x_{M2}, \ldots, x_{Mn}) \) denote the market portfolio, and let \( \bar{r}_M = \bar{r}^T x_M \) denote the expected return on the market portfolio. The first-order optimality conditions (64) imply a relationship between a security’s expected return and a measure of its risk. This relationship is called the Security Market Line (SML) and is often used to determine an appropriate expected return for a security (or asset).

To derive this relationship from (64), we begin by noting that the \( i^{th} \) equation of (64) in non-matrix notation is simply

\[
\mu(\bar{r}_i - r_f) = \sum_j \text{Cov}(r_i, r_j)x_{Mj} = \text{Cov}(r_i, \sum_j r_jx_{Mj}) = \text{Cov}(r_i, r_M).
\]

(68)

To pin down the value of \( \mu \), substitute \( x_M \) into (64), multiply both sides by \( x_M \), and use the fact that

\[
x_M^T \hat{r} = x_M^T(\bar{r} - r_f e) = x_M^T \bar{r} - r_f (x_M^T e) = \bar{r}_M - r_f
\]

to derive that

\[
\text{Var}(r_M) = \mu(\bar{r}_M - r_f).
\]

(69)

From (68) and (69), we derive the SML

\[
\bar{r}_i = r_f + \frac{\text{Cov}(r_i, r_M)}{\text{Var}(r_M)}(\bar{r}_M - r_f) := r_f + \beta_i(\bar{r}_M - r_f).
\]

(70)

In (70) the parameter

\[
\beta_i := \frac{\text{Cov}(r_i, r_M)}{\text{Var}(r_M)} = \frac{\rho_{iM}\sigma_i\sigma_M}{\sigma_M^2} = \frac{\rho_{ij}\sigma_i}{\sigma_M}
\]

(71)

is called the security’s beta. In the portfolio context it is considered an appropriate measure of a security’s “risk”. The term \( \bar{r}_M - r_f \) is called the market premium. So, the Security Market Line states that

Expected return of a security = the risk-free rate + (security’s beta)*(market premium).

(72)

That is, there is a linear relationship between a security’s expected return and its risk (as measured by its \( \beta \)). This relationship is what economists call an equilibrium relationship in this sense: If a security’s expected return, say, exceeded what is predicted by the SML, then there would be more demand for it, thereby raising its price and ultimately lowering its expected return to place it back on the SML.

**Examples.** Historically, the market premium is 8.4%. With a risk-free rate of 2% the SML becomes

\[
\bar{r} = 0.02 + 0.084 \times \beta.
\]
• If $\beta = 0$, then $\bar{r} = 0.02$. So when a security is uncorrelated with the market, it only “deserves” an expected return equal to the risk-free rate.

• If $\beta = 1$, then $\bar{r} = \bar{r}_M$. So when a security’s $\beta$ equals one, then its expected return should equal the market portfolio’s expected return.

• Suppose $\beta < 0$, i.e., a security is negatively correlated with the market portfolio? The SML tells us its expected rate of return should be lower than the risk-free rate! The reason for this is that this security provides an additional benefit by lowering the standard deviation of the market portfolio due to its negative correlation.