1 Consumption with a Perfect Capital Market

Consider a simple two-period world in which a single consumer must decide between consumption \( c_0 \) today (in period 0) and consumption \( c_1 \) tomorrow (in period 1). The consumer is endowed with money \( m_0 \) today and \( m_1 \) tomorrow. Consistent with his endowment, the consumer has the opportunity to borrow or lend \( b_0 \) today at interest rate \( r \). Borrowing corresponds to a positive \( b_0 \), whereas lending corresponds to a negative \( b_0 \).

1.1 Budget constraint

The equations governing the consumer’s feasible actions today and tomorrow are, as follows. Consumption in period 0 equals the original endowment adjusted for the borrowing or lending:

\[
c_0 = m_0 + b_0. \tag{1}
\]

We assume the consumer prefers more consumption to less, and so we write this constraint as an equality. Since consumption cannot be negative, the consumer cannot lend more than \( m_0 \), i.e.,

\[
b_0 \geq -m_0.
\]

Consumption in period 1 equals the original endowment adjusted for the repayment or receipt of the principal and interest associated with the borrowing in period 0, i.e.,

\[
c_1 = m_1 - (1 + r)b_0. \tag{2}
\]

(Once again, since the consumer is assumed to prefer more consumption to less, we write this constraint as an equality.) Since consumption cannot be negative, there is a limit as to how much the consumer can borrow in period 0, namely,

\[
b_0 \leq m_1/(1 + r).
\]

After substituting \( c_0 - m_0 \) for \( b_0 \) in (2), the consumer’s budget constraint is

\[
c_0 + \frac{c_1}{1 + r} = m_0 + \frac{m_1}{1 + r} := W_0. \tag{3}
\]

The symbol \( W_0 \) represents the consumer’s present value of wealth.

Example 1. Suppose \( m_0 = 100 \), \( m_1 = 990 \), and \( r = 10\% \). The present value of wealth is \( W_0 = 100 + 990/1.1 = 1000 \). The consumption plan \( (c_0, c_1) = (500, 550) \) satisfies the budget constraint (3). To achieve this consumption plan, the consumer borrows 400 in period 0 and pays back 440 in period 1. The consumption plan \( (c_0, c_1) = (800, 220) \) also satisfies the budget constraint (3). To achieve this consumption plan, the consumer borrows 700 in period 0 and pays back 770 in period 1.
1.2 Optimal consumption plan

Which consumption plan will the consumer choose? To determine the optimal consumption plan, we assume existence of a utility function $U(c_0, c_1)$ such that the consumer prefers $(c_0^1, c_1^1)$ to $(c_0^2, c_1^2)$ if and only if

$$U(c_0^1, c_1^1) > U(c_0^2, c_1^2).$$  \(4\)

If the utility values are equal, then the consumer is said to be indifferent to the two consumption plans. Under reasonable assumptions on consumer preferences, existence of a utility function that reflects the consumer preferences is guaranteed to exist. For the remainder of this handout, we shall assume that

$$U(c_0, c_1) = \sqrt{c_0 + \beta \sqrt{c_1}}.$$ \(5\)

**Example 2.** We continue with our example. We shall set $\beta = 0.6$. The utility of the consumption plan $(c_0, c_1) = (500, 550)$ is 36.43, the utility of the consumption plan $(c_0, c_1) = (800, 220)$ is 37.18, and the utility of the original endowment $(c_0, c_1) = (100, 990)$ is 28.88. Of these three choices, the consumer prefers $(800, 220)$. The ability to borrow definitely helps our consumer.

Formally, the consumer’s optimization problem is

$$\text{MAX} \{U(c_0, c_1) : c_0 + \frac{c_1}{1+r} = W_0\}.$$ \(6\)

Define

$$c_1(c_0) := (1 + r)(W_0 - c_0),$$ \(7\)

$$\psi(c_0) := U(c_0, c_1(c_0)) = \sqrt{c_0 + \beta \sqrt{c_1(c_0)}}.$$ \(8\)

The consumer’s optimization problem can be equivalently expressed as

$$\text{MAX} \{\psi(c_0) : 0 \leq c_0 \leq W_0\}.$$ \(9\)

Note how the parameter $\beta$ serves as a discount factor on future consumption. Smaller values of $\beta$ imply a larger discount factor on future consumption, which implies that our consumer prefers more consumption today. Consider, for example, the two extreme values for $\beta$, namely, 0 and $+\infty$.

How do we obtain the optimal first period consumption $c_0^*$ and hence the optimal second period consumption $c_1(c_0^*)$? The form of the utility function implies that consumption in both periods must be positive. (This is because the derivative of $\sqrt{x}$ at zero is infinite.) Consequently, the optimal choice for $c_0$ necessarily lies strictly between 0 and $W_0$. Accordingly, to find the optimal choice of $c_0$, we set the derivative of $\psi(\cdot)$ to zero. Using the chain-rule, optimality conditions imply that

$$0 = \psi'(c_0^*) = \frac{\partial U}{\partial c_0} + \frac{\partial U}{\partial c_1} \frac{dc_1}{dc_0} = \frac{1}{2\sqrt{c_0^*}} - \frac{\beta(1+r)}{2\sqrt{c_1(c_0^*)}},$$ \(10\)
It follows directly from (10) that the optimal consumption plan necessarily satisfies the condition
\[ \sqrt{\frac{c_1}{c_0}} = \beta(1 + r), \]  
(11)
or, equivalently,
\[ c_1^* = \beta^2(1 + r)c_0^*. \]  
(12)
Substituting (12) into the budget constraint (3) and solving for \( c_0^* \) and \( c_1^* \), the optimal consumption plan is
\[ c_0^* = \left[ \frac{1}{1 + \beta^2(1 + r)} \right] W_0 := \rho_0 W_0 \]  
(13)
\[ c_1^* = \left[ \frac{\beta^2(1 + r)^2}{1 + \beta^2(1 + r)} \right] W_0 := \rho_1 W_0. \]  
(14)
The constants \( \rho_0 \) and \( \rho_1 \) are independent of present wealth \( W_0 \); that is, they are known parameters strictly determined by the two discount factors \( \beta \) and \( r \). We shall use this important fact later. Note further that the optimal utility is of the form
\[ U^*(W_0) := [\sqrt{\rho_0 + \beta\sqrt{\rho_1}}] \sqrt{W_0}. \]  
(15)

Example 3. In our example, \( c_0^* = 0.716W_0 \), \( c_1^* = 0.312W_0 \), and \( U^*(W_0) = 1.182\sqrt{W_0} \). Here Thus, \( c_0^* = 716 \), \( c_1^* = 312 \), and \( U^*(1000) = 37.36 \). Due to the availability of a capital market at which to borrow or loan, the consumer has increased his utility by almost 30% above the level corresponding to the initial endowment \( U(100,990) \). To obtain the optimal consumption plan, the consumer must borrow \( b_0^* = 716 - 100 = 616 \) today, pay back 678 tomorrow, thereby leaving him with \( 990 - 678 = 312 \) to consume in the final period.

2 Consumption and Investment with a Perfect Capital Market

We now consider a world in which the consumer has the opportunity to invest \( I_0 \) today in production from which he will receive \( f(I_0) \) tomorrow. The function \( f(I_0) \) encapsulates the return on the investment opportunities in production available to the consumer. For example, the consumer may wish to obtain an education while he is young, expecting a return on this investment in his working years. It is generally assumed that (i) \( f(0) = 0 \), (ii) \( f(\cdot) \) is strictly increasing (more investment leads to more return), and (iii) \( f(\cdot) \) exhibits diminishing returns in that the marginal return on an incremental rise in investment declines as the total investment increases. When \( f(\cdot) \) is differentiable, these assumptions imply the first derivative is positive and the second derivative is negative. (Such a function is called concave.) To ensure at least some investment will be made (to make our subsequent calculations easier), we also assume that the derivative \( f'(0) \) is infinite.

The equations governing the consumer’s feasible actions today and tomorrow are now:
\[ c_0 + I_0 = m_0 + b_0 \]  
(16)
\[ c_1 = m_1 + f(I_0) - (1 + r)b_0. \]  

Rearranging terms as we did before, the new budget constraint is

\[ c_0 + \frac{c_1}{1 + r} = W_0 + \left[ \frac{f(I_0)}{1 + r} - I_0 \right] := W_0(I_0). \]  

The new optimization problem facing the consumer is

\[ \text{MAX} \{ \psi(c_0) : 0 \leq c_0 \leq W_0(I_0), \ I_0 \geq 0 \}. \]  

An examination of (18) and (19) reveals a fundamental property: *All consumers, regardless of their utility function, should first determine the optimal investment plan to increase their wealth!* That is, they should select the value of \( I_0 \) to maximize \( W_0(I_0) \). This is achieved by equating the marginal return of investment \( f'(I_0) \) to \( 1 + r \). When the function \( f(I_0) \) represents the investment opportunities for a firm in which consumers hold stock, then each consumer that holds stock should insist that the firm optimize its investment opportunity, *regardless* of each consumer’s different desires for consumption today versus tomorrow. Because there exists a capital market for each consumer to borrow or lend, each consumer can redistribute the increases in wealth as they desire. This principle (in various forms) is known as the **Fisher Separation Theorem of Finance**.

**Example 4.** Suppose \( f(I_0) = 33\sqrt{I_0} \). Now

\[ f'(I_0) = \frac{33}{2\sqrt{I_0}}, \]

and so the optimal choice for investment is \( I_0^* = 225 \). The additional wealth created through investment equals \( 495/1.1 - 225 = 225 \) so that \( W_0(225) = 1225 \). From (13) and (14), the optimal consumption plan is \( c_0^* = 877 \) and \( c_1^* = 382 \) with \( U^*(1225) = 41.34 \). The utility has increased by about 10.7\%, which also corresponds to \( 100(\sqrt{1225}/1000 - 1) \). To obtain the optimal consumption plan, the consumer must borrow \( b_0^* = 877 + 225 - 100 = 1002 \) today, pay back 1103 tomorrow, thereby leaving him with 990 + 495 − 1102 = 382 to consume in the final period.

### 3 Consumption and Investment Without a Capital Market

We now consider the situation in which the consumer has investment opportunities as described in the previous section, but no longer has the opportunity to borrow or loan, i.e., \( b_0 = 0 \). We shall see that without access to a capital market our consumer is far worse off.

The equations governing the consumer’s feasible actions today and tomorrow are now

\[ \begin{align*}
  c_0 + I_0 &= m_0 \\
  c_1 &= m_1 + f(I_0).
\end{align*} \]
The new budget constraint can be represented as
\[ c_1(c_0) = m_1 + f(m_0 - c_0). \] (22)

The optimality conditions (10) imply that
\[ \sqrt{\frac{c_1}{c_0}} = \beta f'(I_0), \] (23)

or, equivalently, that the optimal choice for \( I_0 \) must satisfy the identity
\[ \sqrt{\frac{m_1 + f(I_0)}{m_0 - I_0}} = \beta f'(I_0). \] (24)

After substituting the specific choice for \( f(\cdot) \) and performing simple algebra, the optimal choice for \( I_0 \) must satisfy the identity
\[ 990 + 33\sqrt{I_0} = \frac{9801}{I_0} - 98.01. \] (25)

Since the left-hand side of (25) is an increasing function of \( I_0 \) that is finite when \( I_0 = 0 \) and the right-hand side of (25) is a decreasing function of \( I_0 \) that is infinite when \( I_0 = 0 \), a unique solution exists, which can be obtained by bisection search. (Alternatively, identity (25) can be transformed into a cubic equation, which has a closed-form solution.) The optimal value \( I_0^* \) is about 8.25 with a corresponding consumption plan of \( c_0^* = 91.75 \) and \( c_1^* = 1085 \) with \( U^* = 29.34 \). The utility has dropped considerably to almost the level corresponding to the original endowment.

4 Homework Problems

1. Jones is endowed with money \( m_0 = 55,000 \) today and \( m_1 = 88,000 \) tomorrow. He desires to consume \( c_0 = 80,000 \) today and \( c_1 = 66,000 \) tomorrow.

(a) If there is no opportunity to borrow or lend can Jones achieve his consumption objective? Explain.

(b) Suppose there is a perfect capital market in which Jones may borrow or lend as much as he desires at the market interest rate of 10%. Can Jones now achieve his consumption objective? Explain.

(c) Suppose that in addition to a perfect capital market Jones has an opportunity to invest \( I_0 = 100,000 \) today and receive \( f(I_0) \) tomorrow. Determine the minimum value for \( f(I_0) \) for which Jones will be able to exactly achieve his consumption objective.

(d) Explain exactly what Jones must do using the capital market and investment opportunity available to him so that he may exactly achieve his consumption objective.
2. Jones is endowed with money \( m_0 = 40,000 \) today and \( m_1 = 99,000 \) tomorrow. He desires to consume \( c_0 = 100,000 \) today and \( c_1 = 55,000 \) tomorrow.

(a) If there is no opportunity to borrow or lend can Jones achieve his consumption objective? Explain.

(b) Suppose there is a perfect capital market in which Jones may borrow or lend as much as he desires at the market interest rate of 10%. Can Jones now achieve his consumption objective? Explain.

(c) Suppose that in addition to a perfect capital market Jones has an opportunity to invest \( I_0 = 50,000 \) today and receive \( f(I_0) \) tomorrow. Determine the minimum value for \( f(I_0) \) for which Jones will be able to exactly achieve his consumption objective.

(d) Explain exactly what Jones must do using the capital market and investment opportunity available to him so that he may exactly achieve his consumption objective.

3. Smith’s utility function is \( U(c_0, c_1) = \ln c_0 + 0.4 \ln c_1 \). Smith is endowed with money \( m_0 = 90,000 \) today and \( m_1 = 500,000 \) tomorrow. There is a perfect capital market for borrowing and lending at the market rate of interest of 25% per period.

(a) Determine the optimal consumption plan for Smith. Explain exactly what Smith must do each period to achieve his optimal consumption plan. (Recall that \( \frac{d}{dx} \ln x = \frac{1}{x} \).)

(b) Smith has an opportunity to invest \( I_0 = 80,000 \) today and receive \( f(I_0) = 135,000 \) tomorrow. Should he take advantage of this opportunity? If not, explain why not. If so, explain exactly what he should do each period to achieve his new optimal consumption plan.

(c) If Smith no longer has access to the capital market (he cannot borrow or lend), then should he take advantage of the investment opportunity presented in (b)? Explain your reasoning.

5 Homework Solutions

Problem 1

a. No. Desired consumption \( c_0 \) exceeds his endowment \( m_0 \).

b. No. The \( PV \) of his wealth \( 55+88/1.1 = 135 \) is less than the \( PV \) of his desired consumption \( 80 + 66/1.1 = 140 \).

c. Jones needs to add 5 to the \( PV \) of his wealth. Consequently, \( \frac{f(I_0)}{1.1} - 100 = 5 \), which means that the smallest value for \( f(I_0) \) is 115.5.

d. In the first period, Jones consumes 80, invests 100 and borrows 125. In the second period, his endowment of 88 plus his investment payout of 115.5 generates a supply of 203.5 of which 137.5 is needed to pay back the loan, thereby leaving 66 for consumption, as desired.
Problem 2

a. No. Desired consumption $c_0$ exceeds his endowment $m_0$.

b. No. The PV of his wealth $40 + 99/1.1 = 130$ is less than the PV of his desired consumption $100 + 55/1.1 = 150$.

c. Jones needs to add 20 to the PV of his wealth. Consequently, $f(I_0) - 50 = 20$, which means that the smallest value for $f(I_0)$ is 77.

d. In the first period, Jones consumes 100, invests 50 and borrows 110. In the second period, his endowment of 99 plus his investment payout of 77 generates a supply of 176 of which 121 is needed to pay back the loan, thereby leaving 55 for consumption, as desired.

Problem 3

a. The first-order optimality conditions imply that

$$\frac{1}{c_0} + \frac{0.4}{c_1}(-1.25) = 0,$$  \hspace{1cm} (26)

which implies that $c_1 = 0.5c_0$. Since $c_0 + 0.8c_1 = W_0 = 490$, we conclude that $c_0 = W_0/1.4$; thus, Smith should consume 350 now and 175 next period. To achieve this plan, Smith needs to borrow 260 today, and pay back 325 next period from the 500 supply, which leaves 175 to consume, as desired.

b. Since the $NPV$ created by the investment opportunity is $135/1.25 - 80 = 28 > 0$, Smith should invest in the project. His new $PV$ of wealth will be 518. He should consume $518/1.4 = 370$ now and 185 next period. To achieve this plan, Smith needs to borrow 360, and pay back 450 next period from the 500 + 135 total supply, which leaves 175 to consume, as desired.

c. Since Smith cannot use a capital market he is faced with 2 choices: either invest in the project, which will give him a utility of $\ln(10) + 0.4 \ln(635) = 4.884$, or simply consume his initial endowment, which will give him a utility of $\ln(90) + 0.4 \ln(500) = 6.986$. Smith should not invest.