Real Options Analysis
ISyE 4803

In these notes we show how to apply Real Options Analysis (ROA). We will develop the ideas by examining five “classic” Real Options problems. We will also illustrate the potential pitfalls of applying Decision Tree Analysis (DTA), a close companion of ROA.

1 The Delay Option

1.1 Background information

Relevant information pertaining to a project under consideration is as follows:

1. In year 1, the project’s present value (of discounted expected subsequent future) cash flow will be 1500 if the market goes up next year and will be 666.6 if the market goes down next year. The project’s present value cash flow in year 2 will be as follows: if the market goes up two consecutive years, then it will be 2250; if the market goes up and then down or down and then up, then it will be 1000; if the market goes down two consecutive years, then it will be 444.4.

2. The project pays no dividends.

3. The probability of the market going up each year is 0.70.

4. The project’s initial investment $I_0$ (at time 0) is 1500.

5. It is possible to delay the project one year. To do so will cost the company 175 today, and the required investment next year will increase by 10%.

6. In year 1 the company can spend an additional 25 (at time 1) to delay the project one more year. The required investment will increase by another 10%.

7. There is a tradable (non-dividend) security $S$ whose current price is 100. Its price process follows a binomial lattice with $u = 1.5$ and $d = u^{-1} = 0.6$.

8. The risk-free rate is 5%.
1.2 Analysis of the project without the delay option

The first step is to determine the project’s value today. The project’s payoffs $P$ are perfectly correlated with the tradable security, i.e., $P = 10S$. Thus, the market value of $P$ today must be 1000. This is the present value of the project.

The project’s initial investment $I_0$ (at time 0) is 1050. The project’s Net Present Value (NPV) is given by

$$NPV = PV - I_0 = 1000 - 1050 = -50 < 0.$$  \hspace{1cm} (1)

Since the NPV is negative, the project without the delay option should be rejected.

1.3 Analysis of the one year delay option

Consider the project with the one year delay option. Its value next year is determined, as follows. If the market goes up, the company should make the investment, and its value is

$$1500 - (1.1)(1050) = 345;$$ \hspace{1cm} (2)

if, however, the market goes down, then its value is zero since the company would not make the investment.

What is the value today for such a risky payoff? We examine two approaches.

DTA analysis

Let’s first examine how traditional DTA would analyze this opportunity. The project’s cost of capital is 25% since

$$\frac{0.7(1500) + 0.3(666.6)}{1.25} = 1000.$$ \hspace{1cm} (3)

Accordingly, using DTA the NPV would be determined as follows:

$$NPV = \frac{(0.7)(345) + (0.3)(0)}{1.25} - 175 = 18.2 > 0.$$ \hspace{1cm} (4)

Since the NPV is positive, traditional DTA says the project with the one year delay option should be undertaken and implemented next year only if the market goes up.
**ROA analysis**

*The use of DTA is not correct given the present setup.* Let’s use Real Options Analysis (ROA) to see why. The risk-neutral probability of the market going up is 0.46, since

\[
0.46(150) + 0.54(66.6) = 100. 
\]

(5)

Therefore, the correct value of the project today is computed as

\[
\frac{0.46(345) + 0.54(0)}{1.05} - 175 = -23.86 < 0, 
\]

(6)

and so the correct NPV says the delay option and the project should be **rejected**.

How can we verify/explain this? The PV of the project is 151.14 since

\[
151.14 - 175 = -23.86. 
\]

(7)

Since

\[
\frac{\Delta P}{\Delta S} = \frac{345 - 0}{150 - 66.6} = 4.14, 
\]

(8)

the replicating portfolio of \( S \) and the money market \( M \) is

\[
4.14S - 262.86M, 
\]

(9)

which, not surprisingly, costs 151.14 to purchase. This portfolio’s value next year will either be 345 if the market goes up or 0 if the market goes down, which exactly matches the value of the project with the delay option.

If someone insists the correct PV is 193.2 (193.2 = 18.2 + 175), then, in principle, you sell this person a “similar” project for 193.2 and buy the replicating portfolio for 151.14. The replicating portfolio will fully hedge your position and you will make 193.2 − 151.14 = 42.06 today, risk-free.

**Remark.** Using DTA can work, but only if the cost of capital is properly adjusted. The expected value of next year’s project value using the objective probabilities is \( (0.7)(345) = 241.5 \). Consequently, if one uses a cost of capital of 59.79%, then the PV will be 151.14, as required. As we have remarked in class, it is often difficult to arrive at correct values for the cost of capital at each project state since the project’s risk characteristics typically change over the life of the project. Recall there is another way to arrive at the cost of capital of 59.79%. The replicating portfolio weights are

\[
w_S = \frac{4.14(100)}{151.14} = 2.739 \quad \text{(10)}
\]

\[
w_M = \frac{-262.86}{151.14} = -1.739 \quad \text{(11)}
\]

respectively. Since \( r_S = 25\% \) and \( r_M = 5\% \), the replicating portfolio’s expected return is

\[
2.739(25) - 1.739(5) = 59.79. 
\]

(12)
Table 1: Project value event tree without flexibility

<table>
<thead>
<tr>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>1500.00</td>
<td>2250.00</td>
</tr>
<tr>
<td>666.66</td>
<td>1000.00</td>
<td>444.44</td>
</tr>
</tbody>
</table>

Sensitivity analysis

Let $\rho$ denote the “premium” cost today to delay the project, and let $100\theta\%$ denote the percentage increase in the required investment. As a function of $\rho$ and $\theta$, it is possible to determine the acceptance region for the project.

In what follows we assume that $\theta > 0$ and $(1 + \theta)(1050) < 1500$. As a function of $\rho$ and $\theta$, the project’s correct $NPV$ is

$$
\frac{0.46[1500 - (1 + \theta)(1050)]}{1.05} - \rho,
$$

which must be positive if the project with the delay option is to be accepted. The acceptance region is therefore

$$
\{(\theta, \rho) : 197.14 \geq 460\theta + \rho\}.
$$

1.4 Analysis of the two year delay option

Project value event tree without flexibility

Table 1 records the “Project value event tree without flexibility,” which we take as our underlying tradable security.

Project value event tree with flexibility

The project with flexibility may be viewed as a collection of options on the underlying security. Table 2 records the “Project NPV event tree with flexibility” according to ROA. Here are how these numbers were obtained.

The project will not be undertaken next year if the market goes down. We need to assess the correct project value (with the delay options) if the market goes up. If the project is not delayed one more year, then the project’s value is 345, as before. However, the company does
Table 2: Project NPV event tree with flexibility (* = delay)

<table>
<thead>
<tr>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.04</td>
<td>*404.11</td>
<td>979.50</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

have the option to delay one more year, and this option must be considered at this point in time. If the company chooses to delay one more year, the discounted expected project value using the risk-neutral probability is

\[
\frac{0.46[1.5(1500) - (1.1)^2(1050)]}{1.05} = 429.11. \tag{15}
\]

After subtracting the cost of 25 the value in this state is 404.11. Since 404.11 > 345 it is optimal to delay the project 1 more year should the market go up next year. We conclude that the project’s value next year is thus 404.11 if the market goes up and 0 if the market goes down. The correct NPV for this project (with the delay options) is therefore equal to

\[
\frac{0.46(404.11)}{1.05} - 175 = 2.04 > 0, \tag{16}
\]

and so the project with the two year delay option should be accepted.
2 Option to Contract Operations

2.1 Background information

A company is considering investing two projects. Relevant information is as follows:

1. Each project requires an initial investment of 80 million.
2. Each project has a present value without flexibility of 100 million.
3. Each project pays no dividends.
4. The annual volatility for the present value of project 1 is 40% (i.e. \( \sigma = 0.40 \)) whereas the annual volatility for the present value of project 2 is 20%.
5. The appropriate cost of capital for each project is 12%.
6. The company has only 80 million to spend on new investment, so only one project may be selected.
7. With each project it is possible to contract operations by 40% at any time during the next two years. If operations are contracted, the salvage value (cash received) for project 1 is 33 million and 42 million for project 2.
8. The risk-free rate is 5%.

2.2 Analysis of the projects without the option to contract

Tables 3 and 4 record the respective project value event trees without flexibility. For project 1 the value for \( u = e^{\sigma(\Delta t)^{1/2}} = e^{0.40(1)^{1/2}} = 1.4918 \) (and so \( d = 1/u = 0.6703 \)) and for project 2 the value for \( u = e^{\sigma(\Delta t)^{1/2}} = e^{0.20(1)^{1/2}} = 1.2214 \) (and so \( d = 1/u = 0.8187 \)).

With an initial investment of 80 million, each project has an NPV of 20 million, which is positive. Therefore, the company should invest in one of the projects.

For subsequent reference the value event tree for each project has 6 “nodes”, which we shall label, respectively, as \( \{ 0, U, D, UU, UD = DU, DD \} \).
Table 3: Project 1 value event tree without flexibility

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>149.18</td>
<td>222.55</td>
</tr>
<tr>
<td></td>
<td></td>
<td>67.03</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>44.93</td>
</tr>
</tbody>
</table>

Table 4: Project 2 value event tree without flexibility

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>122.14</td>
<td>149.18</td>
</tr>
<tr>
<td></td>
<td></td>
<td>81.87</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>67.03</td>
</tr>
</tbody>
</table>

2.3 DTA Analysis

DTA uses the objective probabilities and the cost of capital for the project without flexibility. Let $p_i$ denote the probability for the “up” state for project $i$, $i = 1, 2$. Since the cost of capital for each project is 12%, it follows that

\[
\frac{149.18p_1 + 67.03(1 - p_1)}{1.12} = 100 \implies p_1 = 0.547, \tag{17}
\]

\[
\frac{122.14p_2 + 81.87(1 - p_2)}{1.12} = 100 \implies p_2 = 0.748. \tag{18}
\]

Analysis of project 1

Table 5 records the NPV event tree with flexibility using DTA. Here are how these numbers were obtained.

Table 5: Project 1 NPV event tree with flexibility using DTA (* = contract)

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>22.47</td>
<td>149.18</td>
<td>222.55</td>
</tr>
<tr>
<td></td>
<td>*73.22</td>
<td>100.00</td>
<td>*59.96</td>
</tr>
</tbody>
</table>
We must work backwards through the value event tree. Assume the company has reached year 2 and has not yet exercised its option to contract. It does not pay to contract in states $UU$ and $UD$, and so their respective values remain the same at 222.55 and 100, respectively. In state $DD$ it does pay to contract since
\[ 0.6(44.93) + 33 = 59.96 > 44.93, \]
and so the value here is 59.96.

Now let’s move back to year 1, and suppose the company has not yet exercised its option to contract. In state $U$ it would not pay to contract. Now consider state $D$. If the company chooses to contract now, then the value would be
\[ 0.6(67.03) + 33 = 73.22. \]
The company can, however, choose not to contract, and then the value would be
\[ \frac{0.547(100) + 0.453(59.96)}{1.12} = 73.09. \]
The best choice at state $D$ is therefore to contract, and the associated value is 73.22.

Now let’s move back to year 0. Its value is
\[ \frac{0.547(149.18) + 0.453(73.22)}{1.12} = 102.47, \]
which gives an $NPV$ of $102.47 - 80 = 22.47$.

According to $DTA$ the option to contract is worth 2.47. Moreover, according to $DTA$ the option to contract should be exercised in year 1 if state $D$ occurs.

**Analysis of project 2**

Table 6 records the NPV event tree with flexibility using DTA. Here are how these numbers are obtained.

<table>
<thead>
<tr>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>22.36</td>
<td>122.58</td>
<td>149.18</td>
</tr>
<tr>
<td>*91.09</td>
<td>*102.00</td>
<td>*82.22</td>
</tr>
</tbody>
</table>
We must work backwards through the value event tree. Assume the company has reached year 2 and has not yet exercised its option to contract. It does not pay to contract in state \( UU \), and so its value remains the same at 222.55. In state \( UD \) it does pay to contract since
\[
0.6(100) + 42 = 102 > 100, \quad (23)
\]
and so the value in state \( UD \) is 102. In state \( DD \) it also pays to contract since
\[
0.6(44.93) + 42 = 82.22 > 67.03, \quad (24)
\]
and so the value in state \( DD \) is 82.22.

Now let’s move back to year 1, and suppose the company has not yet exercised its option to contract. In state \( U \) it would not pay to contract. Its (new) \( PV \) is
\[
\frac{0.748(149.18) + 0.252(102)}{1.12} = 122.58. \quad (25)
\]
Now consider state \( D \). If the company chose to contract \textit{now}, then the value would be
\[
0.6(81.87) + 42 = 91.09. \quad (26)
\]
The company can, however, choose \textit{not} to contract, and then the value would be
\[
\frac{0.748(102) + 0.252(82.22)}{1.12} = 85.29. \quad (27)
\]
The best choice is therefore to contract in state \( D \), and the associated value is 91.09.

Now let’s move back to year 0. Its value according to \textit{DTA} is
\[
\frac{0.748(122.58) + 0.252(91.09)}{1.12} = 102.36, \quad (28)
\]
which gives an \textit{NPV} of 102.36 − 80 = 22.36.

According to \textit{DTA} the option to contract for project 2 is worth 2.36. Moreover, according to \textit{DTA} the option to contract should be exercised in year 1 if state \( D \) occurs or in year 2 if state \( UD \) occurs.

Since the \textit{NPV} for project 1 (2.47) is higher than that for project 2 (2.36), the company should invest in project 1 \textit{according to DTA}.

### 2.4 ROA Analysis

\textit{ROA} uses the \textit{risk-neutral} probabilities and the risk-free rate to perform the discounted expectation for valuation purposes when working backwards through the \textit{PV} tree. Using \textit{ROA} requires that an underlying marketable security be identified that can be used to construct the replicating portfolio; here, we take the project without flexibility as such a security. The project and the money complete the market, i.e., span all possible valuations on the event tree, which then implies risk-neutral valuation will work and provide the \textit{correct} valuation.
Table 7: Project 1 NPV event tree with flexibility using ROA (* = contract)

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>23.92</td>
<td>149.18</td>
<td>222.55</td>
</tr>
<tr>
<td></td>
<td>74.72</td>
<td>100.00</td>
<td>*59.96</td>
</tr>
</tbody>
</table>

Analysis of project 1

Table 7 records the Project 1 NPV event tree with flexibility using $ROA$. Here are how these numbers are obtained.

For project 1 the risk-neutral probability $q_1$ must satisfy

$$\frac{149.18q_1 + 67.03(1 - q_1)}{1.05} = 100 \implies q_1 = 0.462.$$  \hspace{1cm} (29)

We now proceed, as before.

Assume the company has reached year 2 and has yet to have exercised its option to contract. The terminal values in year 2 are unaffected here by the change in probabilities and discount rate. The values for states $\{UU, UD, DD\}$ are, respectively, $\{222.55, 100, 59.96\}$. (If the operations have not been contracted by year 2, then the company should contract them in state $DD$.)

Now let’s move back to year 1, and suppose the company has not yet exercised its option to contract. In state $U$ it would not pay the company to contract. In this state the value will still be

$$149.18 = \frac{0.462(222.55) + 0.538(100)}{1.05}. \hspace{1cm} (30)$$

Now consider state $D$. If the company chooses to contract now, then the value would be

$$0.6(67.03) + 33 = 73.22,$$  \hspace{1cm} (31) which has not changed. The company can, however, choose not to contract, and then the value according to $ROA$ would be

$$\frac{0.462(100) + 0.538(59.96)}{1.05} = 74.72.$$  \hspace{1cm} (32)

The best choice is therefore not to contract, and the associated value is 74.72. (Here is an example of a difference between recommendations using $DTA$ and $ROA$.)
Table 8: Project 2 NPV event tree with flexibility using ROA (* = contract)

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>24.16</td>
<td>122.93</td>
<td>149.18</td>
</tr>
<tr>
<td>*</td>
<td>91.09</td>
<td>102.00</td>
<td>82.22</td>
</tr>
</tbody>
</table>

Now let’s move back to year 0. Its value according to $ROA$ is

$$\frac{0.462(149.18) + 0.538(74.72)}{1.05} = 103.92,$$

which gives an $NPV$ of $103.92 - 80 = 23.92$. The correct value of the option to contract is 3.92 for Project 1. The company should only contract operations if it reaches state $DD$ in year 2.

Analysis of project 2

Table 8 records the Project 2 NPV event tree with flexibility using $ROA$. Here are how these numbers are obtained.

For project 2 the risk-neutral probability $q_2$ must satisfy

$$\frac{122.14q_2 + 81.87(1 - q_2)}{1.05} = 100 \implies q_2 = 0.574.$$  \hfill (34)

We now proceed, as before.

Assume the company has reached year 2 and has not yet exercised its option to contract. The terminal values in year 2 are unaffected here by the change in probabilities and discount rate. The values for states $\{UU, UD, DD\}$ are, respectively, $\{149.18, 102, 82.22\}$. (If the company has not contacted operations by year 2, then it should contract them if either state $UD$ or $DD$.)

Now let’s move back to year 1, and suppose the company has not yet exercised its option to contract. In state $U$ it would not pay the company to contract. In this state the value will be

$$\frac{0.574(149.18) + 0.426(102)}{1.05} = 122.93.$$  \hfill (35)

(This value will change since the value for state $UD$ changed.) Now consider state $D$. If the company chooses to contract now, then the value would be

$$0.6(81.87) + 42 = 91.09,$$

\hfill (36)
which has not changed. The company can, however, choose not to contract, and then the value according to ROA would be

\[
\frac{0.574(102) + 0.426(82.22)}{1.05} = 89.12. \tag{37}
\]

The best choice is therefore to contract, and the associated value is still 91.09.

Now let’s move back to year 0. Its value according to ROA is

\[
\frac{0.574(122.93) + 0.426(91.09)}{1.05} = 104.16, \tag{38}
\]

which gives an NPV of 104.16 – 80 = 24.16. The correct value of the option to contract is 4.16 for Project 1. The company should contract operations if it reaches state D in year 1.

### 2.5 Comparison of DTA to ROA

DTA recommends Project 1 since 22.47 > 22.36, whereas ROA recommends Project 2 since 24.16 > 23.92. (They are close.) The optimal contract policy changes, too.
Table 9: Project value event tree without flexibility

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
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<tr>
<td>100</td>
<td>74.08</td>
<td>100.00</td>
<td>134.99</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>54.88</td>
<td>74.08</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>40.66</td>
<td></td>
</tr>
</tbody>
</table>

3 Option to Abandon Operations

3.1 Background information

A company is considering investing in a project. Relevant information is as follows:

1. The project’s present value without flexibility is 100 million.
2. The project pays no dividends.
3. The project’s present value without flexibility has an annual volatility of 30%.
4. The project requires an initial investment of 108 million.
5. The appropriate cost of capital for the project is 15%.
6. The company has the option to abandon operations at any time during the next 3 years, and receive an after-tax cash flow (salvage value) of 90 million.
7. The risk-free rate is 5%.

3.2 Analysis of project without flexibility

The project’s NPV is $100 - 108 = -8 < 0$, and so this the company should not invest in the project without flexibility.

For subsequent reference Table 9 records the Project value event tree for the project without flexibility. It has 10 “nodes”, which we shall label, respectively, as \{0, U, D, UD = DU, DD, UU, UUD = UDU = DDU = UDD = DUD = DDU, DDD\}. The value for $u$ is $e^{0.30} = 1.3499$ and $d$ is $e^{-0.30} = 0.7408$. 
Table 10: Project NPV event tree with flexibility using DTA (* = abandon)

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-2.69</td>
<td>136.29</td>
<td>104.55</td>
<td>245.96</td>
</tr>
<tr>
<td>*90.00</td>
<td></td>
<td>*90.00</td>
<td></td>
<td>*90.00</td>
</tr>
<tr>
<td></td>
<td>*90.00</td>
<td></td>
<td>*90.00</td>
<td></td>
</tr>
</tbody>
</table>

3.3 DTA analysis

DTA uses the objective probabilities and the cost of capital of the project without flexibility. Let $p$ denote the probability for the “up” state for the project. We must have

$$\frac{134.99p + 74.08(1 - p)}{1.15} = 100 \quad \Rightarrow \quad p = 0.672. \quad (39)$$

Table 10 records the NPV event tree with flexibility using DTA. Here are how these numbers were obtained.

We work backwards through the value event tree. Suppose it is year 3 and the company has yet to exercise its option to abandon. This option will not be executed in states $UUU$ and $UUD$, and so their values remain the same. In states $UDD$ and $DDD$ it does pay to execute the option, and the values in both states are 90.

Now let’s go back to year 2. In state $UU$ it does not pay to execute the option, and its value will remain the same. It also does not pay to execute the option in state $UD$; its value is

$$\frac{0.672(134.99) + 0.328(90)}{1.15} = 104.55. \quad (40)$$

In state $DD$ it should be clear that the option should be executed, as it is better to receive the 90 now instead of next year. (In both future states from state $DD$ the value is 90.)

Now let’s go back to year 1. In state $U$ the option will not be executed; its value is

$$\frac{0.672(182.21) + 0.328(104.55)}{1.15} = 136.29. \quad (41)$$

Now consider state $D$. If the option to abandon is executed the company receives 90. However, the company can choose not to abandon now; the value if the company choose not to abandon is

$$\frac{0.672(104.55) + 0.328(90)}{1.15} = 86.76. \quad (42)$$

Thus, the option to abandon should be executed in state $D$, and its value in this state is 90.
Table 11: Project NPV event tree with flexibility using ROA (* = abandon)

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3.14</td>
<td>138.52</td>
<td>182.21</td>
<td>245.96</td>
</tr>
<tr>
<td></td>
<td>94.17</td>
<td>107.88</td>
<td>*90.00</td>
<td>*90.00</td>
</tr>
<tr>
<td></td>
<td>*90.00</td>
<td>*90.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now let’s go back to year 0. The value is

$$
\frac{0.672(136.29) + 0.328(90)}{1.15} = 105.31. \tag{43}
$$

The $NPV$ of the project is therefore $105.31 - 108 = -2.69 < 0$, and so the project should be rejected.

According to $DTA$ the value of the option to abandon is worth 5.31 since the change in $NPV$ is $-2.69 - (-8) = 5.31$.

3.4 ROA analysis

For this project the risk-neutral probability $q$ must satisfy

$$
\frac{134.99p + 74.08(1 - p)}{1.05} = 100 \implies q = 0.508. \tag{44}
$$

Table 11 records the NPV event tree with flexibility using $ROA$. Here are how these numbers were obtained.

We work backwards through the value event tree. Assume the company has reached year 3 and has not yet exercise its option to abandon. The terminal values in year 3 are unaffected here by the change in probabilities and discount rate. The values for states \{UUU, UUD, UDD, DDD\} are, respectively, \{245.96, 134.99, 90, 90\}. (If the company has not abandoned operations by year 3 it should abandon them if it reaches state $UDD$ or $DDD$.)

Now let’s go back to year 2, and suppose the company has not yet exercised its option to abandon. In state $UU$ it does not pay to abandon, and its value will remain the same. It also does not pay to abandon in state $UD$; its value is

$$
\frac{0.508(134.99) + 0.492(90)}{1.05} = 107.48. \tag{45}
$$

In state $DD$ it should be clear that the company should abandon, as it is better to receive the 90 now instead of next year. (In both future states from state $DD$ the value is 90.)
Now let’s go back to year 1. In state $U$ the option will not be executed; its value is
\[
\frac{0.508(182.21) + 0.492(107.48)}{1.15} = 138.52. \tag{46}
\]
Now consider state $D$. If the company abandons it receives 90. However, the company can choose not to abandon now; the value if the company continues the project is
\[
\frac{0.508(107.48) + 0.492(90)}{1.05} = 94.17. \tag{47}
\]
Thus, the option to abandon should \textit{not} be executed in state $D$, and its value in this state is 94.17.

Now let’s go back to year 0. The value is
\[
\frac{0.508(138.52) + 0.492(94.17)}{1.05} = 111.14. \tag{48}
\]
The \textit{NPV} of the project is therefore $111.14 - 108 = 3.14 > 0$, and so the project should be accepted.

It may be seen that the abandonment option here is equivalent to a three year American put option with strike price equal to 90.

### 3.5 Comparison of DTA to ROA

\textit{ROA} accepts the project; \textit{DTA} rejects the project. \textit{ROA} would not abandon the project if state $D$ is reached in year 1, whereas \textit{DTA} would.
4 The Installment Option

4.1 Background information

A company is considering investing in two projects. Background information is as follows.

1. The present value of each project without flexibility is 100 million.
2. Each project pays no dividends.
3. The project value for each project without flexibility has an annual volatility of 80%.
4. The risk-free rate is 5%.
5. The present value (at the risk-free rate) of the required investment for each project is 102 million.
6. The required investment for project 1 is staged over time, as follows: 52 million at time 0, 21 million at time 1 and 33.075 million at time 2. The required investment for project 2 is staged over time, too, as follows: 22 million at time 0, 31.5 million at time 1 and 55.125 million at time 2.
7. For both projects their respective investments must be made to continue the project for the upcoming year. These investments may be viewed as planned. However, at any time the company has the option to “default” on its planned investment at which point the project is terminated.
8. The appropriate cost of capital for the project (without flexibility) is 20%.

4.2 Analysis of the projects without flexibility

Both projects have an NPV of $100 - 102 = -2 < 0$, and so the company should not invest in either project.

Table 12 records the project value event tree for both projects. Here, $u = e^{0.80} = 2.2255$ and $d = 1/u = 0.4493$. 

4.3 ROA analysis

Analysis of project 1

Let \( q \) denote the risk-neutral probability for the “up” state. We must have

\[
\frac{222.55q + 44.93(1-q)}{100} = 1.05 \implies q = 0.338. \tag{49}
\]

Table 13 records the Project 2 NPV event tree with flexibility using ROA. Here are how these numbers were obtained.

We must work backwards through the value event tree. Assume the company has reached year 2 and has not yet defaulted on its planned investments. In states \( UU \) and \( UD \) it pays the company to make its final planned investment, and so their values in these states become, respectively, 462.22 and 66.92. It does not pay to make the final investment in state \( DD \), and so its value becomes 0.

Now let’s move back to year 1, and suppose the company has not yet defaulted on its planned investments. In state \( U \) it pays the company to make the investment. Its (new) value would be

\[
\frac{0.338(462.22) + 0.662(66.92)}{1.05} - 21 = 169.98. \tag{50}
\]

Now consider state \( D \). If the company chooses to continue with the project now, then the value at this state would be

\[
\frac{0.338(66.92) + 0.662(0)}{1.05} - 21 = 0.54. \tag{51}
\]
Table 14: Project 2 NPV event tree with flexibility using ROA (* = default)

<table>
<thead>
<tr>
<th></th>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>22.58</td>
<td>138.48</td>
<td>440.17</td>
</tr>
<tr>
<td>*</td>
<td>0.00</td>
<td>*0.00</td>
<td>44.87</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>*0.00</td>
</tr>
</tbody>
</table>

Since this value is positive, the company should continue the project in state $D$.

Now let’s move back to year 0. Its value is

$$0.338(169.98) + 0.662(0.54) = 55.06,$$

which gives an $NPV$ of $55.06 - 52 = 3.06$. As the $NPV$ is positive, the project is accepted.

The value of the default option is $3.06 - (-2) = 5.06$. If state $DD$ is reached, then the final investment should not be made.

**Analysis of project 2**

Table 14 records the Project 2 NPV event tree with flexibility using ROA. Here are how these numbers were obtained.

We must work backwards through the value event tree. Assume the company has reached year 2 and has not yet defaulted on its planned investments. In states $UU$ and $UD$ it pays the company to make the final planned investment, and so their values in these states become, respectively, 440.17 and 44.87. It does not pay to make the final investment in state $DD$, and so its value becomes 0.

Now let’s move back to year 1, and suppose the company has not yet defaulted on its planned investments. In state $U$ it would pay the company to make the investment. Its (new) value would be

$$\frac{0.338(440.17) + 0.662(44.87)}{1.05} - 31.5 = 138.48.$$

Now consider state $D$. If the company chooses to continue with the project now, then the value at this state would be

$$\frac{0.338(44.87) + 0.662(0)}{1.05} - 31.5 = -16.61.$$

Since this value is negative, the company should not make the planned investment in state $D$. 19
Now let’s move back to year 0. Its value is

\[
\frac{0.338(138.48) + 0.662(0)}{1.05} = 44.58,
\]

which gives an \( NPV \) of 44.58 − 22 = 22.58. As the \( NPV \) is positive, the project should be accepted. The value of the default option is 22.58 − (−2) = 24.58. If state \( D \) is reached, then the project should be terminated.

\textit{Remark.} Project 2 has a much higher value than project 1 when the default option is in place. This is because its planned investments are “back loaded” as opposed to the first project’s planned investments, which are “front loaded”. If it is possible to delay investment while learning about the project’s value, then this will create extra value, as this example illustrates.
5 Switch Use Option

A chemical company currently uses technology X to manufacture its final product, which has recently experienced an increase in demand. The company wants to build a new factory to satisfy demand for the next two years. It is considering using technology X again. It is also considering adopting two other technologies, Y and Z.

5.1 Background information

1. Technology X. If the company adopts technology X the current net cash flow is 100 with an annual volatility of 33.65%. The objective probability of going “up” from each node is 0.5. The appropriate cost of capital for this type of risky cash flows is 9%. The initial investment required to build the factory with technology X is 100.

2. Technology Y. The cost of material inputs used in Y are less volatile than those used in X, but the quality of the final good is less, and so it commands a lower (albeit less volatile) price. If technology Y is adopted, its current net cash flow would also be 100, but its annual volatility would only be 9.53%. The initial investment required to build the factory would also be 100 if technology Y were adopted.

3. Technology Z. Should it pay to do so, the chemical company also has the opportunity to acquire “flexible” technology Z, which can switch (at a cost) back and forth between technologies X and Y. The changeover cost from X to Y is 15, whereas it only costs 10 to switch from Y to X. The initial investment required to build the factory would be 110 if the flexible technology Z were adopted.

4. The risk-free rate is 5%.

5.2 ROA Analysis

Technology X

Since technology X has been in use for some time we can use it to model the underlying security. But, we have to be careful in what we model here.

The first step here is to determine the cash flow event tree. The values in this tree are recorded in Table 15. Here, \( u = e^{0.3365} = 1.4000 \) and \( d = 1/u = 0.7142 \).

The cash flow event tree is not the value event tree. The value event tree at each node represents the present values of future discounted expected cash flows. We shall use the “Running PV” approach. We shall also use the objective probabilities with the appropriate cost of capital,
Table 15: Technology X cash flow event tree

<table>
<thead>
<tr>
<th></th>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>140.00</td>
<td>196.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>71.42</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>51.02</td>
</tr>
</tbody>
</table>

Table 16: Technology X value event tree

<table>
<thead>
<tr>
<th></th>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>291.05</td>
<td>275.78</td>
<td>196.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>140.70</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>51.02</td>
</tr>
</tbody>
</table>

since the cash flows with technology X are assumed to be marketed. (We cannot use risk-neutral probabilities here as there is no other underlying security from which to make this calculation.)

Table 16 records the value event tree. Here are how these numbers were obtained. The PV’s at states \{UU, UD, DD\} in year 2 correspond to their net cash flows, namely, they are \{196, 100, 51.02\}, respectively. Now consider state U in year 1. Its value is

$$\frac{0.5(196) + 0.5(100)}{1.09} + 140 = 275.78.$$  \hspace{1cm} (56)

Similarly, the value in state D in year 2 is

$$\frac{0.5(100) + 0.5(51.02)}{1.09} + 71.42 = 140.70.$$  \hspace{1cm} (57)

In year 0 the PV is

$$\frac{0.5(275.78) + 0.5(140.70)}{1.09} + 100 = 291.05.$$  \hspace{1cm} (58)

To perform ROA one needs an underlying marketable security. We shall use the value event tree for X as our marketed security (along with the money market) upon which to build a replicating portfolio.

Constructing portfolios with X and the associated calculation of the risk-neutral probability is different here because the value event tree for X pays dividends. For example, one cannot meaningfully represent the value at time 0, 291.05, as a discounted expectation of the respective
future values 275.78 and 140.70 at time 1. Examining the calculation of 291.05 in (58), it could just as easily been determined via
\[ \frac{q(275.78) + (1-q)(140.70)}{1.05} + 100 = 291.05 \implies q = 0.443. \]

The cash flow at time 0, 100, does not figure into the calculation of the risk-neutral probability. (It did not figure into the discounted expectation when we used the objective probabilities with the cost of capital.) When constructing a portfolio with X, one assumes it is being valued ex-dividend; that is, after the dividend payout of 100. The ex-dividend cost of X at time 0 is 191.05. The number 291.05 is called the cum-dividend value.

You may verify that the risk-neutral probability of 0.443 will be the same at every node. For example,
\[ \frac{0.443(196.00) + 0.557(100.00)}{1.05} + 140 = 275.78 \]
\[ \frac{0.443(100.00) + 0.557(51.02)}{1.05} + 71.42 = 140.70. \]

For the specific type of cash flow event tree here this fact will always be true. Note that the risk-neutral probability that would be calculated using the cash flow event tree directly is
\[ \frac{105 - 71.42}{140 - 71.42} = 0.4896, \]
which is definitely not equal to the correct value 0.443. One must always calculate the risk-neutral probability on the event tree coinciding with the tradable/marketable security.

**Technology Y**

Once again to determine the value event tree we first must determine the cash flow event tree. Table 17 records the cash flow event tree. Here, \( u = e^{0.0953} = 1.1000 \) and \( d = 1/u = 0.9091 \).

It is tempting at this point to proceed as we did for technology X; however, the cash flow stream for Y is clearly less volatile than that for X so the cost of capital should be lower. In lieu
Table 18: Technology Y value event tree

<table>
<thead>
<tr>
<th></th>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>284.19</td>
<td>214.10</td>
<td>121.00</td>
</tr>
<tr>
<td></td>
<td>176.94</td>
<td>100.00</td>
<td>82.64</td>
</tr>
</tbody>
</table>

of estimating the cost of capital for \( Y \), we will use the value event tree for \( X \) as our marketed security (along with the money market) upon which to build a replicating portfolio.

Now we are in position to properly value the present values of future discounted expected cash flows for technology \( Y \). We use the “Running PV” approach using the risk-neutral probabilities. Table 18 records the value event tree. Here are how these numbers were obtained.

The \( PV \)'s at states \{\(UU\), \(UD\), \(DD\)\} in year 2 correspond to their net cash flows, namely, they are \{121, 100, 82.64\}, respectively. Now consider state \( U \) in year 1. Its value is

\[
\frac{0.443(121) + 0.557(100)}{1.05} + 110 = 214.10. \tag{63}
\]

Similarly, the value in state \( D \) in year 2 is

\[
\frac{0.443(100) + 0.557(82.64)}{1.05} + 90.91 = 176.94. \tag{64}
\]

In year 0 the \( PV \) is

\[
\frac{0.443(214.10) + 0.557(176.94)}{1.05} + 100 = 284.19. \tag{65}
\]

\textit{Remark.} To determine the appropriate cost of capital \( k_Y \) for technology \( Y \) we use the objective probabilities; that is, \( k_Y \) satisfies

\[
\frac{0.5(214.10) + 0.5(176.94)}{1 + k_Y} = 184.19 \implies k_Y = 6.15\%. \tag{66}
\]

It will be the same at any node along the tree. (If one uses the risk-neutral probabilities, the cost of capital would be simply 5\%.)

**Technology Z**

Table 19 records the value event tree corresponding to technology \( Z \) \textit{assuming the switching costs are zero}. (Assume the value event trees for \( X \) and \( Y \) correspond to the same states of nature.) Here are how these numbers were calculated.
Table 19: Technology Z value event tree with no switching costs. The letter(s) in parentheses indicate the optimal technology to employ at each state.

<table>
<thead>
<tr>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X/Y) 310.22</td>
<td>(X) 275.78</td>
<td>(X) 196.00</td>
</tr>
<tr>
<td></td>
<td>(Y) 176.94</td>
<td>(X/Y) 100.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(Y) 82.64</td>
</tr>
</tbody>
</table>

Table 20: Technology Z value event tree with switching costs (* = switch)

<table>
<thead>
<tr>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>X Y</td>
<td>X Y</td>
<td>X Y</td>
</tr>
<tr>
<td>302.26</td>
<td>306.00</td>
<td>275.78</td>
</tr>
<tr>
<td></td>
<td></td>
<td>*265.78</td>
</tr>
<tr>
<td>*161.94</td>
<td>176.94</td>
<td>196.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>*186.00</td>
</tr>
<tr>
<td></td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td>*67.64</td>
<td>82.64</td>
</tr>
</tbody>
</table>

With no changeover costs it is clearly optimal to use the technology with the highest net cash flow each period depending on the state of nature. In year 2 the PV’s corresponding to states \{UU, UD, DD\} are, respectively, \{196, 100, 82.64\}. Now consider state U in year 1. Its value is

\[
\frac{0.443(196) + 0.557(100)}{1.05} + 140 = 275.78. \tag{67}
\]

The value value for state D in year 1 is

\[
\frac{0.443(100) + 0.557(82.64)}{1.05} + 90.91 = 176.94. \tag{68}
\]

In year 0 the PV value is

\[
\frac{0.443(275.78) + 0.557(176.94)}{1.05} + 100 = 310.22. \tag{69}
\]

When there are changeover costs the calculations become more involved, as it is necessary to know what technology is currently in operation in order to decide if it is optimal to switch use. Table 20 records the value event tree with switching costs. Here are how these numbers were obtained.

Let \( V_{tX}(\cdot) \), resp. \( V_{tY}(\cdot) \), denote the value function in year \( t \) given the company enters year \( t \) using technology \( X \) (resp. \( Y \)). In state \( UU \) in year 2 it pays the company to switch from \( Y \) to \( X \), and in state \( DD \) in year 2 it pays the company to switch from \( X \) to \( Y \); otherwise, it
does not pay to switch use. Thus, \( V_{2X}(UU) = 196; \ V_{2X}(UD) = 100; \ V_{2X}(DD) = 67.64, \) and \( V_{2Y}(UU) = 186; \ V_{2Y}(UD) = 100; \ V_{2Y}(DD) = 82.64. \)

Now consider year 1. In each state \( U \) or \( D \) the company must decide whether to switch assuming that it entered each state using either technology \( X \) or \( Y \). That is, four separate calculations must be performed, as follows:

\[
\begin{align*}
V_{1X}(U) &= \text{Max}\{140 + \frac{0.443V_{2X}(UU) + 0.557V_{2X}(UD)}{1.05}; \ (110 - 15) + \frac{0.443V_{2Y}(UU) + 0.557V_{2Y}(UD)}{1.05} \} \\
&= 275.78; \tag{70} \\
V_{1Y}(U) &= \text{Max}\{110 + \frac{0.443V_{2Y}(UU) + 0.557V_{2Y}(UD)}{1.05}; \ (140 - 10) + \frac{0.443V_{2X}(UU) + 0.557V_{2X}(UD)}{1.05} \} \\
&= *265.78; \tag{71} \\
V_{1X}(D) &= \text{Max}\{71.42 + \frac{0.443V_{2X}(DU) + 0.557V_{2X}(DD)}{1.05}; \ (90.91 - 15) + \frac{0.443V_{2Y}(DU) + 0.557V_{2Y}(DD)}{1.05} \} \\
&= *161.94; \tag{72} \\
V_{1Y}(D) &= \text{Max}\{90.91 + \frac{0.443V_{2Y}(DU) + 0.557V_{2Y}(DD)}{1.05}; \ (71.42 - 10) + \frac{0.443V_{2X}(DU) + 0.557V_{2X}(DD)}{1.05} \} \\
&= 176.94. \tag{73}
\end{align*}
\]

The asterisk symbol denotes the decision to switch use.

Now in year 0 one computes:

\[
\begin{align*}
V_{0X} &= \text{Max}\{100 + \frac{0.443V_{1X}(U) + 0.557V_{1X}(D)}{1.05}; \ (100 - 15) + \frac{0.443V_{1Y}(U) + 0.557V_{1Y}(D)}{1.05} \} = 302.26; \tag{74} \\
V_{0Y} &= \text{Max}\{100 + \frac{0.443V_{1Y}(U) + 0.557V_{1Y}(D)}{1.05}; \ (100 - 10) + \frac{0.443V_{1X}(U) + 0.557V_{1X}(D)}{1.05} \} = 306.00. \tag{75}
\end{align*}
\]

At time 0 the value of technology \( Z \) is 302.26 if its initial state is technology \( X \) or 306.00 if its initial state is technology \( Y \). Obviously, if one has a choice, it would be preferable to begin in technology \( Y \).

The \( NPV \) for technology \( X \) is 191.05, the \( NPV \) for technology \( Y \) is 184.19, and the \( NPV \) for technology \( Z \) is either 192.26 or 196.00. In any case, technology \( X \) is preferred to technology \( Y \), and technology \( Z \) is preferred to \( X \) regardless of its initial state.