A Verification Based Method to Generate Cutting Planes for IPs

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SIAM Conference on Optimization, 2011

Outline

1
Cutting Plane Operators

A basic question in integer programming

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Given a closed convex set $P \subseteq \mathbb{R}^n$, obtain

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2. Typically we are happy to obtain relaxation of the convex hull of $P \cap \mathbb{Z}^n$, i.e., $T \subseteq \mathbb{R}^n$ s.t.

$$P \supseteq T \supseteq P_I. \tag{1}$$

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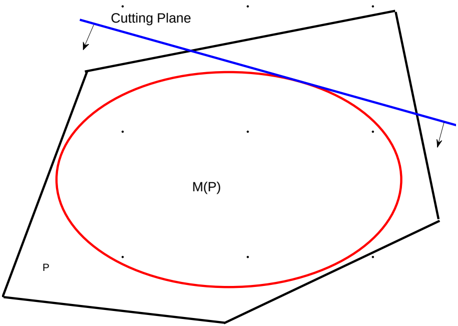
$$P \supseteq T \supseteq P_I. \quad (1)$$
3. Let M be an *cutting plane operator* applied to P to obtain a (closed) convex relaxation of $(P \cap \mathbb{Z}^n)$, i.e.
$$P \supseteq M(P) \supseteq P_I. \quad (2)$$

Cutting planes: the way we use these operators

- If $\langle a, x \rangle \leq b$ is a valid inequality for $M(P)$, then it is a valid inequality for $(P \cap \mathbb{Z}^n)$.



Examples of some well-known cutting-plane operators

- In this presentation, “$P \subseteq \mathbb{R}^n$" is always a rational polytope.
- Some operators for general integer programs:
  1. Gomory-Chvátal closure (GC)
  2. Split disjunctive closure (SC)
- Some operators for $0-1$ integer programs, i.e, $P \subseteq [0,1]^n$ (Lovász-Schrijver operators):
  1. Lift-and-project operator (N_0)
  2. N
  3. N_+

Admissible cutting-plane procedures for 0-1 polytopes

All known '*reasonable*' cutting plane operators satisfy the following properties:

Definition

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5. SINGLE COORDINATE ROUNDING: If $x_i \leq \epsilon < 1$ (or $x_i \geq \epsilon > 0$) is valid for P, then $x_i \leq 0$ (or $x_i \geq 1$) is valid for $\mathrm{M}(P)$.

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6. COMMUTING WITH COORDINATE FLIPS AND DUPLICATIONS: $\tau_i(\mathrm{M}(P)) = \mathrm{M}(\tau_i(P))$, where τ_i is either one of the following two operations: (i) Coordinate flip: $\tau_i : [0,1]^n \to [0,1]^n$ with $(\tau_i(x))_i = (1 - x_i)$ and $(\tau_i(x))_j = x_j$ for $j \in [n] \setminus \{i\}$; (ii) Coordinate Duplication: $\tau_i : [0,1]^n \to [0,1]^{n+1}$ with $(\tau_i(x))_{n+1} = x_i$ and $(\tau_i(x))_j = x_j$ for $j \in [n]$.

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7. SHORT VERIFICATION: There exists a polynomial p such that for any inequality $\langle c, x \rangle \leq d$ that is valid for $M(P)$ there is a set $I \subseteq [m]$ with $|I| \leq p(n)$ such that $\langle c, x \rangle \leq d$ is valid for $M(\{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \leq b_i, i \in I\})$.

Definition is modified from [Pokutta and Schulz (2009)]

Admissible cutting-plane procedures for general polytopes

Definition

If M is defined for general rational polytopes $P \subseteq \mathbb{R}^n$, then we say M is admissible if

- M satisfies (1.)-(7.) when restricted to polytopes contained in $[0, 1]^n$ and
- for general polytopes $P \subseteq \mathbb{R}^n$, M satisfies VALIDITY, INCLUSION PRESERVATION, SHORT VERIFICATION and HOMOGENEITY is replaced by

  8. STRONG HOMOGENEITY: If $P \subseteq F^{\leq} := \{x \in \mathbb{R}^n \mid \langle a, x \rangle \leq b\}$ and $F = \{x \in \mathbb{R}^n \mid \langle a, x \rangle = b\}$ where $(a, b) \in \mathbb{Z}^n \times \mathbb{Z}$, then $\mathrm{M}(F \cap P) = \mathrm{M}(P) \cap F$.

Definition

If M satisfies all required properties for being admissible except SHORT VERIFICATION, then we say M is *almost admissible*.

Definition is modified from [Pokutta and Schulz (2009)]

2

Generating Cutting Plane Differently: Design and Verify

The Computation Scheme

Input :		Black box		*Output* :
$P := \{x \in \mathbb{R}^n \mid Ax \le b\}$	$\rightarrow$	operator M	$\rightarrow$	$\langle c, x\rangle \le d$ valid for $M(P)$

Computation vs. Verification Scheme

Computation Scheme

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Computation vs. Verification Scheme

Computation Scheme

Input : $P := \{x \in \mathbb{R}^n \,|\, Ax \leq b\}$ $\rightarrow$ Black box operator M $\rightarrow$ *Output* : $\langle c, x \rangle \leq d$ valid for $M(P)$

Verification Scheme

Input : $P := \{x \in \mathbb{R}^n \,|\, Ax \leq b\}$ $\rightarrow$ Unfaithful Oracle to generate cuts $\rightarrow$ Inequality *claimed* to be valid : $\langle c, x \rangle \leq d$

$\downarrow$

Use Black Box operator to verify validity

$\downarrow$

If valid, then Output : $\langle c, x \rangle \leq d$

A similar idea discussed in Cook, Coullard, Turán (1987).

How to verify validity of cut using black box operator

Assuming $c \in \mathbb{Z}^n$, $d \in \mathbb{Z}$:

$$\langle c, x \rangle \leq d \text{ is valid for } P_I$$

$$\Updownarrow$$

$$\left(P \cap \{x \in \mathbb{R}^n \mid \langle c, x \rangle \geq d + 1\}\right)_I = \emptyset$$

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$$\Uparrow$$

$$M\left(P \cap \{x \in \mathbb{R}^n \,|\, \langle c, x \rangle \geq d + 1\}\right) = \emptyset$$

So if $M(P \cap \{x \in \mathbb{R}^n \,|\, \langle c, x \rangle \geq d + 1\}) = \emptyset$, then we declare $\langle c, x \rangle \leq d$ is a valid inequality.

Question:

How much do we gain (if at all?) from having to only verify that a given inequality is valid for P_I, rather than actually computing it.

Why is verification scheme interesting

Sufficient condition for $\langle c, x \rangle \leq d$ to be valid for P_I:

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(i) $M(P) \;\cap\; \{x \mid \langle c, x\rangle \geq d + 1\} = \emptyset$ (computation)

Why is verification scheme interesting

Sufficient condition for $\langle c, x \rangle \leq d$ to be valid for P_I:

(i) $M(P) \cap \{x \mid \langle c, x \rangle \geq d + 1\} = \emptyset$ (computation)

(ii) $M(P \cap \{x \mid \langle c, x \rangle \geq d + 1\}) = \emptyset$ (verification)

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The strength of the verification scheme lies in the following inclusion that is satisfied by every "reasonable" cutting plane operator.

$$M\left(P \cap \{x \in \mathbb{R}^n \mid \langle c, x\rangle \geq d+1\}\right) \subseteq M(P) \cap \{x \in \mathbb{R}^n \mid \langle c, x\rangle \geq d+1\}.$$

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This inclusion can be strict!

1. That is, suppose there exists $(c^0, d^0) \in \mathbb{Z}^{n+1}$ such that

$$M(P \cap \{x \mid \langle c^0, x\rangle \geq d^0 + 1\}) = \emptyset \quad \text{and} \tag{3}$$

$$M(P) \cap \{x \mid \langle c^0, x\rangle \geq d^0 + 1\} \neq \emptyset \tag{4}$$

2. Then $\langle c^0, x\rangle \leq d^0$ is verifiable but not computable.

3
Verification Closure and Basic Properties

Verification Closure: Definition and Basic Property

Verification Closure

Definition

Let M be admissible. Then

$$\partial M(P) := \bigcap_{(c,d)\in\mathbb{Z}^{n+1}} \{\langle c, x\rangle \le d \mid M(P \cap \{\langle c, x\rangle \ge d+1\}) = \emptyset\} \tag{5}$$

is the *verification scheme closure of M*.

We add all inequalities whose validity can be verified with application of M.

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Theorem

If M is admissible, then ∂M is almost admissible.

Comparing closures

Properties of general admissible M

Theorem

1. $\partial M(P) \subseteq M(P)$ *for all rational polytopes* P*. There exists a rational polytope* Q*, such that* $\partial M(Q) \subsetneq M(Q)$.
2. $\partial M(P) \subseteq GC(P) \cap N_0(P)$ *for all rational polytopes* P*. There exists a rational polytope* Q*, such that* $\partial M(Q) \subsetneq GC(Q) \cap N_0(Q)$.

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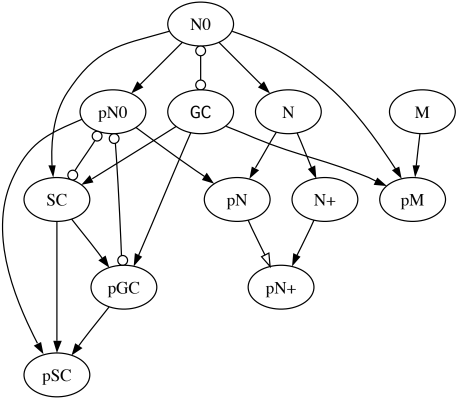
Comparing specific closures

Theorem

Let L *and* M *be admissible cutting plane operators such that* $L(P) \subseteq M(P)$ *for all rational polytopes* P*. Then* $\partial L(P) \subseteq \partial M(P)$ *for all rational polytopes* P*. Moreover,*

1. $\partial GC(P) \subseteq SC(P)$ *for all rational polytopes* P*. There exists a rational polytope* Q*, such that* $\partial GC(Q) \subsetneq SC(Q)$.
2. *There exist polytopes* Q^1 *and* Q^2*, such that* $\partial N_0(Q^1) \subsetneq \partial GC(Q^1)$ *and* $\partial N_0(Q^2) \supsetneq \partial GC(Q^2)$.
3. *There exist polytopes* Q^1 *and* Q^2*, such that* $\partial N_0(Q^1) \subsetneq SC(Q^1)$ *and* $\partial N_0(Q^2) \supsetneq SC(Q^2)$.
4. *There exists a rational polytope* Q*, such that* $\partial N(Q) \subsetneq \partial N_0(Q)$.

Comparing closures



4.1
Upper Bound on Ranks using Verification Cuts

∂GC in $\mathbb{R}^2$

Theorem

$\partial GC(P) = P_I$ for all rational polytopes $P \subseteq \mathbb{R}^2$, that is $rk_{\partial M}(P) = 1$ for all rational polytopes $P \subseteq \mathbb{R}^2$.

"Mimicking" the N_+ operator

Lemma

Let M be admissible and $P \subseteq [0,1]^n$. Further let $(c,d) \in \mathbb{Z}_+^{n+1}$. If $\langle c, x\rangle \leq d$ is valid for $P \cap \{x \mid x_i = 1\}$ for every $i \in [n]$ with $c_i > 0$, then $\langle c, x\rangle \leq d$ is valid for $\partial M(P)$.

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Stable Set Polytope

Given a graph $G := (V, E)$, let $\mathrm{FSTAB}(G) := \{x \in [0,1]^n \mid x_u + x_v \leq 1 \; \forall (u,v) \in E\}$.

Theorem

Clique Inequalities, odd hole inequalities, odd anti-hole inequalities, and odd wheel inequalities are valid for $\partial \mathrm{M}(FSTAB(G))$ with M being an admissible operator.

Proof uses previous lemma and ideas from [Lovász Schrijver (1991)].

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Monotone Polytope

We say $P \subseteq [0,1]^n$ is *monotone* (or *of anti-blocking type*) if $y \in P$ whenever $y \leq x$ and $x \in P$.

Theorem

Let M be admissible and $P \subseteq [0,1]^n$ be a monotone polytope with $\max_{x \in P_I} ex = k$. Then $rk_{\partial M}(P) \leq k + 1$.

Proof uses previous lemma and ideas from [Cook Dash (2001)].

4.2
Lower Bound on Ranks using Verification Cuts

Not the paragon of perfection: lower bound on rank of A_n

$$A_n := \left\{ x \in [0,1]^n \mid \sum_{i \in I} x_i + \sum_{i \notin I} (1 - x_i) \geq \frac{1}{2} \quad \forall I \subseteq [n] \right\}.$$

Lemma

Let M be admissible and let $\ell \in \mathbb{N}$ such that $rk_M(A_n) \geq \lfloor \frac{n}{\ell} \rfloor$. If $n > 2\ell + 1$, then

$$rk_{\partial M}(A_n) \geq \left\lfloor \frac{n-1}{2\ell+1} \right\rfloor.$$

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Corollary

Let $M \in \{GC, N_0, N, N_+, SC\}$ and $n \in \mathbb{N}$ with $n \geq 4$. Then $rk_{\partial M}(A_n) \geq \left\lfloor \frac{n-1}{3} \right\rfloor$.

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Traveling Salesman Polytope

Let $G = (V, E)$ be the undirected complete graph on n vertices. The *subtour elimination polytope* $H_n \subseteq [0, 1]^{|E|}$ is the set

$$
\begin{aligned}
x(\delta(\{v\})) &= 2 && \forall\, v \in V \\
x(E(W)) &\leq |W| - 1 && \forall\, \emptyset \subsetneq W \subsetneq V \\
x_e &\in [0, 1] && \forall e \in E.
\end{aligned}
$$

Theorem

Let $\mathrm{M} \in \{GC, SC, N_0, N, N_+\}$. *For* $n \in \mathbb{N}$ *and* H_n *as defined above we have*

$$\left\lfloor \frac{\lfloor n/8 \rfloor - 1}{3} \right\rfloor \leq \mathrm{rk}_{\partial \mathrm{M}}(H_n) \leq n + 1.$$

Proof uses results from [Chvátal, Cook, Hartmann (1989)] [Cook Dash (2001)].