Facets for Two-Dimensional Mixed Integer Infinite Group Problem

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Motivation

- Generation of strong cutting planes will help solve general MIPs faster.
- All know group based cutting planes using a single constraint to derive cut.
- We present first know strong cuts that use two constraints.
- We hope these will be stronger since they use information from two constraints concurrently.

Outline

Infinite Group Relaxation of Integer Programs

- Standard IP:

$$Ax = b \quad x \in \mathbb{Z}_+,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{m \times 1}$.

- Relaxation step 1: Consider each row modulo 1.

$$\sum_{i=1}^{n} (A_{ij})(mod1)x_i \equiv b_j(mod1) \quad \forall 1 \leq j \leq m \tag{1}$$

- Rewrite in Group Space: $\sum_{i=1}^{n}(a_i)x_i = r$

  Each a_i belongs to the group $I^m = \{x \in \mathbb{R}^m | 0 \leq x_i < 1 \quad \forall 1 \leq i \leq m\}$.

  Note that $a_i = (A_{i1}(mod1), \ldots, A_{im}(mod1))$.

- Relaxation step 2: Introduce new variables.

$$\sum_{a \in I^m} ax(a) = r \tag{2}$$

Definition: Group Problem and Valid Inequalities

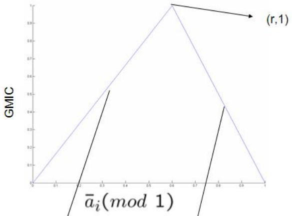
Definition (Integer Group Problem $PI(r, m)$, Johnson 1974)

For $r \in I^m$ and $r \neq o$, the group problem $PI(r, m)$ is the set of functions $t : I^m \longrightarrow \mathbb{R}$ such that

1. $\sum_{u \in I^m} ut(u) = r,\ r \in I^m$,
2. $t(u)$ is a non-negative integer for $u \in I^m$,
3. t has a finite support, i.e., $t(u) > 0$ for a finite subset of I^m.

Definition (Valid Inequality, Johnson 1974)

A function $\phi : I^m \longrightarrow \mathbb{R}_+$ is defined as a valid inequality for PI(r,m) if $\phi(o) = 0$, $\phi(r) = 1$ and $\sum_{u \in I^m} \phi(u)t(u) \geq 1$, $\forall$ $t \in PI(r, m)$.

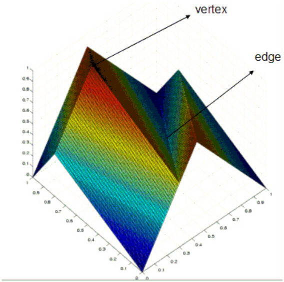


$$\sum_{f_i \leq r} \frac{f_i}{r} x_i + \sum_{f_i > r} \frac{1 - f_i}{1 - r} x_i \geq 1$$

Piecewise Linear Functions

Definition

ϕ is piecewise linear, i.e. I^2 can be decomposed into finitely many polytopes with non-empty interiors $P_1, ..., P_k$, such that $\phi(u) = \alpha_t^T u + \beta_t$, $\forall u \in P_t$, where $\alpha_t \in \mathbb{R}^{2\times 1}, \beta_t \in \mathbb{R}\ \forall t = \{1, 2, ...k\}$.



Step to Prove Function Represents Facet-defining Inequality

1. Prove function is subadditive, i.e., $\phi(u) + \phi(v) \geq \phi(u + v)$ $\forall u, v \in I^2$.
   - Develop methods to efficiently prove a function is subadditive over I^2.
2. Prove function is minimal (un-dominated). This is easy to do. (Gomory and Johnson Theorem 1972).
3. Define an additive equality as $\phi(u) + \phi(v) = \phi(u + v)$. Let $E(\phi)$ be the set of all additive equalities. Then if the ϕ is the only function that satisfies all the equalities $E(\phi)$, then ϕ is a facet.
   - Prove a result called Interval Lemma in two dimension. This result is used to prove $E(\phi)$ is unique.
   - Prove a homomorphism result to generate new facets from older ones.

Checking subadditivity for Functions Defined on I^2

Theorem (Checking Subadditivity)

Let ϕ be a continuous, piecewise linear and nonnegative function on I^2. Then ϕ is subadditive iff

$$\phi(v_1)+\phi(v_2)\geq\phi(v_1+v_2)\quad\forall v_1,v_2\in\mathbb{V}(\phi)\cup\mathbb{V}'(\phi)\tag{3}$$

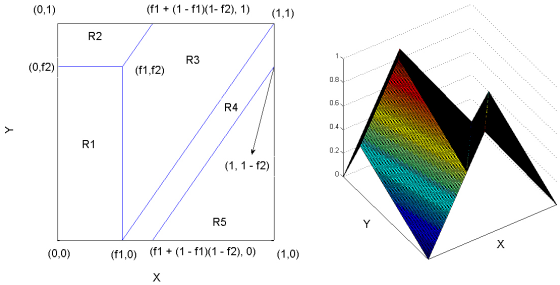
$$\phi(v_1)+\phi(v_3-v_1)\geq\phi(v_3)\quad\forall v_1,v_3\in\mathbb{V}(\phi)\cup\mathbb{V}'(\phi)\tag{4}$$

$$\phi(v_1)+\phi(e_2)\geq\phi(e_3)\ \textit{where}\ e_2\in q_2, e_3\in q_3, v_1+e_2=e_3,\quad\forall v_1\in\mathbb{V}(\phi)\cup\mathbb{V}'(\phi),\quad\forall q_2,q_3\in\mathbb{Q}(\phi)\tag{5}$$

$$\phi(e_1)+\phi(e_2)\geq\phi(v_3)\ \textit{where}\ e_1\in q_1, e_2\in q_2, e_1+e_2=v_3,\quad\forall v_3\in\mathbb{V}(\phi)\cup\mathbb{V}'(\phi),\quad\forall q_1,q_2\in\mathbb{Q}(\phi)\ .\tag{6}$$

Furthermore, if e_2 and e_3 (resp. e_1 and e_2) belong to identical or parallel edges, then (5) (resp. (6)) is redundant.

Discussion on Subadditivity Result



Interval lemma in Two Dimensions

- Interval Lemma is a key tool used to prove that a function is facet-defining by Gomory and Johnson[2003].
- The following a generalization we introduced in two dimensions.

Theorem (Interval Lemma in Two Dimensions)

Let U and V be closed sets in $\mathbb{R}^2$. Let g be a real-valued function defined over U, V and $U+V$. Assume that

1. *U is star-shaped with respect to the origin, and U has a non-empty interior.*
2. *V is path connected.*
3. $g(u) + g(v) = g(u+v)$, $\forall u \in U, \forall v \in V$.
4. $\sum_{i \in S} g(u_i) = g(\sum_{i \in S} u_i)$ $\forall u_i \in U$ *such that* $\sum_{i \in S} u_i \in U$ *and* $\forall S$ *with* $|S| \leq 3$.
5. $g(u) \geq 0, \forall u \in U$.

Then g is a linear function with the same gradient in U, V and $U+V$.

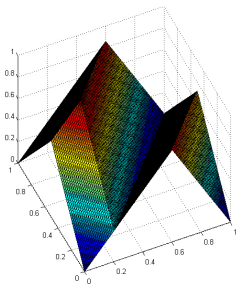
Creating New Facets

Definition (λ Homomorphism)

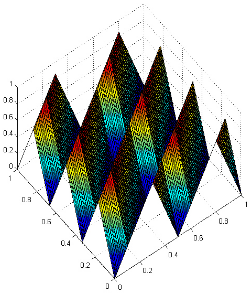
The homomorphism $\lambda : I^2 \rightarrow I^2$ is defined as $\lambda(x, y) = (\lambda_1 x(mod1), \lambda_2 y(mod1))$, where λ_1, λ_2 are positive integers.

Theorem (Homomorphism Theorem)

ϕ is facet-defining with respect to right-hand-side r iff $\phi \circ \lambda$ is facet-defining with respect to right-hand-side v, where $\lambda(v) = r$.



a. Three-gradient facet.



b. Homomorphism of the three gradient facet.

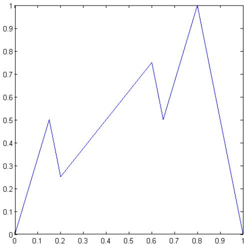
Family 1: Constraint Aggregation

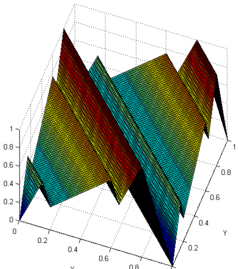
Construction

Given ζ a piecewise linear and continuous valid inequality for one dimensional integer infinite group problem $PI(c,1)$, we construct the function τ for PI(r,2) with right-hand-side $r \equiv (f_1, f_2)$ where $\lambda_1 f_1 + \lambda_2 f_2 = c$ as $\tau(x,y) = \zeta(\lambda_1 x + \lambda_2 y)(mod 1)$, and $\lambda_1, \lambda_2 \in \mathbb{Z}$ and are not both zero.

Theorem (Aggregation Theorem)

τ is facet-defining for PI(r,2) iff ζ is facet-defining for 1DIIGP.





Results on Family - I

Theorem (Two Gradient Theorem)

Any continuous piecewise linear two-gradient facet of PI(r,2) can be derived from a facet of PI(r',1) using Construction 1.

Some observations:

- Gives a complete characterization of continuous functions with only two gradients.
- All two slope functions for 1DIIGP are facet-defining [Gomory and Johnson 1972, 2003]. This is a two-dimensional analog for a similar result for the one dimensional infinite group problem.

Family 2: Three-Gradient Facet

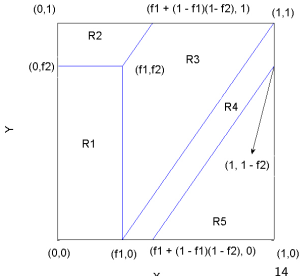
Construction

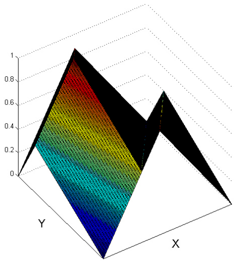
We divide I^2 into five polytopes $R1, R2, R3, R4, R5$ as shown in figure.

We construct ψ to be the only continuous piecewise linear function with $\psi(f_1, f_2) = 1$ and $\psi(0,0) = 0$, whose gradients in $R2$ and $R4$ are equal and whose gradients in $R3$ and $R5$ are equal.

Theorem

ψ is facet-defining for $PI(r, 2)$.





Three-Gradient Functions Yield Facets of IPs

Example

Consider the set of nonnegative integer solutions to

$$\begin{bmatrix} 8 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 7 \end{bmatrix} x_3 + \begin{bmatrix} 5 \\ 2 \end{bmatrix} x_4 + \begin{bmatrix} 6 \\ 3 \end{bmatrix} x_5 + \begin{bmatrix} 4 \\ 1 \end{bmatrix} x_6 + \begin{bmatrix} 0 \\ 8 \end{bmatrix} x_7 = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$$

where $x_i \in \mathbb{Z}_+ \ \forall i \in \{1, \ldots, 7\}$. This system has 3 feasible solutions: $\{0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0\}$, $\{0 \quad 1 \quad 0 \quad 2 \quad 0 \quad 0 \quad 1\}$ and $\{0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1\}$. Now consider the constraints divided by 8.

Observations

1. The three-gradient inequality ψ is generates a facet of the feasible region of the IP.
2. The GMIC generates a different facet of this problem.

Results on Family - II

Using result from Johnson [1974] cuts for integer infinite groups can be extended to mixed integer infinite group problems.

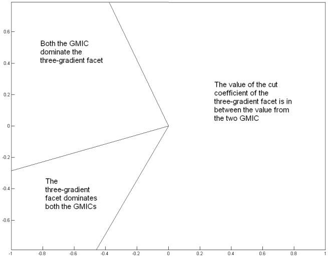
Proposition

Among all the facets of one-dimensional mixed integer infinite group problem (1DMIIGP), the coefficients of continuous variables are strongest in GMIC.

Proposition

The coefficients for continuous variables of the three-gradient inequality ψ are not dominated by GMIC based on the single constraint.

Figure: Three-gradient not dominated by GMIC



- Presented Tools for proving facet-defining tools for the two -dimensional group problem.
- Presented two-families of first known facets of two-dimensional group problem.
- These new families have interesting generate stronger coefficients for continuous variable.

Thank You.