Some representability and duality results for convex mixed-integer programs.

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Joint work with Diego Morán and Juan Pablo Vielma

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Motivation

Mixed integer linear program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b, x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned}$$

A, b are rational.

A number of structural results are known about Integer Linear Programs:

1. Representability results of integer hulls (“Fundamental Theorem of Integer programming")
2. Subaddtive dual
3. Cutting plane closure, ranks, lengths, etc.
4. ...

Motivation

Mixed integer convex program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x \in B, x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned}$$

where $B \subseteq \mathbb{R}^{n_1+n_2}$ is a closed convex set.
In this case:

1. Representability results ?
2. Dual ?
3. Cutting plane closures ?

Motivation

Mixed integer linear program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b, x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned}$$

A, b are rational.

Results hold usually because:

1. Finite number of extreme points of the LP relaxation each of which is rational, existence of basic feasible solutions for LPs.
2. Strong duality results for the Linear Programming (LP) relaxation,
3. Existence of finite generator of the Hilbert basis of a rational polyhedral cone.

Motivation

Mixed integer convex program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x \in B, x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned}$$

where $B \subseteq \mathbb{R}^{n_1+n_2}$ is a closed convex set.

In this case:

1. *Finite number of extreme points of the LP relaxation each of which is rational, existence of basic feasible solutions for LPs.* Does not hold
2. Strong duality results for the Linear Programming (LP) relaxation.
3. *Existence of finite generator of the Hilbert basis of a rational polyhedral cone.* Not always relevant in the convex case

About this talk

In this talk:

- Polyhedrality of integer hulls. (Representability-type result)
- Duality for Conic Mixed-integer programs.

Generalizing the results for integer linear program to integer convex program.

About this talk

In this talk:

- Polyhedrality of integer hulls. (Representability-type result)
- Duality for Conic Mixed-integer programs.

  Generalizing the results for integer linear program to integer convex program.
- But first: A simple, but interesting property of convex mixed-integer programs: Finiteness Property.

1 Finiteness Property.

Finiteness property

Fact

$$\inf_{x \in B} c^T x > -\infty \Rightarrow \inf_{x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})} c^T x > -\infty.$$

Finiteness property

Fact

$$\inf_{x \in B} c^T x > -\infty \Rightarrow \inf_{x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})} c^T x > -\infty.$$

Question $(\star)$

$$\inf_{x \in B} c^T x > -\infty \Leftarrow \inf_{x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})} c^T x > -\infty ???$$

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We know:

- B is polyhedron with rational data $\Rightarrow$ $(\star)$ is true.

Example 1: polyderal set with irrational data

B is a line with irrational slope

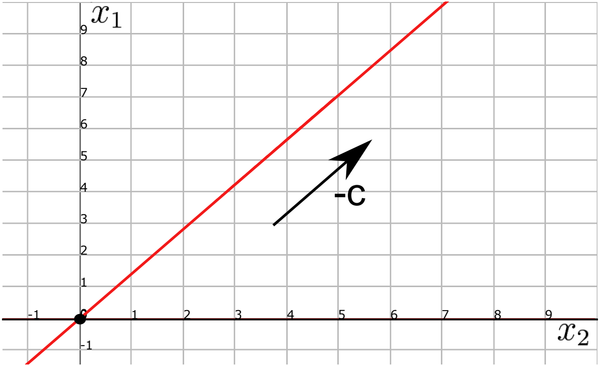


Figure: $(\star)$ is not true.

Example 2: Second order conic representable (SOCR) set with rational data

We will show a convex set $B \subseteq \mathbb{R}^3$ such that:

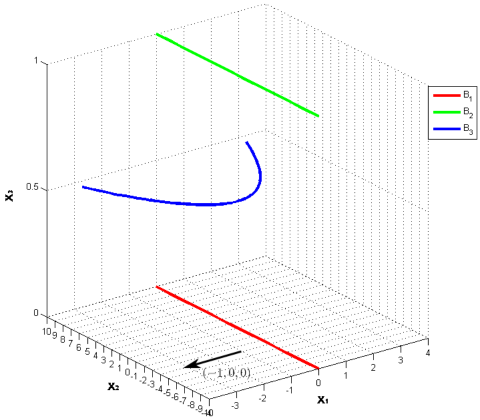
- $\text{conv}(B \cap \mathbb{Z}^3)$ is a polyhedron.
- B is conic representable with rational data.
  - There exists $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ such that

$$B' = \left\{ x \in \mathbb{R}^3 : \ \exists u, \ A \begin{pmatrix} x \\ u \end{pmatrix} - b \in L \right\},$$

    where L is direct product of Lorentz cones.
- However, B does not satisfies the finiteness property $(\star)$.

Example: (SOCP) (cont.)

$B = \text{conv}(B_1 \cup B_2 \cup B_3)$



1.1 Main result:

A sufficient condition for the finiteness property.

A sufficient condition

(for the finiteness property)

Let $B \subseteq \mathbb{R}^{n_1+n_2}$ be a closed convex set.

Proposition

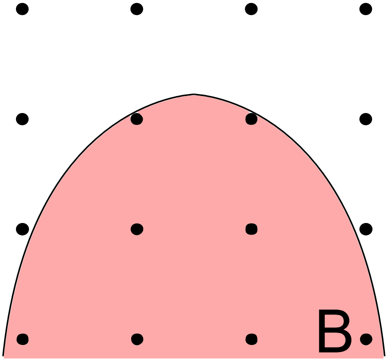
If $\text{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$*, then* $(\star)$ *is true, that is*

$$\inf_{x \in B} c^T x > -\infty \Leftrightarrow \inf_{x \in B \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})} c^T x > -\infty.$$

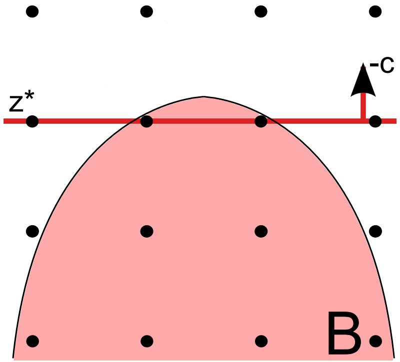
1.3 Sketch of the proof.

(In the pure integer case ($n_2 = 0$).)

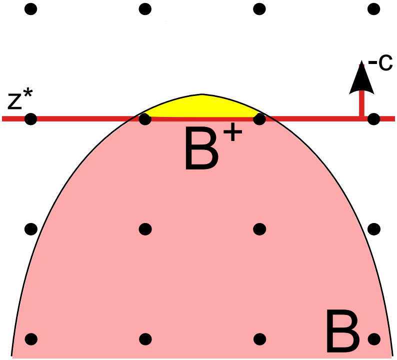
- The convex set B.



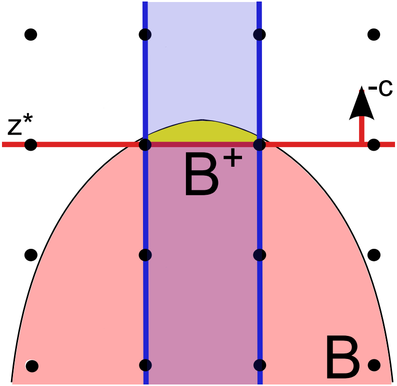
- The convex set B.
- Integer program is finite ($z^* > -\infty$).



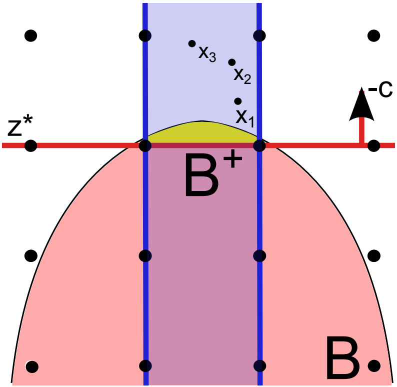
- The convex set B.
- Integer program is finite ($z^* > -\infty$).
- Yellow set B^+ is lattice-free.



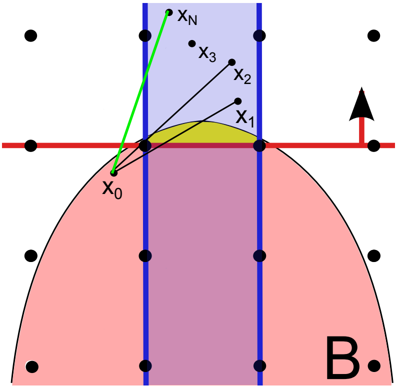
- The convex set B.
- Integer program is finite ($z^* > -\infty$).
- Yellow set B^+ is lattice-free.
- $\Rightarrow$ exists maximal lattice-free convex set Q containing B^+.
- Q = Polytope + Linear subspace.



- Assume there exists $\{x_1, x_2, x_3 \dots\} \subseteq B^+$, with $c^T x_n \to -\infty$.



- Assume there exists $\{x_1, x_2, x_3 \dots\} \subseteq B^+$, with $c^T x_n \to -\infty$.
- B is contains integer point in the interior $\Rightarrow B \not\subseteq Q$.



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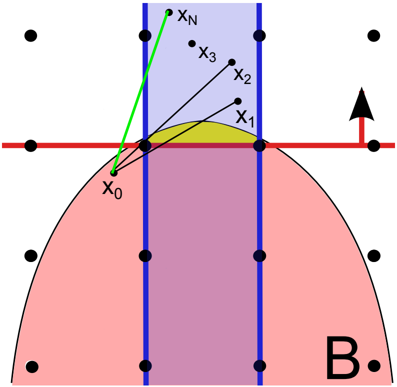
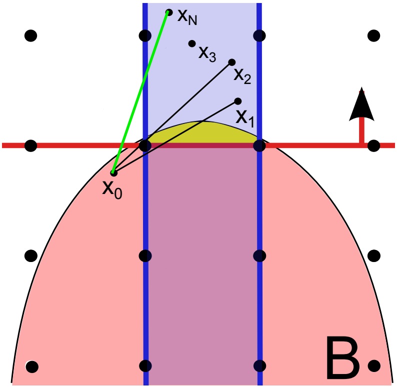
- $B^+ \subseteq Q$



- Assume there exists $\{x_1, x_2, x_3 \dots\} \subseteq B^+$, with $c^T x_n \to -\infty$.
- B is contains integer point in the interior $\Rightarrow B \not\subseteq Q$.
- $B^+ \subseteq Q$
- $\Rightarrow$ for large N we obtain

  Contradiction! $\square$



2 Polyhedrality of integer hulls.

Integer hulls

Theorem (Meyer, 1974)

Let $K \subseteq \mathbb{R}^n$ be a polyhedron with rational polyhedral recession cone, then $\mathrm{conv}(K \cap \mathbb{Z}^n)$ is a rational polyhedron.

This motivates the following questions:

Integer hulls

Theorem (Meyer, 1974)

Let $K \subseteq \mathbb{R}^n$ be a polyhedron with rational polyhedral recession cone, then $\mathrm{conv}(K \cap \mathbb{Z}^n)$ is a rational polyhedron.

This motivates the following questions:

1. Let K be a closed convex set. When is $\mathrm{conv}(K \cap \mathbb{Z}^n)$ a polyhedron?

Examples of conv$(K \cap \mathbb{Z}^n)$ being polyhedral

Ex 1: *K* is a bounded convex set

The easy case

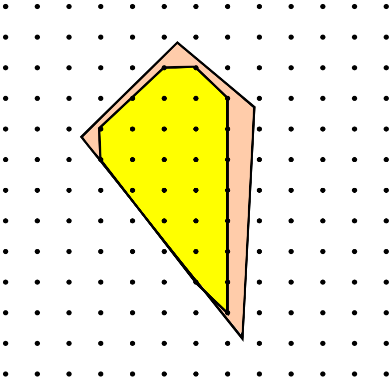


Figure: Polytope

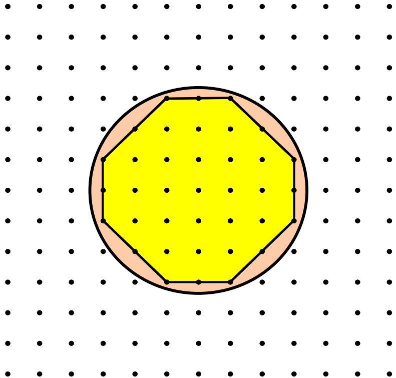
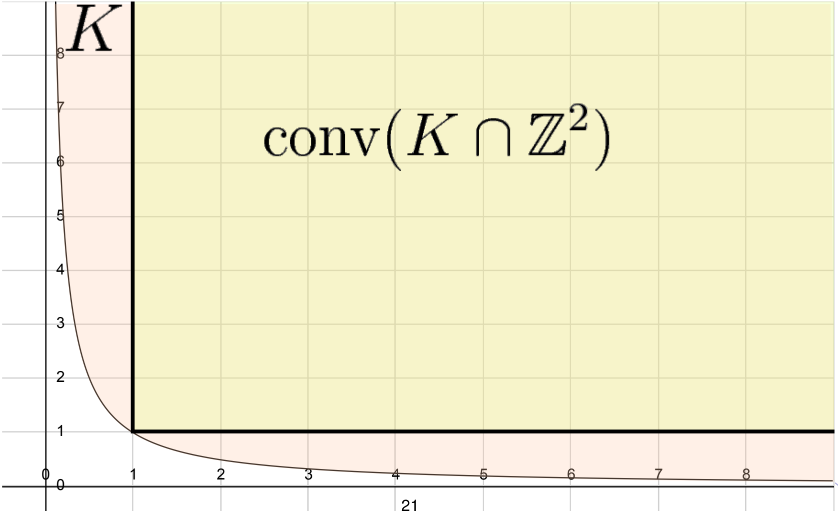


Figure: Bounded convex set

Ex 2: K is an hyperbola



Examples of conv$(K \cap \mathbb{Z}^n)$ not being polyhedral

Ex 1: K is a nonrational polyhedral cone

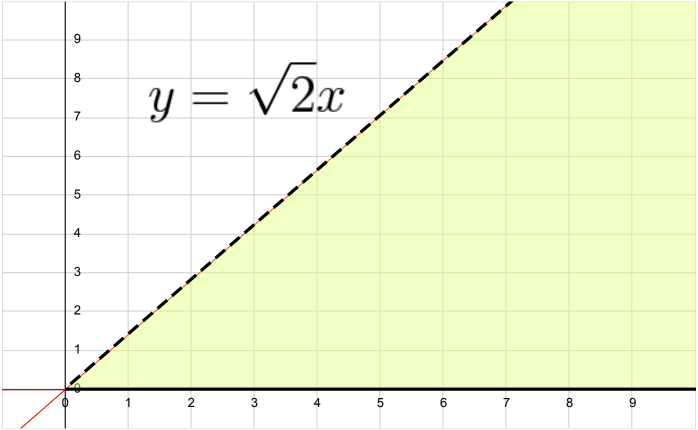


Figure: $\text{conv}(K \cap \mathbb{Z}^n)$ is not even closed

Ex 2: K is a nonrational polyhedron

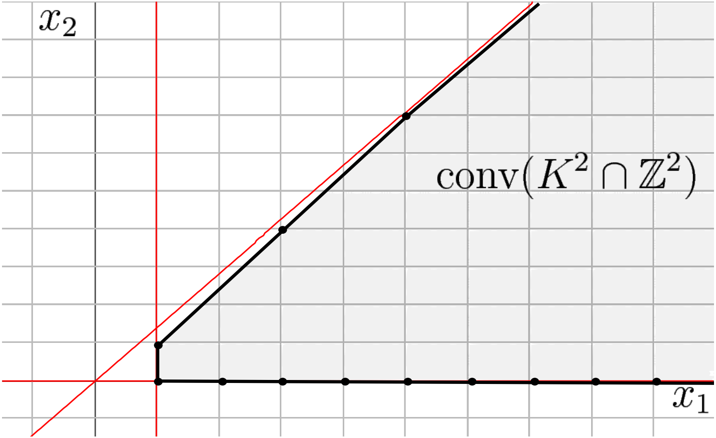


Figure: $\mathrm{conv}(K \cap \mathbb{Z}^n)$ is not a polyhedron

Ex 3: K has a rational polyhedral recession cone

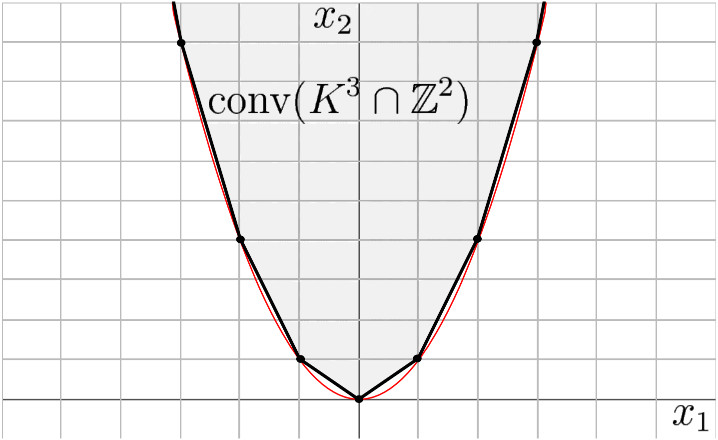


Figure: $\text{conv}(K \cap \mathbb{Z}^n)$ is not a polyhedron

Some remarks

Based on the pictures

What conditions on K led to a polyhedral $\text{conv}(K \cap \mathbb{Z}^n)$?

1. Recession cone of K plays a fundamental role.
   - Rational polyhedra recession cone.

Some remarks

Based on the pictures

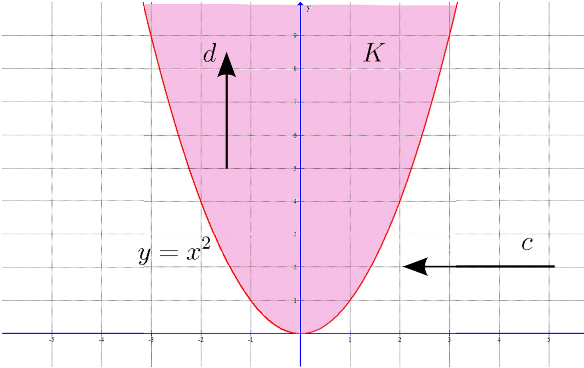
What conditions on K led to a polyhedral $\text{conv}(K \cap \mathbb{Z}^n)$?

1. Recession cone of K plays a fundamental role.
   - Rational polyhedra recession cone.

2. K needs to be ‘similar’ to a polyhedron.
   - Ex: Parabola vs Hyperbola.

Developing intuition about ‘kind of unboundedness’

- Consider $z^* = \sup\{c^t x \,:\, x \in K\}$
- We have $z^* = \infty$.
- $\text{rec.cone}(K) = \{\lambda d,\ \lambda \geq 0\}$.
- $c \perp d$.



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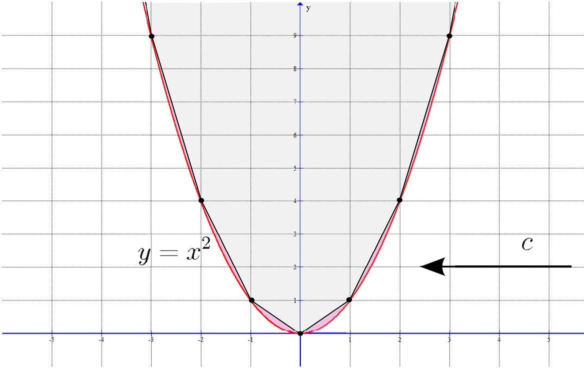

2.1 Main result

Sufficient and ‘necessary’ conditions for polyhedrality.

Definition

Thin convex sets

We say that a closed convex set $K \subseteq \mathbb{R}^n$ is thin if

$$\min\{c^T x \mid x \in K\} = -\infty \;\Leftrightarrow \exists d \in \text{rec.cone}(K) \;\; c^T d < 0.$$

- Every polyhedron is a thin convex set.
- Hyperbola is a thin convex set.

Necessary and sufficient conditions

For polyhedrality

Let $K \subseteq \mathbb{R}^n$ be a closed convex set.

Theorem

1. *If K is thin and* $\text{rec.cone}(K)$ *is a rational polyhedral cone, then* $\text{conv}(K \cap \mathbb{Z}^n)$ *is a polyhedron.*

Necessary and sufficient conditions

For polyhedrality

Let $K \subseteq \mathbb{R}^n$ be a closed convex set.

Theorem

1. *If K is thin and $\text{rec.cone}(K)$ is a rational polyhedral cone, then $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron.*
2. *If $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ and $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron, then K is thin and $\text{rec.cone}(K)$ is a rational polyhedral cone.*

2.2 Sketch of the proof.

Gordan-Dickson Lemma (GDL)

Lemma

Let $m \in \mathbb{N}$ and $S \subseteq \mathbb{Z}^m$ and assume that $\exists s_0 \in \mathbb{Z}^m$ such that

$$\text{for all } s \in S \; s \geq s_0.$$

Then there exists $T \subseteq S$, finite set, satisfying

$$\text{for all } s \in S \; \exists \; t \in T \text{ such that } s \geq t.$$

Sufficient condition

Sketch of Proof

1. rec.cone(K) $= \{d \in \mathbb{R}^n \,:\, Ad \geq 0\}$ for some $A \in \mathbb{Z}^{m \times n}$.
   $\Rightarrow$ rec.cone(conv($K \cap \mathbb{Z}^n$)) $= \{d \in \mathbb{R}^n \,:\, Ad \geq 0\}$

Sufficient condition

Sketch of Proof

1. rec.cone$(K) = \{d \in \mathbb{R}^n : Ad \geq 0\}$ for some $A \in \mathbb{Z}^{m \times n}$.
   $\Rightarrow$ rec.cone$(\text{conv}(K \cap \mathbb{Z}^n)) = \{d \in \mathbb{R}^n : Ad \geq 0\}$
2. We want to show polyhedrality

$$\text{conv}(K \cap \mathbb{Z}^n) = \text{conv}(Y) + \text{rec.cone}(K),$$

for a finite set $Y \subseteq K \cap \mathbb{Z}^n$.

Sufficient condition

Sketch of Proof

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2. We want to show polyhedrality

$$\text{conv}(K \cap \mathbb{Z}^n) = \text{conv}(Y) + \text{rec.cone}(K),$$

   for a finite set $Y \subseteq K \cap \mathbb{Z}^n$.
3. It suffices to show that

$$K \cap \mathbb{Z}^n \subseteq Y + \text{rec.cone}(K),$$

   for a finite set $Y \subseteq K \cap \mathbb{Z}^n$.

Sufficient condition

Sketch of Proof

4. Let $S = \{Ax : x \in K \cap \mathbb{Z}^n\} \subseteq \mathbb{Z}^m$.

Sufficient condition

Sketch of Proof

4. Let $S = \{Ax \,:\, x \in K \cap \mathbb{Z}^n\} \subseteq \mathbb{Z}^m$.
5. K is thin and $\text{rec.cone}(K) = \{d \in \mathbb{R}^n \,:\, Ad \geq 0\}$

   $\Rightarrow \; \exists\, s_0 \in \mathbb{Z}^m$ such that $Ax \geq s_0 \;\forall\, x \in K \cap \mathbb{Z}^n$.

   $\Rightarrow \; \exists\, s_0 \in \mathbb{Z}^m$ such that $s \geq s_0 \;\forall\, s \in S$.

Sufficient condition

Sketch of Proof

4. Let $S = \{Ax \,:\, x \in K \cap \mathbb{Z}^n\} \subseteq \mathbb{Z}^m$.
5. K is thin and $\text{rec.cone}(K) = \{d \in \mathbb{R}^n \,:\, Ad \geq 0\}$
   $\Rightarrow \exists\, s_0 \in \mathbb{Z}^m$ such that $Ax \geq s_0 \;\forall\, x \in K \cap \mathbb{Z}^n$.
   $\Rightarrow \exists\, s_0 \in \mathbb{Z}^m$ such that $s \geq s_0 \;\forall\, s \in S$.
6. We can apply GDL to S to obtain a finite set $T \subset S$.
   $\Rightarrow \exists\, Y \subseteq K \cap \mathbb{Z}^n$, finite set, such that $T = AY$.
   $\Rightarrow$ for all $x \in K \cap \mathbb{Z}^n \;\exists\, y \in Y$ such that $Ax \geq Ay$.
   $\Rightarrow K \cap \mathbb{Z}^n \subseteq Y + \text{rec.cone}(K)$.□

Necessary condition

Sketch of Proof

1. We know that
$$\mathrm{conv}(K \cap \mathbb{Z}^n) = \{x \in \mathbb{R}^n \, : \, Ax \geq b\},$$
for some $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$.
2. Since $\mathrm{int}(K) \cap \mathbb{Z}^n \neq \emptyset$, by finiteness property, $\exists \, b' \in \mathbb{Z}^m$ such that
$$K \subseteq \{x \in \mathbb{R}^n \, : \, Ax \geq b'\}.$$
3. We obtain that
$$\{x \in \mathbb{R}^n \, : \, Ax \geq b\} \subseteq K \subseteq \{x \in \mathbb{R}^n \, : \, Ax \geq b'\}$$

Necessary condition

Sketch of Proof

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3. We obtain that

$$\{x \in \mathbb{R}^n \,:\, Ax \geq b\} \subseteq K \subseteq \{x \in \mathbb{R}^n \,:\, Ax \geq b'\}$$

- $\text{rec.cone}(K) = \{x \in \mathbb{Z}^n \,:\, Ax \geq 0\}$, a rational polyhedral cone.
- K is a thin set.

3 Duality for conic mixed-integer programs

3.1 Notation and basic ideas

The primal problems

Let $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $\mathcal{I} \subseteq \{1, \ldots, n\}$.

(MILP):

$$(\mathcal{P}_{MILP}) \begin{cases} \inf & c^T x \\ s.t. & Ax \geq b \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I}. \end{cases}$$

$(u \geq v \Leftrightarrow u - v \in \mathbb{R}^n_+)$

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(MICP):

$$(\mathcal{P}) \begin{cases} \inf & c^T x \\ s.t. & Ax \succeq_K b \\ & x_i \in \mathbb{Z}, \forall i \in \mathcal{I}. \end{cases}$$

$$(u \succeq_K v \Leftrightarrow u - v \in K)$$

Some definitions

Subadditivity, conic non-decreasing

Let $\Omega \subseteq \mathbb{R}^m$, and $g : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$ be a function.

Definition (Subadditive function)

g is said to be subadditive if for all $u, v \in \Omega$

$$u + v \in \Omega \Rightarrow g(u + v) \leq g(u) + g(v).$$

Definition (Nondecreasing w.r.t K)

g is said to be nondecreasing w.r.t. K if for $u, v \in \Omega$

$$u \succeq_K v \Rightarrow g(u) \geq g(v).$$

The dual problem $(\mathcal{D}_{MILP})$

The Subadditive dual for $(\mathcal{P}_{MILP})$

$$
(\mathcal{D}_{MILP}) \begin{cases} \sup \ g(b) \\ s.t. \ g\left(A^i\right) = -g\left(-A^i\right) = c_i, \quad \forall i \in \mathcal{I} \\ \qquad \bar{g}\left(A^i\right) = -\bar{g}\left(-A^i\right) = c_i, \quad \forall i \notin \mathcal{I} \\ \qquad g(0) = 0 \\ g : \mathbb{R}^m \to \mathbb{R}, \text{ subadditive, nondecreasing w.r.t. } \mathbb{R}^n_+. \end{cases}
$$

where A^i is the ith column of A, and $\overline{g}(d) = \limsup_{\delta \to 0^+} \frac{g(\delta d)}{\delta}$.

The dual problem $(\mathcal{D})$

The Subadditive dual for $(\mathcal{P})$

$$(\mathcal{D}) \begin{cases} \sup \ g(b) \\ s.t.\ g\left(A^i\right) = -g\left(-A^i\right) = c_i, \quad \forall i \in \mathcal{I} \\ \qquad \bar{g}\left(A^i\right) = -\bar{g}\left(-A^i\right) = c_i, \quad \forall i \notin \mathcal{I} \\ \qquad g(0) = 0 \\ g : \mathbb{R}^m \to \mathbb{R}, \text{ subadditive, nondecreasing w.r.t. to } K. \end{cases}$$

Remark

When $K = \mathbb{R}^m_+$ we retrieve the subadditive dual for $(\mathcal{P}_{MILP})$.

Properties of a 'nice dual'

- **Correct lower bounds**: for all $x \in \mathbb{R}^n$ feasible for $(\mathcal{P})$ and for all $g : \mathbb{R}^m \mapsto \mathbb{R}$ feasible for $(\mathcal{D})$, we have

$$g(b) \leq c^T x \qquad \text{(Weak duality).}$$

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$$g(b) \leq c^T x \qquad \text{(Weak duality).}$$

- **Similar behavior**:
  $(\mathcal{P})$ is feasible and bounded $\Leftrightarrow$ $(\mathcal{D})$ is feasible and bounded
- **Best possible lower bounds**: $(\mathcal{P})$ is feasible and bounded, then there exists $g^* : \mathbb{R}^m \mapsto \mathbb{R}$ feasible for $(\mathcal{D})$ such that

$$g^*(b) = \inf\{c^T x : Ax \succeq_K b,\ x_i \in \mathbb{Z},\ \forall i \in \mathcal{I}\} \qquad \text{(Strong duality).}$$

Theorem

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ (data is rational). Then

$(\mathcal{D}_{MILP})$ is a '**nice dual**' for $(\mathcal{P}_{MILP})$.

Theorem

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ (data is rational). Then

$(\mathcal{D}_{MILP})$ is a '**nice dual**' for $(\mathcal{P}_{MILP})$.

Some applications

(of subadditive duality for $(\mathcal{P}_{MILP})$)

Dual feasible functions give ALL valid linear inequalities for $(\mathcal{P}_{MILP})$:

$$\sum_{i \in \mathcal{I}} g(A^i) x_i + \sum_{i \in \mathcal{C}} \overline{g}(A^i) x_i \geq g(b),$$

where g is a feasible dual function.

Example

Gomory mixed-integer cuts are given by dual feasible functions corresponding to 1-row Mixed integer linear programs.

3.2 Main result:

Extension of the Subadditive duality theory for (MILP) to the case of (MICP).

Strong duality for (MILP)

The well-known result:

Theorem (Strong duality for (MILP))

If $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ (data is rational), then

1. $(\mathcal{P}_{MILP})$ *is feasible and bounded if and only if* $(\mathcal{D}_{MILP})$ *is feasible and bounded.*
2. *If* $(\mathcal{P}_{MILP})$ *is feasible and bounded, then there exists a function* g^* *feasible for* $(\mathcal{D}_{MILP})$ *such that*

$$g^*(b) = \inf\{c^T x : Ax \geq b, \; x_i \in \mathbb{Z}, \forall i \in \mathcal{I}\}.$$

Strong duality for (MICP)

The new result:

Theorem (Strong duality for (MICP))

If there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ *s.t.* $A\hat{x} \succ_K b$*, then*

1. $(\mathcal{P})$ *is feasible and bounded if and only if* $(\mathcal{D})$ *is feasible and bounded.*
2. *If* $(\mathcal{P})$ *is feasible and bounded, then there exists a function* g^* *feasible for* $(\mathcal{D})$ *such that*

$$g^*(b) = \inf\{c^T x : Ax \succeq_K b,\ x_i \in \mathbb{Z},\ \forall i \in \mathcal{I}\}.$$

2 Sketch of the proof.

Basic proof steps

- $\mathcal{P}_R$: continuous relaxation of the primal.
- $\mathcal{D}_R$: dual of $\mathcal{P}_R$.

Basic proof steps

- $\mathcal{P}_R$: continuous relaxation of the primal.
- $\mathcal{D}_R$: dual of $\mathcal{P}_R$.

1. $(\mathcal{P})$ is feasible and bounded $\Leftrightarrow$ $(\mathcal{D})$ is feasible and bounded.
   - $(\Rightarrow)$:
     - $(\mathcal{P})$ is feasible and bounded
     - $\Rightarrow$ $(\mathcal{P}_R)$ is feasible and bounded.
     - $\Rightarrow$ $(\mathcal{D}_R)$ is feasible.
     - $\Rightarrow$ $(\mathcal{D})$ is feasible.
     - Weak duality $\Rightarrow$ $(\mathcal{D})$ bounded.
   - $(\Leftarrow)$: basically same proof as (MILP) case.

Basic proof steps

If $(\mathcal{P}_{MILP})$ is feasible and bounded, then there exists a function g^* feasible for $(\mathcal{D}_{MILP})$ such that

$$g^*(b) = \inf\{c^T x : Ax \geq b,\ x_i \in \mathbb{Z}, \forall i \in \mathcal{I}\}.$$

1. Properties of value function f
   - **Definition**: $f : \Omega \mapsto \mathbb{R} \cup \{-\infty\}$, defined as

     $$f(u) = \inf\{c^T x \ : \ Ax \succeq_K u,\ x_i \in \mathbb{Z},\ \forall\ i \in \mathcal{I}\}.$$

   - **Domain of f**:

     $$u \in \Omega \Leftrightarrow (\mathcal{P}) \text{ with r.h.s. } u \text{ is feasible.}$$

   - **Property of f**: In general, f satisfies all constraints of $(\mathcal{D})$, except by $\Omega = \mathbb{R}^m$.

Basic proof steps

2. If $(\mathcal{P})$ is feasible and bounded, then there exists a function g^* feasible for $(\mathcal{D})$ such that $g^*(b) = z^*$.
   - If $\Omega = \mathbb{R}^m$, we just can take $g^* = f$.
     - Because $f(b) = \inf\{c^T x \ : \ Ax \succeq_K b, \ x_i \in \mathbb{Z}, \ \forall \ i \in \mathcal{I}\}$.

Basic proof steps

2. If $(\mathcal{P})$ is feasible and bounded, then there exists a function g^* feasible for $(\mathcal{D})$ such that $g^*(b) = z^*$.
   - If $\Omega = \mathbb{R}^m$, we just can take $g^* = f$.
     - Because $f(b) = \inf\{c^T x \ : \ Ax \succeq_K b, \ x_i \in \mathbb{Z}, \ \forall \ i \in \mathcal{I}\}$.
   - Otherwise, if $\Omega \neq \mathbb{R}^m$ we extend f to a function g^* such that
     - g^* is feasible for the dual $(\mathcal{D})$.
     - $g^*(b) = f(b)$.

   This extension is very technical and uses the fact that there exists $\hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ s.t. $A\hat{x} \succ_K b$.

Final remarks

- The sufficient condition for strong duality is:
  - In (*MILP*): rational data.
  - In (*MICP*): $\text{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$.

Final remarks

- The sufficient condition for strong duality is:
  - In $(MILP)$: rational data.
  - In $(MICP)$: $\text{int}(B) \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset$.
- The condition: "$\exists \hat{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ s.t. $A\hat{x} \succ_K b$" is used in:
  - Strong duality for Conic programming.
  - Finiteness property.
  - Extension of value function.

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