How Good Are Sparse Cutting Planes?

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Thanks to Marco Molinaro for the figures

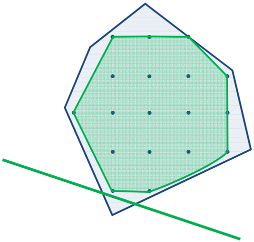
Outline

1
Introduction and Motivation

Cutting-planes

Cutting Plane

1. Cutting-planes in a linear inequality that is valid for all integer feasible points, but may not be valid for the linear programming relaxation.



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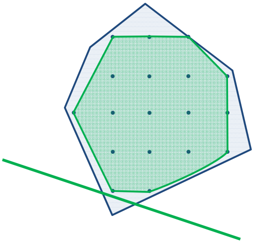
2. The convex hull of integer feasible solutions is a rational polyhedron, so a finite number of cutting-planes gives the convex hull.



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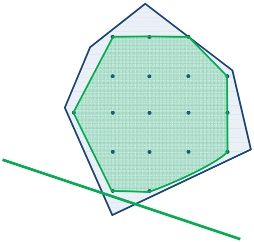
2. The convex hull of integer feasible solutions is a rational polyhedron, so a finite number of cutting-planes gives the convex hull.
3. Huge amount of research in Integer Programming on problem-specific and general purpose cutting-planes.
4. Justified by (or more importantly?) being extremely useful in practice to solve IPs.



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2. "Parallelism" ("Objective function parallelism", "Cutting plane parallelism")
3. "Numerical stability"
4. "Cutting-plane sparsity"

'Cut pool management system', 'Cutting-plane filter'

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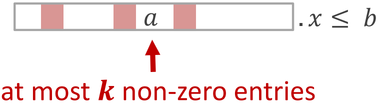
Cutting plane Sparsity

In practice

1. Prefer to use sparse cutting planes.



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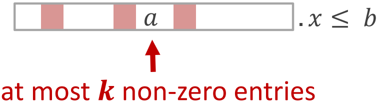

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Pros: It is easy to solve LPs in B&B tree with sparse cutting-planes.

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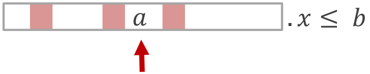

at most k non-zero entries

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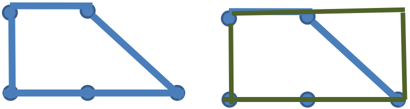

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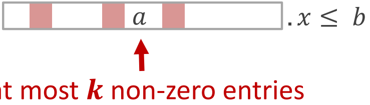

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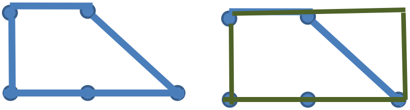

How good are sparse cutting-planes?

2
Setting up the problem

Geometric Problem

Considered abstractly

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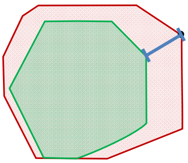
$$\boxed{\text{How well does } P^k \text{ approximate } P?}$$

We wanted a measure that is well-defined in all cases:

$$d(P, P^k) = \max_{x \in P^k} \{\text{distance}(x, P)\}$$



Comments:

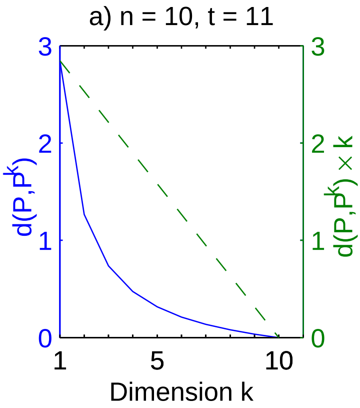
1. At most $\sqrt{n}$.
2. Upper bound on dept of cut measure.

3
Some Examples

Example 1: "Single phase"

$$P := \{x \in [0\ 1]^n | \sum_{i=1}^n x_i \le 1\}$$

$d(P, P^k) \approx \frac{\sqrt{n}}{k}$ This is Good!

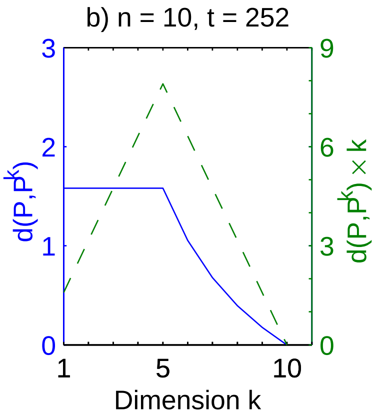


Example 1: "Two phase"

$$P := \{x \in [0\ 1]^n | \sum_{i=1}^n x_i \leq \frac{n}{2}\}$$

Phase 1: $d(P, P^k) \approx \sqrt{n}/2$ This is Bad!

Phase 2: $d(P, P^k) \approx \frac{n\sqrt{n}}{2k}$



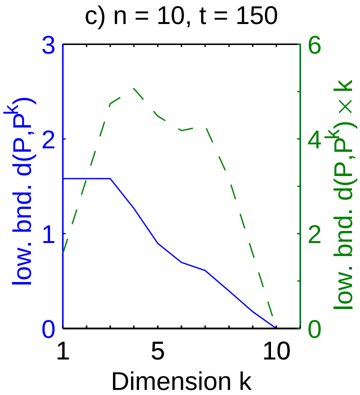
Example 1: "Three phases"

$P :=$ convex hull of random 0/1 points

Phase 1: $d(P, P^k) \approx \sqrt{n}/2$

Phase 2: $d(P, P^k) \propto \frac{1}{\sqrt{k}}$

Phase 3: $d(P, P^k) \propto \frac{1}{k}$



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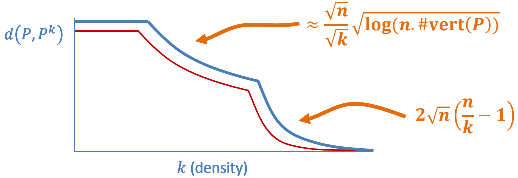
Outline of main results

Main results - Informally

1. Upper bounds on $d(P, P^k)$ depending on n, k, and number of vertices.



Consequences

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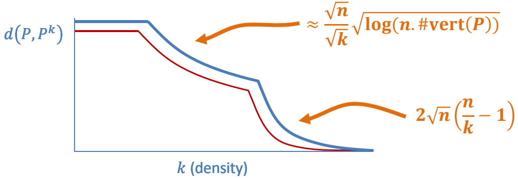
1. Upper bounds on $d(P, P^k)$ depending on n, k, and number of vertices.
2. Matching lower bound: a random 0/1 polytope, with prob $\frac{1}{4}$



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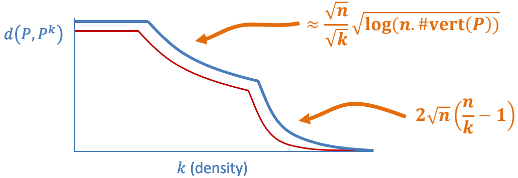

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1. Polynomial number of vertices as a function of dimension with $\sim \frac{1}{2}$ sparsity, implies $d(P, P^k)$ is very small ($\approx \sqrt{\log n}$), i.e. sparse cutting planes are good.

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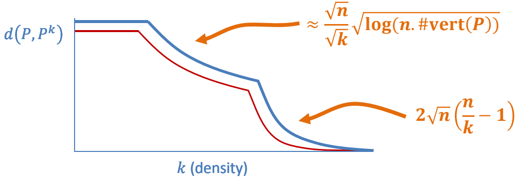
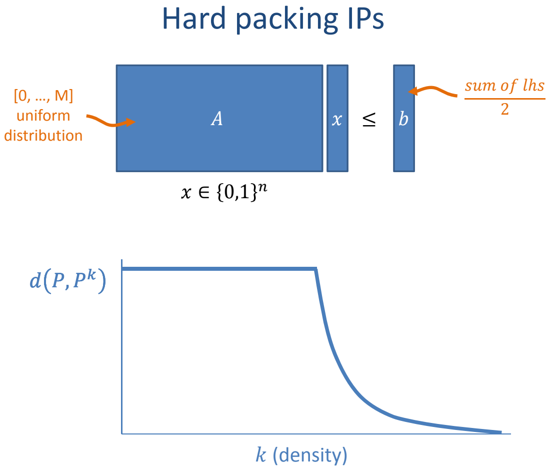


Consequences

1. Polynomial number of vertices as a function of dimension with $\sim \frac{1}{2}$ sparsity, implies $d(P, P^k)$ is very small ($\approx \sqrt{\log n}$), i.e. sparse cutting planes are good.
2. As the number of vertices grow, the location of vertices becomes more important....

Main results - Informally

3. Random packing instances: Sparse cutting-planes are bad.



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4 k-sparse cutting-planes for extended formulation + projection to original space is significantly stronger than k-sparse cutting-planes for original polytope.

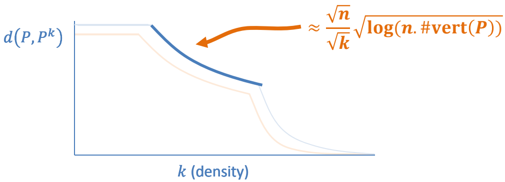
4
Results and outline of proofs

Upper Bound

Theorem 1 [Upper Bound on $d(P, P^k)$]

Let $n \geq 2$. Let $P \subseteq [0,1]^n$ be the convex hull of points $\{p^1, \ldots, p^t\}$. Then

1. $d(P, P^k) \leq 4 \max \left\{ \frac{n^{1/4}}{\sqrt{k}} \sqrt{8 \max_{i \in [t]} \|p^i\|} \sqrt{\log 4tn}, \frac{8\sqrt{n}}{3k} \log 4tn \right\}$
2. $d(P, P^k) \leq 2\sqrt{n} \left(\frac{n}{k} - 1 \right)$. $\square$



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To show: $d(P, P^k) \leq 2\lambda \triangleq \frac{\sqrt{n}}{\sqrt{k}}\sqrt{\log(n \times \text{number of vertices})}$.

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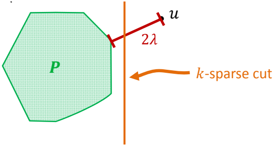

2. How to obtain a k-sparse inequality to separate u?

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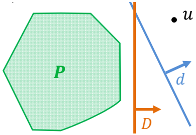

4. Let $\alpha = k/(2\sqrt{n})$. Randomly "sparsify" d to a vector D:

$\alpha\lvert d_i\rvert \leq 1$	$\alpha\lvert d_i\rvert > 1$
$D_i = \begin{cases} \frac{sign(d_i)}{\alpha} & \text{with probability} \alpha\lvert d_i\rvert \\ 0 & \text{with probability} 1 - \alpha\lvert d_i\rvert \end{cases}$	$D_i = d_i$

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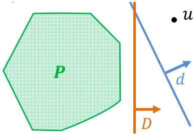

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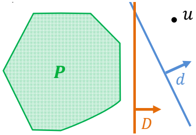

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5. With probability > 0, D is k-sparse, $D^\top x \leq (D^\top v + \lambda)$ is valid for P, and $D^\top u > (D^\top v + \lambda)$.

Upper Bound: "Sparsity of random vectors"

Claim: With high probability D is k-sparse.

Proof

1. $E[\#$ of non-zero components of $D] = \sum_{i=1}^{n} \alpha d_i \leq k/2$.

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3. $\#$ of non-zero components of $D \lesssim k/2 + \sqrt{k/2}$

$$P(\#\text{ of non-zero components of } D \leq k) \underbrace{\geq}_{\text{Bernstein's Inequality}} 1 - \frac{1}{4n}.$$

Upper Bound: "Validity of random inequality"

Observation

Let $w \in \mathbb{R}^n$. Then $E(D^\top w) = d^\top w$, $\text{Var}(D^\top w) = \frac{1}{\alpha} \sum_{i=1}^n w_i^2 |d_i|$.

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3. $Std[D^\top (p - v)] \leq \frac{\lambda}{\sqrt{\log(n \times \text{ number of vertices})}}$
4. $P\left(D^\top p > D^\top v + \lambda\right) \leq P\left(D^\top (p - v) > E[D^\top (p - v)] + \lambda\right) \leq \frac{1}{4n \times \text{ number of vertices}}$.

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5. Probability that atleast one vertex does not satisfy $D^\top x \leq D^\top (v + \lambda)$ $\leq \left(\frac{1}{4n \times \text{ number of vertices}} \times \text{ number of vertices}\right)$. (Union Bound)

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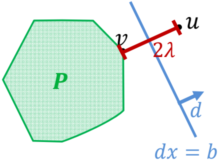
$$P\Big(D^\top x \leq (D^\top v + \lambda) \text{ is a valid inequality}\Big) \geq 1 - \frac{1}{4n}.$$

Upper Bound: "Separation of u"



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Claim: With probability at least $\frac{1}{2n-1}$, $D^\top d > 1/2$.

Proof

1. $E[D^\top d] = 1.$
2. $D^\top d \leq n$

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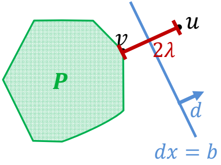

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3. With probability at least $\frac{1}{2n-1}$, $D^\top d > 1/2$. (Using Markov's inequality on the random variable $n - D^\top d$)

Upper Bound: Putting it all together

1. With probability at most $\frac{1}{4n}$, the inequality is not k-sparse.
2. With probability at most $\frac{1}{4n}$, the inequality is not valid.
3. With probability at most $1 - \frac{1}{2n-1}$, the inequality is does not separate u.

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With probablity at most $1 - \frac{1}{(2n-1)(2n)}$, the inequality is either not k-sparse, not valid or does not separate u.

$\Rightarrow$ With probablity at least $\frac{1}{(2n-1)(2n)} > 0$, the inequality is k-sparse, valid inequality and separates u.

Lower Bound

Theorem 2 [‘Matching’ Lower Bound on $d(P, P^k)$]

Let $k, t, n \in \mathbb{Z}_+$ satisfy

1. $64 \leq k \leq n$
2. $(0.5k^2 \log n + 2k + 1)^2 \leq t \leq e^n$.

Let $X^1, X^2, \ldots, X^t$ be independent uniformly random points in $\{0,1\}^n$ and let $P = \text{conv}(X^1, X^2, \ldots, X^t)$. Then with probability at least $1/4$ we have that

$$d(P, P^k) \geq \min\left\{\frac{\sqrt{n}}{\sqrt{k}}\frac{\sqrt{\log t}}{110\sqrt{\log n}}, \frac{\sqrt{n}}{8}\right\}\left(\frac{1}{2} - \frac{1}{k^{3/2}}\right) - 3\sqrt{\log t}. \qquad \square$$

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Upper bound $\approx \frac{\sqrt{n}}{\sqrt{k}} \sqrt{\log(n \times t)}$.

Lower Bound: Proof Outline

We show that there are points in P^k far from P.

1. With high probability the inequality $\sum_{i=1}^{n} x_i \leq \frac{n}{2} + 3\sqrt{n \log t}$ is valid for P.

Lower Bound: Proof Outline

We show that there are points in P^k far from P.

1. With high probability the inequality $\sum_{i=1}^{n} x_i \leq \frac{n}{2} + 3\sqrt{n \log t}$ is valid for P.
2. With high propobablity if the inequality $a^\top x \leq b$ is valid for P^k then $b \geq \left(1 + \frac{\sqrt{\log t}}{110\sqrt{k}\sqrt{\log n}}\left(1 - \frac{1}{k^2}\right)\right)\frac{\sum_{i=1}^{n} a_i}{2} - \frac{||a||_\infty}{2k^2}$

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4. The point $\sim (\frac{1}{2})e$ is closest point to above point in P. $\Rightarrow$

$$d(P, P^k) \approx \frac{\sqrt{n}}{\sqrt{k}}\sqrt{\log t}$$

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    - 2.1 New anti-concentration inequality (approx): $P\left(ax \geq E[ax] + \gamma\frac{|a|_1}{\sqrt{n}}\right) \geq e^{-\gamma^2}$
    - 2.2 If $c^\top x \leq f$ is a facet-defining inequality of a $0/1$ polytope in R^k, then $||c||_\infty \leq k^{k/2}$.
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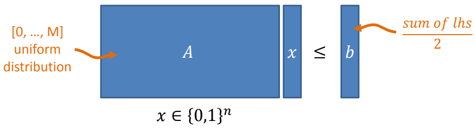
Hard Packing Instances

Hard packing IPs



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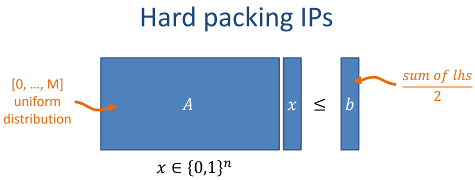

Theorem 3

Consider $n, m, M \in N$ such that $n \geq 50$ and $8 \log 8n \leq m \leq n$. Then with probability at least $1/2$,
$d(P, P^k) \geq \frac{\sqrt{n}}{2}\left(\frac{2}{\max\{\alpha,1\}}(1-\epsilon)^2 - (1+\epsilon')\right)$, where $c = k/n$ and

$$\frac{1}{\alpha} = \frac{M}{2(M+1)}\left[\frac{n - 2\sqrt{n\log 8m}}{c((2-c)n+1) + 2\sqrt{10cnm}}\right], \quad \epsilon = \frac{24\sqrt{\log 4n^2 m}}{\sqrt{n}},$$

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□

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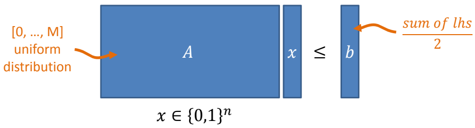

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$\sim$ With probably $1/2$, $d(P, P^k) \gtrsim \sqrt{n}\left(\frac{n}{k} - 1\right)$ for $k \geq n/2$.

Sparse cutting-planes and extended formulations

Let $\text{proj}_x : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denote the projection operator onto the first n coordinates.

Proposition

Consider a polyhedron $P \subseteq \mathbb{R}^n$ and an extended formulation $Q \subseteq \mathbb{R}^n \times \mathbb{R}^m$ for it. Then $\text{proj}_x(Q^k) \subseteq (\text{proj}_x(Q))^k = P^k$.

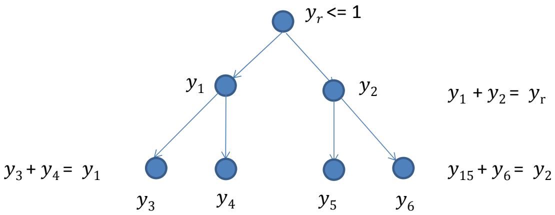
Proposition

Consider $n \in N$ and assume it is a power of 2. Then there is a polytope $P \subseteq \mathbb{R}^n$ such that:

1. *$d(P, P^k) = \sqrt{n/2}$ for all $k \leq n/2$.*
2. *There is an extended formulation $Q \subseteq \mathbb{R}^n \times \mathbb{R}^{2n-1}$ of P such that $\text{proj}_x(Q^3) = P$.*

Extended Formulation

Original Polytope:
$P = \{x \in [0\ 1]^n \mid \sum_{i=1}^n x_i \leq n/2\}$



Extended Formulation:

$$
\begin{aligned}
& y_r \leq 1 \\
& y_v = y_{\text{left}(v)} + y_{\text{right}(v)}, \quad \forall v \in \text{int}(T) \\
& y_v = \frac{2}{n} x_{i(v)}, \quad \forall v \in T \setminus \text{int}(T) \\
& y_v \geq 0, \quad \forall v \in T \\
& x_i \in [0, 1], \quad \forall i \in [n].
\end{aligned} \tag{1}
$$

Thank You!