Extreme Inequalities For Two-Dimensional Group Problem With Minimal Coefficients For Continuous Variables

Santanu S. Dey   Laurence A. Wolsey

Center for Operations Research and Econometrics, UCL, Belgium.

Aussois 2008

Outline

Group Cuts.

Minimal Inequalities for Infinite Group Problem With Only Continuous Variables.

Fill-in Procedure $\equiv$ Lifting.

- Definition
- Strength of Fill-in Procedure

Families of Extreme Inequalities for the Mixed-Integer Infinite Group Problem.

Group Cutting Planes.

General Definition of Group Problem

$F(x_1, ..., x_m) = (x_1(mod1), ..., x_m(mod1))$.

Definition (Gomory and Johnson (1972a,b))

Let U be a subgroup of I^2 and W be any subset of $\mathbb{R}^2$. For $r \in I^2 \setminus \{0\}$, the group Problem $MI(U, W, r)$ is the set of functions $x : U \rightarrow \mathbb{Z}_+$, $y : W \rightarrow \mathbb{R}_+$ such that

1.
$$\sum_{u \in U} ux(u) + \sum_{w \in W} F(wy(w)) = r, \quad (1)$$

2. x has a finite support, i.e., $x(u) > 0$ for a finite subset of U.
3. y has a finite support, i.e., $y(w) > 0$ for a finite subset of W. □

General Definition of Group Problem

$F(x_1, ..., x_m) = (x_1(mod1), ..., x_m(mod1))$.

Definition (Gomory and Johnson (1972a,b))

Let U be a subgroup of I^2 and W be any subset of $\mathbb{R}^2$. For $r \in I^2 \setminus \{0\}$, the group Problem $MI(U, W, r)$ is the set of functions $x : U \to \mathbb{Z}_+$, $y : W \to \mathbb{R}_+$ such that

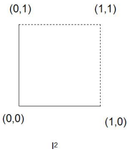
1. $$\sum_{u \in U} ux(u) + \sum_{w \in W} F(wy(w)) = r, \quad (1)$$
2. x has a finite support, i.e., $x(u) > 0$ for a finite subset of U.
3. y has a finite support, i.e., $y(w) > 0$ for a finite subset of W. □

Notation: Let $f = -r(mod1)$. $MI(U, W, r)$ can be equivalently written as:

$$\sum_{u \in U} ux(u) + \sum_{w \in W} wy(w) + f \in \mathbb{Z}^2.$$

(In $\mathbb{R}^2$ instead of Group Space)

Inequalities for Infinite Group Problems



Definition (Valid Inequality, Gomory and Johnson (1972a,b), Johnson 1974)

A pair of functions, $\phi : I^2 \rightarrow \mathbb{R}_+$ and $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is defined as a valid inequality for $MI(I^2, \mathbb{R}^2, r)$ if

1. $\phi(0) = 0$,
2. $\phi(r) = 1$,
3. $\sum_{u \in I^2} \phi(u)x(u) + \sum_{w \in \mathbb{R}^2} \pi(w)y(w) \geq 1$, $\forall\ (x, y) \in MI(I^2, \mathbb{R}^2, r)$. □

A Hierarchy of Valid Cutting Planes

Definition (Minimal Inequality, Gomory and Johnson (1972a,b))

A function (ϕ, π) is defined as a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$ if there exists no valid function $(\phi', \pi') \neq (\phi, \pi)$ such that $\phi'(u) \leq \phi(u)\ \forall u \in I^2$ and $\pi'(w) \leq \pi(w)\ \forall w \in \mathbb{R}^2$. □

Definition (Extreme Inequality, Gomory and Johnson (1972a,b))

We say that an inequality (ϕ, π) is extreme for $MI(I^2, \mathbb{R}^2, r)$ if there does not exist valid inequalities $(\phi_1, \pi_1) \neq (\phi_2, \pi_2)$ such that $(\phi, \pi) = \frac{1}{2}(\phi_1, \pi_1) + \frac{1}{2}(\phi_2, \pi_2)$. □

GMIC is a Lifted Inequality Starting With Minimal Inequalities for $MI(\emptyset, \mathbb{R}^1, r)$ in the One-Row Case

1. Let $0 < f < 1$. First fix integer variables in $MI(I^1, \mathbb{R}, r)$ to zeros, i.e., consider the problem $MI(\emptyset, \mathbb{R}, r)$:

$$\sum_{w \in \mathbb{R}} wy(w) + f \in \mathbb{Z} \tag{2}$$

2. First generate the minimal inequality for (2)

$$\pi(w) = \begin{cases} \frac{w}{1-f} & w > 0 \\ \frac{-w}{f} & \text{otherwise} \end{cases} \tag{3}$$

3. Next lift the integer variables into (4). There is a an unique lifting function: Gomory Mixed Integer Cut.

$$\phi(u) = \begin{cases} \frac{u}{1-f} & \text{if } u < 1 - f \\ \frac{1-u}{f} & \text{if } u \geq 1 - f \end{cases} \tag{4}$$

Minimal Inequalities for Infinite Group Problem With Continuous Variables For Two Rows.

‘Minimal Inequality $\equiv$ Maximal Lattice-Free Convex Set’

[Borozan and Cornuéjols (2007), Andersen, Louveaux, Weismantel, and Wolsey (2007)]

Definition

A set S is called a maximal lattice-free convex set in $\mathbb{R}^2$ if it is closed, convex, and

1. $\text{interior}(S) \cap \mathbb{Z}^2 = \emptyset$,
2. There exists no convex set S' satisfying (1), such that $S \subsetneq S'$. □

Theorem

For the system $f + \sum_{w \in \mathbb{Q}^2} w y(w) \in \mathbb{Z}^2, \quad y(w) \geq 0$, where y has a finite support, an inequality of the form $\sum_{w \in \mathbb{Q}^2} \pi(w) y(w) \geq 1$ is minimal, if the closure of

$$P(\pi) = \{w \in \mathbb{Q}^2 | \pi(w - f) \leq 1\} \tag{5}$$

is a maximal lattice-free convex set. Moreover, given a maximal lattice-free convex set $P(\pi)$ such that $f \in \mathit{interior}(P(\pi))$, the function

$$\pi(w) = \begin{cases} 0 & \textit{if } w \in \textit{recession cone of } P(\pi) \\ \lambda & \textit{if } f + \frac{w}{\lambda} \in \textit{Boundary}(P(\pi)) \end{cases} \tag{6}$$

is a minimal valid inequality. □

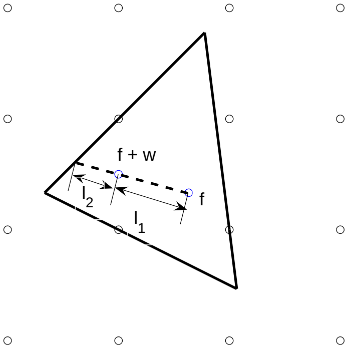
Finding the Value of $\pi(w)$

Idea: π is positively homogenous and value of $\pi(u) = 1$ if $u + f \in Bnd(P(\pi))$

Given: a vector w,

$$\vec{f} + \frac{\vec{w}}{l_1/(l_1+l_2)} \in \text{Boundary}P(\pi).$$

Therefore, $\pi(w) = \frac{l_1}{l_1+l_2}$



Maximal Bounded Lattice-Free Convex Sets are Triangles and Quadrilaterals

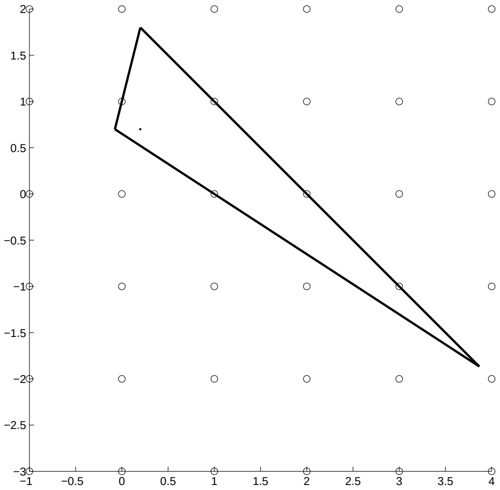
Proposition

Let P be a maximal lattice-free set in $\mathbb{R}^2$ that is bounded. Then,

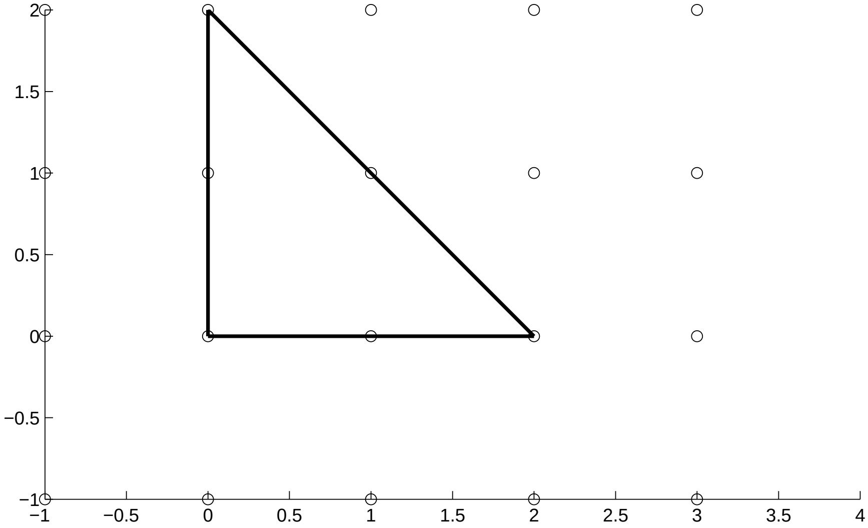
1. *If P is a maximal lattice-free triangle in $\mathbb{R}^2$, then exactly one of the following is true:*
   1.1 *One side of P contains more than one integral point in its interior[1] .*
   1.2 *All the vertices are integral and each side contains one integral point in its interior.*
   1.3 *The vertices are non-integral and each side contains one integral point in its interior.*
2. *If P is a lattice-free quadrilateral, then each of its sides contains exactly one integral point in its interior.* □

[1] Interior implies Relative Interior

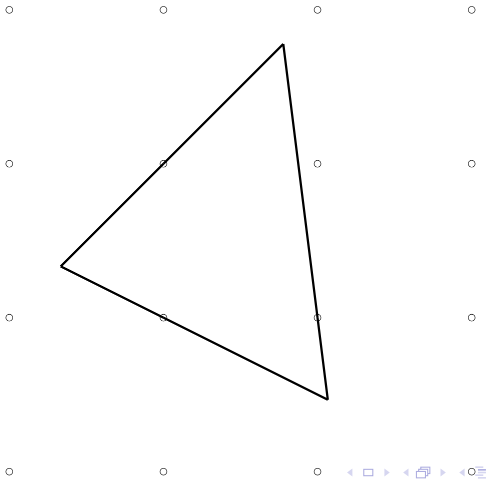
Example: One Side of Triangle has Multiple Integral Points.



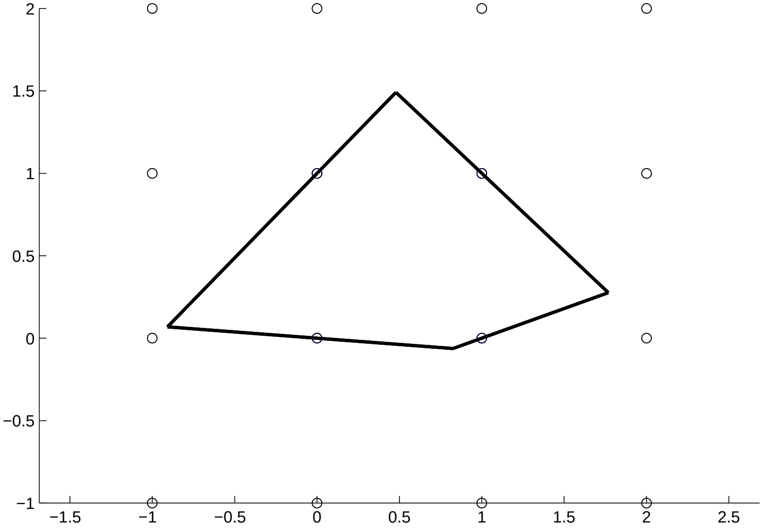
Example: Triangle Whose Vertices are Integral and Each Side Contains One Integral Point in Its Interior.



Example: Triangle Whose Vertices are Non-Integral and Each Side Contains One Integral Point in Its Interior.



Example: Quadrilateral.



Lifting integer variables in the minimal inequalities for continuous variables in the two-rows case.

Fill-in Procedure $\equiv$ Lifting Integer Variables

Modified from Gomory and Johnson (1972), Johnson (1974).

Definition (Fill-in Procedure)

- Let π be a inequality for $MI(\emptyset, \mathbb{R}^2, r)$.

Fill-in Procedure $\equiv$ Lifting Integer Variables

Modified from Gomory and Johnson (1972), Johnson (1974).

Definition (Fill-in Procedure)

- Let π be a inequality for $MI(\emptyset, \mathbb{R}^2, r)$.
- Let G be any subgroup of I^2. Let (V, π) be a valid subadditive function for $MI(G, \mathbb{R}^2, r)$.

Fill-in Procedure $\equiv$ Lifting Integer Variables

Modified from Gomory and Johnson (1972), Johnson (1974).

Definition (Fill-in Procedure)

- Let π be a inequality for $MI(\emptyset, \mathbb{R}^2, r)$.
- Let G be any subgroup of I^2. Let (V, π) be a valid subadditive function for $MI(G, \mathbb{R}^2, r)$.
  - For example: Let $u_1 \in G$ and we want to lift $x(u_1)$ in the inequality π. We solve the problem:

$$V(u_1) = \sup_{z \in \mathbb{Z}_+, z \geq 1} \{ \frac{1 - \pi(w)}{z} | u_1 z + w - r \in \mathbb{Z}^2 \} \quad (7)$$

Fill-in Procedure $\equiv$ Lifting Integer Variables

Modified from Gomory and Johnson (1972), Johnson (1974).

Definition (Fill-in Procedure)

- Let π be a inequality for $MI(\emptyset, \mathbb{R}^2, r)$.
- Let G be any subgroup of I^2. Let (V, π) be a valid subadditive function for $MI(G, \mathbb{R}^2, r)$.
  - For example: Let $u_1 \in G$ and we want to lift $x(u_1)$ in the inequality π. We solve the problem:

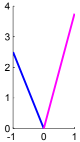
$$V(u_1) = \sup_{z\in\mathbb{Z}_+, z\geq 1}\{\frac{1-\pi(w)}{z} | u_1 z + w - r \in \mathbb{Z}^2\} \tag{7}$$

- Define function $\phi^{G,V} : I^2 \to \mathbb{R}_+$ as follows:

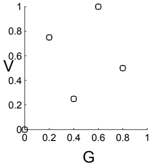
$$\phi^{G,V}(u) = \inf_{v\in G, w\in\mathbb{R}^2} \{V(v) + \pi(w) \mid v + w \equiv u\} . \tag{8}$$

Example of Fill-in Procedure in One Dimension

π is a valid inequality for the continuous group problem with r = 0.6

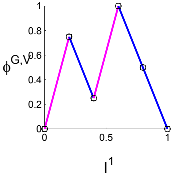


G = {0, 0.2, 0.4, 0.6, 0.8}
V(0) = 0, V(0.2) = 0.75, V(0.4) = 0.25, V(0.6) = 1, V(0.8) = 0.5



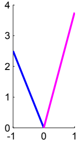
(V, π) is a valid inequality for MI(G,R^1, 0.6)

($\phi^{G,V}$, π) is extreme inequality for MI(I^1, R^1, 0.6)



The Strength Of Fill-in Function Depends On The Choice of *G*, *V*

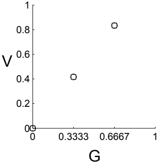
Same π as before



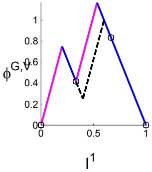
A different choice of G and V:

G = {0, 1/3, 2/3}
V(0) = 0, V(1/3) = 5/12, V(2/3) = 5/6

Again (V, π) is a valid inequality for $MI(G, R^1, 0.6)$

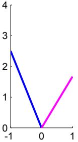


$(\phi^{G,V}, \pi)$ is not minimal.



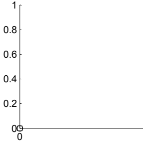
Deriving GMIC As A Fill-in Function: *G* Is The Trivial Subgroup

π is a minimal valid inequality for the continuous group problem.

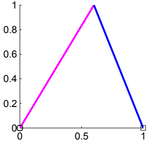


G is the trivial subgroup:

G = {0}
V(0) = 0

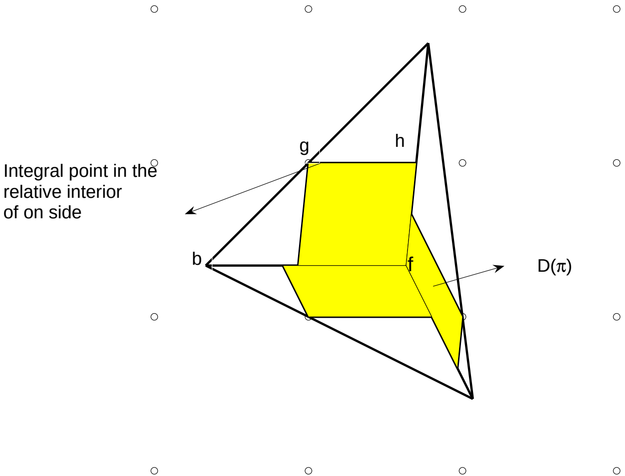


$(\phi^{\{0\},\{0\}}, \pi)$ is the Gomory Mixed Integer Cut



Strength of Fill-in Inequalities: ‘For what choice of G and V do we get strong inequalities for $MI(I^2, \mathbb{R}^2, r)$?’

Towards a Framework to Analyze Strength of Fill-in Procedure: $D(\pi)$.



‘Strength’ of Fill-in function With Trivial Subgroup Depends on ‘Area’ of $D(\pi)$

Key Idea: Given $u \in I^2$, if $\exists\, w \in D(\pi)$ such that $F(w) = u$ then $\phi^{\{0\},\{0\}}(u)$ is the best possible coefficient corresponding to the variable $x(u)$.

Proposition

If π is a valid and minimal function for $MI(\emptyset, \mathbb{R}^2, r)$ and $\phi^{G,V}(u) + \phi^{G,V}(r-u) = 1$ $\forall u \in I^2$, then $(\phi^{G,V}, \pi)$ is minimal for $MI(I^2, \mathbb{R}^2, r)$. □

Proposition

Let π is a valid and minimal function for $MI(\emptyset, \mathbb{R}^2, r)$. The function $(\phi^{\{0\},\{0\}}, \pi)^2$ is minimal for $MI(I^2, \mathbb{R}^2, r)$ iff $F(D(\pi)) = I^2$. □

[2] Note that $\phi^{\{0\},\{0\}}(u) = inf_{w \in \mathbb{R}^2}\{\pi(w) \mid w \equiv u\}$.

Uniqueness and Extremity of Fill-in Functions

Lemma (Uniqueness)

Let $(\phi^{G,V}, \pi)$ be minimal for MI$(I^2, \mathbb{R}^2, r)$. If (ϕ', π) is any valid minimal function for MI$(I^2, \mathbb{R}^2, r)$ such that $\phi'(u) = V(u)\ \forall u \in G$, then $\phi'(v) = \phi^{G,V}(v)\ \forall v \in I^2$. □

Implications:

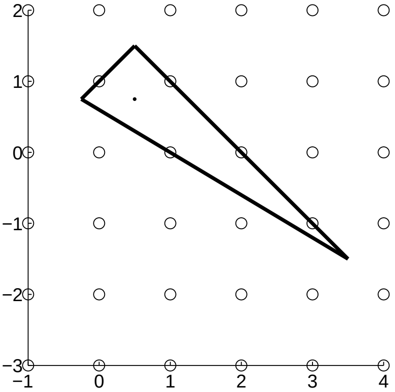
1. If $(\phi^{\{0\},\{0\}}, \pi)$ is minimal, this is the unique minimal function: the behavior is similar to the one-dimensional case.
2. If $(\phi^{\{0\},\{0\}}, \pi)$ is not minimal, then by selecting different subgroups G, and corresponding functions V for the subgroup, we may obtain different minimal functions: this behavior is not seen in the one-dimensional case.

Theorem (Extremity)

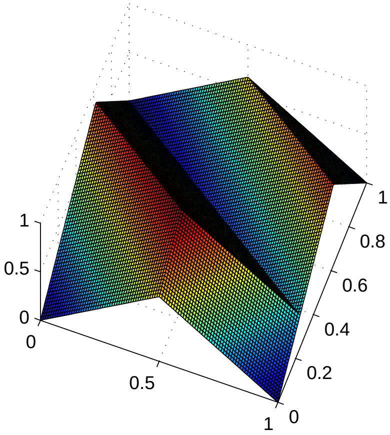
Let (V, π) be minimal for MI$(G, \mathbb{R}^2, r)$. $(\phi^{G,V}, \pi)$ is an extreme valid inequality for MI$(I^2, \mathbb{R}^2, r)$ iff (V, π) is extreme for MI$(G, \mathbb{R}^2, r)$ and $(\phi^{G,V}, \pi)$ is minimal for MI$(I^2, \mathbb{R}^2, r)$.

Families of Extreme Inequalities.

$P(\pi)$ is a Triangle With Multiple Integral Points in the Interior of One Side: $F(D(\pi)) = I^2$

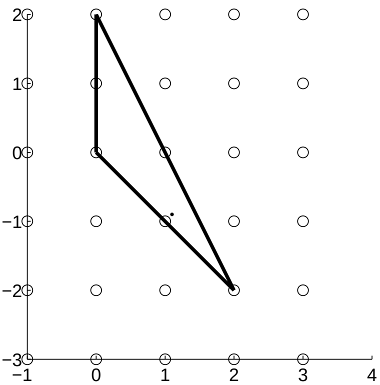


Maximal Lattice free triangle with $f_1 = 0.5$, $f_2 = 0.75$

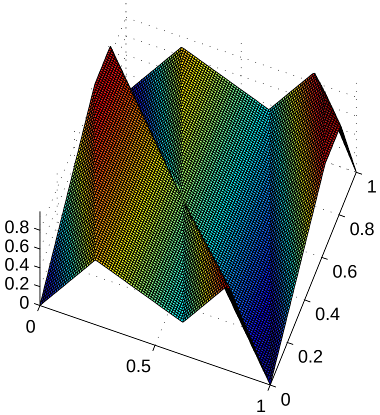


Fill in function $(\phi^{\{0\},\{0\}}, \pi)$ is extreme

$P(\pi)$ is a Triangle With Integral Vertices and One Integral Point in the Interior of Each Side: $F(D(\pi)) = I^2$

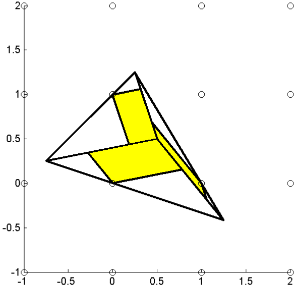


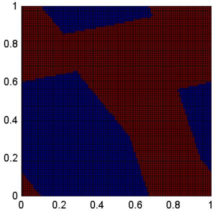
Maximal Lattice free triangle with integral vertices and one integral spoint in the interior of each side



Fill in function $(\phi^{\{0\},\{0\}}, \pi)$ is extreme

Triangle With Non-Integral Vertices and One Integral Point in the Interior of Each Side: $F(D(\pi)) \subsetneq I^2$

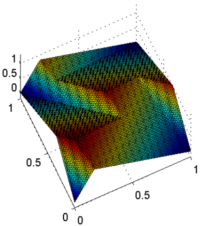




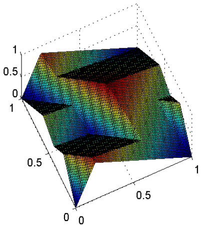
Brown region corresponds to D(π)

Example Where $P(\pi)$ is a Triangle With Non-Integral Vertices and One Integral Point in the Interior of Each Side

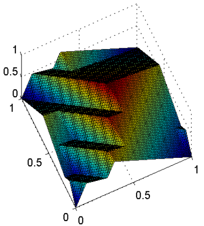
Fill-in function with the trivial subgroup. This function is not minimal.



(ϕ^{V}_{0}, π) is extreme for the two-dimensional group problem.



Another extreme inequality.



$P(\pi)$ is a Quadrilateral

Theorem

Let π correspond to maximal lattice-free quadrilateral $P(\pi)$. Then $(\phi^{\{0\},\{0\}}, \pi)$ is not extreme for $MI(I^2, \mathbb{R}^2, r)$. □

Not Unique: It is not necessary that there are unique extreme functions.

Discussion

1. Techniques for lifting integer variables into minimal inequalities for continuous variables.
2. Lifting functions are unique for
   - Triangles with multiple integer points in the interior of each side.
   - Triangles with one integral point in the interior of each side, and integral vertices.
3. Lifting functions are not unique for
   - Triangles with one integral point in the interior of each side, and non-integral vertices.
   - Quadrilaterals.
4. The class of new inequalities are 'like GMIC' since their continuous coefficients are strong.

Challenges:

1. Separation.
2. Closed-form of some of the inequalities [Sequential-Merge Inequalities, Mixing....]

Thank You.