# Gomory Functions

Santanu S. Dey Jean-Philippe P. Richard

ISyE - Georgia Tech; ISE- University of Florida, Gainsville.
Based on the review “The Group-Theoretic Approach in Mixed Integer Programming”

Multiple Row Cuts in Integer Programming
December, 2009

# Outline

# Getting Generic Cuts: Basic Idea

- Consider the set
$$P = \{x \in \mathbb{Z}^n_+ \mid Ax = b\}.$$
We would like to find all it's facet-defining inequalities of the convex hull of feasible solutions.

# Getting Generic Cuts: Basic Idea

- Consider the set

$$P = \{x \in \mathbb{Z}_+^n \mid Ax = b\}.$$

  We would like to find all it's facet-defining inequalities of the convex hull of feasible solutions.
- Without knowing structure, this is an impossible task.

# Getting Generic Cuts: Basic Idea

- Consider the set
$$P = \{x \in \mathbb{Z}^n_+ \mid Ax = b\}.$$
We would like to find all it's facet-defining inequalities of the convex hull of feasible solutions.
- Without knowing structure, this is an impossible task.
- So we resort to the following strategy:
  - Generate a "generic" relaxation of MIPs which is “easy to handle"/structured.
  - Derive valid inequalities for the convex hull of this relaxation.

1
Corner and Master Group Relaxation

# The Corner Relaxation

1. Consider the set
$$P = \{x \in \mathbb{Z}^n_+ \mid Ax = b\}$$
where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$.

# The Corner Relaxation

1. Consider the set

$$P = \{x \in \mathbb{Z}^n_+ \mid Ax = b\}$$

   where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$.

2. Given a basis $B$ of the LP relaxation of $P$, we rewrite

$$P = \{x \in \mathbb{Z}^n_+ \mid x_B = A_B^{-1}b - A_B^{-1}A_N x_N\}.$$

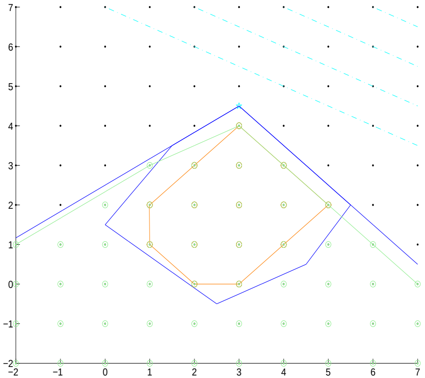
# The Corner Relaxation

1. Consider the set
$$P = \{x \in \mathbb{Z}^n_+ \mid Ax = b\}$$
where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$.
2. Given a basis $B$ of the LP relaxation of $P$, we rewrite
$$P = \{x \in \mathbb{Z}^n_+ \mid x_B = A_B^{-1} b - A_B^{-1} A_N x_N\}.$$
3. Drop the non-negativity constraints at the current solution. (If Basis is not degenerate this is equivalent to dropping all the constraints that are not binding at the current solution)
$$\text{Corner}_B(P) = \{(x_B, x_N) \in (\mathbb{Z}^m \times \mathbb{Z}^{n-m}_+) \mid x_B = A_B^{-1} b - A_B^{-1} A_N x_N\}.$$

# The Corner Relaxation

1. Consider the set
$$P = \{x \in \mathbb{Z}^n_+ \mid Ax = b\}$$
where $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$.
2. Given a basis $B$ of the LP relaxation of $P$, we rewrite
$$P = \{x \in \mathbb{Z}^n_+ \mid x_B = A_B^{-1}b - A_B^{-1}A_N x_N\}.$$
3. Drop the non-negativity constraints at the current solution. (If Basis is not degenerate this is equivalent to dropping all the constraints that are not binding at the current solution)
$$\text{Corner}_B(P) = \{(x_B, x_N) \in (\mathbb{Z}^m \times \mathbb{Z}^{n-m}_+) \mid x_B = A_B^{-1}b - A_B^{-1}A_N x_N\}.$$
4. Rewrite as [Gomory (1965)]
$$\text{Corner}_B(P) = \{x \in \mathbb{Z}^{n-m}_+ \mid A_B^{-1}A_N x = A_B^{-1}b (\text{mod } \mathbf{1})\}.$$

# Example of Corner Relaxation



# Corner Relaxation to Master Group Relaxation

$$\text{Corner}_B(P) = \{x \in \mathbb{Z}_+^{n-m} \mid \sum_{i=1}^{n-m} \tilde{A}^i x_i = \tilde{b} (\text{mod } \mathbf{1})\}$$

Disadvantage of Corner Relaxation: It is dependent on data.

# Corner Relaxation to Master Group Relaxation

$$\text{Corner}_B(P) = \{x \in \mathbb{Z}_+^{n-m} \mid \sum_{i=1}^{n-m} \tilde{A}^i x_i = \tilde{b} (\text{mod } \mathbf{1})\}$$

Disadvantage of Corner Relaxation: It is dependent on data.

Master Group Relaxation:

1. Aggregate all the variables that correspond to the same column.

# Corner Relaxation to Master Group Relaxation

$$\text{Corner}_B(P) = \{x \in \mathbb{Z}_+^{n-m} \mid \sum_{i=1}^{n-m} \tilde{A}^i x_i = \tilde{b} (\text{mod } \mathbf{1})\}$$

Disadvantage of Corner Relaxation: It is dependent on data.

Master Group Relaxation:

1. Aggregate all the variables that correspond to the same column.
2. Let $\mathcal{G}$ be a group such that
   - Group operation is addition modulo 1 componentwise,
   - $\tilde{A}^i \in \mathcal{G} \; \forall i \in \{1, ..., n-m\}$,
   - $\tilde{b} \in \mathcal{G}$,

# Corner Relaxation to Master Group Relaxation

$$\text{Corner}_B(P) = \{x \in \mathbb{Z}_+^{n-m} \mid \sum_{i=1}^{n-m} \tilde{A}^i x_i = \tilde{b} (\text{mod } \mathbf{1})\}$$

Disadvantage of Corner Relaxation: It is dependent on data.

Master Group Relaxation:

1. Aggregate all the variables that correspond to the same column.
2. Let $\mathcal{G}$ be a group such that
   - Group operation is addition modulo 1 componentwise,
   - $\tilde{A}^i \in \mathcal{G} \, \forall i \in \{1, ..., n-m\}$,
   - $\tilde{b} \in \mathcal{G}$,
3. Add one non-negative variable for each new column in $\mathcal{G}$.

# Corner Relaxation to Master Group Relaxation

$$\text{Corner}_B(P) = \{x \in \mathbb{Z}_+^{n-m} \mid \sum_{i=1}^{n-m} \tilde{A}^i x_i = \tilde{b} (\text{mod } \mathbf{1})\}$$

Disadvantage of Corner Relaxation: It is dependent on data.

Master Group Relaxation:

1. Aggregate all the variables that correspond to the same column.
2. Let $\mathcal{G}$ be a group such that
   - Group operation is addition modulo 1 componentwise,
   - $\tilde{A}^i \in \mathcal{G} \; \forall i \in \{1, ..., n - m\}$,
   - $\tilde{b} \in \mathcal{G}$,
3. Add one non-negative variable for each new column in $\mathcal{G}$.

[Gomory (1969)]

## Definition (Master Group Relaxation)

The master group relaxation is the set of all $x : \mathcal{G} \to \mathbb{Z}_+$ such that

- $x(g) \in \mathbb{Z}_+ \forall g \in \mathcal{G}$,
- x has a finite support,
- $\sum_{g \in \mathcal{G}} g x(g) = r$.

(I am going to assume $r \neq \mathbf{0}$)

# What Group to Use?

1. The Infinite Group:
   $I^m = \{x \in \mathbb{R}^m \,|\, 0 \leq x_i < 0\}$:

# What Group to Use?

1. The Infinite Group:
   $I^m = \{x \in \mathbb{R}^m \,|\, 0 \leq x_i < 0\}$: Gives the infinite group relaxation.

# What Group to Use?

1. The Infinite Group:
   $I^m = \{x \in \mathbb{R}^m \,|\, 0 \leq x_i < 0\}$: Gives the infinite group relaxation.
2. Direct product of Cyclic Groups:
   $C^m_{\mathbf{K}} = \{(x_1, ..., x_m) \,|\, x_i = \frac{n}{\mathbf{K}_i},\ n \in \{0, ..., \mathbf{K}_i - 1\}\}$:

# What Group to Use?

1. The Infinite Group:
   $I^m = \{x \in \mathbb{R}^m \,|\, 0 \leq x_i < 0\}$: Gives the infinite group relaxation.
2. Direct product of Cyclic Groups:
   $C^m_{\mathbf{K}} = \{(x_1, ..., x_m) \,|\, x_i = \frac{n}{\mathbf{K}_i},\ n \in \{0, ..., \mathbf{K}_i - 1\}\}$: Gives the finite group relaxation.

# Mixed Integer Group Relaxation

Notation: $\mathbb{P} : \mathbb{R}^m \longrightarrow I^m$ is the function
$\mathbb{P}(u) = [u_1 (\text{mod } 1), u_2 (\text{mod } 1), ..., u_m (\text{mod } 1)]^T$.

# Mixed Integer Group Relaxation

Notation: $\mathbb{P} : \mathbb{R}^m \to I^m$ is the function
$\mathbb{P}(u) = [u_1(\text{mod } 1), u_2(\text{mod } 1), ..., u_m(\text{mod } 1)]^T$.

[Gomory (1969)], [Gomory and Johnson (1972a,b)]

## Definition

Let $\mathcal{G}$ be a subgroup of $I^m$ and let $\mathcal{W}$ be a subset of $\mathbb{R}^m$.

Then the Mixed Integer Group Relaxation $MI(\mathcal{G}, \mathcal{W}, r)$ is set of functions $x : \mathcal{G} \to \mathbb{Z}_+$ and $y : \mathcal{W} \to \mathbb{R}_+$ such that

- $x$, $y$ have finite support,
- $\sum_{g\in\mathcal{G}} gx(g) + \mathbb{P}(\sum_{w\in\mathcal{W}} wy(w)) = r$.

# Mixed Integer Group Relaxation

Notation: $\mathbb{P} : \mathbb{R}^m \rightarrow I^m$ is the function
$\mathbb{P}(u) = [u_1(\text{mod } 1), u_2(\text{mod } 1), ..., u_m(\text{mod } 1)]^T$.

[Gomory (1969)], [Gomory and Johnson (1972a,b)]

## Definition

Let $\mathcal{G}$ be a subgroup of $I^m$ and let $\mathcal{W}$ be a subset of $\mathbb{R}^m$.

Then the Mixed Integer Group Relaxation $MI(\mathcal{G}, \mathcal{W}, r)$ is set of functions $x : \mathcal{G} \rightarrow \mathbb{Z}_+$ and $y : \mathcal{W} \rightarrow \mathbb{R}_+$ such that

- $x, y$ have finite support,
- $\sum_{g \in \mathcal{G}} gx(g) + \mathbb{P}(\sum_{w \in \mathcal{W}} wy(w)) = r$.

This presentation is going to be mostly about the 'pure' integer group relaxation $MI(\mathcal{G}, \emptyset, r)$.

## 2
## Valid Inequalities

# Valid Inequalities

[Gomory (1969)], [Gomory and Johnson (1972a,b)], [Johnson (1974)]

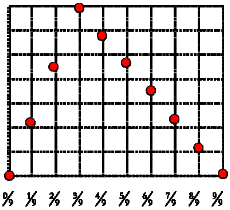
## Definition (Valid Inequality)

Valid inequalities for $MI(\mathcal{G}, \mathcal{W}, r)$ are represented by functions $\phi : \mathcal{G} \rightarrow \mathbb{R}_+$ and $\pi : \mathcal{W} \rightarrow \mathbb{R}_+$ such that $\sum_{g\in\mathcal{G}} \phi(g)x(g) + \sum_{w\in\mathcal{W}} \pi(w)y(w) \geq 1$.

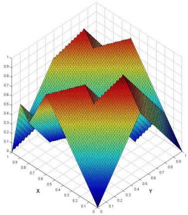
Also see [Basu, Cornuejols, Conforti, and Zambelli (2009)]

# Example of Valid Inequalities

A valid inequality For a finite group relaxation




A valid inequality for a two row infinite group relaxation



# Subadditive Valid Inequality

[Gomory (1969)], [Gomory and Johnson (1972a,b)]

## Theorem

*Given a valid inequality* $\phi : \mathcal{G} \longrightarrow \mathbb{R}_+$ *for MI*$(\mathcal{G}, \emptyset, r)$, *there exists a function* $\bar{\phi} : \mathcal{G} \longrightarrow \mathbb{R}_+$ *such that*

- $\bar{\phi}$ *represents a valid inequality.*
- $\bar{\phi}$ *is subadditive, i.e.,* $\phi(u) + \phi(v) \geq \phi(u + v)\ \forall u, v \in \mathcal{G}$.
- $\bar{\phi}(u) \leq \phi(u)\ \forall u \in \mathcal{G}$.

*Proof Sketch:*

# Subadditive Valid Inequality

[Gomory (1969)], [Gomory and Johnson (1972a,b)]

## Theorem

*Given a valid inequality $\phi : \mathcal{G} \rightarrow \mathbb{R}_+$ for $MI(\mathcal{G}, \emptyset, r)$, there exists a function $\bar{\phi} : \mathcal{G} \rightarrow \mathbb{R}_+$ such that*

- *$\bar{\phi}$ represents a valid inequality.*
- *$\bar{\phi}$ is subadditive, i.e., $\phi(u) + \phi(v) \geq \phi(u + v)\ \forall u, v \in \mathcal{G}$.*
- *$\bar{\phi}(u) \leq \phi(u)\ \forall u \in \mathcal{G}$.*

*Proof Sketch:*

1. Construct the function

$$\begin{aligned} \bar{\phi}(v) := \quad & \inf \quad \sum_{g \in \mathcal{G}} \phi(g)x(g) \\ & \text{s.t.} \quad \sum_{g \in \mathcal{G}} g x(g) = v \\ & \qquad\quad x(g) \in \mathbb{Z}_+, x \text{ has a finite support.} \end{aligned}$$

# Subadditive Valid Inequality

[Gomory (1969)], [Gomory and Johnson (1972a,b)]

## Theorem

*Given a valid inequality $\phi : \mathcal{G} \rightarrow \mathbb{R}_+$ for MI$(\mathcal{G}, \emptyset, r)$, there exists a function $\bar{\phi} : \mathcal{G} \rightarrow \mathbb{R}_+$ such that*

- *$\bar{\phi}$ represents a valid inequality.*
- *$\bar{\phi}$ is subadditive, i.e., $\phi(u) + \phi(v) \geq \phi(u + v)\ \forall u, v \in \mathcal{G}$.*
- *$\bar{\phi}(u) \leq \phi(u)\ \forall u \in \mathcal{G}$.*

*Proof Sketch:*

1. Construct the function

$$\begin{aligned} \bar{\phi}(v) := \quad & \inf \quad \sum_{g \in \mathcal{G}} \phi(g)x(g) \\ & \text{s.t.} \quad \sum_{g \in \mathcal{G}} gx(g) = v \\ & \qquad x(g) \in \mathbb{Z}_+, x \text{ has a finite support.} \end{aligned}$$

2. Show that the function above is a subadditive and represents a valid inequality.

# Minimal Valid Inequality

[Gomory (1969)], [Gomory and Johnson (1972a,b)]

## Definition

We say that a valid inequality $\phi$ for $MI(\mathcal{G}, \emptyset, r)$ is minimal, if there does not exist a valid inequality $\bar{\phi}$ such that $\bar{\phi}(u) \leq \phi(u)\ \forall u \in \mathcal{G}$ and $\phi \neq \bar{\phi}$.

## Proposition

*Every minimal inequality is a subadditive function.*

# Characterization of Minimal Inequalities

[Gomory and Johnson (1972a,b)]

## Theorem

*A function $\phi$ is valid minimal inequality for MI$(\mathcal{G}, \emptyset, r)$ if and only if*

- *$\phi$ is subadditive,*
- $\phi(u) + \phi(r - u) = \phi(r) = 1$.

# Characterization of Minimal Inequalities

[Gomory and Johnson (1972a,b)]

## Theorem

*A function $\phi$ is valid minimal inequality for MI$(\mathcal{G}, \emptyset, r)$ if and only if*

- *$\phi$ is subadditive,*
- $\phi(u) + \phi(r - u) = \phi(r) = 1$.

*Proof Sketch:* $\Leftarrow$ We assume conditions above are satisfied.

- $\phi$ is valid: Let $\bar{x}$ be a feasible solution to *MI*$(\mathcal{G}, \emptyset, r)$. The $\sum_{g\in\mathcal{G}} \phi(g)\bar{x}(g) \geq \phi(\sum_{g\in\mathcal{G}} g\bar{x}(g)) = \phi(r) = 1$.

# Characterization of Minimal Inequalities

[Gomory and Johnson (1972a,b)]

## Theorem

*A function $\phi$ is valid minimal inequality for $MI(\mathcal{G}, \emptyset, r)$ if and only if*

- *$\phi$ is subadditive,*
- $\phi(u) + \phi(r - u) = \phi(r) = 1$.

*Proof Sketch:* $\Leftarrow$ We assume conditions above are satisfied.

- $\phi$ is valid: Let $\bar{x}$ be a feasible solution to $MI(\mathcal{G}, \emptyset, r)$. The $\sum_{g\in\mathcal{G}} \phi(g)\bar{x}(g) \geq \phi(\sum_{g\in\mathcal{G}} g\bar{x}(g)) = \phi(r) = 1$.
- $\phi$ is minimal valid inequality: Assume by contradiction that $\phi$ is not minimal. Then let $\phi'$ be a valid inequality and let $\phi'(\bar{u}) < \phi(\bar{u})$. Then $\phi'(\bar{u}) + \phi'(r - \bar{u}) < \phi(\bar{u}) + \phi(r - \bar{u}) = 1$, a contradiction.

# Proof Contd.

$\Rightarrow$ Let $\phi$ be minimal.

# Proof Contd.

$\Rightarrow$ Let $\phi$ be minimal.

1. Show that $\phi$ is subadditive: Assume by contradiction that $\phi$ is not subadditive. Then $\phi(u^0) + \phi(v^0) < \phi(u^0 + v^0)$. Then construct a new function

$$\phi'(w) = \begin{cases} \phi(w) & w \neq u^0 + v^0 \\ \phi(u^0) + \phi(v^0) & w = u^0 + v^0. \end{cases}$$

   Show that $\phi'$ is a valid inequality to obtain a contradiction.

# Proof Contd.

$\Rightarrow$ Let $\phi$ be minimal.

1. Show that $\phi$ is subadditive: Assume by contradiction that $\phi$ is not subadditive. Then $\phi(u^0) + \phi(v^0) < \phi(u^0 + v^0)$. Then construct a new function

$$\phi'(w) = \begin{cases} \phi(w) & w \neq u^0 + v^0 \\ \phi(u^0) + \phi(v^0) & w = u^0 + v^0. \end{cases}$$

   Show that $\phi'$ is a valid inequality to obtain a contradiction.

2. Show that $\phi(u) + \phi(r - u) = 1$: Assume by contradiction $\phi(u^0) + \phi(r - u^0) = 1 + \delta$ where $\delta > 0$. WLOG let $\phi(u^0) > 0$. Construct the function

$$\phi'(w) = \begin{cases} \frac{1}{1+\delta}\phi(v^0) & w = u^0 \\ \phi(w) & w \neq u^0. \end{cases}$$

   Show that $\phi'$ is a valid inequality to obtain a contradiction (Proof uses the fact that $\phi$ is subadditive and non-negative).

# Extreme Valid Inequality

[Gomory and Johnson (1972a,b)]

## Definition

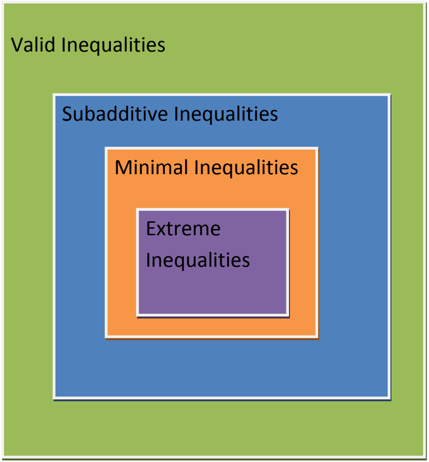
A function $\phi$ is called an extreme valid inequality, if there do not exist valid functions $\phi^1, \phi^2 : \mathcal{G} \to \mathbb{R}_+$ such that $\phi^1 \neq \phi^2$ and $\phi = \frac{1}{2}\phi^1 + \frac{1}{2}\phi^2$.

## Theorem

*Every extreme function is minimal.*

*Proof:* Suppose that $\phi$ is extreme and not minimal. Then there is a valid function $\phi'$ such that $\phi' < \phi$. Then $\phi = \frac{1}{2}\phi' + \frac{1}{2}(2\phi - \phi')$, a contradiction.

# Hierarchy of Valid Inequalities



3

# Extreme Inequalities for Finite Group Relaxation.

# Generating All Extreme Inequalities Using ‘First Principle’

[Gomory (1965)]

## Theorem

*Let $\mathcal{G}$ be a finite subgroup of $I^m$ and let $r \neq o$. The extreme inequalities of $MI(\mathcal{G}, \emptyset, r)$ are $x_i \geq 0$ for $i \in \mathcal{G}$ and $\sum_{g \in \mathcal{G}} \phi(g)x(g) \geq 1$ where $\phi \in \mathbb{R}_+^{|\mathcal{G}|}$ are the extreme points of $P^*(\mathcal{G}, \emptyset, r)$, where*

$$P^*(\mathcal{G}, \emptyset, r) = \left\{ \phi \in \mathbb{R}^{|\mathcal{G}|} \;\middle|\; \begin{array}{ll} \phi(u) + \phi(v) \geq \phi(u+v), & \forall u, v \in \mathcal{G} \\ \phi(u) + \phi(r-u) = \phi(r), & \forall u \in \mathcal{G} \\ \phi(r) = 1, & \\ \phi(u) \geq 0, & \forall u \in \mathcal{G} \end{array} \right\}.$$

# Example

For every ``red point i" we have a subadditive relation satisfied at equality

For every ``blue point i" we have a subadditive relation satisfied at equality.

.

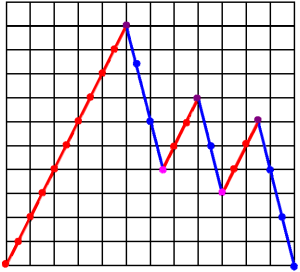
.

.

.

---

K -2 linearly independent inequalities are satisfied.



# Shooting Experiment

[Gomory and Johnson (2003)]

## Theorem

*The extreme inequalities of $MI(\mathcal{G}, \emptyset, r)$ hit first by a random ray $v$ (when shot from the origin) are the extreme inequalities $\pi^*$ that correspond to optimal solutions of the following linear program:*

$$
\begin{aligned}
min \quad & v^T\pi \\
s.t. \quad & \pi(r) = 1 \\
& \pi(g_i) + \pi(g_j) \geq \pi(g_i + g_j) \quad \forall g_i, g_j \in G \\
& \pi(g_i) + \pi(r - g_i) = 1 \quad \forall g_i \in G \\
& \pi(g_i) \geq 0 \quad \forall g_i \in G.
\end{aligned} \tag{1}
$$

Also see [Gomory, Johnson, and Evans (2003)][Dash and Günlük (2006a)]

# Key Observations in Shooting Experiment

(Underlying Cyclic Group has order atmost 30)

- Less than 10% of all the extreme inequalities are hit. Further, 50% of the hits are collected by a very small number of extreme inequalities.
- The extreme inequalities that receive the most hits are: GMIC, homomorphisms and automorphisms of the GMIC, and other two-slope inequalities.
- In addition to the structure corresponding to subgroups, there is significant persistence in the shape of extreme inequalities for group problems corresponding to different sizes. For example, the two extreme inequalities that are hit most often for $P(C_{13}, 12/13)$ are similar in shape to the extreme inequalities that are hit most often for $P(C_{19}, 18/19)$.

Similar observation in [Dash and Günlük (2006a)] using larger order cyclic groups.

# Results Based on Algebraic Structure

## Definition

- Homomorphism: Given two groups $(\mathcal{G}^1, \oplus)$ and $(\mathcal{G}^2, \otimes)$, a function $\varsigma : \mathcal{G}^1 \to \mathcal{G}^2$ is said to be a homomorphism if $\varsigma(g_1 \oplus g_2) = \varsigma(g_1) \otimes \varsigma(g_2)$ for all $\mathcal{G}^1$, $g_2 \in \mathcal{G}^1$.
- Kernel: We define the kernel of a homomorphism $\varsigma : \mathcal{G}^1 \to \mathcal{G}^2$ as the set of elements of $\mathcal{G}^1$ that are mapped to the zero of $\mathcal{G}^2$, i.e. $Kern(\varsigma) := \{g \in \mathcal{G}^1 \,|\, \varsigma(g) = 0\}$.
- Isomorphism: Given two groups $(\mathcal{G}^1, \oplus)$ and $(\mathcal{G}^2, \otimes)$, a function $\omega : \mathcal{G}^1 \to \mathcal{G}^2$ is said to be an isomorphism if $\omega$ is a homomorphism and if $\omega$ is also a bijection. We say that $\mathcal{G}^1$ is isomorphic to $\mathcal{G}^2$.
- An automorphism $\phi : \mathcal{G} \to \mathcal{G}$ is an isomorphism of a group to itself.

# Results Based on Algebraic Structure

## Definition

- Homomorphism: Given two groups $(\mathcal{G}^1, \oplus)$ and $(\mathcal{G}^2, \otimes)$, a function $\varsigma : \mathcal{G}^1 \to \mathcal{G}^2$ is said to be a homomorphism if $\varsigma(g_1 \oplus g_2) = \varsigma(g_1) \otimes \varsigma(g_2)$ for all $\mathcal{G}^1$, $g_2 \in \mathcal{G}^1$.
- Kernel: We define the kernel of a homomorphism $\varsigma : \mathcal{G}^1 \to \mathcal{G}^2$ as the set of elements of $\mathcal{G}^1$ that are mapped to the zero of $\mathcal{G}^2$, i.e. $Kern(\varsigma) := \{g \in \mathcal{G}^1 \,|\, \varsigma(g) = 0\}$.
- Isomorphism: Given two groups $(\mathcal{G}^1, \oplus)$ and $(\mathcal{G}^2, \otimes)$, a function $\omega : \mathcal{G}^1 \to \mathcal{G}^2$ is said to be an isomorphism if $\omega$ is a homomorphism and if $\omega$ is also a bijection. We say that $\mathcal{G}^1$ is isomorphic to $\mathcal{G}^2$.
- An automorphism $\phi : \mathcal{G} \to \mathcal{G}$ is an isomorphism of a group to itself.

[Gomory (1965)]

## Theorem (Automorphism)

*Let $\omega : \mathcal{G} \to \mathcal{G}$ be an automorphism. Then $\phi$ is an extreme inequality for $MI(\mathcal{G}, \emptyset, r)$ iff $\phi \circ \omega$ is extreme for $MI(\mathcal{G}, \emptyset, \omega^{-1}(r))$.*

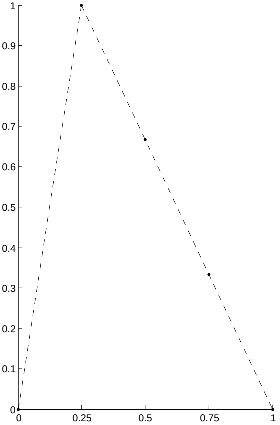
## Theorem (Homomorphism)

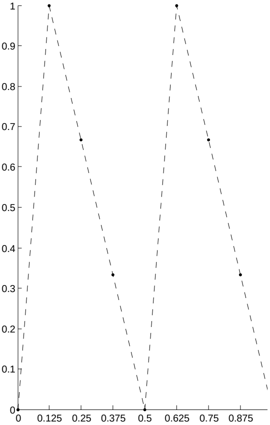
*Let $\mathcal{G}^1, \mathcal{G}^2$ be finite groups, and let $r \in \mathcal{G}^1 \setminus \{o\}$. Let $\varsigma : \mathcal{G}^1 \to \mathcal{G}^2$ be a surjective homomorphism. Assume that $r \notin Kern(\varsigma)$ and that $\phi : \mathcal{G}^2 \to \mathbb{R}_+$ is an extreme inequality for $MI(\mathcal{G}^2, \emptyset, \varsigma(r))$. Then $\phi \circ \varsigma$ is extreme for $MI(\mathcal{G}^1, \emptyset, r)$.*

(K-cuts [Cornuéjols, Li, Vandenbussche (2003)])

# Example of the Homomorphism Result

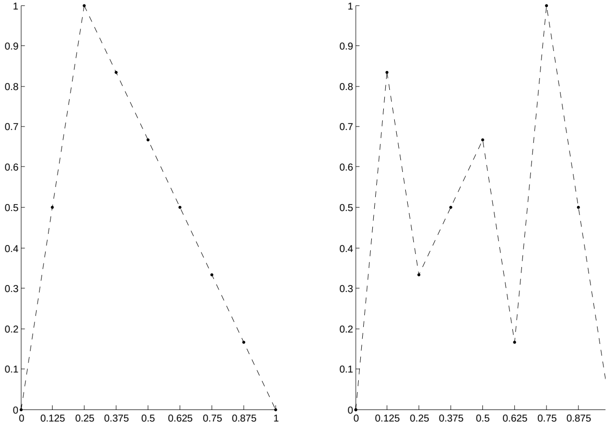
$\tau(x) = 2x \pmod 1$



# Example of the Automorphism Result

$\tau(x) = 3x (\text{mod } 1)$



# Results Specific to Cyclic Group Relaxation -I

[Gomory and Johnson (1972a,b)]

## Theorem

*If $\phi : I^1 \to \mathbb{R}_+$ is an extreme function for MI$(I^1, \emptyset, r)$ that is continuous and piecewise linear, then for any cyclic group $\mathcal{G}$ containing all the points at which $\phi$ is non-differentiable, the valid function $\phi|_{\mathcal{G}}$ obtained by restricting $\phi$ to $\mathcal{G}$ is extreme for MI$(\mathcal{G}, \emptyset, r)$.*

# Results Specific to Cyclic Group Relaxation -I

[Gomory and Johnson (1972a,b)]

## Theorem

*If $\phi : I^1 \to \mathbb{R}_+$ is an extreme function for $MI(I^1, \emptyset, r)$ that is continuous and piecewise linear, then for any cyclic group $\mathcal{G}$ containing all the points at which $\phi$ is non-differentiable, the valid function $\phi|_{\mathcal{G}}$ obtained by restricting $\phi$ to $\mathcal{G}$ is extreme for $MI(\mathcal{G}, \emptyset, r)$.*

Proof uses the important concept of interpolation.

[Gomory and Johnson (1972a,b)]

## Definition (Interpolation)

The interpolation $\phi : I^1 \to \mathbb{R}_+$ of $\hat{\phi} : C_n \to \mathbb{R}_+$ is defined as

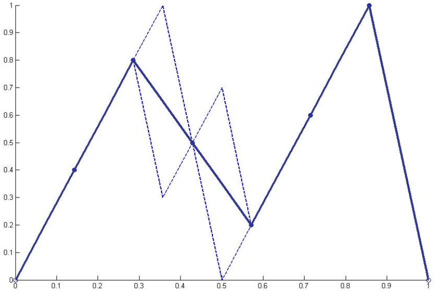
$$\phi(u) = \begin{cases} \hat{\phi}(u) & u \in C_n \\ \frac{(\mathbb{P}^{-1}(u_2) - \mathbb{P}^{-1}(u))\hat{\phi}(u_1) + (\mathbb{P}^{-1}(u) - \mathbb{P}^{-1}(u_1))\hat{\phi}(u_2)}{\mathbb{P}^{-1}(u_2) - \mathbb{P}^{-1}(u_1)} & u \notin C_n. \end{cases} \quad (2)$$

## Proposition

*If $\hat{\phi} : C_n \to \mathbb{R}_+$ is a minimal inequality for $MI(C_n, \emptyset, r)$, then $\phi : I^1 \to \mathbb{R}_+$ is a minimal inequality for $MI(I^1, \emptyset, r)$.*

See also [Dash and Günlük (2006)]

# Example of Interpolation



# Results Specific to Cyclic Group Relaxation-II

## Definition (Master Equality Knapsack)

The feasible region of a master equality knapsack problem of size r, denoted as $K_r$ is:

$$\left\{ x \in \mathbb{Z}^r_+ \;\middle|\; \sum_{i=1}^{r} i x_i = r \right\}. \tag{3}$$

## Theorem ([Aráoz (1974)])

*Extreme inequalities $\sum_{i=1}^{r} \rho_i x_i \geq \rho_r$ of the master knapsack polytope $K_r$ are the extreme rays of the cone defined by*

1. $\rho_i + \rho_j \geq \rho_{i+j} \; \forall i, j, i+j \in \{1, ..., r\}$.
2. $\rho_i + \rho_{r-i} = \rho_r \; \forall i \in \{1, ..., \lfloor \frac{r}{2} \rfloor\}$.

## Theorem (Tilting [Aráoz, Evans, Gomory, Johnson (2003)])

*Let $\rho$ be a facet of $K_r$ such that $\rho$ is nonnegative, $\rho_r = 1$ and $\rho_{i'} = 0$ for some $i' \in \{1, ..., r\}$. Set $\alpha \in \mathbb{R}$ as*

$$\alpha = max \begin{cases} \frac{r}{n}(\rho_k - \rho_i - \rho_j) & for\ 1 \leq i, j, k \leq r \\ \frac{r}{n(k-r)}((-\rho_i + \rho_j)(n-r) + (n-k)) & for\ 1 \leq i, j \leq r\ and\ r \leq k < n \\ \frac{r}{n}((\rho_k - \rho_i)\frac{n-r}{n-j} - 1) & for\ 1 \leq i, k \leq r\ and\ r \leq j < n \end{cases}$$

*where $k \equiv (i+j)\,(mod\, n)$. If $\alpha$ given by (4) is positive, then $\phi = \frac{\bar{\rho} + \alpha\xi}{1+\alpha}$ is extreme for $MI(C_n, \emptyset, r)$.*

## 4
## Extreme Inequalities for Infinite Group Relaxation.

# Using 'First Principle'

Table: Proving a function $\phi : I^m \rightarrow \mathbb{R}_+$ is extreme for $MI(I^m, \emptyset, r)$.

---

0 Input: A function $\phi : I^m \rightarrow \mathbb{R}_+$ and $r \neq o$ such that $\phi(r) = 1$ and $\phi(o) = 0$.

# Using 'First Principle'

Table: Proving a function $\phi : I^m \rightarrow \mathbb{R}_+$ is extreme for $MI(I^m, \emptyset, r)$.

---

0 Input: A function $\phi : I^m \rightarrow \mathbb{R}_+$ and $r \neq o$ such that $\phi(r) = 1$ and $\phi(o) = 0$.

1. Prove $\phi$ is subadditive, i.e., $\phi(u) + \phi(v) \geq 1 \; \forall u, v \in I^m$. This shows that $\phi$ is a valid inequality for $MI(I^m, \emptyset, r)$.

# Using ‘First Principle’

Table: Proving a function $\phi : I^m \rightarrow \mathbb{R}_+$ is extreme for $MI(I^m, \emptyset, r)$.

---

0 Input: A function $\phi : I^m \rightarrow \mathbb{R}_+$ and $r \neq o$ such that $\phi(r) = 1$ and $\phi(o) = 0$.

1. Prove $\phi$ is subadditive, i.e., $\phi(u) + \phi(v) \geq 1 \ \forall u, v \in I^m$. This shows that $\phi$ is a valid inequality for $MI(I^m, \emptyset, r)$.

2. Prove $\phi$ satisfies complementarity conditions, i.e., $\phi(u) + \phi(r - u) = 1$ $\forall u \in I^m$. This shows that $\phi$ is a minimal valid inequality for $MI(I^m, \emptyset, r)$.

# Using 'First Principle'

Table: Proving a function $\phi : I^m \rightarrow \mathbb{R}_+$ is extreme for $MI(I^m, \emptyset, r)$.

---

0 Input: A function $\phi : I^m \rightarrow \mathbb{R}_+$ and $r \neq o$ such that $\phi(r) = 1$ and $\phi(o) = 0$.

1. Prove $\phi$ is subadditive, i.e., $\phi(u) + \phi(v) \geq 1 \ \forall u, v \in I^m$. This shows that $\phi$ is a valid inequality for $MI(I^m, \emptyset, r)$.
2. Prove $\phi$ satisfies complementarity conditions, i.e., $\phi(u) + \phi(r - u) = 1 \ \forall u \in I^m$. This shows that $\phi$ is a minimal valid inequality for $MI(I^m, \emptyset, r)$.
3. Assume by contradiction that $\phi$ is not extreme, i.e., $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$ such that $\phi_1 \neq \phi_2$ and $\phi_1, \phi_2$ are valid inequalities for $MI(I^m, \emptyset, r)$. It can be proven that $\phi_1$ and $\phi_2$ are minimal valid inequalities.

# Using 'First Principle'

Table: Proving a function $\phi : I^m \to \mathbb{R}_+$ is extreme for $MI(I^m, \emptyset, r)$.

---

0 Input: A function $\phi : I^m \to \mathbb{R}_+$ and $r \neq o$ such that $\phi(r) = 1$ and $\phi(o) = 0$.

1. Prove $\phi$ is subadditive, i.e., $\phi(u) + \phi(v) \geq 1 \; \forall u, v \in I^m$. This shows that $\phi$ is a valid inequality for $MI(I^m, \emptyset, r)$.
2. Prove $\phi$ satisfies complementarity conditions, i.e., $\phi(u) + \phi(r - u) = 1$ $\forall u \in I^m$. This shows that $\phi$ is a minimal valid inequality for $MI(I^m, \emptyset, r)$.
3. Assume by contradiction that $\phi$ is not extreme, i.e., $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$ such that $\phi_1 \neq \phi_2$ and $\phi_1, \phi_2$ are valid inequalities for $MI(I^m, \emptyset, r)$. It can be proven that $\phi_1$ and $\phi_2$ are minimal valid inequalities.
4. Define the *equality set* of $E(\phi) = \{(u, v) \in I^m \times I^m \,|\, \phi(u) + \phi(v) = \phi(u + v)\}$. Since $\phi_1$ and $\phi_2$ are minimal functions, they are subadditive. Thus $E(\phi_1) \supseteq E(\phi)$ and $E(\phi_2) \supseteq E(\phi)$.

# Using 'First Principle'

Table: Proving a function $\phi : I^m \to \mathbb{R}_+$ is extreme for $MI(I^m, \emptyset, r)$.

0 Input: A function $\phi : I^m \to \mathbb{R}_+$ and $r \neq o$ such that $\phi(r) = 1$ and $\phi(o) = 0$.

1. Prove $\phi$ is subadditive, i.e., $\phi(u) + \phi(v) \geq 1 \; \forall u, v \in I^m$. This shows that $\phi$ is a valid inequality for $MI(I^m, \emptyset, r)$.
2. Prove $\phi$ satisfies complementarity conditions, i.e., $\phi(u) + \phi(r - u) = 1$ $\forall u \in I^m$. This shows that $\phi$ is a minimal valid inequality for $MI(I^m, \emptyset, r)$.
3. Assume by contradiction that $\phi$ is not extreme, i.e., $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$ such that $\phi_1 \neq \phi_2$ and $\phi_1, \phi_2$ are valid inequalities for $MI(I^m, \emptyset, r)$. It can be proven that $\phi_1$ and $\phi_2$ are minimal valid inequalities.
4. Define the *equality set* of $E(\phi) = \{(u, v) \in I^m \times I^m \mid \phi(u) + \phi(v) = \phi(u + v)\}$. Since $\phi_1$ and $\phi_2$ are minimal functions, they are subadditive. Thus $E(\phi_1) \supseteq E(\phi)$ and $E(\phi_2) \supseteq E(\phi)$.
5. Obtain a contradiction by showing that if $E(\phi_1) \supseteq E(\phi)$, then $\phi_1 = \phi$.

# Using 'First Principle'

Table: Proving a function $\phi : I^m \to \mathbb{R}_+$ is extreme for $MI(I^m, \emptyset, r)$.

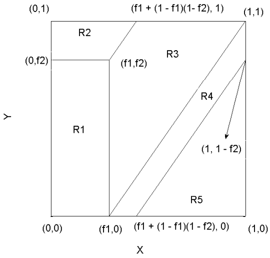
0 Input: A function $\phi : I^m \to \mathbb{R}_+$ and $r \neq o$ such that $\phi(r) = 1$ and $\phi(o) = 0$.

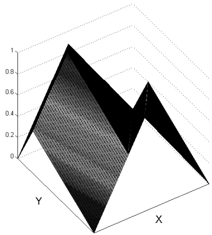
1. Prove $\phi$ is subadditive, i.e., $\phi(u) + \phi(v) \geq 1$ $\forall u, v \in I^m$. This shows that $\phi$ is a valid inequality for $MI(I^m, \emptyset, r)$.
2. Prove $\phi$ satisfies complementarity conditions, i.e., $\phi(u) + \phi(r - u) = 1$ $\forall u \in I^m$. This shows that $\phi$ is a minimal valid inequality for $MI(I^m, \emptyset, r)$.
3. Assume by contradiction that $\phi$ is not extreme, i.e., $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$ such that $\phi_1 \neq \phi_2$ and $\phi_1, \phi_2$ are valid inequalities for $MI(I^m, \emptyset, r)$. It can be proven that $\phi_1$ and $\phi_2$ are minimal valid inequalities.
4. Define the *equality set* of $E(\phi) = \{(u, v) \in I^m \times I^m \,|\, \phi(u) + \phi(v) = \phi(u + v)\}$. Since $\phi_1$ and $\phi_2$ are minimal functions, they are subadditive. Thus $E(\phi_1) \supseteq E(\phi)$ and $E(\phi_2) \supseteq E(\phi)$.
5. Obtain a contradiction by showing that if $E(\phi_1) \supseteq E(\phi)$, then $\phi_1 = \phi$.

# Verifying Subadditivity: Piecewise Linear Function

## Definition (Boundary)

For a continuous and piecewise linear function $\phi$, we say that a point $l$ belongs to the boundary of $\phi$, denoted $\mathbb{B}(\phi)$, if $\phi$ is not differentiable at $l$.



# Verifying Subadditivity: Piecewise Linear Function

[Gomory and Johnson (2003)], [D. and Richard (2007)]

## Proposition

*Let $\phi$ be a continuous, piecewise linear and nonnegative function over $I^m$. The function $\phi$ is subadditive if and only if*

$$\phi(l_1) + \phi(l_2) \geq \phi(l_1 + l_2) \tag{5}$$

$$\phi(l_1) + \phi(l_2 - l_1) \geq \phi(l_2) \tag{6}$$

$\forall l_1, l_2 \in \mathbb{B}(\phi)$.

See [Richard, Li, Miller (2007)] for extensions to discontinuous functions.

# Using ‘First Principle’

Table: Proving a function $\phi : I^m \rightarrow \mathbb{R}_+$ is extreme for $MI(I^m, \emptyset, r)$.

0 Input: A function $\phi : I^m \rightarrow \mathbb{R}_+$ and $r \neq o$ such that $\phi(r) = 1$ and $\phi(o) = 0$.

1. Prove $\phi$ is subadditive, i.e., $\phi(u) + \phi(v) \geq 1 \; \forall u, v \in I^m$. This shows that $\phi$ is a valid inequality for $MI(I^m, \emptyset, r)$.
2. Prove $\phi$ satisfies complementarity conditions, i.e., $\phi(u) + \phi(r - u) = 1$ $\forall u \in I^m$. This shows that $\phi$ is a minimal valid inequality for $MI(I^m, \emptyset, r)$.
3. Assume by contradiction that $\phi$ is not extreme, i.e., $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$ such that $\phi_1 \neq \phi_2$ and $\phi_1, \phi_2$ are valid inequalities for $MI(I^m, \emptyset, r)$. It can be proven that $\phi_1$ and $\phi_2$ are minimal valid inequalities.
4. Define the *equality set* of $E(\phi) = \{(u, v) \in I^m \times I^m \,|\, \phi(u) + \phi(v) = \phi(u + v)\}$. Since $\phi_1$ and $\phi_2$ are minimal functions, they are subadditive. Thus $E(\phi_1) \supseteq E(\phi)$ and $E(\phi_2) \supseteq E(\phi)$.
5. Obtain a contradiction by showing that if $E(\phi_1) \supseteq E(\phi)$, then $\phi_1 = \phi$.

# $E(\phi_1) \supseteq E(\phi) \Rightarrow \phi_1 = \phi$

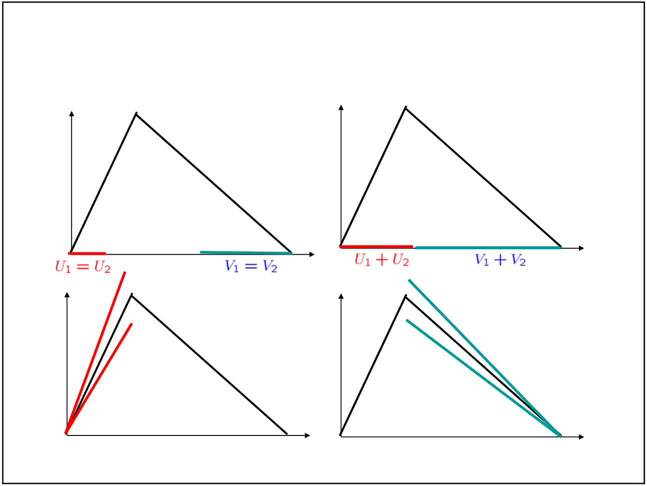
1. First verify that if the function $\phi$ is linear over some set $S \subseteq I^m$, then $\phi_1$ is also linear over this set $S$.
2. Then verify that this linear function is exactly the same function.

The first step is accomplished using 'Interval Lemma':
Example of one such result: [Gomory and Johnson (2003)]

## Lemma

*Let $U_1 \equiv [u_1, u_2] \subset I^1$, $U_2 \equiv [v_1, v_2] \subset I^1$ and $U_1 + U_2 \equiv [u_1 + v_1, u_2 + v_2]$ such that $u_1 \neq u_2$ and $v_1 \neq v_2$. If there exists a continuous real-valued function $\phi$ defined over $U_1$, $U_2$ and $U_1 + U_2$ such that $\phi(u) + \phi(v) = \phi(u + v)$ $\forall u \in U, v \in V$, then $\phi$ must be a straight line with constant slope $s$ over $U_1$, $U_2$ and $U_1 + U_2$.*

# Illustration of the Last Step



# Interval Lemma in $\mathbb{R}^2$

[D. and Richard (2007)]

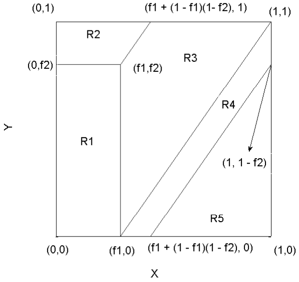
## Proposition

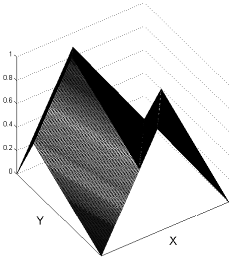
*Let $U$ and $V$ be closed sets in $\mathbb{R}^2$. Let $g$ be a real-valued function defined over $U$, $V$ and $U + V$. Assume that*

1. *$U$ is star-shaped with respect to the origin, and $U$ has a non-empty interior.*
2. *$V$ is path connected.*
3. $g(u) + g(v) = g(u + v), \forall u \in U, \forall v \in V.$
4. $\sum_{i \in S} g(u_i) = g(\sum_{i \in S} u_i)$ $\forall u_i \in U$ *such that* $\sum_{i \in S} u_i \in U$ *and* $\forall S$ *with* $|S| \leq 3$.
5. $g(u) \geq 0, \forall u \in U.$

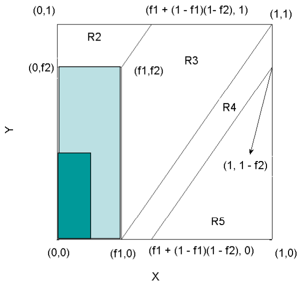
*Then $g$ is a linear function with the same gradient in $U$, $V$ and $U + V$.*

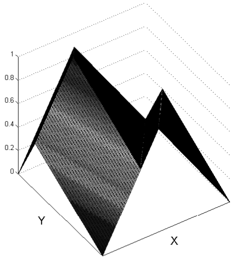
# Illustration of Interval Lemma in Two Dimensions



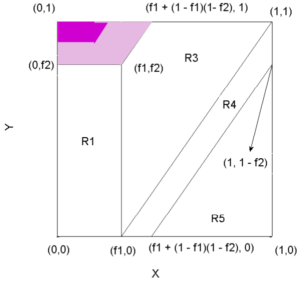



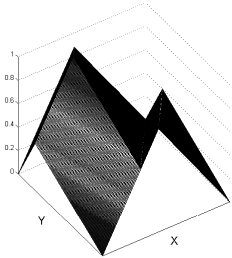
# Illustration of Interval Lemma in Two Dimensions



# Illustration of Interval Lemma in two Dimensions



# A Conjecture of Gomory and Johnson

- Gomory and Johnson (2003) conjectured that all extreme inequalities of infinite group relaxation are piecewise linear.
- Recently [Basu, Cornuéjols, Conforti, and Zambelli (2009)] gave a counterexample.
- The proof involves a new variant of Interval Lemma: Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded periodic function with period one. Let $U = [u_1, u_2]$, $V = [v_1, v_2]$, and $U + V = [u_1 + v_1, u_2 + v_2]$ be three intervals of the real line such that $u_1 < u_2$ and $v_1 < v_2$. If $f(u) + f(v) = f(u + v)$ for every $u \in U$ and $v \in V$, then the graph of $f$ above $U$, $V$, and $U + V$ is a straight line with some constant slope $s$.

# Discussion: Constructing Subadditive Functions

- Unlike the finite case, we do not have a complete characterization of all extreme inequalities.
- One way of thinking of extreme functions is to first understand different approaches to generate subadditive valid inequalities (and then studying when they yield extreme inequalities).
- Rest of the talk is structured on the following approaches to generate subadditive functions:

1. Interpolation in one dimension (discussed already)
2. Using algebraic results.
3. Sequences of subadditive functions.
4. Value functions of MIPs.
5. If $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is subadditive and non-decreasing and $\psi : \mathcal{D} \longrightarrow \mathbb{R}$ is subadditive, then $\phi \circ \psi$ is a subadditive function.
6. Support function of closed convex sets are positively homogenous subadditive functions.

# Results Based on Algebraic Structure

[Gomory and Johnson (1972a,b)], [Johnson (1974)]

## Theorem

*Let $\omega : I^m \rightarrow I^m$ be an automorphism and let $\phi$ be an extreme inequality for $MI(I^m, \emptyset, r)$. Then $\phi \circ \omega$ is extreme for $MI(I^m, \emptyset, \omega^{-1}(r))$.*

The only automorphisms for $MI(I^m, \emptyset, r)$ are rotations, reflections and their combinations.

# Results Based on Algebraic Structure

[Gomory and Johnson (1972a,b)], [Johnson (1974)]

## Theorem

*Let $\omega : I^m \rightarrow I^m$ be an automorphism and let $\phi$ be an extreme inequality for $MI(I^m, \emptyset, r)$. Then $\phi \circ \omega$ is extreme for $MI(I^m, \emptyset, \omega^{-1}(r))$.*

The only automorphisms for $MI(I^m, \emptyset, r)$ are rotations, reflections and their combinations.

## Definition

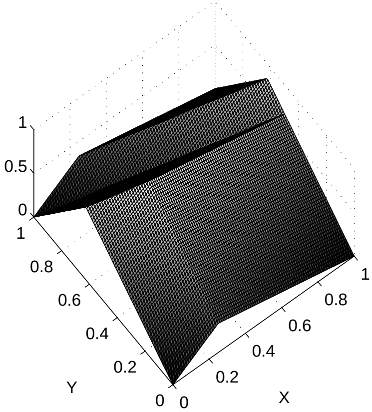
Let $\varsigma : I^m \rightarrow I^m$ be the homomorphism defined as $\varsigma(x) = (n_1x_1, n_2x_2, \ldots, n_mx_m)$ where $n_i \in \mathbb{Z}_+ \setminus \{0\}$ for $i = 1, \ldots, m$. We refer to $\varsigma(x)$ as *multiplicative homomorphism*.

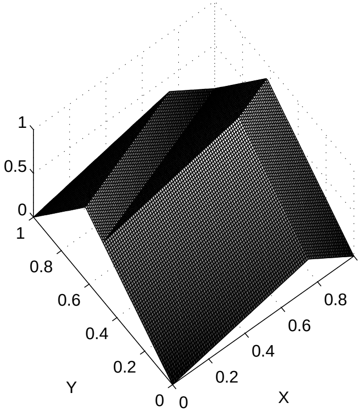
[Gomory and Johnson (2003)], [D. and Richard (2007)]

## Theorem

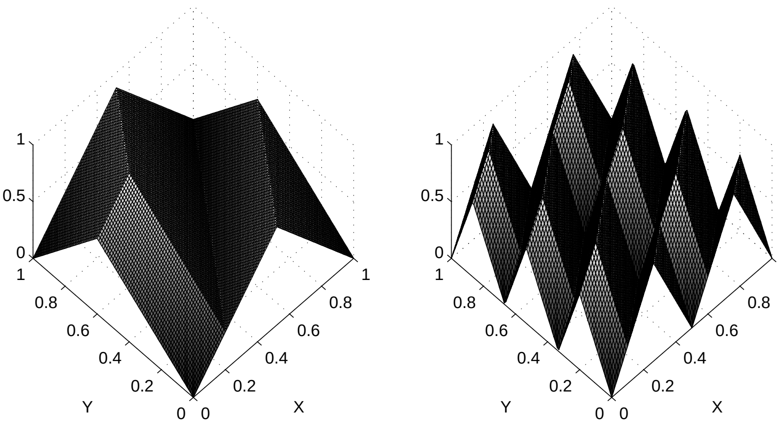
*Let $\varsigma$ be a multiplicative homomorphism. Then $\phi$ is extreme for $MI(I^m, \emptyset, r)$ iff $\phi \circ \varsigma$ is extreme for $MI(I^m, \emptyset, v)$, where $\varsigma(v) = r$.*

# Example of Reflection Automorphism Reflection



# Example of Homomorphism Result




a. Three−gradient facet

b. Homomorphism of the three−gradient facet

# Sequence of Functions

[Dash and Günlük (2006)] [D., Richard, Li and Miller (2007)]

## Proposition

*Let $\phi_i : I^m \rightarrow \mathbb{R}_+$ be valid, subadditive and minimal functions of $MI(I^m, \emptyset, r)$ for $i = 1, 2, \ldots$. If the sequence of functions $\{\phi_i\}_{i=1}^{\infty}$ converges to $\phi$ pointwise on $I^m$, then $\phi$ is a valid, subadditive and minimal function of $MI(I^m, \emptyset, r)$.*

In general, the limit of a converging sequence of extreme functions is not necessarily extreme.

## Theorem

*Let $\phi_i : I^1 \rightarrow \mathbb{R}_+$ be piecewise linear, continuous extreme functions of $MI(I^m, \emptyset, r)$ for $i \geq 1$. Assume that the sequence of functions $\{\phi_i\}_{i=1}^{\infty}$ converges pointwise to $\phi$ and that*

1. *$\phi$ is piecewise linear,*
2. *The right-derivative of $\phi$ at zero, $\phi'_+(0)$, exists and satisfies $0 < \phi'_+(0) < \infty$,*
3. *There exists a sequence of finite subgroups $C_{k_i}$, where $lim_{i \rightarrow +\infty} k_i = +\infty$, that satisfy*
   3.1 *$\phi_i(u) = \phi(u)\ \forall u \in C_{k_i}$, and*
   3.2 *all the points at which the function $\phi_i$ is non-differentiable belong to $C_{k_i}$.*

*Then $\phi$ is an extreme function of $MI(I^1, \emptyset, r)$.*

# Fill-in Function is Related to Value Function

[Gomory and Johnson (1972a,b)][Johnson (1974)]

Table: Fill-in Procedure

---

1. Input: A subadditive valid function $(\phi, \theta)$ for $MI(\mathcal{G}, \mathcal{W}, r)$, where $\mathcal{W} \subseteq \mathbb{R}^m$ and $\mathcal{G}$ is a subgroup of $I^m$.
2. Construct the function $\Theta : \mathbb{R}^m \rightarrow \mathbb{R}_+$ as follows:

$$\Theta(w) = \min \left\{ \sum_{v \in \mathcal{W}} \theta(v) y(v) \;\middle|\; w = \sum_{v \in \mathcal{W}} v y(v), y(v) \geq 0 \right\}, \tag{7}$$

# Fill-in Function is Related to Value Function

[Gomory and Johnson (1972a,b)][Johnson (1974)]

Table: Fill-in Procedure

1. Input: A subadditive valid function $(\phi, \theta)$ for $MI(\mathcal{G}, \mathcal{W}, r)$, where $\mathcal{W} \subseteq \mathbb{R}^m$ and $\mathcal{G}$ is a subgroup of $I^m$.
2. Construct the function $\Theta : \mathbb{R}^m \rightarrow \mathbb{R}_+$ as follows:
$$\Theta(w) = \min \left\{ \sum_{v \in \mathcal{W}} \theta(v) y(v) \;\middle|\; w = \sum_{v \in \mathcal{W}} v y(v), y(v) \geq 0 \right\}, \qquad (7)$$
3. Construct the function $\Phi : I^m \rightarrow \mathbb{R}_+$ as follows:
   - Compute $\Phi : I^m \rightarrow \mathbb{R}_+$ as follows:
$$\Phi(u) = \min \left\{ \phi(v) + \Theta(w) \;\middle|\; v \in \mathcal{G}, w \in \mathcal{W}, v + \mathbb{P}(w) = u \right\}. \qquad (8)$$
4. Output: A subadditive valid function $(\Phi, \Theta)$ for $MI(I^m, \mathbb{R}^m, r)$.

Fill-in does not always generate extreme inequalities.

# Two Slope Theorem and Extensions

[Gomory and Johnson (1972a,b)]

## Theorem

*Continuous, piecewise linear, subadditive and minimal functions that have only two slopes are extreme for* $MI(I^1, \emptyset, r)$.

# Two Slope Theorem and Extensions

[Gomory and Johnson (1972a,b)]

## Theorem

*Continuous, piecewise linear, subadditive and minimal functions that have only two slopes are extreme for* $MI(I^1, \emptyset, r)$.

One (Incomplete) Interpretation:

- Let $u^1, ..., u^k$ be the points at which the function $\phi$ is not differentiable. (I am assuming these are rational points)
- Let $C_n$ be the cyclic subgroup of $I^1$ that is generated by $u^1, ..., u^k$.
- Let $\pi^+ = \lim_{h \downarrow 0} \frac{\phi(h)}{h}$ and $\pi^- = \lim_{h \downarrow 0} \frac{\phi(1-h)}{h}$.
- The inequality $\sum_{u \in C_n} \phi(u)x(u) + \pi^+ y(1) + \pi^- y(-1) \geq 1$ is extreme for $MI(C_n, \{+1, -1\}, r)$.
- The function $\phi : I^1 :\rightarrow \mathbb{R}_+$ is obtained as a fill-in function.

# Two Slope Theorem and Extensions

[Gomory and Johnson (1972a,b)]

## Theorem

*Continuous, piecewise linear, subadditive and minimal functions that have only two slopes are extreme for $MI(I^1, \emptyset, r)$.*

One (Incomplete) Interpretation:

- Let $u^1, ..., u^k$ be the points at which the function $\phi$ is not differentiable. (I am assuming these are rational points)
- Let $C_n$ be the cyclic subgroup of $I^1$ that is generated by $u^1, ..., u^k$.
- Let $\pi^+ = \lim_{h \downarrow 0} \frac{\phi(h)}{h}$ and $\pi^- = \lim_{h \downarrow 0} \frac{\phi(1-h)}{h}$.
- The inequality $\sum_{u \in C_n} \phi(u)x(u) + \pi^+ y(1) + \pi^- y(-1) \geq 1$ is extreme for $MI(C_n, \{+1, -1\}, r)$.
- The function $\phi : I^1 :\rightarrow \mathbb{R}_+$ is obtained as a fill-in function.

[D. and Wolsey (2007)]

## Theorem

*Let $(\phi, \pi)$ be minimal for $MI(\mathcal{G}, \mathbb{R}^m, r)$ where $\mathcal{G}$ is a finite subset of $I^m$. Then the fill-in inequality $(\Phi, \pi)$ is extreme for $MI(I^m, \mathbb{R}^m, r)$ if and only if $(\phi, \pi)$ is extreme for $MI(\mathcal{G}, \mathbb{R}^m, r)$ and $(\Phi, \pi)$ is minimal for $MI(I^m, \mathbb{R}^m, r)$.*

# Using Simple Sets: Gomory Mixed Integer Cut from Mixed Integer Rounding Inequalities

[Nemhauser and Wolsey (1990)][Wolsey (1998)]

- Analyze a simple mixed integer set $Q$. E.g.

$$X^{\geq} := \{(x, y) \in \mathbb{Z} \times \mathbb{R}_{+} \mid x + y \geq b\}.$$

# Using Simple Sets: Gomory Mixed Integer Cut from Mixed Integer Rounding Inequalities

[Nemhauser and Wolsey (1990)][Wolsey (1998)]

- Analyze a simple mixed integer set $Q$. E.g.

$$X^{\geq} := \{(x, y) \in \mathbb{Z} \times \mathbb{R}_+ \mid x + y \geq b\}.$$

- Obtain a valid inequality (Mixed Integer Rounding Inequalities). E.g.

$$x + \frac{y}{b - \lfloor b \rfloor} \geq \lceil b \rceil$$

- Using aggregation of variables in the group relaxation to rewrite it in the form of $Q$
- Using the valid inequality for $Q$, obtain a valid inequality for the group problem. E.g.: yields the Gomory Mixed Integer Cut.

# Two Step MIR inequality

[Dash and Günlük (2006)]

- Consider the following Set

$$Q^2 = \{(z_1, z_2, y) \in \mathbb{Z} \times \mathbb{Z}_+ \times \mathbb{R}_+ \mid z_1 + \alpha z_2 + y \geq b\},$$

where $0 < \alpha < b - \lfloor b \rfloor$ and $\frac{1}{\alpha} \geq \left\lceil \frac{b - \lfloor b \rfloor}{\alpha} \right\rceil$. Assume for simplicity that $0 < b < 1$, i.e., $\lfloor b \rfloor = 0$.

- The following *two-step MIR inequality* is proven to be facet-defining for $Q^2$:

$$\left\lceil \frac{b}{\alpha} \right\rceil z_1 + z_2 + \frac{y}{b - \alpha \lfloor \frac{b}{\alpha} \rfloor} \geq \left\lceil \frac{b}{\alpha} \right\rceil.$$

- Consider the following single row

$$x_0 + \sum_{i=1}^{n-1} \frac{i}{n} x_i = r,$$

where $x_0 \in \mathbb{Z}$ and $x_i \in \mathbb{Z}_+ \; \forall i$. Suppose $\alpha > 0$ is chosen such that $r > \alpha$ and $\frac{1}{\alpha} \geq \lceil \frac{r}{\alpha} \rceil = 2$. The following relaxation is constructed

$$\left(x_0 + \sum_{i \mid i > nr} x_i\right) + \alpha \left(\sum_{i \mid (r-\alpha)n \leq i \leq nr} x_i\right) + \left(\sum_{i \mid i < n(r-\alpha)} \frac{i}{n} x_i + \sum_{i \mid \alpha n < i \leq nr} \left(\frac{i}{n} - \alpha\right) x_i\right) \geq r.$$

# Two-Step MIR inequality

[Dash and Günlük (2006)]

## Theorem

*Let $0 < \alpha < r$ and $\frac{1}{\alpha} \geq \lceil \frac{r}{\alpha} \rceil > \frac{r}{\alpha}$. Then the function $g^{r,\alpha} : I^1 \rightarrow \mathbb{R}_+$ defined as*

$$g^{r,\alpha}(u) = \begin{cases} \frac{u(1-\rho\tau)-k(u)(\alpha-\rho)}{\rho\tau(1-r)} & \text{if } u - k(u)\alpha < \rho \\ \frac{k(u)+1-\tau u}{\tau(1-r)} & \text{if } u - k(u)\alpha \geq \rho \end{cases}, \quad (9)$$

*where $\rho = r - \alpha\lfloor \frac{r}{\alpha} \rfloor$, $\tau = \lceil \frac{r}{\alpha} \rceil$, and $k(u) = min\{\lceil \frac{u}{\alpha} \rceil, \tau\} - 1$ represents an extreme inequality for the one-row group problem with right-hand side $r$.*

[Kianfar and Fathi (2008)] generalized the above to the $n$-step MIR inequalities.

# Aggregation

## Definition

Let $\phi : I^1 \to \mathbb{R}_+$ be a valid inequality for $MI(I^1, \emptyset, r)$ and let $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{Z}^m \backslash \{0\}$. We define the aggregation function $\phi^\lambda : I^m \to \mathbb{R}_+$ as $\phi^\lambda(x) = \phi(\sum_{i=1}^m \lambda_i x_i)$.

[D. and Richard (2007)]

## Theorem

*If $\phi$ is an extreme inequality for $MI(I^1, \emptyset, r)$, then $\phi^\lambda$ is an extreme inequality for $MI(I^m, \emptyset, r')$ where $r' \in I^m$ satisfies $(\sum_{i=1}^m \lambda_i r'_i)(mod\, 1) = r$.*

# Sequential-Merge Inequalities

- Given a valid inequality $\phi$ for $MI(I^m, \emptyset, r)$, define $[\phi]_r : \mathbb{R}^m \to \mathbb{R}$ as $[\phi]_r(x) = \sum_{i=1}^m x_i - \sum_{i=1}^m \mathbb{P}^{-1}(r_i)\phi(\mathbb{P}(x))$.
- Given a function $\psi : \mathbb{R}^m \to \mathbb{R}$ that satisfies $\psi(x + e_i) = \psi_r(x) + 1$, define $[\psi]_r^{-1} : I^m \to \mathbb{R}$ as $[\psi]_r^{-1}(x) = \frac{\sum_{i=1}^m \mathbb{P}(x_i) - \psi(x)}{\sum_{i=1}^m \mathbb{P}^{-1}(r_i)}$.

## Proposition

*If $\phi$ is valid function for $PI(r, m)$,*

1. $[\phi]_r(x + e_i) = [\phi]_r(x) + 1$*, where $e_i$ is the $i^{th}$ unit vector of* $\mathbb{R}^m$.
2. $[\phi]_r$ *is superadditive iff $\phi$ is subadditive.* $\square$

# Explanation of Sequential-Merge Operator

Given: (1)$m+1$ Tableau Rows:

$$\begin{aligned} \sum_i a_i^1 x_i &= b^1 \quad \}\textbf{... First Row} \\ \sum_i a_i^2 x_i &= b^2 \quad \}\textbf{... Next m Rows} \end{aligned}$$

(2) Valid group functions $g$ and $h$.

# Explanation of Sequential-Merge Operator

Given: (1)$m+1$ Tableau Rows:

$$\sum_i a_i^1 x_i = b^1 \quad \}\textbf{... First Row}$$

$$\sum_i a_i^2 x_i = b^2 \quad \}\textbf{... Next m Rows}$$

(2) Valid group functions $g$ and $h$.

1. Generate valid inequality [h] for the last $m$-rows:

$$\sum_i [h](a_i^2)x_i \leq [h](b^2) \tag{10}$$

# Explanation of Sequential-Merge Operator

Given: (1)$m + 1$ Tableau Rows:

$$\sum_i a_i^1 x_i = b^1 \quad \}\text{... **First Row**}$$

$$\sum_i a_i^2 x_i = b^2 \quad \}\text{... **Next m Rows**}$$

(2) Valid group functions $g$ and $h$.

1. Generate valid inequality [h] for the last $m$-rows:

$$\sum_i [h](a_i^2) x_i \leq [h](b^2) \tag{10}$$

2. Add (10) to the first row of tableau, i.e.,

$$\sum_i ([h](a_i^2) + a^1) x_i \leq [h](b^2) + b^1 \tag{11}$$

# Explanation of Sequential-Merge Operator

Given: (1) $m+1$ Tableau Rows:

$$\begin{aligned} \sum_i a_i^1 x_i &= b^1 \quad \}\text{... **First Row**} \\ \sum_i a_i^2 x_i &= b^2 \quad \}\text{... **Next m Rows**} \end{aligned}$$

(2) Valid group functions $g$ and $h$.

1. Generate valid inequality [h] for the last $m$-rows:

$$\sum_i [h](a_i^2)x_i \leq [h](b^2) \tag{10}$$

2. Add (10) to the first row of tableau, i.e.,

$$\sum_i ([h](a_i^2) + a^1)x_i \leq [h](b^2) + b^1 \tag{11}$$

3. Generate the valid cut [g] for (11):

$$\sum_i [g]([h](a_i^2) + a^1)x_i \leq [g]([h](b^2) + b^1) \tag{12}$$

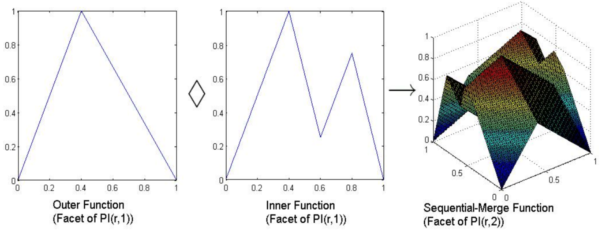
# Sequential-Merge Procedure

Proof of validity uses the fact that the function $\varpi(a, b) := \varphi(a + \psi(b))$ is a superadditive function if $\psi$ is superadditive and $\varphi$ is non-decreasing and superadditive.
[D. and Richard (2007)]

## Theorem

*Assume that $g$ and $h$ are continuous, piecewise linear valid functions for $MI(I^1, \emptyset, r^1)$ and $MI(I^m, \emptyset, r^2)$ respectively. Assume also that $g$ and $h$ are unique solutions of $E(g)$ and $E(h)$ respectively and that $[g]_{r^1}$ and $[h]_{r^2}$ are nondecreasing. Then $g\diamondsuit h$ is an extreme inequality for $MI(I^{m+1}, \emptyset, (r^1, r^2)')$.*

# Illustration of Sequential Merge Inequalities



# Searching for Extreme Inequalities Using Approximate Lifting

[Richard, Li, and Miller (2007)], [Miller, Li, and Richard (2007)] Basic Idea:

- Decide on a specific template of inequalities.
- Find extreme inequalities within a specific class.
- Verify these inequalities are extreme in general.

## Definition

Let $K \in \mathbb{R}_+$ and $r \in (0, K)$. Let $n \in \mathbb{Z}_+$, $z = (z_1, z_2, \cdots, z_n) \in \mathbb{R}^n_+$, and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{R}^n_+$ be such that $\sum_{j=1}^n z_j = \frac{K-r}{2}$ and $\sum_{j=1}^n \gamma_j z_j = \frac{1}{2}$. A function $\psi : \mathbb{R}^1 \rightarrow \mathbb{R}_+$ is said to be a $\mathrm{CPL}_n(K; r; z; \gamma)$ function if, when $u$ is restricted to $[0, K)$,

$$\psi(u) = \begin{cases} 0, & \text{if } u \in [0, r], \\ \Gamma_{i-1} + \gamma_i(u - r - Z_{i-1}) & \text{if } u \in (r + Z_{i-1}, r + Z_i), \\ \Gamma_i & \text{if } u = r + Z_i, \\ 1 - \Gamma_i & \text{if } u = K - Z_i, \\ 1 - \Gamma_{i-1} - \gamma_i(K - u - Z_{i-1}) & \text{if } u \in (K - Z_i, K - Z_{i-1}), \end{cases}$$

for $i = 1, \ldots, n$ where $Z_0 = 0$, $\Gamma_0 = 0$, $Z_i = \sum_{j=1}^i z_j$ and $\Gamma_i = \sum_{j=1}^i \gamma_j$ for $i = 1, \ldots, n$. For the sake of brevity, we call a $\mathrm{CPL}_n(K; r; z; \gamma)$ function a $\mathrm{CPL}_n$ function.

# Approximate Lifting Contd.

## Proposition

*Let $z \in \mathbb{R}^n_+$ be such that $Z_n = \frac{K-r}{2}$. Parameter $\gamma$ defines a superadditive $\mathcal{CPL}_n$ function if and only if $\gamma$ belongs to the polyhedron*

$$
\begin{aligned}
P\Theta_n(z) = \{\gamma \in \mathbb{R}^{n-1}_+ \mid\ & \Gamma_i + \Gamma_j \leq \psi(2r + Z_i + Z_j), && 0 \leq i,j \leq n-1, \\
& \Gamma_i - \Gamma_j \leq \psi(r + K + Z_i - Z_j) - 1, && 0 \leq i,j \leq n-1, \\
& \Gamma_i + \Gamma_j \geq \psi(r + Z_i + Z_j), && 0 \leq i,j \leq n-1\}.
\end{aligned}
$$

# Approximate Lifting Contd.

## Proposition

*Let $z \in \mathbb{R}^n_+$ be such that $Z_n = \frac{K-r}{2}$. Parameter $\gamma$ defines a superadditive $CPL_n$ function if and only if $\gamma$ belongs to the polyhedron*

$$
\begin{aligned}
P\Theta_n(z) = \{\gamma \in \mathbb{R}^{n-1}_+ \mid\ & \Gamma_i + \Gamma_j \leq \psi(2r + Z_i + Z_j), && 0 \leq i,j \leq n-1,\\
& \Gamma_i - \Gamma_j \leq \psi(r + K + Z_i - Z_j) - 1, && 0 \leq i,j \leq n-1,\\
& \Gamma_i + \Gamma_j \geq \psi(r + Z_i + Z_j), && 0 \leq i,j \leq n-1\}.
\end{aligned}
$$

## Theorem

*The following are the only extreme points of $P\Theta_2(z_1)$:*

$$
\gamma_1^1 = \left\{ \begin{array}{lll} \frac{z_1}{K-r}, & z_1 \in [0, \frac{K-r}{2}] & \text{(GMIC)} \end{array} \right.
$$

$$
\gamma_1^2 = \left\{ \begin{array}{lll} \frac{z_1+r}{K+r}, & z_1 \in [0, \frac{K-2r}{3}) & \text{(3-slope inequality)} \\ \frac{z_1}{K-2r}, & z_1 \in [\frac{K-2r}{3}, \frac{K-2r}{2}) & \text{(new inequality)} \\ \frac{1}{2}, & z_1 \in [\frac{K-2r}{2}, \frac{K-r}{2}] & \text{(2-slope inequality).} \end{array} \right.
$$

# Some Questions

- More than 4 slopes in 1 dimension?
- A number of techniques used to prove inequalities are extreme (especially for the infinite group problems) are technical in nature. Better methods to search and prove that inequalities are extreme are needed. Example:
  - Understanding Interpolation in higher dimensions.
  - Interval lemma in higher dimension.
- Given the large number of extreme inequalities, new methods, other than based on merit index or strength of continuous variables coefficients, are needed to predict the usefulness of extreme inequalities for group problems.
- Good cut selection rules when using multi-row Cuts.
- Generating safe group cuts [Cook, Dash, Fukasawa, Goycoolea].