Continuous and Discontinuous Extreme Inequalities for Infinite Group Problems

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Difference Between Group Relaxation of Pure Integer and Mixed Integer Programs

Infinite Group Relaxation of Integer Programs

- Standard IP:

$$Ax = b \quad x \in \mathbb{Z}_+,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{m \times 1}$.

- Relaxation step 1: Consider each row modulo 1.

$$\sum_{i=1}^{n} (A_{ij})(mod 1)x_i \equiv b_j(mod 1) \quad \forall 1 \leq j \leq m \tag{1}$$

- Rewrite $\sum_{i=1}^{n} (a_i)x_i = r$

Each a_i belongs to the group $\mathbb{I}^m = \{x \in \mathbb{R}^m | 0 \leq x_i < 1 \quad \forall 1 \leq i \leq m\}$.

Note that $a_i = (A_{i1}(mod 1), \ldots, A_{im}(mod 1))$.

- Relaxation step 2: Introduce new variables.

$$\sum_{a \in \mathbb{I}^m} ax(a) = r \tag{2}$$

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$$\sum_{i=1}^{n} (A_{ij})(mod 1)x_i \equiv b_j(mod 1) \quad \forall 1 \leq j \leq m \tag{3}$$

- Rewrite $\sum_{i=1}^{n}(a_i)x_i = r$

  Each a_i belongs to the group $\mathbb{G} = C_{|k|} \times C_{|k|} \ldots C_{|k|}$ where $C_{|k|}$ is the cyclic group of order k.
- Relaxation step 2: Introduce new variables.

$$\sum_{a \in \mathbb{G}} a x_a = r \tag{4}$$

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Each a_i belongs to the group $\mathbb{G} = C_{|k|} \times C_{|k|} \dots C_{|k|}$ where $C_{|k|}$ is the cyclic group of order k.

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Definition: Group Problem and Valid Inequalities

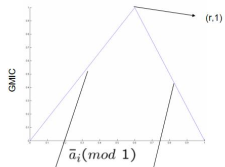
Definition (Integer Group Problem $PI(r, m)$, Johnson 1974)

For $r \in \mathbb{I}^m$ and $r \neq o$, the group problem $PI(r, m)$ is the set of functions $t : \mathbb{I}^m \longrightarrow \mathbb{R}$ such that

1. $\sum_{u \in \mathbb{I}^m} ut(u) = r, r \in \mathbb{I}^m$,
2. $t(u)$ is a non-negative integer for $u \in \mathbb{I}^m$,
3. t has a finite support, i.e., $t(u) > 0$ for a finite subset of $\mathbb{I}^m$.

Definition (Valid Inequality, Johnson 1974)

A function $\phi : \mathbb{I}^m \longrightarrow \mathbb{R}_+$ is defined as a valid inequality for PI(r,m) if $\phi(o) = 0$, $\phi(r) = 1$ and $\sum_{u \in \mathbb{I}^m} \phi(u)t(u) \geq 1, \forall\, t \in PI(r, m)$.



$$\sum_{f_i \leq r} \frac{f_i}{r} x_i + \sum_{f_i > r} \frac{1 - f_i}{1 - r} x_i \geq 1$$

Hierarchy of cutting planes

- Valid Inequality.
- **Subadditive Valid Inequality:** A function f is a subadditive valid inequality if $f(x) + f(y) \geq f(x+y)\ \forall x, y \in \mathbb{I}^m$.
- **Minimal Inequality:** A valid inequality ϕ is minimal for $PI(r, m)$ if there does not exist a valid function ϕ^* for $PI(r, m)$ different from ϕ such that $\phi^*(u) \leq \phi(u)\ \forall u \in \mathbb{I}^m$.
- **Extreme Inequality:** A valid inequality ϕ is extreme for $PI(r, m)$ if whenever $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$ for some valid inequalities ϕ_1 and ϕ_2 of $PI(r, m)$ then $\phi = \phi_1 = \phi_2$.

Theorem (Gomory & Johnson 1972a,1972b, Johnson 1974)

Valid Inequality $\supset$ *Subadditive Valid Inequality* $\supset$ *Minimal Inequality* $\supset$ *Extreme Inequality*

Main Contributions

All known extreme inequalities for $PI(r, 1)$ are continuous functions over $\mathbb{I}^1$.

Theorem (Discontinuous Extreme Inequality)

There exists extreme inequalities for $PI(r, 1)$ that are discontinuous.

Other Results:

1. It is well known that piecewise linear extreme inequalities for continuous inequalities yield extreme inequalities for finite group problem. We prove a weak converse.
2. Point-wise limit of a sequences of functions that represent extreme inequalities is also extreme.

Key Result

Proposition (Continuity Proposition)

Let $\phi : \mathbb{I}^1 \rightarrow \mathbb{R}_+$ *be*

- *A piecewise linear, subadditive and valid function for* $PI(r, 1)$*,*
- $\phi(u) = cu \ \forall u \in U$*, where* $c > 0$, $U \equiv [0, w]$*,* w *is a non-zero number less than* 1*.*

Assume that

$$\phi = (1 - \lambda)\phi_1 + \lambda\phi_2,$$

where $0 < \lambda < 1$ *and* ϕ_1 *and* ϕ_2 *are some subadditive valid inequalities. Then* ϕ_1 *and* ϕ_2 *are continuous at all points at which* ϕ *is continuous.*

One Slope Extreme Inequality

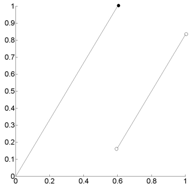


Figure: π with $r = 0.6$

Theorem

The function $\pi : \mathbb{I}^1 \rightarrow \mathbb{R}_+$ is defined for a right-hand-side r with $r \geq 0.5$ as

$$\pi(x) = \begin{cases} \frac{x}{r} & 0 \leq x \leq r \\ \frac{x}{r} - \frac{1}{2r} & r < x < 1. \end{cases} \tag{5}$$

is extreme for $PI(r, 1)$.

The function π was discovered by Letchford and Lodi(2002).

Overview of Proof: Use of *Continuity Proposition*

- Assume by contradiction that π is not extreme

$$\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$$

where $\pi_1 \neq \pi_2$

- Using *Continuity Proposition* we know that π_1, π_2 is continuous in the intervals $[0, r]$ and $(r, 1)$.
- Use Interval Lemma (Gomory and Johnson 2003):

Proposition (Interval Lemma)

Let f be a continuous function over intervals U, V and U + V. If $f(u) + f(v) = f(u + v)\ \forall u \in U, v \in V$, f is a linear function with the same slope everywhere.

Then use this to prove that π_1 and π_2 have same slopes in the intervals $[0, r]$ and $(r, 1)$. Some algebraic manipulations lead to result. So need to know that π_1 and π_2 are continuous in the intervals $[0, r]$ and $(r, 1)$.

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Another Interpretation: Sequence of Converging Extreme Inequalities

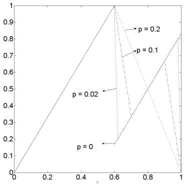

Figure: π^p with different values of p

$$\pi = \pi^0$$

$$\pi^p(x) = \begin{cases} \frac{x}{r} & x \le r \\ 1 + \frac{2p-1}{2pr}(x-r) & r < x \le r+p \\ \frac{x}{r} - \frac{1}{2r} & r+p < x \le 1-p \\ \frac{2p-1}{2pr}(x-1) & 1-p < x < 1 \end{cases} \quad (6)$$

See Dash and Günlük (2006) for an exposition of this interpretation.

General Result on Converging Extreme Inequalities

Theorem (Limit Extreme Inequality)

Let $f_i : \mathbb{I}^1 \to \mathbb{R}_+$

- *Be piecewise linear, continuous extreme functions of* $PI(r, 1)$ *for* $i \geq 1$.
- *The sequence of functions* $\{f_i\}_{i=1}^{\infty}$ *converges to* ϕ *pointwise on* $\mathbb{I}^1$
- ϕ *satisfies the conditions of Continuity Proposition*
- *Let* $\mathbb{G}$ *be a finite subgroup of* $\mathbb{I}^1$ *such that if* ϕ *is discontinuous at* u *then* $u \in \mathbb{G}$. *Assume that for every* $i \in \mathbb{Z}_+$, *there is* $k(i) \in \mathbb{Z}_+$, *such that the non-differentiable points of* f_i *belong to* $2^k\mathbb{G}$ *and* $f_i(u) = \phi(u)\ \forall u \in 2^k\mathbb{G}$.

Then ϕ *is an extreme function for* $PI(r, 1)$.

Two Slope Extreme Inequality - I

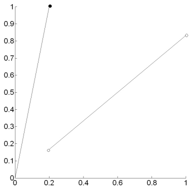

Figure: κ with $r = 0.2$

Theorem

The function $\kappa : \mathbb{I}^1 \rightarrow \mathbb{R}_+$ *is defined for* $r < 0.5$ *as*

$$\kappa(u) = \begin{cases} \frac{u}{r} & u \in [0, r] \\ \frac{u}{r+1} & u \in (r, 1) \end{cases} \tag{7}$$

is extreme for $PI(r, 1)$.

The function κ was discovered by Richard, Miller and Li(2006).

Two Slope Extreme Inequality - II

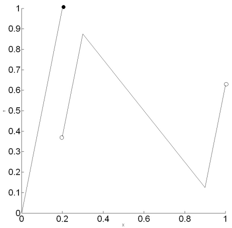

Figure: π with $r = 0.2$, $\theta = 0.1$

Theorem

The function $\zeta^{\theta} : \mathbb{I}^1 \to \mathbb{R}_+$ *is defined for* $\hat{\theta} \leq \min\left\{\frac{\hat{r}}{2}, \frac{1-\hat{r}}{4}\right\}$ *as*

$$\zeta^{\theta}(x) = \begin{cases} \frac{x}{r} & 0 \leq x \leq r \\ \frac{1-r-\theta}{1-r} - \frac{r+\theta-x}{r} & r < x \leq r+\theta \\ \frac{1-x}{1-r} & r+\theta \leq x \leq 1-\theta \\ \frac{\theta}{1-r} + \frac{x-1+\theta}{r} & 1-\theta \leq x < 1 \end{cases} \quad (8)$$

is extreme for $PI(r, 1)$.

What is the Merit of These Inequalities?

Definition (Gomory & Johnson 2003)

Let C_2 be the unit square in two dimensions. The merit index $\mathbb{M}(\phi)$ of a given inequality ϕ is equal to twice the area of the set of points $q \equiv (u_1, u_2) \in C_2$ such that $\phi(u_1) + \phi(u_2) = \phi(u_1 + u_2)$.

Merit Index was empirically shown to be strongly correlated to the results of the shooting experiment.

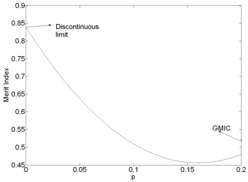

Figure: Merit Index for π^p

Similar graphs for κ and ζ^θ.

Conclusions

1. There exist discontinuous functions that are extreme for the integer infinite group problem. In particular we show that three different families of discontinuous valid inequalities are extreme for $PI(r, 1)$ and and the proof use different tools that we introduced.
2. The limiting inequality of a sequence of continuous piecewise linear extreme inequalities is extreme under some conditions.

Another Consequence of *Continuity Proposition*

Proposition (Finite To Infinite Group Extreme Inequality)

Let $\hat{\phi}$ be a valid subadditive extreme inequality for a finite group problem $P(C_{|\mathbb{G}|,r})$. Consider the linear interpolation *of $\hat{\phi}$, $\phi : \mathbb{I}^1 \rightarrow \mathbb{R}_+$, defined as*

$$\phi(u) = \begin{cases} \hat{\phi}(u) & u \in \mathbb{G} \\ \frac{(\hat{u}_2 - \hat{u})\hat{\phi}(u_1) + (\hat{u} - \hat{u}_1)\hat{\phi}(u_2)}{\hat{u}_2 - \hat{u}_1} & u \notin \mathbb{G}. \end{cases} \quad (9)$$

Suppose that ϕ restricted to $2^k\mathbb{G}$ is an extreme valid inequality for $P(C_{|2^k\mathbb{G}|,r})$ for all $k \in \mathbb{Z}_+$, then ϕ is extreme for the infinite group problem.

Note on Notation:

- Here $2^k\mathbb{G}$ is a subgroup of $\mathbb{I}^1$ such that $\mathbb{G}$ is a subgroup of $2^k\mathbb{G}$ and

$$\frac{|2^k\mathbb{G}|}{|\mathbb{G}|} = 2^k$$

- u_1 and u_2 are the closest points of $\mathbb{G}$ to u such that $\hat{u}_1 < \hat{u} < \hat{u}_2$.

Introduction to Group Relaxation for MIPs

Let $\mathbb{S}^m$ represent the set of real m-dimensional vectors $w = (w_1, w_2 \ldots w_m)$, such that $max\{|w_i||1 \leq i \leq m\} = 1$.

Definition (Mixed Integer Group Problem MI(r,m), Johnson 1974)

The mixed integer infinite group problem, $MI(r, m)$, is defined as a set of functions $t : \mathbb{I}^m \to \mathbb{R}$ and $s : \mathbb{S}^m \to \mathbb{R}$ that satisfy

1. $\sum_{u \in \mathbb{I}^m} ut(u) + F(\sum_{v \in \mathbb{S}^m} vs(v)) = r$, $r \in \mathbb{I}^m$,
2. $t(u)$ is a non-negative integer for $u \in \mathbb{I}^m$, $s(v)$ is a non-negative real number for $v \in \mathbb{S}^m$,
3. t and s have finite supports.

Definition (Valid Inequality)

A valid inequality for MI(r,m) is defined as a pair of functions, $\phi : \mathbb{I}^m \to \mathbb{R}_+$ and $\mu_\phi : \mathbb{S}^m \to \mathbb{R}_+$, such that $\sum_{u \in \mathbb{I}^m} \phi(u)t(u) + \sum_{v \in \mathbb{S}^m} \mu_\phi(v)s(v) \geq 1$, $\forall$ $(t, s) \in MI(r, m)$, where $\phi(o) = 0$ and $\phi(r) = 1$.

Difference Between Integer and Mixed Integer Infinite Group Problem

Theorem (Johnson 1974)

Let $\phi : \mathbb{I}^m \rightarrow \mathbb{R}_+$ and let $\tau_\phi : \mathbb{S}^m \rightarrow \mathbb{R}_+$. For any $r \in I \setminus \{0\}$ and any $W \subseteq \mathbb{S}^m$, (ϕ, τ_ϕ) is a minimal valid inequality if and only if

$$\begin{aligned} \phi(u) + \phi(v) &\geq \phi(u+v) && \forall u, v \in \mathbb{I}^m \\ \tau_\phi(w) &= \lim_{h \downarrow 0} \frac{\phi(F(hw))}{h} && \forall w \in W \\ \phi(u) + \phi(u_0 - u) &= 1 && \forall u \in \mathbb{I}^m. \end{aligned} \quad (10)$$

Theorem

Let (ϕ, μ_ϕ) be a valid inequality for $MI(r, m)$. If ϕ satisfies the first two conditions in (10), then ϕ is continuous.

A related result was presented by Zhang (1992) assuming bounded function and a slight variant of the infinite group problem.

Final Comments

1. New method for constructing extreme inequalities for infinite group problem from extreme inequalities of finite group problems.
2. There exist discontinuous functions that are extreme for the integer infinite group problem. In particular we show that three different families of discontinuous valid inequalities are extreme for $PI(r, 1)$ and and the proof use different tools that we introduced.
3. The limiting inequality of a sequence of continuous piecewise linear extreme inequalities is extreme under some conditions.
4. We provide a proof of the fact that extreme functions for mixed integer infinite group problem are always continuous.

Thank You.