Some Properties of Convex Hulls of Integer Points Contained in General Convex Sets

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June, 2011.

Fundamental theorem of integer programming

Theorem (Meyer, 1974)

If $K \subseteq \mathbb{R}^n$ be a rational polyhedron[1]*, then* $\mathrm{conv}(K \cap \mathbb{Z}^n)$ *is a rational polyhedron.*

This motivates the following questions:

[1] It is sufficient that K has a rational polyhedral recession cone

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This motivates the following questions:

1. Let K be a closed convex set. When is $\text{conv}(K \cap \mathbb{Z}^n)$ a polyhedron?
2. A more basic question: Let K be a closed convex set. When is the set $\text{conv}(K \cap \mathbb{Z}^n)$ closed?

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1

Closedness of $\mathrm{conv}(K \cap \mathbb{Z}^n)$

1.0

Lets look at some example

Theme: *Recession cone plays a role, however there are other factors.*

Example A

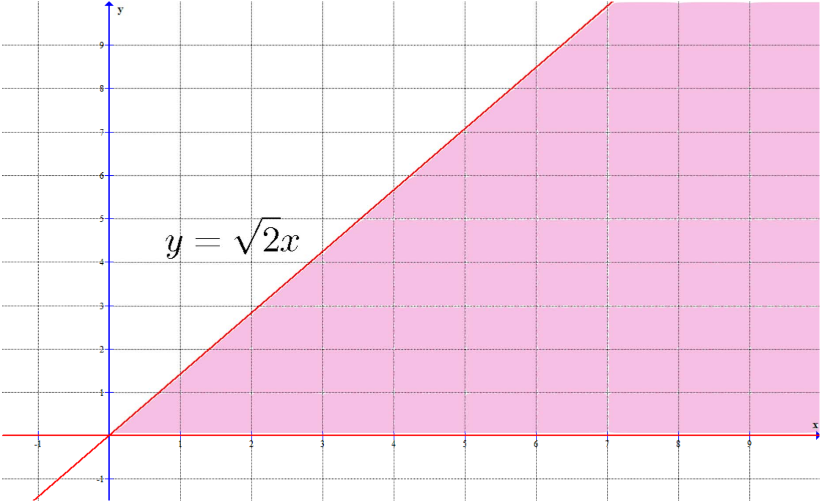

Figure: conv($K \cap \mathbb{Z}^n$) is not closed.

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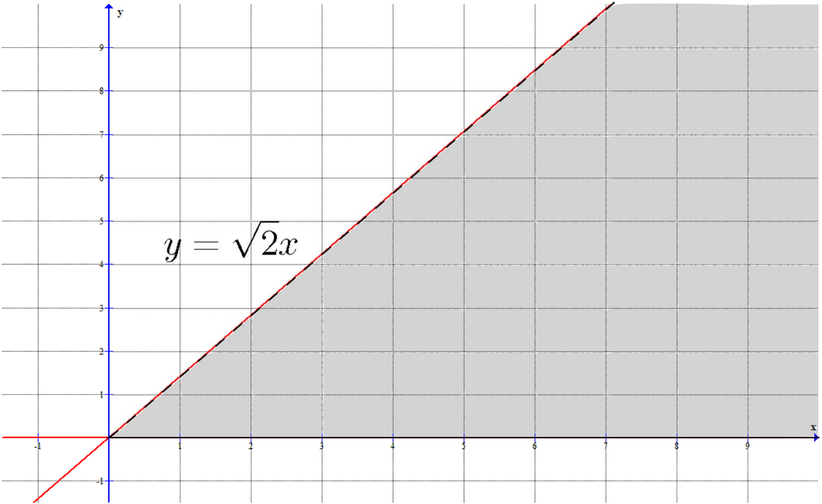

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Example B

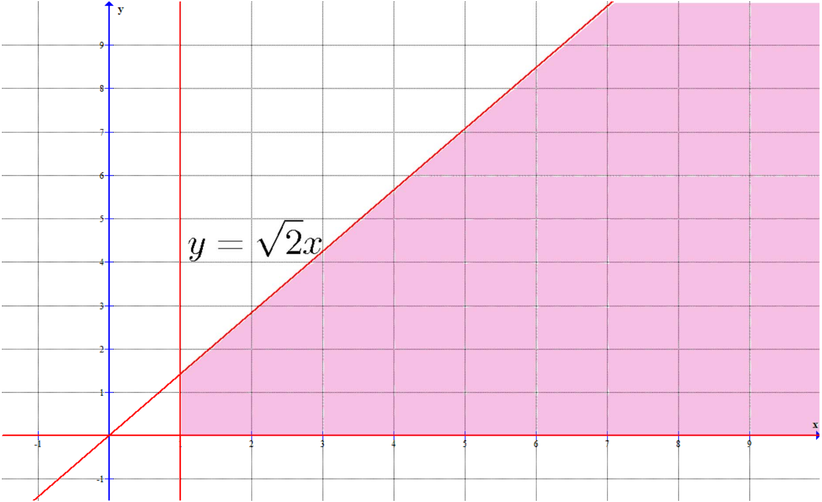

Figure: conv$(K \cap \mathbb{Z}^n)$ is closed!

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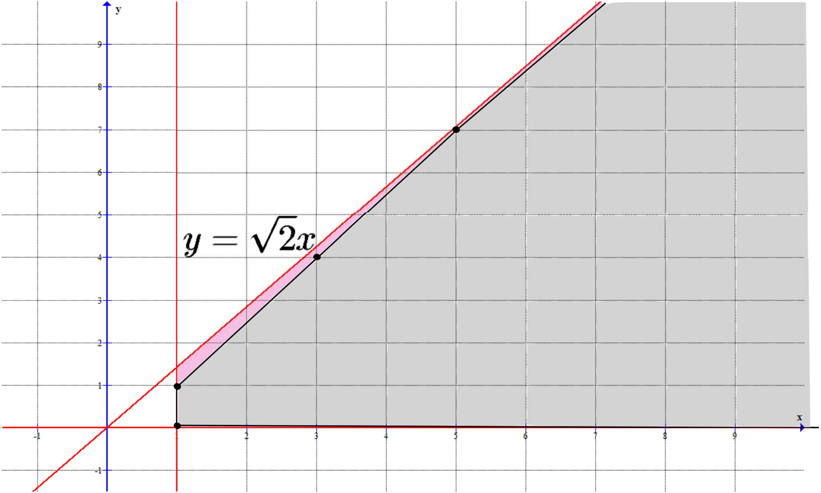

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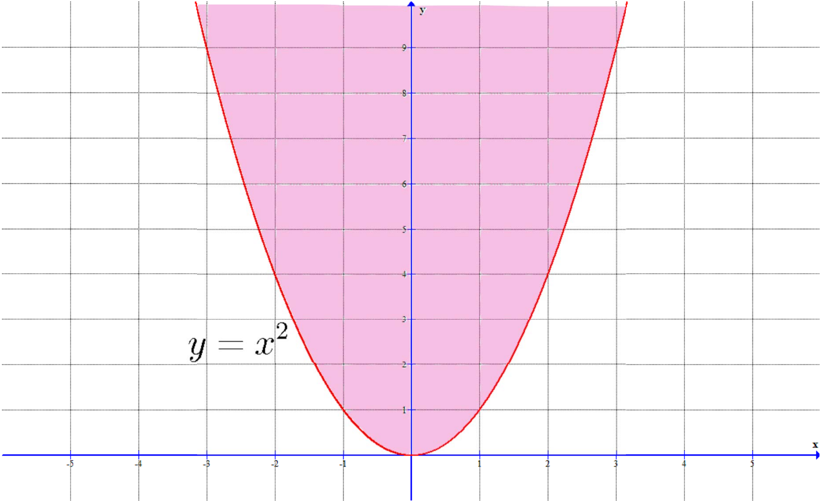
Example C



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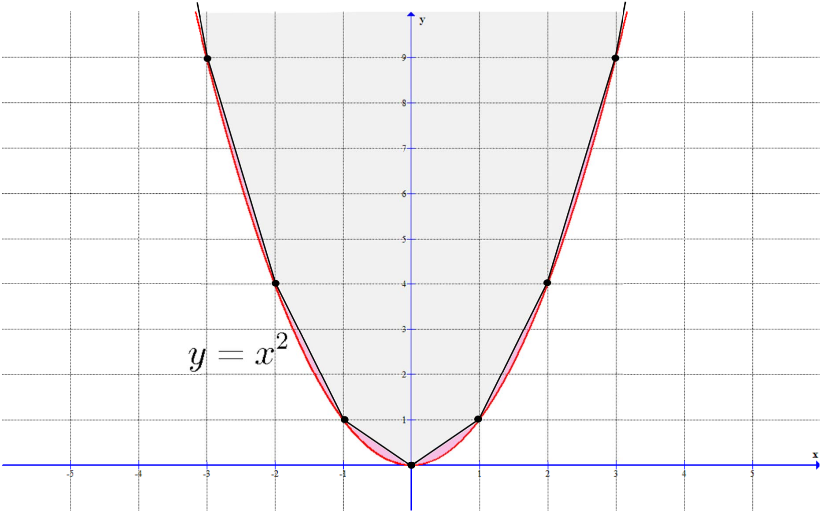

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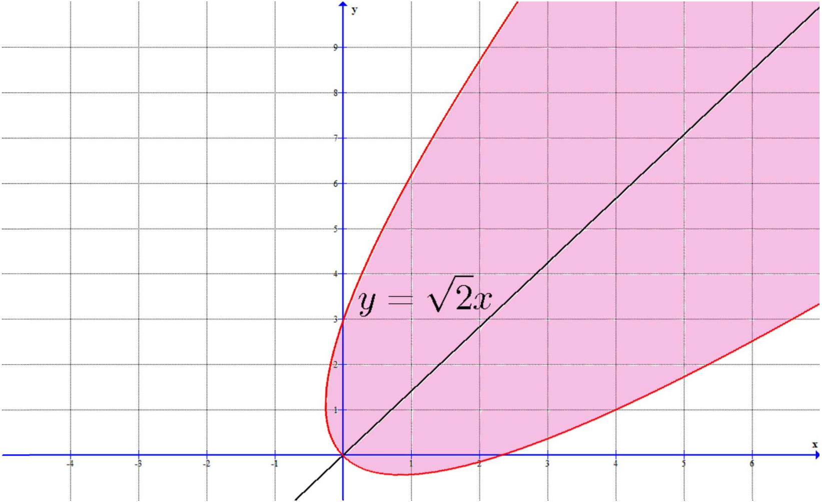

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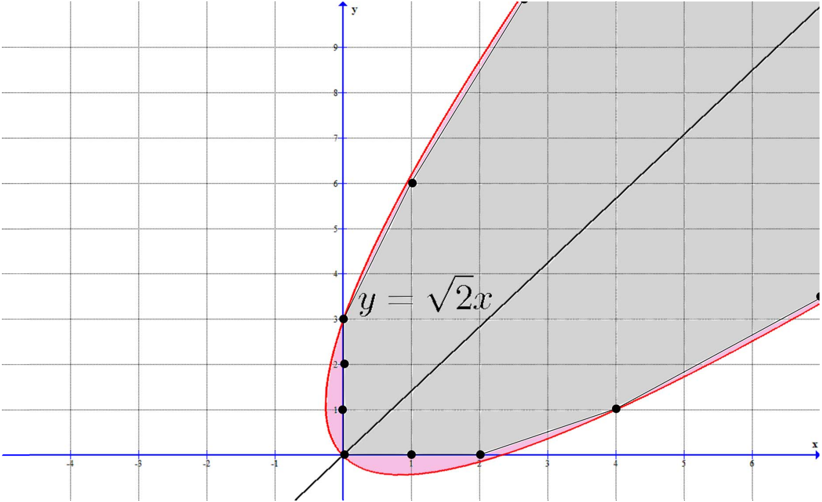


Figure: conv($K \cap \mathbb{Z}^n$) is closed!

Literature review: general polyhedron

Theorem (Moussafir, 2000)

Let $K \subseteq \mathbb{R}^n$ be a (not necessarily rational) polyhedron not containing a line such that:

- $\text{int}(\text{rec.cone}(K)) \neq \emptyset$.
- *For every proper face F of K: If $F \cap \mathbb{Z}^n \neq \emptyset$, then for all $u \in F \cap \mathbb{Z}^n$ and for all $r \in \text{rec.cone}(F)$, $\{u + \lambda r \mid \lambda \geq 0\} \subseteq \text{conv}(F \cap \mathbb{Z}^n)$.*

Then $\text{conv}(K \cap \mathbb{Z}^n)$ is closed.

Notation and definition

Definition ($u(K)$)

Given a convex set $K \subseteq \mathbb{R}^n$ and $u \in K \cap \mathbb{Z}^n$, we define:

$$u(K) = \{d \in \mathbb{R}^n \mid u + \lambda d \in \text{conv}(K \cap \mathbb{Z}^n) \; \forall \lambda \geq 0\}$$

Illustration of $u(K)$ (1)

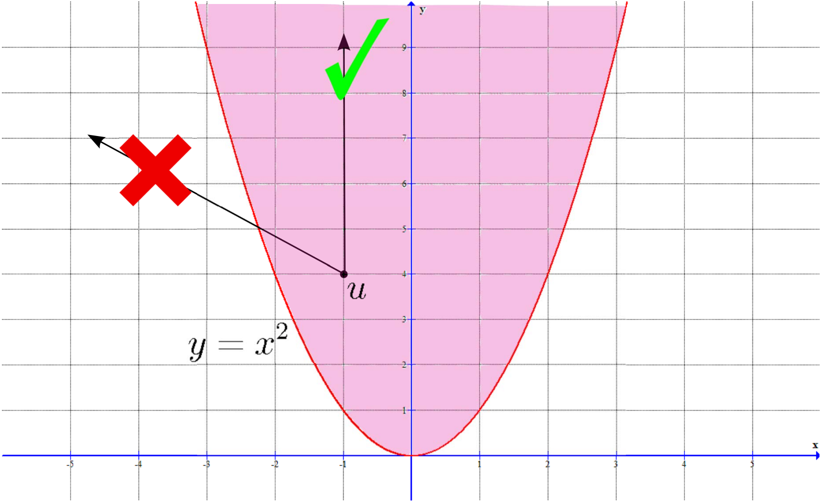
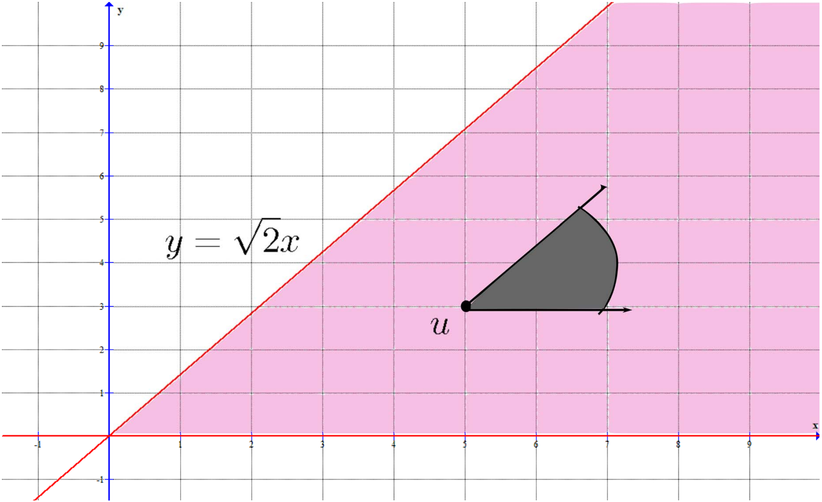


Illustration of $u(K)$ (2)



A sufficient condition for a direction to be in $u(K)$

Definition (Rational Linear Subspace)

A linear subspace $L \subseteq \mathbb{R}^n$ is said to be *rational* if there exists a basis of L contained in $\mathbb{Q}^n$.

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Definition (Rational Linear Subspace)

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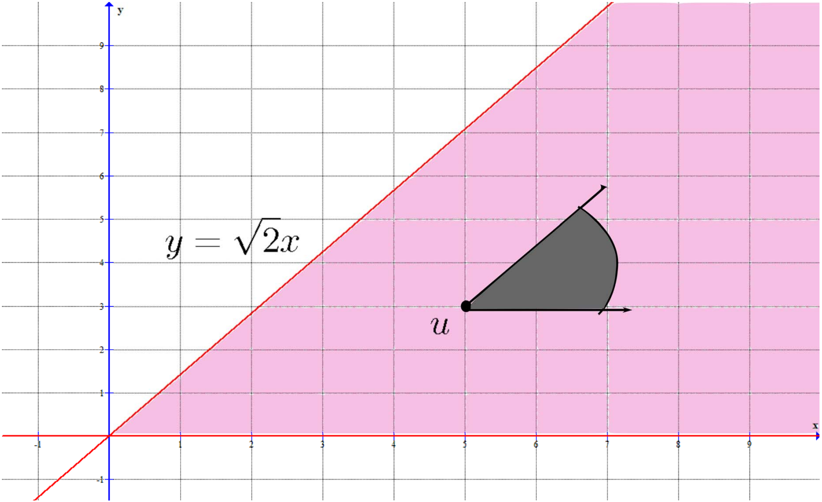
Lemma

Let $K \subseteq \mathbb{R}^n$ be a closed convex set such that $\text{aff}(K)$ *is a rational affine set. Let $u \in K \cap \mathbb{Z}^n$. If $\{u + \lambda d | \lambda > 0\} \subseteq \text{rel.int}(K)$, then $\{u + d\lambda | \lambda \geq 0\} \subseteq conv(K \cap \mathbb{Z}^n)$.*

This property is useful to compute the set $u(K)$ in the case $u \in \text{rel.int}(K)$.

Illustration of $u(K)$ (2)



1.1
Convex sets not containing lines

Convex sets not containing lines: necessary and sufficient conditions

Theorem

Let $K \subseteq \mathbb{R}^n$ be a closed convex set not containing a line. Then the following are equivalent:

1. $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed.*
2. $u(K)$ *is identical for every* $u \in K \cap \mathbb{Z}^n$*.*

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Interpretation:

- It is not difficult to verify that,

$$\operatorname{conv}(K \cap \mathbb{Z}^n) = \operatorname{conv}\left(\bigcup_{u \in K \cap \mathbb{Z}^n} (u + u(K))\right). \tag{1}$$

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- ## Lemma (Rockafeller (1970))

  If K_1, ..., K_m are non-empty closed convex sets in $\mathbb{R}^n$ all having the same recession cone, then conv$(K_1 \cup \ldots \cup K_m)$ *is closed.*

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  - The union in (1) can be over infinite terms. (There are straightforward counterexamples to lemma, when the union is not over finite terms.)

Proof outline

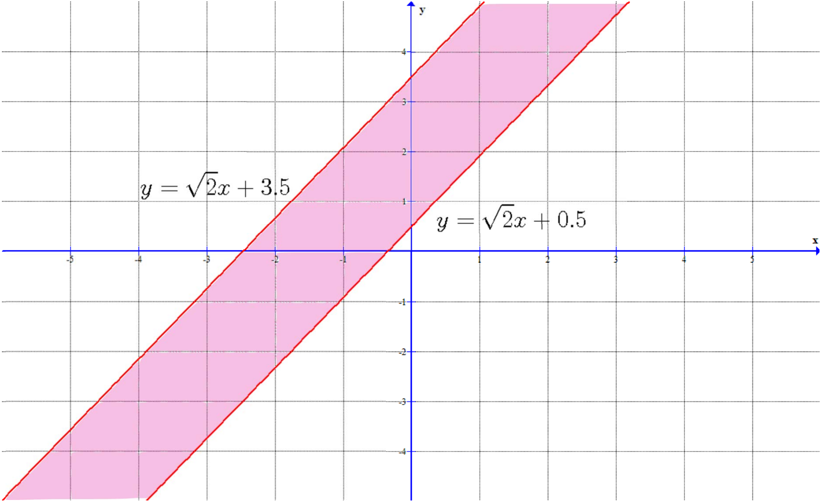
Case 1: rel.int(conv($K \cap \mathbb{Z}^n$)) $\cap \mathbb{Z}^n \neq \emptyset$.

0.1 $u(K)$ identical for all $u \in K \cap \mathbb{Z}^n \Rightarrow u(K) =$ rec.cone(conv($K \cap \mathbb{Z}^n$).
0.2 For some $u \in$ rel.int(conv($K \cap \mathbb{Z}^n$)), we also have have $u(K) =$ rec.cone($\overline{\text{conv}}$($K \cap \mathbb{Z}^n$).
0.3 Use result [*Husseinov*(1999)]: If $T \subseteq \mathbb{R}^n$ is closed, then every extreme point of $\overline{\text{conv}}(T)$ belongs to T.

Case 2: rel.int(conv($K \cap \mathbb{Z}^n$)) $\cap \mathbb{Z}^n = \emptyset$

0.1 Proof by induction on dimension. (Proof straightforward for dimension 0,1)
0.2 WLOG assume that conv($K \cap \mathbb{Z}^n$) is full-dimensional. There exists a full-dimensional maximal lattice-free convex set Q containing conv($K \cap \mathbb{Z}^n$). Let $F_1, \ldots, F_N$ be facets of Q.
0.3 Prove that $u(K \cap F_i)$ is identical. (using the fact that $u(K)$ are identical)
0.4 Observe conv($K \cap \mathbb{Z}^n$) $=$ conv$[\cup_{i \in \{1,\ldots,N\}}$conv$(K \cap F_i \cap \mathbb{Z}^n)]$

"Convex set not containing line" condition is crucial



1.2
Convex sets containing lines

Convex sets containing lines: necessary and sufficient conditions

Definition (Coterminal)

Given a set K and a half-line $d := \{u + \lambda r \mid \lambda \geq 0\}$ we say K is *coterminal with* d if $\sup\{\mu \mid \mu > 0, u + \mu r \in K\} = \infty$.

Theorem

Let $K \subseteq \mathbb{R}^n$ be a closed convex set such that the lineality space $L = \text{lin.space}(\overline{\text{conv}}(K \cap \mathbb{Z}^n))$ is not trivial. Then, $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed if and only if*

1. *L is a rational subspace.*
2. *The set $K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n)$ is co-terminal with every extreme facial ray of* $\overline{\text{conv}}(K \cap L^\perp \cap P_{L^\perp}(\mathbb{Z}^n))$.

1.3 Applications

Application 1: $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$

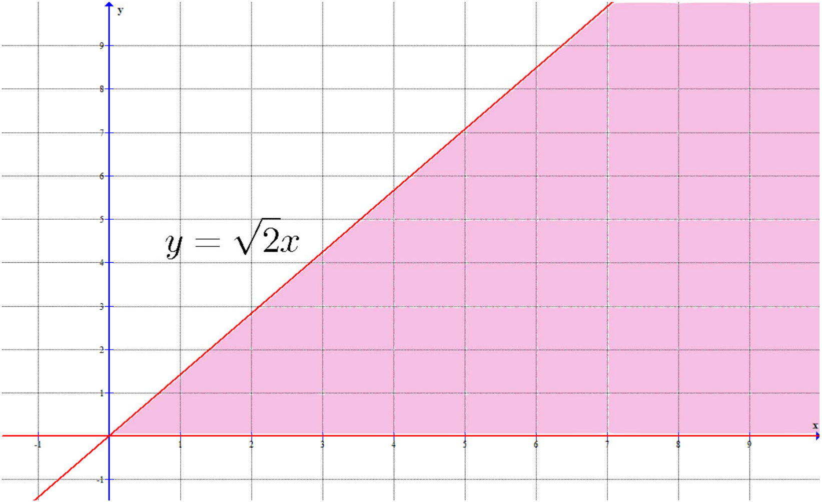
Proposition

Let $K \subseteq \mathbb{R}^n$ be a closed convex set such that $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$. Then the following are equivalent.

1. $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed.*
2. $u(K) = \text{rec.cone}(K), \forall u \in K \cap \mathbb{Z}^n$.
3. *For every proper exposed face F of K: If $F \cap \mathbb{Z}^n \neq \emptyset$, then for all $u \in F \cap \mathbb{Z}^n$ and for all $r \in \text{rec.cone}(F)$, $\{u + \lambda r \mid \lambda \geq 0\} \subseteq \text{conv}(F \cap \mathbb{Z}^n)$.*

This is a generalization of result from [Moussafir, (2000)]

Example A revisited (not closed)



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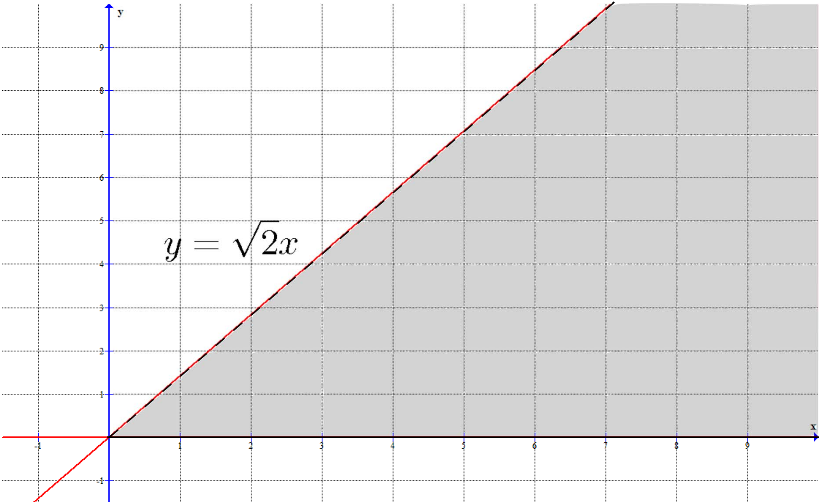
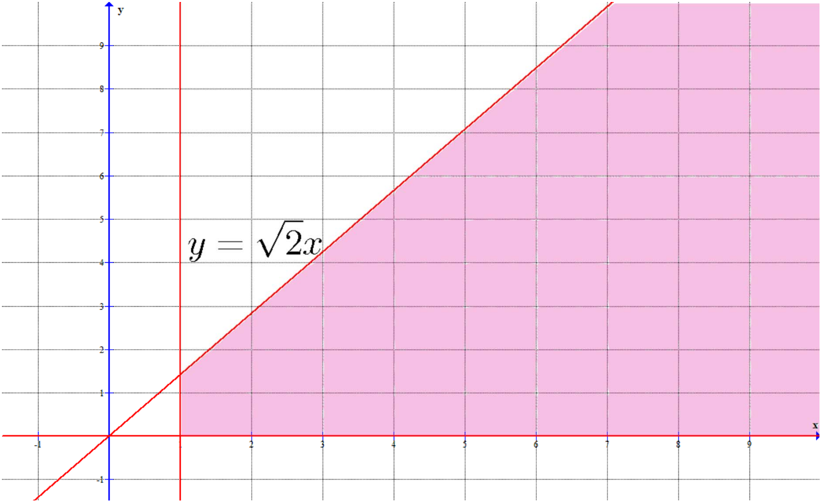


Example B revisited (closed)



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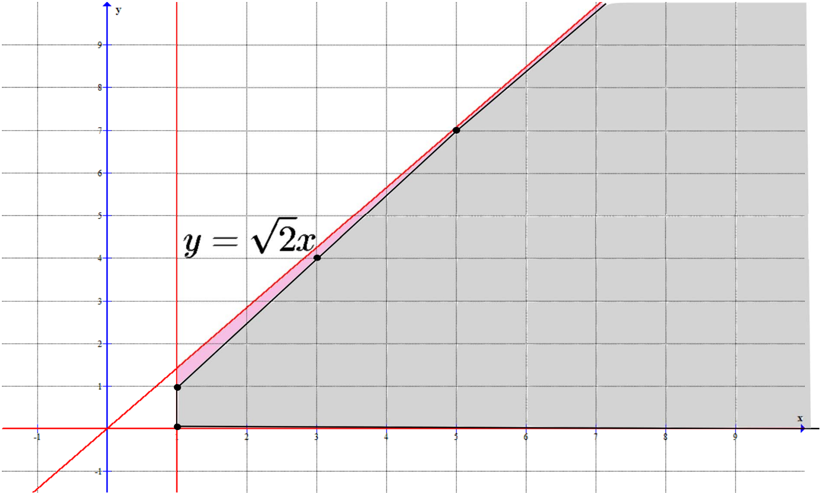


Application 2: Strictly convex sets

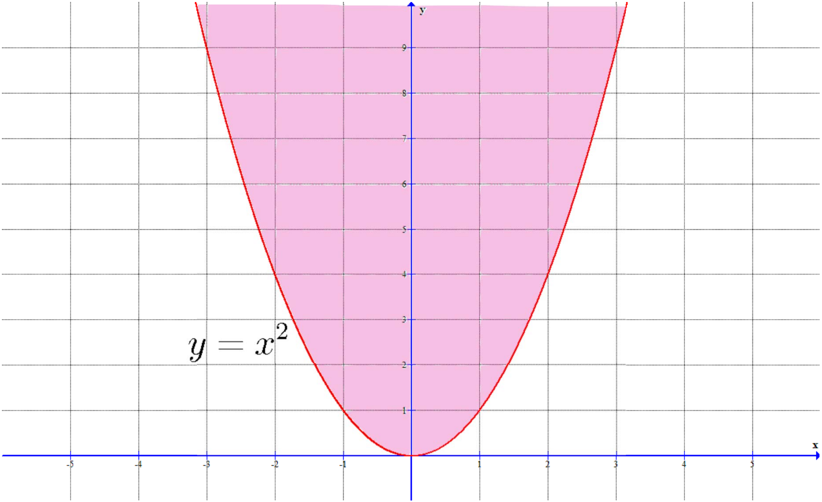
Definition (strictly convex set)

A set $K \subseteq \mathbb{R}^n$ is called a *strictly convex set*, if K is a convex set and for all $x, y \in K$, $\lambda x + (1 - \lambda)y \in \text{rel.int}(K)$ for $\lambda \in (0, 1)$.

Proposition

If $K \subseteq \mathbb{R}^n$ is a closed strictly convex set, then $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed.*

Example C revisited (closed)



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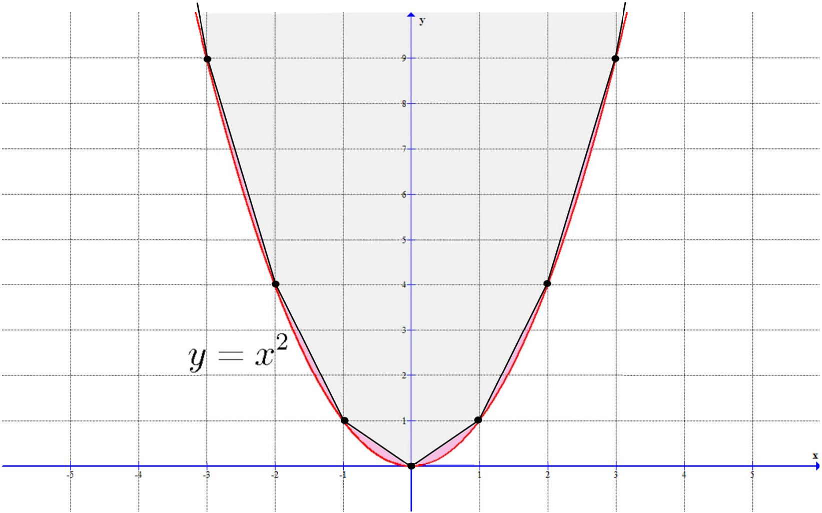
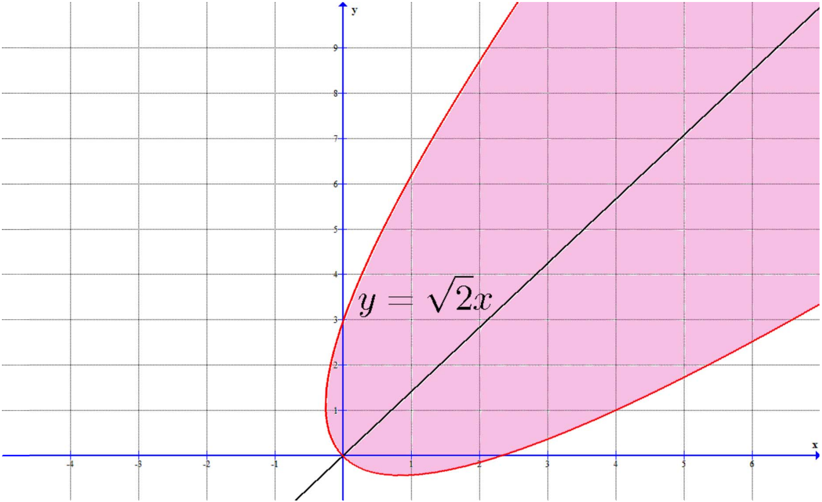


Example D revisited (closed)



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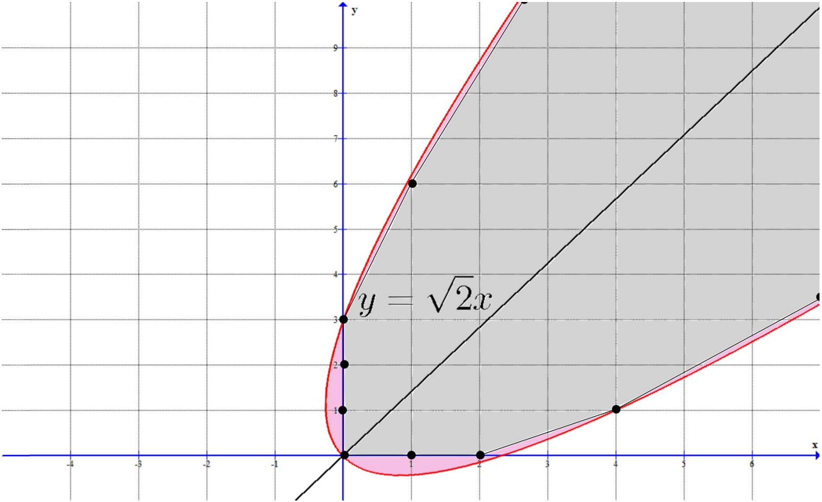

Application 3: Closed convex cones

Definition (rational scalable)

A vector $r \in \mathbb{R}^n$ is said to be *rational scalable* if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\lambda r \in \mathbb{Z}^n$

Application 3: Closed convex cones

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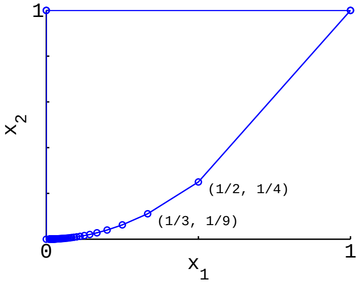
A vector $r \in \mathbb{R}^n$ is said to be *rational scalable* if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\lambda r \in \mathbb{Z}^n$

Proposition

Let K be a full-dimensional pointed closed convex cone in $\mathbb{R}^n$. Then,

1. $\overline{\text{conv}}(K \cap \mathbb{Z}^n) = K$.
2. *Moreover,* $\text{conv}(K \cap \mathbb{Z}^n)$ *is closed if and only if every extreme ray of K is rational scalable.*

Example: Non-polyhedral cone (closed)



- Consider the set $C = \{(0, 0, 1)\} \cup \{(0, 1, 1)\} \cup \{(\frac{1}{n}, \frac{1}{n^2}, 1)\}_{n \geq 1}$.
- $K = \text{conv}\left(\{\sum_{u \in C} \lambda_u u \mid \lambda_u \geq 0 \; \forall u \in C\}\right)$ is a closed convex cone.
- By previous result, $\text{conv}(K \cap \mathbb{Z}^3) = K$ is closed.

Application 4: sets containing lines

Proposition

Let $K \subseteq \mathbb{R}^n$ be a closed convex set and let rec.cone(K) *be a rational polyhedral cone. Then* conv($K \cap \mathbb{Z}^n$) *is closed.*

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Proposition

Let $K \subseteq \mathbb{R}^n$ *be a closed convex set such that* $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$*. If* $\text{lin.space}(K)$ *is not a rational subspace, then* $\text{conv}(K \cap \mathbb{Z}^n)$ *is not closed.*

Application 5: Closedness of conic quadratic IP

Definition

A polyhedral cone C is a lattice-cone wrt a lattice $\mathcal{L}$ if all the extreme rays of C belong to $\mathcal{L}$.

Proposition

Let

1. *L be the Ice-cream cone, i.e., $L := \{x \in \mathbb{R}^m \mid \|(x_1, \dots, x_{m-1})\| \leq x_m\}$.*
2. *$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be defined as $T(x) = Ax - b$.*
3. *$P = T(\mathbb{R}^n)$.*

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3. *$P = T(\mathbb{R}^n)$.*

Then $\mathrm{conv}(\{x \in \mathbb{R}^n \,|\, Ax \succeq_L b\} \cap \mathbb{Z}^n)$ *is closed iff one of the following holds:*

1. $0 \notin L \cap P$*, or*
2. $\dim(L \cap P) \leq 1$*, or*
3. $[\dim(L \cap P) = 2$ *and $L \cap P$ is a lattice-cone w.r.t. $T(\mathbb{Z}^n)$*$]$.

A calculus of operations on K preserving closed-ness of conv($K \cap \mathbb{Z}^n$)?

Suppose $K_1, K_2 \subseteq \mathbb{R}^n$ be closed convex sets such that conv($K_1 \cap \mathbb{Z}^n$) and conv($K_2 \cap \mathbb{Z}^n$) is closed. Then:

1. Intersection: Is conv($K_1 \cap K_2 \cap \mathbb{Z}^n$) always closed?
2. Products: Is conv($K_1 \times K_2 \cap (\mathbb{Z}^n \times \mathbb{Z}^n)$) always closed?
3. Rational Affine Image: Let $A \in \mathbb{Q}^{m \times n}$ and let $b \in \mathbb{Q}^m$. Is conv $(\{Ax + b \,|\, x \in K_1\} \cap \mathbb{Z}^m)$ always closed?
4. Rational Affine Pre-Image: Let $A \in \mathbb{Q}^{n \times m}$ and let $b \in \mathbb{Q}^n$. Is conv $(\{x \,|\, Ax + b \in K_1\} \cap \mathbb{Z}^m)$ always closed?
5. Sums: Is conv($K_1 + K_2 \cap \mathbb{Z}^n$) always closed?

A calculus of operations on K preserving closed-ness of conv$(K \cap \mathbb{Z}^n)$?

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5. Sums: Is conv$(K_1 + K_2 \cap \mathbb{Z}^n)$ always closed? No

2

Polyhedrality of $\mathrm{conv}(K \cap \mathbb{Z}^n)$

Developing intuition about recession cone

- Suppose $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ and that $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron.

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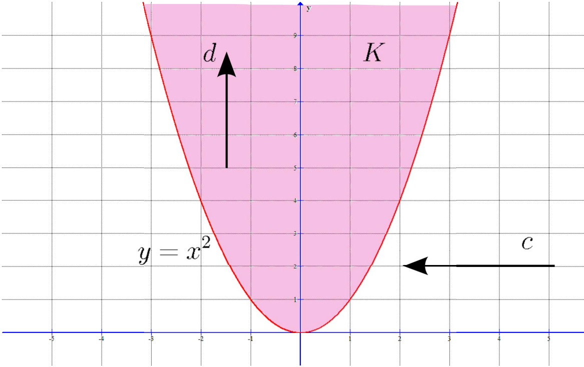
- This implies $\mathrm{rec.cone}(\mathrm{conv}(K \cap \mathbb{Z}^n)) = \mathrm{rec.cone}(K)$.

Developing intuition about recession cone

- Suppose $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ and that $\text{conv}(K \cap \mathbb{Z}^n)$ is a polyhedron.
- This implies $\text{rec.cone}(\text{conv}(K \cap \mathbb{Z}^n)) = \text{rec.cone}(K)$.
- Therefore, in this case $\text{rec.cone}(K)$ must be a rational polyhedron.

Developing intuition about 'kind of unboundedness'

- Consider $z^* = \sup\{c^t x \; : \; x \in K\}$
- We have $z^* = \infty$.
- $\text{rec.cone}(K) = \{\lambda d, \; \lambda \geq 0\}$.
- $c \perp d$.



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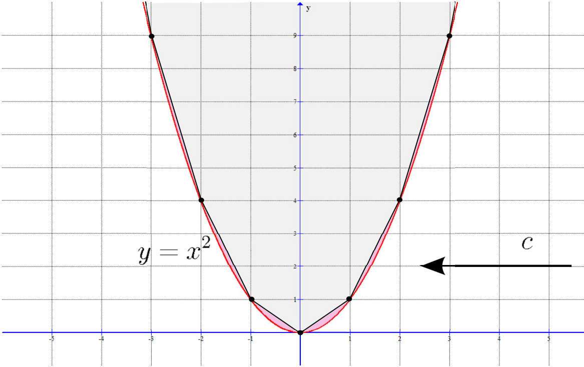

Thin convex sets

Definition (Thin Convex set)

Let $K \subseteq \mathbb{R}^n$ be a closed convex set. We say K is *thin* if the following holds: $\sup\{c^t x \,:\, x \in K\} = \infty$ if and only if there exist $d \in \text{rec.cone}(K)$ such that $c^t d > 0$.

Example

Every Polyhedron is thin.

Necessary and sufficient conditions

Theorem

Let $K \subseteq \mathbb{R}^n$ be a convex set. Then,

- *If K is thin and* $\text{rec.cone}(K)$ *is a rational polyhedral cone, then* $\text{conv}(K \cap \mathbb{Z}^n)$ *is a polyhedron.*

Necessary and sufficient conditions

Theorem

Let $K \subseteq \mathbb{R}^n$ *be a convex set. Then,*

- *If* K *is thin and* $\text{rec.cone}(K)$ *is a rational polyhedral cone, then* $\text{conv}(K \cap \mathbb{Z}^n)$ *is a polyhedron.*
- *Moreover, if* $\text{int}(K) \cap \mathbb{Z}^n \neq \emptyset$ *and* $\text{conv}(K \cap \mathbb{Z}^n)$ *is a polyhedron, then* K *is thin and* $\text{rec.cone}(K)$ *is a rational polyhedral cone.*

Proof outline for sufficient case

Let $K \subseteq \mathbb{R}^n$ be a thin convex set with a rational polyhedral recession cone.
Notation:

- Then $\sigma_K : \mathbb{R}^n \to \mathbb{R}$ is defined as
$$\sigma_K(d) = \sup\{d^t x \mid x \in K\}$$
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   1.3 $P_v \cap \mathbb{Z}^n \supseteq K \cap \mathbb{Z}^n$.
2. By compactness argument, we select a finite set of polyhedra $P_1, \ldots, P_l$ such that
$$P' := \cap_{i=1}^{l} P_v \subseteq K$$
Then P' satisfies (A.) and (B.)

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4. Finally, prove that if K is contained within two thin sets with same recession cone, it must be thin.

Alternative proof

1. We recently discovered an alternative proof: Starting from a result in [Hemmecke, Weismantel (2007)]
2. Depends on the following alternative characterization of thin convex sets: A convex set K is thin iff $K \subseteq B + \text{rec.cone}(K)$ where B is a bounded set.
3. This proof essentially uses triangulation of the recession cone and Gordan-Dickson's Lemma.