Some Relationships between Disjunctive Cuts and Cuts based on S-free Convex Sets

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July 2010.

A Fundamental Relationship in Mixed Integer Programming

Theorem ([Nemhauser, Wolsey (1990)], [Cournuéjols, Li (2002)])

$$\begin{aligned} &\text{Mixed Integer Rounding (MIR) Closure} \\ &= \text{Gomory Mixed Integer Cuts (GMIC) Closure} \\ &= \text{Split Closure} \end{aligned} \tag{1}$$

Each of the cutting planes above is based on "information from $\mathbb{Z}^1$".

Observation

The convex hull of a MIP with one integer variable is given by the split closure.

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- The goal here: To understand and establish generalization of (1) with respect to cuts based on “information from $\mathbb{Z}^2$".

Definitions: S-free and Lattice-free convex set

Definition (Maximal S-free Convex Sets)

Let $S = P \cap \mathbb{Z}^m$ where P is a rational polyhedron.

- A convex set K is called S-free (resp. lattice-free) convex set, if $\text{int}(K) \cap S = \emptyset$ (resp. $\text{int}(K) \cap \mathbb{Z}^m = \emptyset$).
- An S-free (resp. lattice-free) convex set K is called a maximal S-free (resp. lattice-free) convex set if K is not contained properly in another S-free (resp. lattice-free) convex set.

Theorem (Lovász (1989), D., Wolsey (2009), Basu et al. (2009), Fukasawa, Günlük (2009), Moran, D. (2010))

Maximal S-free convex sets are polyhedron.

1
MIR Procedure

Looking at MIR cuts

Traditional MIR Set

$$x + y^{+} \geq b, x \in \mathbb{Z}, y^{+} \in \mathbb{R}_{+}$$

Looking at MIR cuts

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$$x + y^+ \geq b, x \in \mathbb{Z}, y^+ \in \mathbb{R}_+$$

$$x = b - y^+ + y^-$$
$$x \in \mathbb{Z}, y^+, y^- \in \mathbb{R}_+$$

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The MIR cut (written differently) is:

$$\frac{1}{f(b)} y^+ + \frac{1}{1 - f(b)} y^- \geq 1$$

Looking at MIR cuts

Set introduced by [Andersen et al. (2007)]

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Two rows 'Canonical Set':

$$x_1 = b_1 + \sum_{i=1}^{n} a_{1i} y_i$$
$$x_2 = b_2 + \sum_{i=1}^{n} a_{2i} y_i$$
$$x \in \mathbb{Z}^2, y_i \in \mathbb{R}_+ \forall i \in \{1, ...n\}$$

The cuts are based on lattice-free convex sets....
[Cornuéjols, Margot (2008)]
[Borozan, Cornuéjols (2008)]

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Both sets obtained from simplex tableau by relaxing: (1) Bounds on basic integer variables (2) Integrality of non-basic variables.

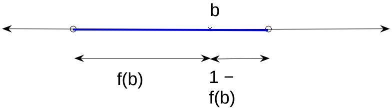
Intersection Cuts

Intersection Cut [Balas (1971)]

$$x = b - y^{+} + y^{-}$$
$$x \in \mathbb{Z}, y^{+}, y^{-} \in \mathbb{R}_{+}$$

The MIR cut (written differently) is:

$$\frac{1}{f(b)} y^{+} + \frac{1}{1 - f(b)} y^{-} \geq 1$$



Intersection Cuts II

Intersection Cut [Balas (1971)]

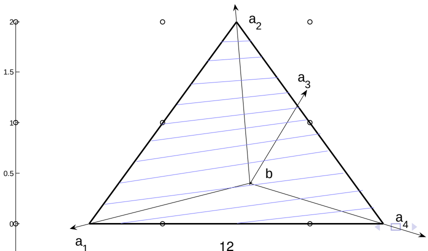
$$x_1 = b_1 + \sum_{i=1}^{n} a_{1i} y_i$$

$$x_2 = b_2 + \sum_{i=1}^{n} a_{2i} y_i$$

$$x \in \mathbb{Z}^2, y_i \in \mathbb{R}_+ \forall i \in \{1, ...n\}$$

Let K be a lattice-free convex set containing b in its interior. The cut $\sum_{i=1}^{n} \alpha_i y_i \geq 1$ obtained using K is of the form:

$$\alpha_i = \begin{cases} 0 & \text{if } a_i \in \text{recc.cone}(K) \\ \lambda & \exists \lambda > 0, \text{ s.t. } b + \frac{1}{\lambda} a_i \in \text{bnd}(K) \end{cases}$$



2D lattice-free closure

Definition (2D Lattice-free Cut Closure)

- Rewrite the MIP set as $P := \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_{+} \mid Ax + Gy = b\}$ (by possibly adding slacks).
- Let $\lambda_1, \lambda_2 \in \mathbb{R}^m$ such that $\lambda_1 A, \lambda_2 A \in \mathbb{Z}^{n_1}$
- $\underbrace{\lambda_1 Ax}_{z_1} + \underbrace{\lambda_1 G y}_{g_1} = \underbrace{\lambda_1 b}_{b_1}$

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- $\underbrace{\lambda_1 Ax}_{z_1} + \underbrace{\lambda_1 G}_{g_1} y = \underbrace{\lambda_1 b}_{b_1}$
- $$P_2(\lambda_1, \lambda_2) = \{(z, y) \in \mathbb{Z}^2 \times \mathbb{R}^{n_2}_+ \mid z_1 + g_1 y = b_1,\ z_2 + g_2 y = b_2,\ y \geq 0\}$$
- A *general 2D lattice-free cut* for the mixed-integer set P is an inequality $\alpha y \geq 1$ which can be obtained as a intersection cut from a 2D maximal lattice-free convex set applied to above set.
- The set of points of P^{LP} that satisfy all general 2D lattice-free cut is called the 2D lattice-free closure.

Taking Stock...

One row/one integer variable based	Two row/two integer variables based
MIR Set (MIR closure)	Canonical Set (2D Lattice-free cut closure)
GMIC Closure	?
Split Cut Closure	?

2

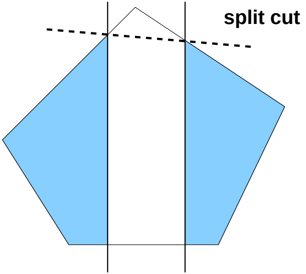
Split Procedure

Split cuts

[Balas(1979)], [Cook et al.(1990)]

- Let $P = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_{+} \mid Ax + Gy = b\}$ be a mixed integer linear set and let P^{LP} be its linear relaxation.
- Let $\pi \in \mathbb{Z}^{n_1}$ and $\gamma \in \mathbb{Z}$.
- A split cut is an inequality valid for

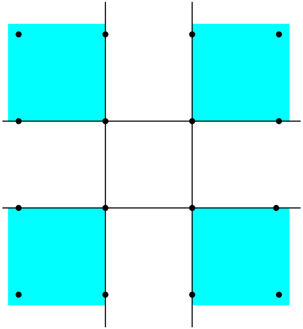
$$P^{LP} \cap \{(x, y) \mid \pi x \leq \gamma\} \cup P^{LP} \cap \{(x, y) \mid \pi x \geq \gamma + 1\}$$



Towards a generalization of split cuts

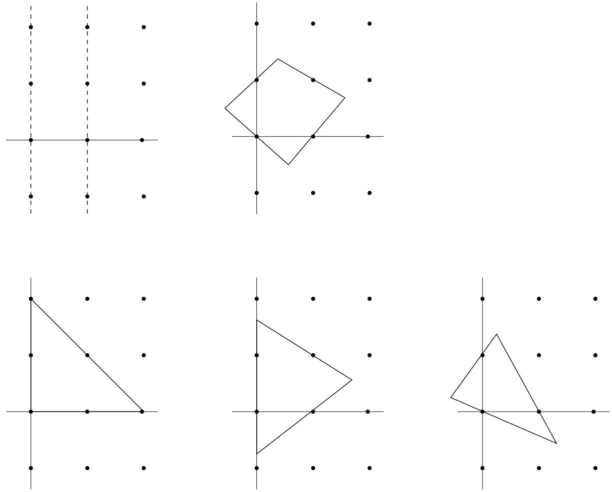
- Let $\pi_1, \pi_2 \in \mathbb{Z}^{n_1}$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$.
- A cross cut is an inequality valid for

$$\begin{aligned} P^{LP} \cap \{(x,y) \,|\, \pi_1 x \leq \gamma_1, \pi_2 x \leq \gamma_2\} \\ \cup P^{LP} \cap \{(x,y) \,|\, \pi_1 x \leq \gamma_1, \pi_2 x \geq \gamma_2 + 1\} \\ \cup P^{LP} \cap \{(x,y) \,|\, \pi_1 x \geq \gamma_1 + 1, \pi_2 x \leq \gamma_2\} \\ \cup P^{LP} \cap \{(x,y) \,|\, \pi_1 x \geq \gamma_1 + 1, \pi_2 x \geq \gamma_2 + 1\} \end{aligned}$$



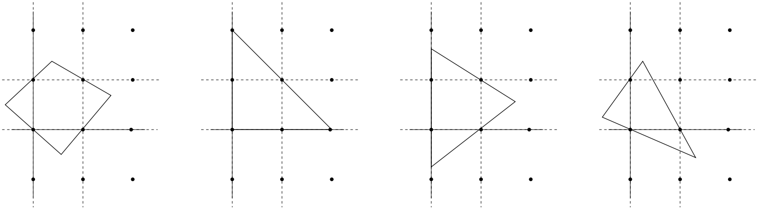
[Balas (ISMP 2009)] [Balas and Qualizza (2009)] [Li and Richard (2008)]

Maximal Lattice-free convex Sets



[Lovász (1989)]
[D., Wolsey (2007)]

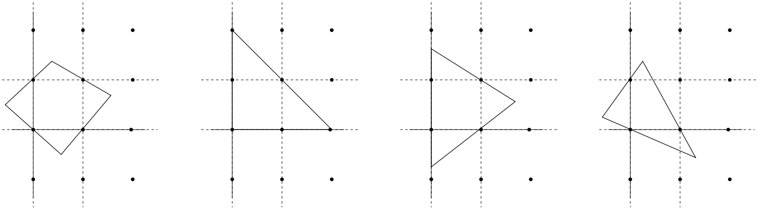
2D Lattice-free convex cuts vs Unimodular Cross Cut



Definition

A unimodular cross cut is one where $\pi_1, \pi_2 \in \mathbb{Z}^2$ and form a unimodular matrix.

2D Lattice-free convex cuts vs Unimodular Cross Cut



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Related result first shown by [Balas (ISMP 2009)], [Balas and Qualizza (2009)]

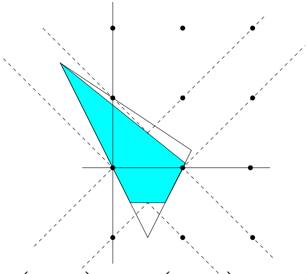
Proposition

For the canonical set, all unimodular cross cuts are either split cuts, quadrilateral cuts or triangle cuts of type 1 or 2.

Proposition

For the canonical set, split cuts, quadrilateral cuts and triangle cuts of type 1 or 2 are dominated by unimodular cross cuts.

Unimodular vs Nonunimodular Crosses



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} y_1 + \begin{pmatrix} -1 \\ 1 \end{pmatrix} y_2 + \begin{pmatrix} 11 \\ -6 \end{pmatrix} y_3 + \begin{pmatrix} 1 \\ -2 \end{pmatrix} y_4 .$$

Now consider the non-unimodular cross set

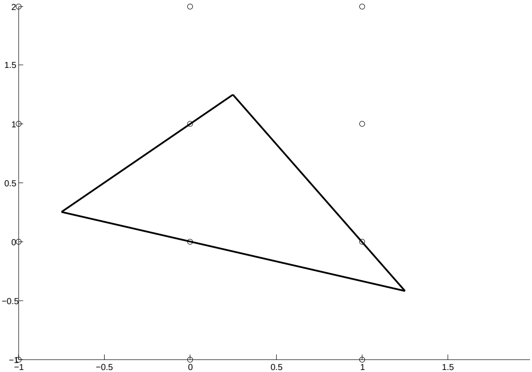
$$\{(x_1, x_2) \in \mathbb{R}^2 \ : \ 0 \le x_1 + x_2 \le 1\} \cup \{(x_1, x_2) \in \mathbb{R}^2 \ : \ 0 \le x_1 - x_2 \le 1\}. \quad (2)$$

The inequality $y_1 + y_2 + 14y_3 + 2y_4 \ge 1$ is obtained by the above disjunction and cannot be obtained by any single unimodular cross cut.

Observation

Some cross cuts are not unimodular cross cuts.

Type 3 Triangles vs Cross Disjunctions



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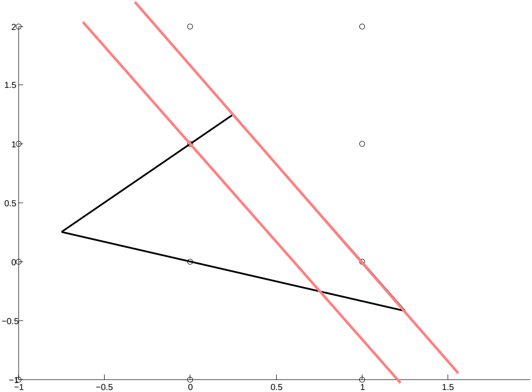

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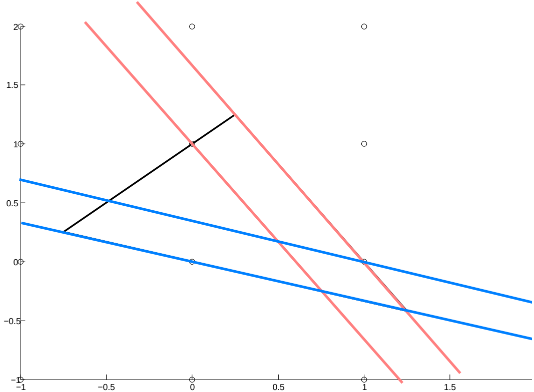

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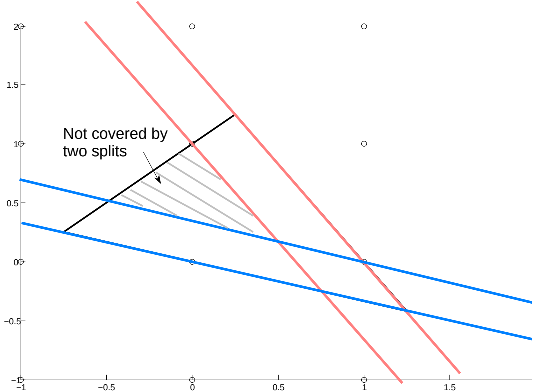
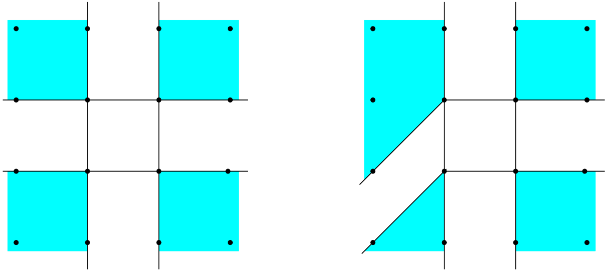


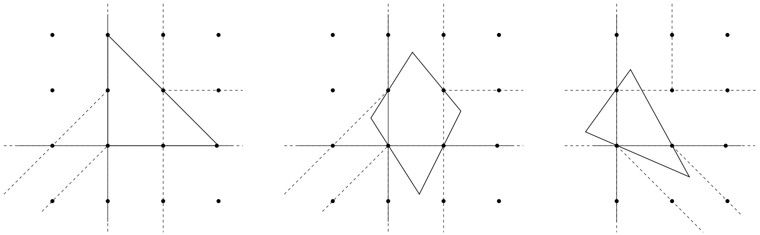
Crooked Cross Cut



- Let $P = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_{+} \mid Ax + Gy = b\}$ be a mixed integer linear set and let P^{LP} be its linear relaxation.
- Let $\pi_1, \pi_2 \in \mathbb{Z}^{n_1}$ and $\gamma_1, \gamma_2 \in \mathbb{Z}$.
- A crooked cross cut is an inequality valid for

$$\begin{aligned} &P^{LP} \cap \{(x, y) \mid \pi_1 x \leq \gamma_1, (\pi_2 - \pi_1)x \leq \gamma_2 - \gamma_1\} \\ \cup &P^{LP} \cap \{(x, y) \mid \pi_1 x \leq \gamma_1, (\pi_2 - \pi_1)x \geq \gamma_2 - \gamma_1 + 1\} \\ \cup &P^{LP} \cap \{(x, y) \mid \pi_1 x \geq \gamma_1 + 1, \pi_2 x \leq \gamma_2\} \\ \cup &P^{LP} \cap \{(x, y) \mid \pi_1 x \geq \gamma_1 + 1, \pi_2 x \geq \gamma_2 + 1\} \end{aligned}$$

Crooked Cross Cut



Proposition

All maximal lattice-free convex sets in $\mathbb{R}^2$ are contained in crooked cross sets.

Consequences for canonical set

Theorem

All valid inequalities of the canonical set are obtained by disjunctions based on crooked cross set.

A generalization of Split Closure

Definition (Crooked Cross Closure)

For the mixed-integer set $P := \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_{+} \mid Ax + Gy = b\}$ the crooked cross closure is defined as

$$\bigcap_{\pi_1 \in \mathbb{Z}^{n_1}, \pi_2 \in \mathbb{Z}^{n_1}, \gamma_1 \in \mathbb{Z}, \gamma_2 \in \mathbb{Z}} \text{conv} \left(\begin{array}{c} P^{LP} \cap \{(x, y) \mid \pi_1 x \leq \gamma_1, (\pi_2 - \pi_1)x \leq \gamma_2 - \gamma_1\} \\ \cup P^{LP} \cap \{(x, y) \mid \pi_1 x \leq \gamma_1, (\pi_2 - \pi_1)x \geq \gamma_2 - \gamma_1 + 1\} \\ \cup P^{LP} \cap \{(x, y) \mid \pi_1 x \geq \gamma_1 + 1, \pi_2 x \leq \gamma_2\} \\ \cup P^{LP} \cap \{(x, y) \mid \pi_1 x \geq \gamma_1 + 1, \pi_2 x \geq \gamma_2 + 1\} \end{array} \right)$$

Theorem

Let $P := \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_{+} \mid Ax + Gy = b\}$. If $\text{rank}(A) = 2$, then the crooked cross closure is the convex hull of P.

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Corollary

The convex hull of a MIP with two integer variable is given by the crooked cross closure.

Corollary

Let $P := \{x \in \mathbb{Z}^n \mid Ax \leq b\}$ where A and b are integral and on removing two columns from A the remaining matrix is totally unimodular, then the convex hull of P is obtained by the crooked cross closure.

Some Other Sets in the Literature

D., Wolsey (2009), Basu et al. (2009), Fukasawa, Günlük (2009)

- Two row 'Canonical Set' + Constraints:

$$x_1 = b_1 + \sum_{i=1}^{n} a_{1i} y_i$$

$$x_2 = b_2 + \sum_{i=1}^{n} a_{2i} y_i$$

$$x \in P \cap \mathbb{Z}^2, y_i \in \mathbb{R}_+ \forall i \in \{1, ...n\},$$

  where P is a rational polyhedron.

- Let $S = P \cap \mathbb{Z}^2$. Let K be a maximal S-free convex set containing b in its interior, then we can generate facet-defining inequalities as follows: Let $K - f$ be written as a set $\{x \,|\, (g^j)^T x \leq 1, j \in \{1, ..., l\}\}$. Let $\pi^K(u) = \max_{1 \leq j \leq l}\{(g^j)^T u\}$. Then the inequality

$$\sum_{i=1}^{n} \pi^K(r^i) y_i \geq 1$$

Corollary

The inequalities obtained by S-free convex sets are dominated by crooked cross cuts.

Taking Stock...

One row/one integer variable based	Two row/two integer variables based
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GMIC Closure	?
Split Cut Closure	Crooked Cross Closure

Convex hull of canonical set = 2D lattice-free closure = Crooked Cross Closure.

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Convex hull of canonical set = 2D lattice-free closure = Crooked Cross Closure.

Question: Is 2D lattice-free closure = Crooked Cross Closure for general MILPs?

A partial answer

1. Crooked cross closure $\subseteq$ 2D lattice-free closure.

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2. Let $(\pi_1, \gamma_1),\ (\pi_2, \gamma_2) \in \mathbb{Z}^{n_1+1}$ and consider the *crooked cross* disjunction
$$\begin{aligned} D_1 &= \{(x,y) \in \mathbb{R}^{n_1+n_2} \ : \ -\pi_1 x \geq -\gamma_1,\ -(\pi_2 - \pi_1)x \geq -(\gamma_2 - \gamma_1)\}, \\ D_2 &= \{(x,y) \in \mathbb{R}^{n_1+n_2} \ : \ -\pi_1 x \geq -\gamma_1,\ (\pi_2 - \pi_1)x \geq \gamma_2 - \gamma_1 + 1\} \\ D_3 &= \{(x,y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi_1 x \geq \gamma_1 + 1,\ -\pi_2 x \geq -\gamma_2\}, \text{ and,} \\ D_4 &= \{(x,y) \in \mathbb{R}^{n_1+n_2} \ : \ \pi_1 x \geq \gamma_1 + 1,\ \pi_2 x \geq \gamma_2 + 1\}. \end{aligned}$$
3. We say that an inequality $cx + dy \geq f$ is a translation of $c'x + d'y \geq f'$ w.r.t. $P := \{(x,y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_+ \mid Ax + Gy = b\}$, if there exists a vector $\mu \in R^m$ and a positive scalar θ such that $[c,d,f] = \mu[A,G,b] + \theta[c',d',f']$.

A partial answer

1. Crooked cross closure $\subseteq$ 2D lattice-free closure.
2. Let (π_1, γ_1), $(\pi_2, \gamma_2) \in \mathbb{Z}^{n_1+1}$ and consider the *crooked cross* disjunction
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Theorem (Three Rows)

Let $cx + dy \geq f$ be a non-trivial crooked cross cut for P derived from the disjunction $\vee_{i=1}^4 D_i$. Then a translation of $cx + dy \geq f$ can be obtained as a crossed cross cut using the same disjunction from a 3-row relaxation of P, namely

$$P_3(\lambda_1, \lambda_2, \lambda_3) = \{(x,y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} : \pi_1 x + g_1 y = b_1, \pi_2 x + g_2 y = b_2, g_3 y = b_3, \ y \geq 0\},$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^m$, $\pi_3 = 0$ and $\pi_i = \lambda_i A$, for $i = 1, 2, 3$ and $g_i = \lambda_i G$, $b_i = \lambda_i b$ for $i = 1, 2, 3$.

A Corollary

Corollary

Let $P = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_{+} \mid Ax + Gy = b\}$*. If A has* *<u>full row rank</u>*, *then 2D lattice-free closure* $=$ *Crooked Cross Closure.*

A Corollary

Corollary

Let $P = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_{+} \mid Ax + Gy = b\}$. If A has full row rank, then 2D lattice-free closure = Crooked Cross Closure.

If P is a Corner relaxation of an MILP, then 2D lattice-free closure = Crooked Cross Closure for P.

2.5
Split Closure Again

A slightly different view of split closure

1. Let $P = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_{+} \mid Ax + Gy = b\}$ and P^{LP} the LP relaxation.

A slightly different view of split closure

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Two row/two integer variable case

Parametric Cross Closure ($t \in \mathbb{Z}_+$)

$$P_t := \bigcap_{\pi_1 \in \mathbb{Z}^{n_1}, \pi_2 \in \mathbb{Z}^{n_1}, \gamma_1 \in \mathbb{Z}, \gamma_2 \in \mathbb{Z}} \text{conv} \left(\begin{array}{c} P^{LP} \cap \{(x,y) \,|\, \pi_1 x \leq \gamma_1, (\pi_2 - t\pi_1)x \leq \gamma_2 - t\gamma_1\} \\ \cup P^{LP} \cap \{(x,y) \,|\, \pi_1 x \leq \gamma_1, (\pi_2 - t\pi_1)x \geq \gamma_2 - t\gamma_1 + 1\} \\ \cup P^{LP} \cap \{(x,y) \,|\, \pi_1 x \geq \gamma_1 + 1, \pi_2 x \leq \gamma_2\} \\ \cup P^{LP} \cap \{(x,y) \,|\, \pi_1 x \geq \gamma_1 + 1, \pi_2 x \geq \gamma_2 + 1\} \end{array} \right)$$

P_1 is crooked cross closure.

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$$P_{\sqcap} := \cap_{\pi_1, \pi_2 \in \mathbb{Z}^{n_1}} \text{conv}(P^{LP} \cap \{(x,y) \mid \pi_1 x, \pi_2 x \in \mathbb{Z}\})$$

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$$P_\Pi := \cap_{\pi_1, \pi_2 \in \mathbb{Z}^{n_1}} \text{conv}(P^{LP} \cap \{(x,y) \,|\, \pi_1 x, \pi_2 x \in \mathbb{Z}\})$$

Theorem

$\forall t \in \mathbb{Z}_+, P_t \supseteq P_1 = P_\Pi$.

3

Gomory Mixed Integer Cut

Monoidal Strengthening/Trivial Fill-in: From MIR to GMIC

[Balas and Jeroslow (1984)]

$$x_B = b + \sum_{i=1}^{n_1} a_i z_i + \sum_{i=1}^{n_2} c_i y_i$$
$$x_B \in \mathbb{Z}, z_i \in \mathbb{Z}_+, y_i \in \mathbb{R}_+$$

Let $t_i = \text{argmin}\{t \in \mathbb{Z} \,|\, \max\{\frac{-t-a_i}{f(b)}, \frac{a_i+t}{1-f(b)}\}\}$.
Then rewrite as

$$x_B - \sum_{i=1}^{n_1} t_i z_i = b + \sum_{i=1}^{n_1} (a_i - t_i) z_i + \sum_{i=1}^{n_2} c_i y_i$$
$$x_B \in \mathbb{Z}, z_i \in \mathbb{Z}_+, y_i \in \mathbb{R}_+$$

- Aggregate all the integer terms on the left-hand-side.

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- Aggregate all the integer terms on the left-hand-side.
- Aggregate all variables with positive coefficients on the right-hand-side.
- Aggregate all variables with negative coefficients on the right-hand-side.

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$$x_B - \sum_{i=1}^{n_1} t_i z_i = b + \sum_{i=1}^{n_1}(a_i - t_i) z_i + \sum_{i=1}^{n_2} c_i y_i$$
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- Aggregate all the integer terms on the left-hand-side.
- Aggregate all variables with positive coefficients on the right-hand-side.
- Aggregate all variables with negative coefficients on the right-hand-side.
- Apply MIR.

Monoidal Strengthening/Trivial Fill-in

[Balas and Jeroslow (1984)]

$$x_1 = b_1 + \sum_{i=1}^{n_1} a_{1i} z_i + \sum_{i=1}^{n_2} c_{1i} y_i$$

$$x_2 = b_2 + \sum_{i=1}^{n_2} a_{1i} z_i + \sum_{i=1}^{n_2} c_{2i} y_i$$

$$x \in \mathbb{Z}^2, z_i \in \mathbb{Z}_+ \ \forall i \in \{1, ...n_1\}, y_i \in \mathbb{R}_+ \ \forall i \in \{1, ...n_2\}$$

Given a lattice-free convex set K containing b, let

$$\pi(u) = \begin{cases} 0 & \text{if } u \in \text{recc.cone}(K) \\ \lambda & \exists \lambda > 0, \text{s.t.} b + \frac{1}{\lambda} u \in \text{bnd}(K) \end{cases}$$

Let $t_i = \text{argmin}\{t \in \mathbb{Z}^2 \,|\, \pi(t + a_i)\}$. Then rewrite:

$$x_1 - \sum_{i=1}^{n_1} t_{1i} z_i = b_1 + \sum_{i=1}^{n_1} (a_{1i} - t_{1i}) z_i + \sum_{i=1}^{n_2} c_{1i} y_i$$

$$x_2 - \sum_{i=1}^{n_1} t_{2i} z_i = b_2 + \sum_{i=1}^{n_1} (a_{2i} - t_{2i}) z_i + \sum_{i=1}^{n_2} c_{2i} y_i$$

Apply 2D lattice-free cut.

2D lattice-free cuts + Monoidal Strengthening closure

1. Rewrite the MILP set as $P := \{(z, y) \in \mathbb{Z}^{n_1}_{+} \times \mathbb{R}^{n_2}_{+} \mid Az + Gy = b\}$.
2. Construct two row relaxation as

$$Ez + Fy + d = 0, z \in \mathbb{Z}^{n_1}_{+}, y \in \mathbb{R}^{n_1},$$

   where $E = [\lambda_1; \lambda_2]A$, $F = [\lambda_1; \lambda_2]G$, $d = -[\lambda_1; \lambda_2]b$ and $\lambda_1, \lambda_2 \in \mathbb{R}^{1\times m}$.
3. Relax the set to

$$x = Ez + Fy + d, x \in \mathbb{Z}^2, z \in \mathbb{Z}^{n_1}_{+}, y \in \mathbb{R}^{n_1}$$

4. Apply all possible 2D lattice-free closure + Monoidal Strengthening cut.

2D lattice-free closure = 2D lattice-free cut + Monoidal Strengthening closure

1. Let $P := \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_{+} \mid Ax + Gy = b\}$. Then (2D lattice-free cut + Monoidal Strengthening closure) $\subseteq$ (2D lattice-free closure), by rewriting P as $P := \{(x^+, x^-, y) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_{+} \mid Ax^+ - Ax^- + Gy = b\}$
2. Let $P := \{(z, y) \in \mathbb{Z}^{n_1}_{+} \times \mathbb{R}^{n_2}_{+} \mid Az + Gy = b\}$. Now

$$x_1 = b_1 + \sum_{i=1}^{n_1} a_{1i} z_i + \sum_{i=1}^{n_2} c_{1i} y_i \quad (3)$$

$$x_2 = b_2 + \sum_{i=1}^{n_1} a_{1i} z_i + \sum_{i=1}^{n_2} c_{2i} y_i \quad (4)$$

$$x \in \mathbb{Z}^2, z_i \in \mathbb{Z}_+ \ \forall i \in \{1, \dots n_1\}, y_i \in \mathbb{R}_+ \ \forall i \in \{1, \dots n_2\} \quad (5)$$

can be rewritten as

$$x_1 = b_1 + \sum_{i=1}^{n_1} a_{1i} z_i + \sum_{i=1}^{n_2} c_{1i} y_i$$

$$x_2 = b_2 + \sum_{i=1}^{n_1} a_{1i} z_i + \sum_{i=1}^{n_2} c_{2i} y_i$$

$$z_i - s_i = 0 \forall i \in \{1, \dots n_1\}$$

$$x_1, x_2 \in \mathbb{Z}, z_i \in \mathbb{Z}, s_i, y_i \geq 0$$

Now by taking suitable combination of above system, every cut from the monoidal strengthening can be obtained using 2D lattice-free closure, i.e. (2D lattice-free closure + Monoidal Strengthening closure) $\supseteq$ (2D lattice-free closure)

Taking Stock...

One row/one integer variable based	Two row/two integer variables based
MIR Set (MIR closure)	Canonical Set (2D Lattice-free cut closure)
GMIC Closure	2D Lattice-free cut + Monoidal Strengthening
Split Cut Closure	Crooked Cross Closure

Theorem

Let P be a mixed integer linear set.

$$\begin{aligned} &\text{2D Lattice-free cut closure} \\ = \; &\text{2D Lattice-free cut + Monoidal Strengthening closure} \\ \supseteq \; &\text{Crooked Cross Closure} \end{aligned} \tag{6}$$

Moreover, for the mixed-integer set $P := \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}_{+} \mid Ax + Gy = b\}$:

1. *If A has full row rank, then (6) holds at equality.*
2. *If $\operatorname{rank}(A) = 2$, then the crooked cross closure is the convex hull of P.*

Questions

1. Is the crooked cross closure or cross closure a polyhedron?
2. Is the crooked cross closure strictly contained in the cross closure?

Thank You.