On the Transportation Problem With Market Choice

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Abstract

We study a variant of the classical transportation problem in which suppliers with limited capacities have a choice of which demands (markets) to satisfy. We refer to this problem as the transportation problem with market choice (TPMC). While the classical transportation problem is known to be strongly polynomial-time solvable, we show that its market choice counterpart is strongly NP-complete. For the special case when all potential demands are no greater than two, we show that the problem reduces in polynomial time to minimum weight perfect matching in a general graph, and thus can be solved in polynomial time. Next, we consider the convex hull of solutions to the problem when a cardinality constraint is introduced on the number of rejected markets. We show that the cardinality constraint does not introduce new fractional extreme points for the case when TPMC is polynomially solvable. We give valid inequalities and coefficient update schemes for general mixed-integer sets that are substructures of TPMC. Finally, we give conditions under which these inequalities define facets, and report our preliminary computational experiments with using them in a branch-and-cut algorithm.

Keywords: Transportation problem, market choice, complexity, cardinality constraint, facet

1 Introduction

We consider a variant of the classical transportation problem in which suppliers with limited capacities have a choice of which demands (markets) to satisfy. In this problem, if a market is selected its demand must be satisfied fully through shipments from the suppliers. If a market is rejected, then the corresponding potential revenue is lost. The objective is to minimize the total cost of shipping and lost revenues. We refer to this problem as the transportation problem with market choice (TPMC).

More formally, we are given a set of supply and demand nodes that form a bipartite graph $G(V_1 \cup V_2, E)$. The nodes in set V_1 represent the supply nodes, where for $i \in V_1$, $s_i \in \mathbb{N}$ represents the capacity of supplier i. The nodes in set V_2 represent the potential markets, where for $j \in V_2$, $d_j \in \mathbb{N}$ represents the demand of market j. The edges between supply and demand nodes have weights that represent shipping costs w_{ij}, where $(i,j) \in E$. For each $j \in V_2$, r_j is the revenue lost if the market j is rejected. For a given vector of parameters γ_j for $j \in S$ and $S' \subseteq S$, we let $\gamma(S') := \sum_{j \in S'} \gamma_j$, throughout the paper.

Let x_{ij} be the amount of demand of market j satisfied by supplier i for $(i,j) \in E$, and let z_j be an indicator variable taking a value 1 if market j is rejected and 0 otherwise. A mixed-integer programming (MIP) formulation of the problem is given where the objective is to minimize the transportation costs and the lost revenues due to unchosen markets:

$$\min \sum_{(i,j)\in E} w_{ij}x_{ij} + \sum_{j\in V_2} r_j z_j \tag{1a}$$

$$\text{s.t.} \sum_{i:(i,j)\in E} x_{ij} = d_j(1-z_j) \qquad \forall j \in V_2 \tag{1b}$$

$$\sum_{j:(i,j)\in E} x_{ij} \le s_i \qquad \forall i \in V_1 \tag{1c}$$

$$z \in \{0,1\}^{|V_2|} \tag{1d}$$

$$x \in \mathbb{R}_+^{|E|}. \tag{1e}$$

We refer to problem description (1a)-(1e) as TPMC. The first set of constraints (1b) is the demand constraint. In TPMC either a demand for a market is fully satisfied or rejected altogether, which necessitates the introduction of the additional binary variables. The second set of constraints (1c) model the supply restrictions.

TPMC is closely related to the capacitated facility location (CFL) problem. In CFL, given a set of potential facilities $j \in V_2$ with capacities $\bar{d}_j, j \in V_2$ and customers $i \in V_1$ with demands $\bar{s}_i, i \in V_1$, we would like to determine which facilities to open so that the demand of all customers can be satisfied from shipments from the open facilities. A MIP formulation of CFL is

$$\sum_{i:(i,j)\in E} \bar{x}_{ij} \le \bar{d}_j \bar{z}_j \qquad \forall j \in V_2 \tag{2a}$$

$$\sum_{j:(i,j)\in E} \bar{x}_{ij} = \bar{s}_i \qquad \forall i \in V_1 \tag{2b}$$

$$\bar{z} \in \{0,1\}^{|V_2|} \tag{2c}$$

$$\bar{x} \in \mathbb{R}_+^{|E|}. \tag{2d}$$

Therefore one may view the CFL problem as a 'complement' of the TPMC problem where the constraints (1b) and (1c) of TPMC change signs in the constraints (2a) and (2b) in CFL respectively. Note that there is no straightforward way of 'complementing' the variables of TPMC in order to construct an instance of CFL or vice versa. While the CFL problem has been extensively studied with respect to its complexity, polyhedral structure, and approximability ([1, 8] and references therein), TPMC is less understood.

Recently, approximation algorithms and heuristics have been proposed for various supply chain planning and logistics problems with market choice [11, 18]. It is assumed that these problems are uncapacitated or that they have *soft* capacities. A two-stage approach is utilized in solving these classes of problems that admit a facility location formulation. In the first stage, the problem is to determine a subset of markets and reject the others. In the second stage, the goal is to minimize the production cost and lost revenues due to unselected markets. In particular, for the *uncapacitated* lot-sizing problem, the facility location formulation is used to model the market choice counterpart. It is shown that the LP relaxation solution can be rounded in a way that guarantees a constant factor approximation algorithm. However, this algorithm relies on scaling continuous variables up, so it does not immediately generalize to our problem with hard capacity constraints (1c). Van den Heuvel et al. [25] consider a maximization version of the same problem and show that no constant factor approximation algorithm exists for this version, unless P=NP. The authors also give several polynomially solvable special cases, and test heuristics for the general case.

The rest of the paper is organized as follows. In Section 2 we explore the complexity of TPMC. We show that while the classical transportation problem admits a strongly polynomial algorithm [16], its market choice counterpart is strongly NP-complete. We also identify a polynomially solvable case when the demands of all potential markets are no more than two. In Section 3 we consider a version of the problem with a service level constraint on the maximum number of markets that can be rejected. We show that for the case in which the original problem is polynomial, its cardinality-constrained version is also polynomial. Furthermore, in this case, we show that adding the cardinality constraint to the convex hull of solutions to the original problem does not create any new fractional extreme points. In Section 4 we present methods for constructing valid inequalities for mixed integer cover sets and mixed-integer knapsack sets with variable upper bound constraints, which appear as substructures of TPMC. We show that these methods are useful for generating valid inequalities for TPMC. We also study the strength of the proposed valid inequalities. Our preliminary computations, summarized in Section 5, show that there is a reduction in the root gap when our valid inequalities are incorporated to the branch-and-cut algorithm. However, we do not give an extensive computational study and the heuristic separation we use needs significant improvement.

2 Complexity

We first show that TPMC is strongly NP-hard in general.

Proposition 1. *The decision version of TPMC is NP-complete even when:*

1. $s_i = 1$ *for all* $i \in V_1$, $d_j = d \geq 3$ *for all* $j \in V_2$, $w_{ij} = 0$ *for all* $(i, j) \in E$ *and* $r_j = 1$ *for all* $j \in V_2$.
2. $|V_1| = 1$ *and* $w_{ij} = 0$ *for all* $(i, j) \in E$.

The proof for Proposition 1 Part 1 is similar to the proof of a related result presented in [22]. For completeness, we provide its proof and the proof of Part 2 in the Appendix. Because the reduction of Part 1 is from the Exact 3-Cover problem, which is strongly NP-complete [10], we conclude that TPMC is strongly NP-hard even for the case where all demands are equal to three. In contrast, Proposition 2 shows that TPMC is polynomially solvable when demands of all markets do not exceed two.

Proposition 2. *Suppose that* $d_j \leq 2$ *for all* $j \in V_2$. *Then there exists a polynomial-time algorithm to solve TPMC.*

This result is proven by a polynomial time reduction to a minimum weight perfect matching problem on a general graph (provided in the Appendix). The key ideas of the reduction are based on those presented in [3]. This result can also be proven by a polynomial time reduction to the b-matching problem [9], see also Theorem 36.1 in [23].

A matrix A is said to have the Edmonds-Johnson property if the sum of the absolute values of the entries in any column of A is less than or equal to 2. Edmonds and Johnson [9] show that the convex hull of integer solutions to a system $Ax \leq b$, where A has this property is given by the so-called blossom inequalities. Note that the constraint matrix defined by inequalities (1b), (1c), (1e), and $z \in \mathbb{R}_+^{|V_2|}$ have the Edmonds-Johnson property when $d_j \leq 2$ for all $j \in V_2$. Hence adding the blossom inequalities to the original formulation is enough to give the convex hull of solutions to TPMC in this case. The blossom inequality for TPMC is

$$\sum_{i \in U_1, j \in U_2 : (i,j) \in E} x_{ij} + \sum_{j \in U_2} \lfloor d_j/2 \rfloor z_j \leq \left\lfloor \frac{s(U_1) + d(U_2)}{2} \right\rfloor, \tag{3}$$

where $U_1 \subseteq V_1$, $U_2 \subseteq V_2$ such that the sum of total supply in U_1 and total demand in U_2, $s(U_1) + d(U_2)$, is odd. The separation of blossom inequalities (3) is polynomial [12, 17, 20]. We propose other classes of valid inequalities for the general case in Section 4.

3 TPMC with a cardinality constraint

An important and natural constraint that one may add to the TPMC problem is that of a service level, i.e., the number of rejected markets is restricted to be at most k. This restriction can be modelled using a *cardinality constraint*, $\sum_{j \in V_2} z_j \leq k$, appended to (1a)-(1e). We call the resulting problem cardinality constrained TPMC (CCTPMC). If we are able to solve CCTPMC in polynomial-time, then we can solve TPMC in polynomial time by solving CCTPMC for all $k \in \{0, \ldots, |V_2|\}$. Therefore by Proposition 1, we obtain that CCTPMC is NP-hard in general. In this section, we examine the specific case where we know that TPMC admits a polynomial-time algorithm.

In light of the proof of Proposition 2, via the reduction to a minimum weight perfect matching problem on a general (non-bipartite) graph $G' = (V', E')$, it is possible to reduce CCTPMC with $d_j \leq 2$ for all $j \in V_2$ to a *minimum weight perfect matching problem with a cardinality constraint on a subset of edges* (specifically the cardinality constraint is applied only on the edges $(j, j') \in E'$ for each $j \in V_2$; see proof of Proposition 2 in Appendix). To the best of our knowledge, the complexity status of minimum weight perfect matching problem on a general graph with a cardinality constraint on a subset of edges is open. This can be seen by observing that if one can solve minimum weight perfect matching problem with a cardinality constraint on a subset of edges, then one can solve the exact perfect matching problem; see discussion in the last section in [6]. On the other hand, we will prove in this section that CCTPMC with $d_j \leq 2$ for all $j \in V_2$, which is a special case of a minimum weight perfect matching problem with cardinality constraint on a specific subset of edges, in fact admits a polynomial-time algorithm. Our approach will be following: We will prove that the TPMC polytope (when $d_j \leq 2$ for all $j \in V_2$) along with the constraint $\sum_{j \in V_2} z_j \leq k$ is integral. Therefore by invoking the ellipsoid algorithm it is possible to solve CCTPMC in polynomial time. This result also allows for solving CCTPMC (when $d_j \leq 2$ for all $j \in V_2$) by a Lagrangian relaxation approach, where we relax the cardinality constraint.

Before we proceed, we briefly note that the intersection of the perfect matching polytope with a cardinality constraint on a strict subset of edges is not always integral.

Example 1. *Consider the bipartite graph $G(V_1 \cup V_2, E)$ with $V_1 = \{1, 2, 3\}, V_2 = \{4, 5, 6\}$, $E = \{(1, 4), (1, 5), (2, 4), (2, 5), (2, 6), (3, 5), (3, 6)\}$, and the cardinality constraint $x_{14} + x_{25} \leq 1$. It is straightforward to show that $x_{14} = x_{15} = x_{24} = x_{25} = 0.5, x_{26} = x_{35} = 0, x_{36} = 1$ is a fractional extreme point of the intersection of the perfect matching polytope with the cardinality constraint.*

Let $X \in \mathbb{R}_+^{|E|} \times \{0, 1\}^{|V_2|}$ be the set of feasible solutions of TPMC. Our main result of this section is presented next.

Theorem 1. *Let $k \in \mathbb{Z}_+$ and $k \leq |V_2|$. Let $X^k := \text{conv}(X \cap \{(x, z) \in \mathbb{R}_+^{|E|} \times \{0, 1\}^{|V_2|} \mid \sum_{j \in V_2} z_j \leq k\})$. If $d_j \leq 2$ for all $j \in V_2$, then $X^k = \text{conv}(X) \cap \{(x, z) \in \mathbb{R}_+^{|E|} \times [0, 1]^{|V_2|} \mid \sum_{j \in V_2} z_j \leq k\}$.*

Corollary 3. *CCTPMC is polynomially solvable when $d_j \leq 2$ for all $j \in V_2$.*

Observation 1. *Theorem 1 is a generalization of the well-known result, Matching Cardinality Theorem: Let $G(V, E)$ be a graph with n vertices and m edges. Let $M \subset \mathbb{R}^m$ be the matching polytope and let $M^k \subset \mathbb{R}^m$ be the convex hull of incidence vectors of matchings with at least k edges. Then $M^k = M \cap \{x \in \mathbb{R}^m \mid \sum_{i=1}^m x_i \geq k\}$. (See [23] for a proof.)*

We construct a bipartite graph $\hat{G}(\hat{V}^1 \cup \hat{V}^2, \hat{E})$ as follows: $\hat{V}^1$ is a set of n vertices corresponding to the n vertices in G. $\hat{V}^2$ corresponds to the set of edges of G, i.e., $\hat{V}^2$ contains m vertices. We use (i, j) to refer to the vertex in $\hat{V}^2$ corresponding to the edge (i, j) in E. The set of edges in $\hat{E}$ are of the form $(i, (i, j))$ and $(j, (i, j))$ for every $i, j \in V$ such that $(i, j) \in E$. Now we can construct (the feasible region of) an instance of TPMC with respect to $\hat{G}(\hat{V}^1 \cup \hat{V}^2, \hat{E})$ as follows:

$$Q = \{(x, z) \in \mathbb{R}^{2m} \times \mathbb{R}^m \mid x_{i,(i,j)} + x_{j,(i,j)} + 2z_{(i,j)} = 2 \ \forall (i, j) \in \hat{V}^2 \tag{4}$$

$$\sum_{j:(i,j) \in E} x_{i,(i,j)} \leq 1 \ \forall i \in \hat{V}^1 \tag{5}$$

$$z_{(i,j)} \in \{0, 1\} \ \forall (i, j) \in \hat{V}^2\}. \tag{6}$$

We can construct an instance of CCTPMC by adding the constraint $\sum_{(i,j) \in E} z_{(i,j)} \leq k$ (call this set Q^k). It is straightforward to verify that the Matching Cardinality Theorem is equivalent to stating $\text{conv}(Q^k) =$

$\mathrm{conv}(Q) \cap \{(x,z) \,|\, \sum_{(i,j)\in E} z_{(i,j)} \leq k\}$. *Thus, the Matching Cardinality Theorem follows from Theorem 1 applied to the bipartite graph* $\hat{G}$.

Now note that the graph $\hat{G}$ *has a very special structure. In particular, the degree of every node in the second set of vertices* ($\hat{V}^2$) *is* 2. *On the other hand, Theorem 1 holds for a general instance of TPMC with* $d_j \leq 2$ *for all* $j \in V_2$, *i.e. in particular for instances corresponding to general bipartite graphs where the degree of the vertices can be more than* 2 *and the value of* d_j *can be either* 1 *or* 2. ☐

To prove Theorem 1, one approach could be to appeal to the reduction to minimum weight perfect matching problem and then use the well-known adjacency properties of the vertices of the perfect matching polytope. However, as illustrated in Example 1, the integrality result does not hold for the perfect matching polytope on a general graph with a cardinality constraint on any subset of edges. Therefore a generic approach considering the perfect matching polytope appears to be less fruitful. We use an alternative approach to prove this result. In particular, we apply a technique similar to that used in [2]. Consider the following desirable property:

Definition 1 (Edge Property). *Let* $T \subseteq \mathbb{R}^p_+ \times \{0,1\}^n$ *be some mixed integer set. We say that* T *satisfies the* edge property *if for all* $(w,r) \in \mathbb{R}^{p+n}$ *such that* $\min\{w^\top x + r^\top z \,|\, (x,z) \in T\}$ *is bounded and has at least two optimal solutions,* (x^1,z^1) *and* (x^2,z^2) *where* $\sum_{j=1}^n z^1_j = k^1$, $\sum_{j=1}^n z^2_j = k^2$ *and* $k^1 \leq k^2 - 2$, *then there is an optimal solution* (x^3,z^3) *such that* $\sum_{j=1}^n z^3_j = k^3$ *and* $k^1 < k^3 < k^2$.

Proposition 4. *Let* $T \subseteq \mathbb{R}^p_+ \times \{0,1\}^n$ *be a mixed integer set such that* $\mathrm{conv}(T)$ *is a pointed polyhedron and let* $T^k := \mathrm{conv}(T \cap \{(x,z) \in \mathbb{R}^p_+ \times \{0,1\}^n \mid \sum_{j=1}^n z_j \leq k\})$. *If* T *satisfies the edge property, then* $T^k = \mathrm{conv}(T) \cap \{(x,z) \in \mathbb{R}^p_+ \times [0,1]^n | \sum_{j=1}^n z_j \leq k\}$.

Proof. Assume by contradiction that

$$T^k \neq \mathrm{conv}(T) \cap \{(x,z) \in \mathbb{R}^p_+ \times [0,1]^n | \sum_{j=1}^n z_j \leq k\},$$

for some $k = k' \in \{0,1,\dots,n\}$. By definition $T^k = \mathrm{conv}(T \cap \{(x,z) \in \mathbb{R}^p_+ \times \{0,1\}^n | \sum_{j=1}^n z_j \leq k\})$ so $T^k \subseteq \mathrm{conv}(T) \cap \{(x,z) \in \mathbb{R}^p_+ \times [0,1]^n | \sum_{j=1}^n z_j \leq k\}$ holds for all $k \in \{0,1,\dots,n\}$. By assumption we obtain $T^{k'} \subset \mathrm{conv}(T) \cap \{(x,z) \in \mathbb{R}^p_+ \times [0,1]^n | \sum_{j=1}^n z_j \leq k'\}$. Since $\mathrm{conv}(T)$ is pointed this implies that there exists a vertex (x',z') of $\mathrm{conv}(T) \cap \{(x,z) \in \mathbb{R}^p_+ \times [0,1]^n | \sum_{j=1}^n z_j \leq k'\}$ such that $(x',z') \notin T^{k'}$. Therefore z' is fractional and $\sum_{j=1}^n z'_j = k'$ (if $\sum_{j=1}^n z'_j < k'$, then this point is also a vertex of $\mathrm{conv}(T)$, therefore integral and belonging to $T^{k'}$ - a contradiction).

Since (x',z') is not a vertex of $\mathrm{conv}(T)$, there exists (w,r) such that the vertex (x',z') is the intersection of the face defined by $\{(x,z) \in \mathbb{R}^p_+ \times [0,1]^n | \sum_{j=1}^n z_j = k'\}$ and an edge of $\mathrm{conv}(T)$ defined as:

$$\{(x,z) \in \mathrm{conv}(T) \,|\, w^\top x + r^\top z = \delta\}, \tag{7}$$

where $\delta = \min\{w^\top x + r^\top z \,|\, (x,z) \in \mathrm{conv}(T)\} = w^\top x' + r^\top z'$. Let (x^1,z^1) and (x^2,z^2) be two feasible points of T that belong to the edge (7) such that (x',z') is a convex combination of (x^1,z^1) and (x^2,z^2). Note that $\delta = w^\top x' + r^\top z' = w^\top x^1 + r^\top z^1 = w^\top x^2 + r^\top z^2$. Hence, (x^1,z^1) and (x^2,z^2) are two optimal solutions corresponding to the objective function (w,r). Furthermore, due to our selection of δ, $\sum_{j\in V_2} z^1_j < k' < \sum_{j\in V_2} z^2_j$. The edge property ensures that there exists an integral optimal solution (x^3,z^3) with $k^3 = \sum_{j\in V_2} z^3_j = k'$ such that $\sum_{j\in V_2} z^1_j < k^3 < \sum_{j\in V_2} z^2_j$. However, this implies that (x^3,z^3) belongs to the edge defined by (7). Thus, (x^3,z^3) must be a convex combination of (x^1,z^1) and (x^2,z^2) or equivalently, we must have $(x^3,z^3) = (x',z')$ with z' integral, a contradiction. ☐

Now, we show how edge property and Proposition 4 can be applied to TPMC with an additional constraint that at most k markets can be rejected. To prove Theorem 1 we use Proposition 4. Similar to the argument in the proof of Proposition 2, we assume that all data are integral, and that $s_i = 1$ for all $i \in V_1$ without loss of generality. It is straightforward to verify that the polyhedron X corresponding to the original instance with $s_i > 1$ for some $i \in V_1$ satisfies the edge property if and only if X corresponding to the corresponding instance with $s_i = 1$ for all $i \in V_1$ satisfies the edge property. We are now ready to present the proof of Theorem 1.

Proof of Theorem 1. By hypothesis $d_j \leq 2$ for all $j \in V_2$. From Proposition 4 it is sufficient to prove that the edge property holds.

Suppose that (x^1, z^1) and (x^2, z^2) are optimal solutions to $\min\{w^\top x + r^\top z \mid (x, z) \in X\}$ and that x^1 is fractional. Then we can solve a simple transportation problem with the set of demand nodes j such that $z^1_j = 0$. Since all data is integral, there exists an optimal solution with integral flows. Therefore, we may assume that x^1 (and similarly x^2) are integral.

Claim 1. *Suppose we have two feasible solutions of X, namely (x^3, z^3) and (x^4, z^4), such that*

1. $\sum_{j \in V_2} z^3_j = k^1 + 1$ *and* $\sum_{j \in V_2} z^4_j = k^2 - 1$ *and*
2. *The objective function value of (x^3, z^3) is $\rho - \delta$ and that of (x^4, z^4) is $\rho + \delta$, where ρ is the objective function value of the solution (x^1, z^1) and $\delta \in \mathbb{R}$,*

then the proof of Theorem 1 is complete.

Proof. Since ρ is the optimal objective function value, we obtain that $\delta = 0$ since otherwise the objective function value of either (x^3, z^3) or (x^4, z^4) is better than that of (x^1, z^1). Therefore (x^3, z^3) is an optimal solution with $k^1 < \sum_{j \in V_2} z^3_j < k^2$. Because edge property is satisfied by Proposition 4, the proof of Theorem 1 is complete. □

Given an integral point $(\tilde{x}, \tilde{z})$ of X, let $S(\tilde{z}) := \{j \in V_2 \mid \tilde{z}_j = 0\}$ be the set of nodes in V_2 whose demands are met. For $j \in S(\tilde{z})$, let $I_j(\tilde{x}, \tilde{z}) = \{i \in V_1 \mid \tilde{x}_{ij} > 0\} = \{i \in V_1 \mid \tilde{x}_{ij} = 1\}$ be the set of suppliers that sends one unit to j.

Given the optimal solutions (x^1, z^1) and (x^2, z^2), let $F := \big(S(z^1) \setminus S(z^2)\big) \cup \big(S(z^2) \setminus S(z^1)\big)$, $P := S(z^1) \cap S(z^2)$ and $R := V_2 \setminus (F \cup P)$. For $j \in F$, observe that only the set $I_j(x^1, z^1)$ or the set $I_j(x^2, z^2)$ is defined. So for $j \in F$, we define I_j as:

$$I_j := \begin{cases} I_j(x^1, z^1) & \text{if } j \in S(z^1) \setminus S(z^2) \\ I_j(x^2, z^2) & \text{if } j \in S(z^2) \setminus S(z^1). \end{cases} \tag{8}$$

As a first step towards constructing (x^3, z^3) and (x^4, z^4) required in Claim 1, we construct a bipartite (conflict) graph $G^*(U_1 \cup U_2, \mathcal{E})$. The set of nodes is constructed as follows:

1. If $j \in S(z^1) \setminus S(z^2)$, then $j \in U_1$ and j is called a *full node*. Let $W_1 = S(z^1) \setminus S(z^2)$ be the set of full nodes of U_1.
2. Similarly, if $j \in S(z^2) \setminus S(z^1)$, then $j \in U_2$ and j is called a *full node*. Let $W_2 = S(z^2) \setminus S(z^1)$ be the set of full nodes of U_2.
3. If $j \in S(z^1) \cap S(z^2)$ and $d_j = 2$ then we place two copies of node j in U_1 (call these j_1 and j_2) and two copies of j in U_2 (call these j_3 and j_4). These nodes are called *partial nodes* of j. Each partial node of j is distinct: If $I_j(x^1, z^1) = \{t_1, t_2\}$, then associate (WLOG) t_1 with j_1 and t_2 with j_2, that is define $I_{j_1} := \{t_1\}$ and $I_{j_2} := \{t_2\}$. Similarly if $I_j(x^2, z^2) = \{t_3, t_4\}$, then associate (WLOG) t_3 with j_3 and t_4 with j_4, that is define $I_{j_3} := \{t_3\}$ and $I_{j_4} := \{t_4\}$. If $j \in S(z^1) \cap S(z^2)$ and $d_j = 1$, then we place one copy of node j in U_1 (call this j_1) and one copy of j in U_2 (call this j_3). Similar to the $d_j = 2$ case these nodes are called *partial nodes* of j. If $I_j(x^1, z^1) = \{t_1\}$ and $I_j(x^2, z^2) = \{t_3\}$, then set $I_{j_1} = \{t_1\}$ and $I_{j_3} = \{t_3\}$. Let $P = P^1 \cup P^2$, where $P^1 = \{j \in P : d_j = 1\}$ and $P^2 = \{j \in P : d_j = 2\}$.

Thus $U_1 = W_1 \cup \left(\bigcup_{j\in P^2}\{j_1, j_2\}\right) \cup \left(\bigcup_{j\in P^1}\{j_1\}\right)$ and for each element $a \in U_1$ the set I_a is well-defined and non-empty. Similarly, $U_2 = W_2 \cup \left(\bigcup_{j\in P^2}\{j_3, j_4\}\right) \cup \left(\bigcup_{j\in P^1}\{j_3\}\right)$ and for each element $b \in U_2$ the set I_b is well-defined and non-empty. Now we construct the edges $\mathcal{E}$ as follows: For all $a \in U_1$ and $b \in U_2$, there is an edge $(a, b) \in \mathcal{E}$ if and only if a and b have at least one common supplier, i.e.,

$$I_a \cap I_b \neq \emptyset \text{ iff } (a, b) \in \mathcal{E}. \tag{9}$$

Let $G'(V', E')$ be a subgraph of $G^*(U_1 \cup U_2, \mathcal{E})$. Since the elements in $V' \cap (W_1 \cup W_2)$ correspond to unique elements in V_2, whenever required we will (with slight abuse of notation) treat $V' \cap (W_1 \cup W_2) \subseteq V_2$.

Claim 2. *Let $G'(V', E')$ be a subgraph of $G^*(U_1 \cup U_2, \mathcal{E})$ satisfying the following properties:*

1. *There are no edges in G^* between the nodes in V' and the nodes in $(U_1 \cup U_2) \setminus V'$.*
2. *For each $j \in P^1$, $|V' \cap \{j_1\}| = |V' \cap \{j_3\}|$ and for each $j \in P^2$, $|V' \cap \{j_1, j_2\}| = |V' \cap \{j_3, j_4\}|$.*
3. *$|W_1 \cap V'| = |W_2 \cap V'| + 1$.*

Now construct

$$z_j^3 = \begin{cases} z_j^1 & \text{if } j \in V_2 \setminus (V' \cap F) \\ 1 & \text{if } j \in V' \cap W_1 \\ 0 & \text{if } j \in V' \cap W_2. \end{cases} \tag{10}$$

$$x_{ij}^3 = \begin{cases} 1 & \text{if } j \in F, z_j^3 = 0, i \in I_j \\ 1 & \text{if } j \in P, j_1 \in (U_1 \cup U_2) \setminus V', i \in I_{j_1} \\ 1 & \text{if } j \in P, j_2 \in (U_1 \cup U_2) \setminus V', i \in I_{j_2} \\ 1 & \text{if } j \in P, j_3 \in V', i \in I_{j_3} \\ 1 & \text{if } j \in P, j_4 \in V', i \in I_{j_4} \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

and

$$z_j^4 = \begin{cases} z_j^2 & \text{if } j \in V_2 \setminus (V' \cap F) \\ 0 & \text{if } j \in V' \cap W_1 \\ 1 & \text{if } j \in V' \cap W_2. \end{cases} \tag{12}$$

$$x_{ij}^4 = \begin{cases} 1 & \text{if } j \in F, z_j^4 = 0, i \in I_j \\ 1 & \text{if } j \in P, j_3 \in (U_1 \cup U_2) \setminus V', i \in I_{j_3} \\ 1 & \text{if } j \in P, j_4 \in (U_1 \cup U_2) \setminus V', i \in I_{j_4} \\ 1 & \text{if } j \in P, j_1 \in V', i \in I_{j_1} \\ 1 & \text{if } j \in P, j_2 \in V', i \in I_{j_2} \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Then (x^3, z^3) and (x^4, z^4) are feasible solutions of X that satisfy the requirements of Claim 1.

Proof. 1. We verify that (x^3, z^3) is a valid solution to X. A similar proof can be given for the validity of (x^4, z^4). Clearly x^3 and z^3 satisfy the variable restrictions. We verify that the constraint $\sum_{i:(i,j)\in E} x_{ij}^3 + d_j z_j = d_j$ is satisfied for all $j \in V_2$. If $j \in R$, then $z_j^3 = z_j^1 = 1$ and $x_{ij}^3 = 0$ for all $(i, j) \in E$; therefore the constraint is satisfied. If $j \in F$, then using the first and last entry in (11), we have $\sum_{i:(i,j)\in E} x_{ij}^3 + d_j z_j^3 = d_j$. If $j \in P$, then $j \in V_2 \setminus (V' \cap F)$. Therefore $z_j^3 = z_j^1 = 0$. Now it is straightforward to verify that $\sum_{i:(i,j)\in E} x_{ij}^3 = 2 = d_j$ for each $j \in P^2$ since $|V' \cap \{j_1, j_2\}| = |V' \cap \{j_3, j_4\}|$ and by the use of the last five entries in (11). For $j \in P^1$ we have $\sum_{i:(i,j)\in E} x_{ij}^3 = 1 = d_j$ since $|V' \cap \{j_1\}| = |V' \cap \{j_3\}|$ and by the use of the second, fourth and sixth entries in (11).

Now we verify that the constraint $\sum_{j:(i,j)\in E} x_{ij} \leq 1$ is satisfied for all $i \in V_1$. Given $i \in V_1$, assume for contradiction that $x^3_{ig} = x^3_{ih} = 1$ for some $g, h \in V_2$ and $g \neq h$. By construction of (x^3, z^3), $x^3_{ij} = 0$ for all $j \in R$. Thus, $g, h \notin R$. Moreover since $\sum_{i:(i,j)\in E} x^3_{ij} + d_j z_j = d_j$ is satisfied for all $j \in V_2$, we have $z^3_g = z^3_h = 0$. Now, there are three cases to consider:

(a) $g, h \in F$. By construction of x^3 we have $i \in I_g \cap I_h$. Now if $g \notin V'$ and $h \notin V'$, then by construction of z^3 (first entry in (10)) we have $z^1_g = z^3_g = 0 = z^3_h = z^1_h$ and thus $g, h \in S(z^1)$. Therefore by the validity of (x^1, z^1) we have $I_g \cap I_h = \emptyset$. This contradicts $i \in I_g \cap I_h$. Now consider the case where $g \in V'$ and $h \in V'$. Since $i \in I_g \cap I_h$ by (9) there is an edge between g and h in $G^*(U_1 \cup U_2, \mathcal{E})$. Thus we may assume without loss of generality that $g \in V' \cap W_1$ and $h \in V' \cap W_2$. However, this implies that $z^3_g = 1$, a contradiction. Now, without loss of generality, assume that $g \in V'$ and $h \notin V'$. Since $i \in I_g \cap I_h$ by (9) there is an edge between g and h in $G^*(U_1 \cup U_2, \mathcal{E})$. On the other hand, by assumption there is no edge between nodes in V' and those not in V', which is the required contradiction.

(b) $g \in F$ and $h \in P$. Without loss of generality we may assume that $g \in W_1$. If $g \in V'$, then $z^3_g = 1$, a contradiction. Therefore, we have $g \notin V'$. Thus $z^1_g = z^3_g = 0$. Therefore by validity of (x^1, z^1) we have $i \notin I_h(x^1, z^1)$ or equivalently $i \in I_h(x^2, z^2)$. Without loss of generality we may assume that $i \in I_{h_3}$. Note that h_3 belongs to V' (by the construction of x^3 and the fact that $x^3_{ih} = 1$ and $i \in I_{h_3}$). Since $i \in I_g$, there exists an edge between g and h_3. However, since $g \notin V'$ and $h_3 \in V'$, we get a contradiction to the fact that there are no edges between the nodes in V' and the nodes in $(U_1 \cup U_2) \setminus V'$.

(c) $g, h \in P$. In this case we may assume without loss of generality that $i \in I_g(x^1, z^1)$ and $i \in I_h(x^2, z^2)$. Therefore without loss of generality, we may assume that $i \in I_{g_1}$ and $i \in I_{h_3}$. Since $x^3_{ig} = x^3_{ih} = 1$, we have $g_1 \notin V'$ and $h_3 \in V'$. By assumption on G', this implies that there is no edge between g_1 and h_3. On the other hand, since $i \in I_{g_1} \cap I_{h_3}$ by (9) we have an edge $(g_1, h_3) \in \mathcal{E}$, a contradiction.

2. Next we verify that the objective function value of (x^3, z^3) is $\rho - \delta$ and that of (x^4, z^4) is $\rho + \delta$ where ρ is the objective function value of the solution (x^1, z^1) and $\delta \in \mathbb{R}$. This result is verified by showing that (x^3, z^3) and (x^4, z^4) are obtained by 'symmetrically' updating demands from (x^1, z^1) and (x^2, z^2) respectively. In particular, we examine each demand node and examine the cost of either satisfying it or not satisfying it in each solution. We consider the different cases next:

(a) $j \in R$. Then $z^4_j = z^3_j = z^1_j = z^2_j = 1$.

(b) $j \in V' \cap W_1$. Then $z^1_j = 0$ and $z^3_j = 1$. On the other hand $z^2_j = 1$ and $z^4_j = 0$. Notice that in each solution where d_j is satisfied, this is done by using the same set of input nodes (and thus using the same arcs). Therefore the difference in objective function value between (x^1, z^1) and (x^3, z^3) due to demand node j is $-\sum_{i\in I_j} w_{ij} + r_j$ and the difference in objective function value between the solutions (x^2, z^2) and (x^4, z^4) due to demand node j is $\sum_{i\in I_j} w_{ij} - r_j$.

(c) $j \in V' \cap W_2$. Similar to the above case the difference in objective function value between (x^1, z^1) and (x^3, z^3) due to demand node j is $\sum_{i\in I_j} w_{ij} - r_j$ and the difference in objective function value between (x^2, z^2) and (x^4, z^4) due to demand node j is $-\sum_{i\in I_j} w_{ij} + r_j$.

(d) $j \in F \setminus V'$, then $z^1_j = z^3_j$ and $z^2_j = z^4_j$.

(e) $j \in P^2$ such that $j_1, j_2 \in (U_1 \cup U_2) \setminus V'$ and $j_3, j_4 \in (U_1 \cup U_2) \setminus V'$. Then the demand d_j is satisfied by the nodes in $I_j(x^1, z^1)$ in (x^1, z^1) and (x^3, z^3). Therefore there is no difference in objective function value between (x^1, z^1) and (x^3, z^3) with respect to demand node j. Similarly, the demand d_j is satisfied by the nodes in $I_j(x^2, z^2)$ in (x^2, z^2) and (x^4, z^4) and there is no difference in objective function value between (x^2, z^2) and (x^4, z^4) with respect to demand node j. We can make a similar argument for $j \in P^1$ such that $j_1 \in (U_1 \cup U_2) \setminus V'$ and $j_3 \in (U_1 \cup U_2) \setminus V'$.

(f) $j \in P^2$ such that $j_1 \in V'$, $j_2 \in (U_1 \cup U_2) \setminus V'$, $j_3 \in (U_1 \cup U_2) \setminus V'$, $j_4 \in V'$ without loss of generality. Then the demand d_j is satisfied by the nodes in $(I_{j_1} \cup I_{j_2})$ in (x^1, z^1) and by nodes $(I_{j_2} \cup I_{j_4})$ in (x^3, z^3). Therefore the difference in objective function value between (x^1, z^1) and (x^3, z^3) with respect to demand node d_j is $\sum_{i \in I_{j_1}} w_{ij} - \sum_{i \in I_{j_4}} w_{ij}$. The demand d_j is satisfied by the nodes in $(I_{j_3} \cup I_{j_4})$ in (x^2, z^2) and by the nodes in $(I_{j_1} \cup I_{j_3})$ in (x^4, z^4). Therefore the difference in objective function value between (x^2, z^2) and (x^4, z^4) with respect to demand node j is $\sum_{i \in I_{j_4}} w_{ij} - \sum_{i \in I_{j_1}} w_{ij}$. We can make a similar argument for the cases: $j_1 \in (U_1 \cup U_2) \setminus V'$, $j_2 \in V'$, $j_3 \in V'$, $j_4 \in (U_1 \cup U_2) \setminus V'$; $j_1 \in V'$, $j_2 \in (U_1 \cup U_2) \setminus V'$, $j_3 \in V'$, $j_4 \in (U_1 \cup U_2) \setminus V'$ and $j_1 \in (U_1 \cup U_2) \setminus V'$, $j_2 \in V'$, $j_3 \in (U_1 \cup U_2) \setminus V'$, $j_4 \in V'$.

(g) $j \in P^2$ such that $j_1 \in V'$, $j_2 \in V'$, $j_3 \in V'$, $j_4 \in V'$. Then the demand d_j is satisfied by the nodes in $(I_{j_1} \cup I_{j_2})$ in (x^1, z^1) and by the nodes in $(I_{j_3} \cup I_{j_4})$ in (x^3, z^3). Therefore, the difference in the objective function value between (x^1, z^1) and (x^3, z^3) with respect to satisfying demand d_j is $\sum_{i \in (I_{j_1} \cup I_{j_2})} (w_{ij} + w_{ij}) - \sum_{i \in (I_{j_3} \cup I_{j_4})} (w_{ij} + w_{ij})$. The demand d_j is satisfied by the nodes in $(I_{j_3} \cup I_{j_4})$ in (x^2, z^2) and by the nodes in $(I_{j_1} \cup I_{j_2})$ in (x^4, z^4). Therefore, the difference in the objective function value between (x^2, z^2) and (x^4, z^4) with regards to satisfying demand d_j is $-\sum_{i \in (I_{j_1} \cup I_{j_2})} (w_{ij} + w_{ij}) + \sum_{i \in (I_{j_3} \cup I_{j_4})} (w_{ij} + w_{ij})$. For $j \in P^1$, we can similarly consider j_1 and j_3 with $j_1 \in V'$, $j_3 \in V'$.

Therefore, the objective function value of (x^3, z^3) is $\rho - \delta$ and that of (x^4, z^4) is $\rho + \delta$ where ρ is the objective function value of the solution (x^1, z^1) and (x^2, z^2) and $\delta \in \mathbb{R}$.

3. Finally we verify that $\sum_{j \in V_2} z_j^3 = k^1 + 1$ and $\sum_{j \in V_2} z_j^4 = k^2 - 1$. We prove this for (x^3, z^3). The proof is similar for the case of (x^4, z^4). Observe that if $j \in R$, then $z_j^1 = z_j^3 = 1$. If $j \in P$, then $z_j^1 = z_j^3 = 0$. If $j \in F \setminus V'$, then $z_j^1 = z_j^3$. If $j \in W_1 \cap V'$, then $z_j^1 = 0$ and $z_j^3 = 1$ and if $j \in W_2 \cap V'$, then $z_j^1 = 1$ and $z_j^3 = 0$. Thus $\sum_{j \in V_2} z_j^1 - \sum_{j \in V_2} z_j^3 = |V' \cap W_2| - |V' \cap W_1| = -1$, where the last equality is by assumption (3) of G'. Thus, $\sum_{j \in V_2} z_j^3 = k^1 + 1$.

□

Now the proof of Theorem 1 is complete by showing that a subgraph $G'(V', E')$ of $G^*(U_1 \cup U_2, \mathcal{E})$ always exists that satisfies the conditions of Claim 2. In order to prove this, we verify a few results.

Claim 3. *Connected components of G^* are paths or cycles of even length and all the cycles involve only full nodes.*

Proof. This is evident from the fact that G^* is bipartite and degree of $a \in (U_1 \cup U_2)$ is bounded from above by $|I_a|$. □

We associate a value v_j to each node $j \in U_1 \cup U_2$. In particular:

1. If $j \in W_1$, then $v_j = 1$.
2. If $j \in U_1$ and j is a partial node, then $v_j = \frac{1}{2}$.
3. If $j \in U_2$ and j is a partial node, then $v_j = -\frac{1}{2}$.
4. If $j \in W_2$, then $v_j = -1$.

For a subgraph $\tilde{G}(\tilde{V}, \tilde{E})$ of G^* we call $v(\tilde{V}) = \sum_{j \in \tilde{V}} v_j$ the *value of the path.*

Claim 4. *$v(U_1 \cup U_2) = k^2 - k^1 \geq 2$.*

Proof. $\sum_{j \in U_1 \cup U_2} v_j = \sum_{j \in W_1} v_j + \sum_{j \in P^2} (v_{j_1} + v_{j_2}) + \sum_{j \in P^1} v_{j_1} + \sum_{j \in W_2} v_j + \sum_{j \in P^2} (v_{j_3} + v_{j_4}) + \sum_{j \in P^1} v_{j_3} = |S(z^1) \setminus S(z^2)| - |S(z^2) \setminus S(z^1)| = |S(z^1)| - |S(z^2)| = k^2 - k^1$. □

Claim 5. *If $\tilde{G}(\tilde{V}, \tilde{E})$ is a cyclic subgraph of $G^*(U_1 \cup U_2, \mathcal{E})$, then $v(\tilde{V}) = 0$.*

Proof. By Claim 3, a cycle has only full nodes. Moreover, since a cycle is of even length, it contains equal number of nodes from W_1 and W_2. □

Note that a partial node must be a leaf node in a path. Using this observation and by some simple case analysis the following three claims can be verified.

Claim 6. *If $\tilde{G}(\tilde{V}, \tilde{E})$ is a path containing exactly one partial node, then $v(\tilde{V}) \in \{-\frac{1}{2}, \frac{1}{2}\}$.*

Claim 7. *If $\tilde{G}(\tilde{V}, \tilde{E})$ is a path containing two partial nodes, then $v(\tilde{V}) = 0$.*

Claim 8. *If $\tilde{G}(\tilde{V}, \tilde{E})$ is a path containing only full nodes, then $v(\tilde{V}) \in \{-1, 0, 1\}$.*

For the subgraph $\tilde{G}(\tilde{V}, \tilde{E})$, consider a $k \in \tilde{V} \setminus F$ such that $k = j_t$ where $t \in \{1, 2, 3, 4\}$ and $j \in P^2$. Suppose $k = j_1$ or j_2, then we say that a path $\tilde{G}(\tilde{V}, \tilde{E})$ is a *mirror path* for j, if $\tilde{V}$ contains either j_3 or j_4. Moreover we call one of j_3 or j_4 (whichever belongs to $\tilde{V}$ or arbitrarily select one of these if both belong to $\tilde{V}$) as the *mirror node*. Similarly if $k = j_3$ or j_4, then we say that a path $\tilde{G}(\tilde{V}, \tilde{E})$ is a *mirror path* for j, if $\tilde{V}$ contains either j_1 or j_2. *Mirror node* is similarly defined in this case. For $j \in P^1$ we consider $k = j_1$ and $k = j_3$. Suppose $k = j_1$, then we say that a path $\tilde{G}(\tilde{V}, \tilde{E})$ is a *mirror path* for j, if $\tilde{V}$ contains j_3 and we call j_3 the *mirror node*. Similarly if $k = j_3$, then we say that a path $\tilde{G}(\tilde{V}, \tilde{E})$ is a *mirror path* for j, if $\tilde{V}$ contains j_1 and we call j_1 the *mirror node*.

Algorithm 1 constructs $G'(V', E')$ that satisfies all the properties of Claim 2. We next verify that Algorithm 1 is well-defined, that is all the steps can be carried out. Moreover we show that the algorithm generates a subgraph $G'(V', E')$ that satisfies the conditions of Claim 2.

Algorithm 1 Construction of $G'(V', E')$

Input: $G^*(U_1 \cup U_2, \mathcal{E})$.
Output: $G'(V', E')$ that satisfies all conditions of Claim 2.

1. If there exists a path $\tilde{G}(\tilde{V}, \tilde{E})$ in $G^*(U_1 \cup U_2, \mathcal{E})$ containing only full nodes with $v(\tilde{V}) = 1$, then set $G' := \tilde{G}$. STOP.
2. Tag all paths in $G^*(U_1 \cup U_2, \mathcal{E})$ as 'unmarked.'
3. Select a path $\tilde{G}(\tilde{V}, \tilde{E})$ from the set of 'unmarked' paths containing a partial node such that $v(\tilde{V}) = \frac{1}{2}$. Tag this path as 'marked.' Note that by Claim 6 and Claim 7, $\tilde{V}$ contains a unique partial node j^*.
4. Select a path from the list of 'unmarked' paths, such that it is a mirror path for j^*. Tag this path as 'marked.'
5. There are three cases:
   (a) The mirror path tagged as 'marked' in (4) contains a unique partial node and its value is $\frac{1}{2}$. GO TO Step 6
   (b) The mirror path tagged as 'marked' in (4) contains a unique partial node and its value is $-\frac{1}{2}$. GO TO Step 3.
   (c) The mirror path tagged as 'marked' in (4) contains two partial nodes (then its value is 0): One of the partial nodes corresponds to the mirror node. Set j^* to be the other partial node. GO TO Step 4.
6. Set $G'(V', E')$ to be disjoint union of the paths tagged as 'marked.' STOP.

Claim 9. *Algorithm 1 is well-defined.*

1. *At the beginning of Step (3), the total value of all marked paths is* 0.
2. *Let* $\hat{V} := \bigcup_{\tilde{G}(\tilde{V},\tilde{E}) \text{ is marked before Step (3)}} \tilde{V}$*. Then* $|\hat{V} \cap \{j_1, j_2\}| = |\hat{V} \cap \{j_3, j_4\}|$ *for all* $j \in P^2$ *and* $|\hat{V} \cap \{j_1\}| = |\hat{V} \cap \{j_3\}|$ *for all* $j \in P^1$*.*
3. *Step (3) is well-defined, that is as long as the algorithm does not terminate, Step (3) can be carried out.*
4. *At the end of Step (3), the total value of all marked paths is* $\frac{1}{2}$*.*
5. *Step (4) is well-defined, that is as long as the algorithm does not terminate, Step (4) can be carried out.*

Proof. We prove Claim 9 by induction on the iteration number (n) of the algorithm visiting Step (5). When $n = 0$:

1. At the beginning of Step (3) there are no 'marked' paths and therefore the total value of all marked paths is 0.
2. $\hat{V} = \emptyset$.
3. By Step (1), we know that there exists no path containing only full nodes with $v(\tilde{V}) = 1$. Moreover by Claim 4 we have $v(U_1 \cup U_2) \geq 2$. Since by Claim 5 all cycles have a value of 0, there must exist at least one path with partial nodes with positive value. Since this is only possible (Claim 6 and Claim 7) if there exists exactly one partial node in the path, we see that Step (3) is well-defined.
4. At Step (3) one path is marked which has a value of half.
5. Since one path is tagged as marked in Step (3), it contains exactly one partial node, $j^* \in P$. Suppose that $j^* \in P^2$ and $j^* = j_i^*$ for some $i \in \{1, \ldots, 4\}$. Then there exists paths (at least two) which contain the other three partial nodes corresponding to j^*. If $j^* \in P^1$ then there exists one path which contains the other partial node. Therefore this step is well-defined.

Now for any $n \in \mathbb{Z}_+$, assuming by the induction hypothesis that the result is true for $n' = 0, \ldots, n-1$:

1. Step (3) is arrived at via Step (5b). Let $n' < n$ be the last iteration when Step (3) is invoked. By the induction hypothesis the total value of all the marked paths at the end of Step (3) in iteration n' is $\frac{1}{2}$. From iterations $n'+1, \ldots, n-1$, the algorithm alternates between Step (4) and Step (5c). The total value of all the marked paths here is 0. Finally, the value of the last path tagged as marked in Step (4) is $-\frac{1}{2}$ (since the algorithm invokes Step (5b)). Hence, the total value of all the marked paths is 0 at the beginning of Step (3) in iteration n.
2. Let $n' < n$ be the last iteration when Step (3) is invoked. By the induction hypothesis $|\hat{V} \cap \{j_1, j_2\}| = |\hat{V} \cap \{j_3, j_4\}|$ for all $j \in P^2$ and $|\hat{V} \cap \{j_1\}| = |\hat{V} \cap \{j_3\}|$ for all $j \in P^1$ where $\hat{V} := \bigcup_{\tilde{G}(\tilde{V},\tilde{E}) \text{ is marked before Step (3) iteration n'}} \tilde{V}$. From iterations $n'+1, \ldots, n-1$, the algorithm alternates between Step (4) and Step (5c). Since in iteration $n-1$ at Step (4), we add one path that contains only the mirror node to j^* (the unique partial node from the previous iteration), we arrive at this result.
3. Proof same as that in the case where $n = 0$.
4. The total value of paths at the end of Step (3) = value of marked path + total value of previously marked path = $\frac{1}{2} + 0$.

5. Step (4) is invoked after either Step (3) or Step (5c). In case we arrive via Step (3), by the induction hypothesis $|\hat{V} \cap \{j_1, j_2\}| = |\hat{V} \cap \{j_3, j_4\}|$ for all $j \in P^2$ and $|\hat{V} \cap \{j_1\}| = |\hat{V} \cap \{j_3\}|$ for all $j \in P^1$ where $\hat{V} := \bigcup_{\tilde{G}(\tilde{V},\tilde{E}) \text{ is marked before Step (3) iteration n}'} \tilde{V}$. Moreover the path marked in step (3) contains exactly a unique partial node j^* then, there must exist an unmarked path containing a mirror node to j^*. In case of we arrive via Step (5c), again the proof is essentially the same by observing that at the start of Step (4), there is a unique partial node j^* that is not paired with a mirror partial node.

□

Claim 10. *Algorithm 1 terminates in finite time.*

Proof. This is true since there are a finite number of edges and at each iteration of the algorithm at least one unmarked path is tagged as marked. □

Claim 11. *Algorithm 1 generates a subgraph $G'(V', E')$ that satisfies the properties of Claim 2.*

Proof. First observe that since the output $G'(V', E')$ of the algorithm is a disjoint union of paths, there exists no edge between V' and $(U_1 \cup U_2) \setminus V'$ in $\mathcal{E}$, so property 1 is satisfied.

By Claim 9, 2. we have $|\hat{V} \cap \{j_1, j_2\}| = |\hat{V} \cap \{j_3, j_4\}|$ for all $j \in P^2$ and $|\hat{V} \cap \{j_1\}| = |\hat{V} \cap \{j_3\}|$ for all $j \in P^1$ where

$$\hat{V} := \bigcup_{\tilde{G}(\tilde{V},\tilde{E}) \text{ is marked before Step (3)}} \tilde{V}.$$

Therefore, it is easily verified that in the last iteration before termination, a path with a unique partial node, which is a mirror node to j^*, is marked in Step (4). This is because before termination we arrive at Step (5a) implying that the value of the path marked in Step (4) is $\frac{1}{2}$. Hence Claim 6 and Claim 7 imply that there is a unique partial node in this path. Thus, $|V' \cap \{j_1, j_2\}| = |V' \cap \{j_3, j_4\}|$ for all $j \in P^2$ and $|V' \cap \{j_1\}| = |V' \cap \{j_3\}|$ for all $j \in P^1$, so property 2 is satisfied.

Finally, since $v(V') = 1$ and $|V' \cap \{j_1, j_2\}| = |V' \cap \{j_3, j_4\}|$ for all $j \in P^2$ and $|V' \cap \{j_1\}| = |V' \cap \{j_3\}|$ for all $j \in P^1$ we have

$$\sum_{j \in V' \cap W_1} v_j + \sum_{j \in V' \cap W_2} v_j = 1.$$

As a result, $|V' \cap W_1| = |V' \cap W_2| + 1$, so property 3 is satisfied. □

We showed that the set of solutions to TPMC satisfies the edge property. Theorem 1 then follows from Proposition 4.

Finally we ask the natural question: Does the edge property hold for TPMC when there exist demands that are greater than 2? The next example illustrates that the edge property can fail to hold even if $d_j > 2$ for only one $j \in V_2$.

Example 2. *Consider an instance of TPMC where $G(V_1 \cup V_2, E)$ is a bipartite graph with $V_1 = \{1, 2, \ldots, 6\}$, $V_2 = \{1, 2, 3, 4\}$, $E = \{(1, 1), (2, 2), (3, 3), (4, 1), (4, 4), (5, 2), (5, 4), (6, 3), (6, 4)\}$, $s_i = 1$, $i \in V_1$, $d_j = 2$, $j = \{1, 2, 3\}$, $d_4 = 3$. For $k = 2$ we obtain a non-integer extreme point of* $\text{conv}(T) \cap \{(x, z) \in \mathbb{R}^p_+ \times [0, 1]^n | \sum_{j=1}^n z_j \leq k\}$*, given by $x_{11} = x_{22} = x_{33} = x_{41} = x_{44} = x_{52} = x_{54} = x_{63} = x_{64} = z_1 = z_2 = z_3 = z_4 = \frac{1}{2}$. Therefore, $T^k \neq \text{conv}(T) \cap \{(x, z) \in \mathbb{R}^p_+ \times [0, 1]^n | \sum_{j=1}^n z_j \leq k\}$ in this example. Next we show how the conflict graph construction fails for this example. In fact, it can be shown that the edge property is not satisfied in this example by using an alternative characterization defined in [2]. Let $w_{11} = w_{22} = w_{33} = w_{41} = w_{44} = w_{52} = w_{54} = w_{63} = w_{64} = 1$ and $r_j = 3$, $j = \{1, 2, 3\}$ and $r_4 = 6$. For $k = 0$ the problem is infeasible. For $k = 1$, an optimal solution is $x_{11} = x_{22} = x_{33} = x_{41} = x_{52} = x_{63} = z_4 = 1$ and all other variables are zero, with an objective function value 12. For $k = 3$, an optimal solution is $x_{44} = x_{54} = x_{64} = z_1 = z_2 = z_3 = 1$ and all other variables are zero, with an objective function value 12. We show that Algorithm 1 fails to find a subgraph $G'(V', E')$ of $G^*(U_1 \cup U_2, \mathcal{E})$ that satisfies the properties given in Claim 2 for this example. We use two feasible solutions, namely solution for $k = 1$ and $k = 3$ to build the bipartite graph given in Figure 1.*

Note that $I_1 = \{1,4\}$, $I_2 = \{2,5\}$, $I_3 = \{3,6\}$ and $I_4 = \{4,5,6\}$. In Step (1) of Algorithm 1 we find a path with $v(\tilde{V}) = 1$ which is $1-4-2$ then the algorithm stops. We have $V' = \{1,4,2\}$ and $(U_1 \cup U_2) \setminus V' = \{3\}$. However, property 1 does not hold since there exists an edge between 3 and 4 but $3 \in (U_1 \cup U_2) \setminus V'$ and $4 \in V'$. Hence, Algorithm 1 fails.

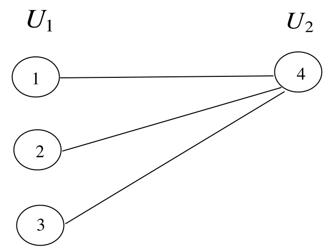

Figure 1: Bipartite Graph $G^*(U_1 \cup U_2, \mathcal{E})$ for Example 2

4 Valid Inequalities

In this section we give valid inequalities for TPMC and study their strength. First, observe that the variable upper bound inequalities (VUB) for $(i,j) \in E$

$$x_{ij} \leq \min\{s_i, d_j\}(1 - z_j) \tag{14}$$

are valid for X.

Proposition 5. *Let $I \subseteq V_1$, $J \subseteq V_2$ such that $d(J) \geq s(V_1 \setminus I)$. The inequality*

$$\sum_{i \in I, j \in J:(i,j) \in E} x_{ij} + \sum_{j \in J} \left(\min\left\{d(J) - s(V_1 \setminus I), d_j\right\}\right) z_j \geq d(J) - s(V_1 \setminus I) \tag{15}$$

is valid for X.

Proof. Given a feasible solution (x,z) we consider two cases.

1. If $z_{j'} = 1$ for some $j' \in J$ such that $\min\{d(J) - s(V_1 \setminus I), d_{j'}\} = d(J) - s(V_1 \setminus I)$, then the feasible solution satisfies inequality (15) because we have

$$\begin{aligned} & \sum_{i \in I, j \in J:(i,j) \in E} x_{ij} + \sum_{j \in J} \left(\min\left\{d(J) - s(V_1 \setminus I), d_j\right\}\right) z_j \\ = & \sum_{i \in I, j \in J \setminus \{j'\}:(i,j) \in E} x_{ij} + \sum_{j \in J \setminus \{j'\}} \left(\min\left\{d(J) - s(V_1 \setminus I), d_j\right\}\right) z_j + d(J) - s(V_1 \setminus I) \\ \geq & \ d(J) - s(V_1 \setminus I) \end{aligned}$$

   where the last inequality holds because $\min\{d(J) - s(V_1 \setminus I), d_j\} \geq 0$ for all $j \in J$, and all x and z variables are non-negative.

2. If $z_j = 0$ for all $j \in J$ satisfying $\min\{d(J) - s(V_1 \setminus I), d_j\} = d(J) - s(V_1 \setminus I)$, then $\sum_{j \in J} \left(\min\left\{d(J) - s(V_1 \setminus I), d_j\right\}\right) z_j = \sum_{j \in J} d_j z_j$. Moreover, observe that $\sum_{i \in I, j \in J:(i,j) \in E} x_{ij} + s(V_1 \setminus$

I) is at least as large as the total flow sent to the demand nodes in J in the solution (x, z), i.e., $\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} + s(V_1 \setminus I) \geq \sum_{j\in J} d_j(1 - z_j)$. Therefore we have

$$\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} + \sum_{j\in J} \left(\min\left\{d(J) - s(V_1 \setminus I), d_j\right\}\right) z_j + s(V_1 \setminus I) \geq \sum_{j\in J} d_j z_j + \sum_{j\in J} d_j(1 - z_j) = d(J),$$

so inequality (15) is valid.

□

Next, we give valid inequalities for general mixed-integer sets that are substructures of TPMC.

4.1 A Coefficient Update Scheme for Mixed-Integer Covers

Consider the mixed integer cover set $\mathcal{S}_1$ defined by

$$t + \sum_{j\in J} \beta_j z_j \geq \beta_0 \tag{16}$$

$$t \geq 0 \tag{17}$$

$$z_j \in \{0, 1\} \quad \forall j \in J, \tag{18}$$

for given $\beta_j \geq 0$ for all $j \in J$ and $\beta_0 \geq 0$. We assume that $\beta_j \leq \beta_0$ for all $j \in J$ without loss of generality. Let $\mathcal{T}_1 = conv(\mathcal{S}_1)$. We refer to inequalities in the form of (16) as type-I base inequalities. Note that inequalities (15) for TPMC are in the form of (16) since we can replace $\sum_{i\in I, j\in J:(i,j)\in E} x_{ij}$ by t and $t \geq 0$. Therefore, (16)-(18) is a relaxation of TPMC.

Proposition 6. *Given a type-I base inequality* (16) *valid for a mixed-integer program (MIP) with* (17)-(18)*, let* $\tilde{J} := \{j_1, j_2, \ldots, j_p\} \subseteq J$ *be a minimal cover, i.e.,* $\sum_{j\in\tilde{J}} \beta_j > \beta_0$ *and* $\sum_{j\in\tilde{J}\setminus\{j_k\}} \beta_j \leq \beta_0$ *for all* $k \in \{1, \ldots, p\}$*. Let* $\beta_{j_p} \geq \beta_{j_k}$ *for all* $k \in \{1, \ldots, p\}$*. Let* $J^* := \tilde{J} \cup \{j \in J : \beta_j \geq \beta_{j_p}\}$, $\beta = \sum_{j\in\tilde{J}} \beta_j - \beta_0$ *and* $\beta_0^{'} := \beta_0 - (p-1)\beta$*. Then,*

$$t + \sum_{j\in J^*} \min\left\{(\beta_j - \beta), \beta_0^{'}\right\} z_j + \sum_{j\in J\setminus J^*} \min\left\{\beta_0^{'}, \beta_j\right\} z_j \geq \beta_0^{'} \tag{19}$$

is a valid inequality for $\mathcal{S}_1$.

Proof. We first claim that $\beta_j \geq \beta$ for all $j \in J^*$. Suppose, without loss of generality, that $\beta_{j_1} \leq \beta_{j_2} \leq \cdots \leq \beta_{j_p}$, and recall that $\beta_j \geq \beta_{j_p}$ for all $j \in J^* \setminus \tilde{J}$. Assume by contradiction that $\beta_{j_1} < \beta$ or equivalently $\beta_{j_1} - (\sum_{k=1}^{p} \beta_{j_k} - \beta_0) < 0$. This is a contradiction to the minimality of the cover $\tilde{J}$.

Next we claim that $\beta_0^{'} \geq 0$: By the previous claim we have $\beta \leq \beta_{j_k}$ for $k = 1, \ldots, p$. Therefore, we obtain

$$\beta_0^{'} = \beta_0 - (p-1)\beta \geq \beta_0 - \sum_{k=1}^{p-1} \beta_{j_k} \geq 0,$$

where the last inequality follows from the fact that $\tilde{J}$ is a minimal cover.

Given a feasible solution (x, z), let $J_1 = \{j \in J : z_j = 1\}$ and $J_1^* = \{j \in J^* : z_j = 1\}$. Consider the following cases:

1. Suppose that there exists $j' \in J_1^*$ such that $\min\left\{\beta_0^{'}, \beta_{j'} - \beta\right\} = \beta_0^{'}$. Then,

$$\begin{aligned} & t + \sum_{j\in J^*} \min\left\{(\beta_j - \beta), \beta_0^{'}\right\} z_j + \sum_{j\in J\setminus J^*} \min\left\{\beta_0^{'}, \beta_j\right\} z_j \\ \geq \quad & t + \sum_{j\in J^*\setminus\{j'\}} \min\left\{(\beta_j - \beta), \beta_0^{'}\right\} z_j + \sum_{j\in J\setminus J^*} \min\left\{\beta_0^{'}, \beta_j\right\} z_j + \beta_0^{'} \geq \beta_0^{'}, \end{aligned}$$

where the last inequality follows from the fact that all variables are non-negative, $\beta_j \geq \beta$ for all $j \in J^*$ and $\beta_0^{'} \geq 0$. The proof for the case where there exists $j' \in J_1 \setminus J_1^*$ such that $\min\left\{\beta_0^{'}, \beta_{j'}\right\} = \beta_0^{'}$ follows similarly.

2. Suppose that for all $j \in J_1^*$, we have $\min\left\{\beta_0^{'}, \beta_j - \beta\right\} = \beta_j - \beta$ and for all $j \in (J_1 \setminus J_1^*)$ we have $\min\left\{\beta_0^{'}, \beta_j\right\} = \beta_j$. There are two cases to consider:

   (a) Suppose that $|J_1^*| \leq p-1$. In this case,

$$\begin{aligned} t + \sum_{j \in J^*} (\beta_j - \beta) z_j + \sum_{j \in J \setminus J^*} \beta_j z_j &= t + \sum_{j \in J_1^*} (\beta_j - \beta) + \sum_{j \in J_1 \setminus J_1^*} \beta_j \\ &= t + \sum_{j \in J_1^*} \beta_j + \sum_{j \in J_1 \setminus J_1^*} \beta_j - |J_1^*|\beta \\ &\geq \beta_0 - |J_1^*|\beta \geq \beta_0 - (p-1)\beta, \end{aligned}$$

   where the first inequality follows because inequality (16) is valid and the second inequality follows because of our assumption $|J_1^*| \leq p-1$.

   (b) Suppose that $|J_1^*| \geq p$. In this case,

$$\begin{aligned} t + \sum_{j \in J^*} (\beta_j - \beta) z_j + \sum_{j \in J \setminus J^*} \beta_j z_j &= t + \sum_{j \in J_1^*} (\beta_j - \beta) + \sum_{j \in J_1 \setminus J_1^*} \beta_j \\ &\geq \sum_{j \in J_1^*} (\beta_j - \beta) \geq \sum_{k=1}^{p} (\beta_{j_k} - \beta) \\ &= \sum_{k=1}^{p} \beta_{j_k} - p\beta = \beta_0 - (p-1)\beta. \end{aligned}$$

   The second inequality holds since $|J_1^*| \geq p$ and since $\beta \leq \beta_{j_1} \leq \beta_{j_2} \leq \cdots \leq \beta_{j_p} \leq \beta_j$ for $j \in J^* \setminus \tilde{J}$.

□

Given type-I base inequalities (16) valid for any MIP with $t \geq 0$, and $z_j \in \{0,1\}$, $j \in J$, we can derive a new class of valid inequalities (19). Similarly, inequality (19) is in the form of (16), so this process can be repeated by letting the valid inequality (19) be the type-I base inequality to derive other classes of valid inequalities.

Inequality (19) is related to the weight inequalities of Weismantel [26] for the 0/1 knapsack polytope. Note that inequality (19) is valid when J^* is replaced with $\tilde{J}$. After complementing the z variables, we can show that inequality (19) where J^* is replaced with $\tilde{J}$ and the condition $\beta_j \leq \beta_0'$ for all $j \in J \setminus \tilde{J}$ is satisfied is equivalent to the weight inequalities for the 0/1 knapsack polytope (ignoring the continuous term t). However, if $J^* \supsetneq \tilde{J}$ then inequality (19) with J^* dominates inequality (19) with $\tilde{J}$. Additionally if $J^* = \tilde{J}$ and there exists $j \in J \setminus \tilde{J}$ such that $\beta_j > \beta_0'$ then inequality (19) dominates the corresponding weight inequality. Weismantel also proposes weight-reduction and extended weight inequalities for the 0/1 knapsack polytope. In Example 3 we show that weight-reduction inequalities and inequalities (19) are not equivalent. We also show that the extended weight inequality is dominated by the inequalities found using Proposition 6 for this example.

Example 3. *Consider the type-I base inequality*

$$3z_1 + 4z_2 + 5z_3 + 6z_4 \geq 6, \tag{20}$$

for $t = 0$. Next, we give examples of inequality (19) for different choices of $\tilde{J}$.

1. *Let* $\tilde{J} = \{1, 4\}$. *Then* $J^* = \tilde{J}$ *and* $\beta = (3 + 6) - 6 = 3$. *Then corresponding inequality* (19) *defined by this choice of* $\tilde{J}$ *is* $\min\{4, 3\}z_2 + \min\{5, 3\}z_3 + 3z_4 \geq 3$, *or*
$$z_2 + z_3 + z_4 \geq 1. \tag{21}$$
2. *Let* $\tilde{J} = \{2, 4\}$. *Then* $J^* = \tilde{J}$ *and* $\beta = (4 + 6) - 6 = 4$ *Then corresponding inequality* (19) *defined by this choice of* $\tilde{J}$ *is* $\min\{3, 2\}z_1 + \min\{5, 2\}z_3 + 2z_4 \geq 2$, *or*
$$z_1 + z_3 + z_4 \geq 1. \tag{22}$$
3. *Let* $\tilde{J} = \{3, 4\}$. *Then* $J^* = \tilde{J}$ *and* $\beta = (5 + 6) - 6 = 5$. *Then corresponding inequality* (19) *defined by this choice of* $\tilde{J}$ *is* $\min\{3, 1\}z_1 + \min\{4, 1\}z_2 + z_4 \geq 1$, *or*
$$z_1 + z_2 + z_4 \geq 1. \tag{23}$$

Inequalities (21)-(23) *dominate the corresponding weight inequalities since for all the inequalities there exists* $j \in J \setminus \tilde{J}$ *such that* $\beta_j > \beta'_0$. *Inequality* (23) *cannot be obtained by weight-reduction inequalities in [26]. On the other hand, the weight-reduction inequality*
$$3z_1 + z_3 + 2z_4 \geq 2,$$
cannot be obtained using Proposition 6. For this example, the only valid extended weight inequality is
$$z_1 + z_2 + 2z_3 + 2z_4 \geq 2,$$
which is dominated by the inequalities (21) *and* (22).

4.2 A Coefficient Update Scheme for Mixed-Integer Knapsacks with Variable Upper Bounds

Next, we consider another substructure of TPMC consisting of a mixed integer knapsack and variable upper bound constraints. We define set $\mathcal{S}_2$ as follows:
$$\sum_{j \in J} t_j + \sum_{j \in J} \alpha_j z_j \leq \alpha_0 \tag{24}$$
$$t_j \leq d_j(1 - z_j) \ \forall j \in J \tag{25}$$
$$z \in \{0, 1\}^{|J|}, t_j \in \mathbb{R}_+^{|J|}, \tag{26}$$
for given $\alpha_j \geq 0$ for all $j \in J$ and $\alpha_0 \geq 0$.

Let $\mathcal{T}_2 = conv(\mathcal{S}_2)$. We refer to inequalities in the form of (24) as type-II base inequalities. If we replace $t_j := \sum_{i \in I:(i,j) \in E} x_{ij}$, $I \subseteq V_1$ then the sum of relaxation of the supply constraints (1c) over I is in the form of (24) (with $\alpha_j = 0$ for all $j \in J$) for TPMC, and (25) is a relaxation of the demand constraints (1b). In this case, we observe that TPMC contains the fixed-charge network flow substructure. Therefore, the lifted flow cover and pack inequalities [4, 5, 13, 21, 24], and submodular inequalities [1, 27] are all valid for TPMC. Furthermore, these inequalities and the blossom inequalities (3) are in the form of (24). Next we describe valid inequalities for the set $\mathcal{S}_2$.

Proposition 7. *Given the mixed-integer set* $\mathcal{S}_2$, *let* $\tilde{J} = \{j_1, j_2, \ldots, j_u\} \subseteq J$ *such that* $d_{j_1} - \alpha_{j_1} \geq d_{j_2} - \alpha_{j_2} \geq \cdots \geq d_{j_u} - \alpha_{j_u}$ *and there exists* $m = \max\{l \in \{0, \ldots, u-1\} : \sum_{k=1}^{l} d_{j_k} + \sum_{k=l+1}^{u} \alpha_{j_k} < \alpha_0 - \sum_{j \in J \setminus \tilde{J}} \max\{d_j, \alpha_j\}\}$. *Let* $M = \{j_1, j_2, \ldots, j_m\}$ ($M = \emptyset$ *if* $m = 0$) *and* $\alpha = \alpha_0 - \sum_{j \in J \setminus \tilde{J}} \max\{d_j, \alpha_j\} - d(M) - \alpha(\tilde{J} \setminus M)$. *Then the inequality given by*
$$\sum_{j \in J} t_j + \sum_{j \in \tilde{J}} (\alpha_j + \alpha) z_j + \sum_{j \in J \setminus \tilde{J}} \alpha_j z_j \leq \alpha_0 + (u - m - 1)\alpha \tag{27}$$
is valid for $\mathcal{S}_2$.

Proof. Given a feasible solution (t, z) to $\mathcal{S}_2$, let $\tilde{J}_1 = \{j \in \tilde{J} : z_j = 1\}$ and $\tilde{J}_0 = \{j \in \tilde{J} : z_j = 0\}$. Consider the following cases:

1. Suppose that $u - m - 1 \geq |\tilde{J}_1|$. In this case,

$$\begin{aligned}\sum_{j\in J} t_j + \sum_{j\in\tilde{J}}(\alpha_j + \alpha)z_j + \sum_{j\in J\setminus\tilde{J}} \alpha_j z_j &= \sum_{j\in J\setminus\tilde{J}_1} t_j + \sum_{j\in\tilde{J}_1}\alpha_j + \sum_{j\in J\setminus\tilde{J}}\alpha_j z_j + |\tilde{J}_1|\alpha \\ &\leq \alpha_0 + |\tilde{J}_1|\alpha \\ &\leq \alpha_0 + (u - m - 1)\alpha.\end{aligned}$$

2. Suppose that $u - m \leq |\tilde{J}_1|$, or equivalently $m \geq u - |\tilde{J}_1| = |\tilde{J}_0|$. Then,

$$\begin{aligned} & \sum_{j\in J} t_j + \sum_{j\in\tilde{J}}(\alpha_j + \alpha)z_j + \sum_{j\in J\setminus\tilde{J}}\alpha_j z_j \\ = \; & \sum_{j\in J\setminus\tilde{J}_1} t_j + \sum_{j\in\tilde{J}_1}\alpha_j + \sum_{j\in J\setminus\tilde{J}}\alpha_j z_j + |\tilde{J}_1|\alpha \\ \leq \; & \sum_{j\in J\setminus\tilde{J}} \max\{d_j, \alpha_j\} + d(\tilde{J}_0) + \sum_{j\in\tilde{J}_1}\alpha_j + |\tilde{J}_1|\alpha \\ = \; & \alpha_0 - \alpha - d(M) - \alpha(\tilde{J}\setminus M) + d(\tilde{J}_0) + \alpha(\tilde{J}_1) + |\tilde{J}_1|\alpha \\ = \; & \alpha_0 - \Big[(d(M) - \alpha(M)) - (d(\tilde{J}_0) - \alpha(\tilde{J}_0))\Big] + (|\tilde{J}_1| - 1)\alpha,\end{aligned}$$

   where the first inequality holds since

$$\begin{aligned} & \sum_{j\in J\setminus\tilde{J}_1} t_j + \sum_{j\in J\setminus\tilde{J}}\alpha_j z_j \\ = \; & \left(\sum_{j\in J\setminus\tilde{J}} t_j + \sum_{j\in J\setminus\tilde{J}}\alpha_j z_j\right) + \sum_{j\in\tilde{J}_0} t_j \leq \sum_{j\in J\setminus\tilde{J}}\max\{d_j, \alpha_j\} + d(\tilde{J}_0),\end{aligned}$$

   and the second equality holds because $\sum_{j\in J\setminus\tilde{J}}\max\{d_j, \alpha_j\} = \alpha_0 - \alpha - d(M) - \alpha(\tilde{J}\setminus M)$. Furthermore, due to the choice of index m, $0 < \alpha \leq d_{j_{m+1}} - \alpha_{j_{m+1}}$. Thus, we have

$$(m - |\tilde{J}_0|)\alpha \leq (m - |\tilde{J}_0|)(d_{j_{m+1}} - \alpha_{j_{m+1}}) \leq \sum_{k=|\tilde{J}_0|+1}^{m}(d_{j_k} - \alpha_{j_k}).$$

   Moreover, $-\Big[(d(M) - \alpha(M)) - (d(\tilde{J}_0) - \alpha(\tilde{J}_0))\Big] \leq -\Big[\sum_{k=|\tilde{J}_0|+1}^{m}(d_{j_k} - \alpha_{j_k})\Big]$. Thus we have

$$\begin{aligned}&\alpha_0 + (|\tilde{J}_1| - 1)\alpha - \Big[(d(M) - \alpha(M)) - (d(\tilde{J}_0) - \alpha(\tilde{J}_0))\Big] \\ &\quad \leq \alpha_0 + (|\tilde{J}_1| - 1)\alpha - (m - |\tilde{J}_0|)\alpha = \alpha_0 + (u - m - 1)\alpha,\end{aligned}$$

   completing the proof.

□

As in Proposition 6, Proposition 7 can be applied recursively to obtain new nontrivial valid inequalities for TPMC.

Next we give an example illustrating the valid inequalities introduced in this section.

Example 4. *Consider an instance of TPMC with a complete bipartite graph,* $V_1 = \{1, 2\}$*,* $V_2 = \{1, 2, 3, 4\}$*,* $s = (31, 20)$ *and* $d = (11, 19, 8, 13)$*. A valid inequality for* X *for this instance is*

$$x_{21} + x_{22} + x_{23} + x_{24} + 11z_1 + 19z_2 + 8z_3 + 13z_4 \geq 20, \tag{28}$$

which corresponds to inequality (15) *with* $I = \{2\}$ *and* $J = \{1, 2, 3, 4\}$*. Note that* $d(J) - s(V_1 \setminus I) = 20 \geq d_j$ *for all* $j \in J$*.*

Using (28) *as the type-I base inequality, we apply the coefficient update in Proposition 6 and let* $\tilde{J} = \{1, 4\}$*,* $J^* = \{1, 2, 4\}$*. Then* $\beta_1 + \beta_4 = 11 + 13 = 24$ *and* $(\beta_1 + \beta_4) - \beta_0 = 24 - 20 = 4 = \beta$*, and we obtain the corresponding inequality* (19)

$$x_{21} + x_{22} + x_{23} + x_{24} + 7z_1 + 15z_2 + 8z_3 + 9z_4 \geq 16, \tag{29}$$

which is valid for X*.*

Using (29) *as the type-I base inequality, we apply the coefficient update in Proposition 6 and let* $\tilde{J} = \{3, 4\}$*,* $J^* = \{2, 3, 4\}$*. Then* $\beta_3 + \beta_4 = 8 + 9 = 17$ *and* $(\beta_3 + \beta_4) - \beta_0 = 17 - 16 = 1 = \beta$ *and again we obtain the corresponding inequality* (19)

$$x_{21} + x_{22} + x_{23} + x_{24} + 7z_1 + 14z_2 + 7z_3 + 8z_4 \geq 15, \tag{30}$$

which is valid for X*.*

Now, consider the supply constraint (1c) *for supplier 2*

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 20. \tag{31}$$

Then using (31) *as the type-II base inequality with* $I = \{2\}$ *and* $J = \{1, 2, 3, 4\}$*, we apply the coefficient update in Proposition 7, where we let* $\tilde{J} = \{2, 4\}$*. Then* $\alpha_0 - \sum_{j \in J \setminus \tilde{J}} \max\{d_j, \alpha_j\} = \alpha_0 - (d_1 + d_3) = 20 - (11 + 8) = 1$*. However, all demand values in set* $\tilde{J}$ *are greater than 1 so* $m = 0$ *and* $\alpha = \alpha_0 - (d_1 + d_3) - \alpha_2 - \alpha_4 = 20 - (11 + 8) - 0 - 0 = 1$*. Then we obtain the corresponding inequality* (27)

$$x_{21} + x_{22} + x_{23} + x_{24} + z_2 + z_4 \leq 21, \tag{32}$$

which is valid for X*.*

4.3 Strength of the Proposed Inequalities

Next we give several facet conditions for inequalities (15). Let V_2' be the set of markets. Observe that if $s(V_1) < d_j$ for some $j \in V_2'$ then the demand of market j can never be met in any feasible solution to TPMC. Therefore, we can set $z_j = 1$ for such markets and let $V_2 = \{j \in V_2' : s(V_1) \geq d_j\}$. In other words, we remove the markets that can never be satisfied from the given set of markets. Therefore, throughout we make the assumption that

$$s(V_1) \geq \max_{j \in V_2} d_j. \tag{33}$$

Let $J^< = \{j \in J : d_j < d(J) - s(V_1 \setminus I)\}$.

Theorem 2. *Inequality* (15) *defines a nontrivial facet of conv(X) only if the following conditions hold:*

1. $d(J) > s(V_1 \setminus I)$*.*
2. *There exists* $j \in J$ *such that* $d_j > d(J) - s(V_1 \setminus I)$*.*
3. $s(V_1) \geq d(J) - \max_{j \in J}\{d_j\} + \max_{j \in V_2 \setminus J}\{d_j\}$*.*
4. *If* $s(V_1) < d(J)$ *and* $I \neq \emptyset$*, then* $|J^<| \geq 2$ *and the sum of the smallest two demands in set* $J^<$ *is not greater than* $d(J) - s(V_1 \setminus I)$*.*

5. $I \neq V_1$.

6. If $|J| = 1$*, then* $|V_1 \setminus I| = 1$.

7. $s(V_1) \geq d(J \setminus J^<) + \max_{j \in J^<}\{d_j\}$.

8. If $s(V_1) = d(J)$ *and* $d_j \geq d(J) - s(V_1 \setminus I)$ *for all* $j \in J$ *then* $|I| \leq 1$.

In addition, if the following conditions hold, then (15) *is a facet of* $conv(TPMC)$*:*

9. $s(V_1) > d(J) - \max_{j \in J}\{d_j\} + \max_{j \in V_2 \setminus J}\{d_j\}$.

10. There exists $\hat{J} \subsetneq J^<$ *such that* $d(J \setminus \hat{J}) > s(V_1 \setminus I)$ *and* $d(J \setminus \hat{J}') > s(V_1 \setminus I)$ *where* $\hat{J}' = \hat{J} \cup \{k_1\}$*, for all* $k_1 \in J^< \setminus \hat{J}$*.*

11. $s(V_1) > \max_{j \in V_2} d_j$.

Proof. Necessity.

1. Assume that $d(J) - s(V_1 \setminus I) \leq 0$.

   From validity of inequality (15) we have $d(J) - s(V_1 \setminus I) \geq 0$ and combined with the assumption we get $d(J) - s(V_1 \setminus I) = 0$. The resulting inequality is implied by the nonnegativity of x_{ij} and z_j for $i \in I$, $j \in J$, $(i,j) \in E$.

2. Assume that $d_j \leq d(J) - s(V_1 \setminus I)$ for all $j \in J$. Under this assumption inequality (15) reduces to

$$\sum_{i \in I, j \in J:(i,j) \in E} x_{ij} + \sum_{j \in J} d_j z_j \geq d(J) - s(V_1 \setminus I). \tag{34}$$

   We add all the demand constraints (1b) in J,

$$\sum_{i \in V_1, j \in J:(i,j) \in E} x_{ij} + \sum_{j \in J} d_j z_j = d(J). \tag{35}$$

   When we subtract (35) from (34) we obtain

$$\sum_{i \in V_1 \setminus I, j \in J:(i,j) \in E} x_{ij} \leq s(V_1 \setminus I). \tag{36}$$

   If $J \subsetneq V_2$ then inequality (36) is weaker than all the supply inequalities (1c) in $V_1 \setminus I$ combined, because $x_{ij} \geq 0$ for all $i \in I, j \in V_2 \setminus J$, $(i,j) \in E$. If $J = V_2$ then inequality (36) is dominated by the supply inequalities $\sum_{j \in V_2:(i,j) \in E} x_{ij} \leq s_i$ for all $i \in V_1 \setminus I$ unless $|V_1 \setminus I| = 1$. However, when $J = V_2$, $|V_1 \setminus I| = 1$ and $d_j \leq d(J) - s(V_1 \setminus I)$ for all $j \in J$ inequality (15) reduces to a trivial facet.

3. Assume that $s(V_1) < d(J) - \max_{j \in J}\{d_j\} + \max_{j \in V_2 \setminus J}\{d_j\}$. Because we have showed that there exists $j \in J$ such that $d_j > d(J) - s(V_1 \setminus I)$ we can conclude that $s(V_1 \setminus I) > d(J) - d_j \geq d(J) - \max_{j \in J}\{d_j\}$. Note that we have to have $s(I) < \max_{j \in V_2 \setminus J}\{d_j\}$ for $s(V_1) < d(J) - \max_{j \in J}\{d_j\} + \max_{j \in V_2 \setminus J}\{d_j\}$ to hold because if $s(I) \geq \max_{j \in V_2 \setminus J}\{d_j\}$, then $s(V_1) = s(V_1 \setminus I) + s(I) > d(J) - \max_{j \in J}\{d_j\} + \max_{j \in V_2 \setminus J}\{d_j\}$ which would contradict our assumption. Let $r^* = \arg\max_{j \in V_2 \setminus J}\{d_j\}$. Because (15) is a non-trivial facet, it is different from $z_{r^*} \leq 1$ and there exists solutions on the face defined by (15) with $z_{r^*} = 0$. Note that $\sum_{j \in J \setminus J^<} z_j \leq 1$ for any point to be on the face defined by inequality (15). We consider the following cases:

(a) $\sum_{j\in J\setminus J^<} z_j = 1 = z_l$ for some $l \in J \setminus J^<$.
In this case, left-hand side of inequality (15) reduces to

$$\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} + \sum_{j\in J\setminus\{l\}} (\min\{d(J) - s(V_1 \setminus I), d_j\})\, z_j + d(J) - s(V_1 \setminus I)$$

since $l \in J \setminus J^<$, $\min\{d(J) - s(V_1 \setminus I), d_l\} = d(J) - s(V_1 \setminus I)$. Thus to satisfy inequality (15) at equality we must have $\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} = 0$, $z_j = 0$ for all $j \in J \setminus \{l\}$ and

$$\sum_{i\in V_1\setminus I, j\in J\setminus\{l\}:(i,j)\in E} x_{ij} = d(J \setminus \{l\}) \le s(V_1 \setminus I) - (d_{r^*} - s(I)) = s(V_1) - d_{r*} \tag{37}$$

where $d_{r^*} - s(I)$ is the amount of demand of market r^* that cannot be satisfied by the suppliers in set I. We obtain a contradiction because (37) implies that $s(V_1) \ge d(J) - d_l + d_{r*} \ge d(J) - \max_{j\in J}\{d_j\} + \max_{j\in V_2\setminus J}\{d_j\}$, since $d_l \le \max_{j\in J}\{d_j\}$.

(b) $\sum_{j\in J\setminus J^<} z_j = 0$.
Let $\hat{J} = \{j \in J^< : z_j = 1\}$. Then a point on the face defined by inequality (15) satisfies

$$\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} + \sum_{j\in\hat{J}} d_j = d(J) - s(V_1 \setminus I).$$

This implies that $\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I) \ge 0$ because otherwise we would not have a feasible solution. Furthermore, $\sum_{i\in V_1\setminus I, j\in J\setminus\hat{J}:(i,j)\in E} x_{ij} = s(V_1 \setminus I)$. Combining the results we observe that because $s(I) < d_{r^*}$ we cannot send all the demand of d_{r^*} from $s(I)$ so some of the supply from $s(V_1 \setminus I)$ should be sent to d_{r^*} but all the supply $s(V_1 \setminus I)$ is sent to markets in $J \setminus \hat{J}$. We reach a contradiction, we cannot have $z_{r^*} = 0$.

4. Suppose that $s(V_1) < d(J)$ and $I \neq \emptyset$, then not all demand in set J can be met, hence $\sum_{j\in J} z_j \ge 1$. Consider the following cases:

(a) $J^< = \emptyset$. Then inequality $\sum_{j\in J}(d(J) - s(V_1 \setminus I))z_j \ge d(J) - s(V_1 \setminus I)$ dominates inequality (15) since inequality (15) has the additional term $\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} \ge 0$.

(b) $|J^<| = 1$. Let $J^< = \{k\}$. We apply the coefficient update in Proposition 6 using inequality (15) as the type-I base inequality. Let $\tilde{J} = \{j, k\}$ where $j \in J \setminus \{k\}$. Therefore, $\beta = \beta_j + d_k - \beta_0 = d(J) - s(V_1 \setminus I) + d_k - (d(J) - s(V_1 \setminus I)) = d_k$ and the corresponding inequality (19) is

$$\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} + \sum_{j\in J\setminus\{k\}} (d(J) - s(V_1 \setminus I) - d_k)z_j + (d_k - d_k)z_k \ge d(J) - s(V_1 \setminus I) - d_k. \tag{38}$$

If we add $\sum_{j\in J} d_k z_j \ge d_k$ to inequality (38) we obtain (15). Hence, (15) cannot be a facet.

(c) $|J^<| \ge 2$ and $d_{j_1} + d_{j_2} > d(J) - s(V_1 \setminus I)$ where d_{j_1} and d_{j_2} are the two smallest demands in set $J^<$. We use the coefficient update in Proposition 6 using inequality (15) as the type-I base inequality. Let $\tilde{J} = \{j_1, j_2\}$. Therefore, $\beta = d_{j_1} + d_{j_2} - (d(J) - s(V_1 \setminus I))$ and the corresponding inequality (19) is

$$\begin{aligned} \textstyle\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} &+ \textstyle\sum_{j\in J\setminus J^<} (2(d(J) - s(V_1 \setminus I)) - d_{j_1} - d_{j_2})z_j \\ &+ \textstyle\sum_{j\in J^<\setminus\{j_1,j_2\}} (d_j - (d_{j_1} + d_{j_2} - (d(J) - s(V_1 \setminus I))))z_j \\ &+ (d(J) - s(V_1 \setminus I) - d_{j_2})z_{j_1} + (d(J) - s(V_1 \setminus I) - d_{j_1})z_{j_2} \\ &\ge 2(d(J) - s(V_1 \setminus I)) - d_{j_1} - d_{j_2}. \end{aligned} \tag{39}$$

Because d_{j_1} and d_{j_2} are the two smallest demands we have $J^* = J$ in Proposition 6. Note that if we add $\sum_{j\in J}(d_{j_1} + d_{j_2} - (d(J) - s(V_1 \setminus I)))z_j \ge d_{j_1} + d_{j_2} - (d(J) - s(V_1 \setminus I))$ to inequality (39) we obtain (15). Hence, (15) cannot be a facet.

5. Assume that $I = V_1$. Then inequality (15) reduces to

$$\sum_{i \in V_1, j \in J:(i,j) \in E} x_{ij} + \sum_{j \in J} d_j z_j \geq d(J). \tag{40}$$

Inequality (40) is a relaxation of the demand equalities (1b) in TPMC. Therefore, if $I = V_1$ then all points in TPMC are on the face defined by inequality (15), therefore this inequality does not define a proper face.

6. Suppose that $J = \{j\}$, but $|V_1 \setminus I| > 1$. Then inequality (15) is

$$\sum_{i \in I:(i,j) \in E} x_{ij} + (d_j - s(V_1 \setminus I)) z_j \geq d_j - s(V_1 \setminus I), \tag{41}$$

where $d_j > s(V_1 \setminus I)$ from facet condition 1. Subtracting the original demand equality (1b) for j from inequality (41), we get

$$\sum_{i \in V_1 \setminus I:(i,j) \in E} x_{ij} \leq s(V_1 \setminus I)(1 - z_j),$$

which is dominated by VUB inequalities (14) for $i \in V_1 \setminus I$.

7. Assume that $s(V_1) < d(J \setminus J^<) + \max_{j \in J^<}\{d_j\}$. Then not all demand for markets in set $J \setminus J^<$ and the largest demand in set $J^<$ can be met at the same time. Hence, $\sum_{j \in J \setminus J^<} z_j + z_m \geq 1$ where $m = \arg\max_{j \in J^<}\{d_j\}$. We use Proposition 6 and inequality (15) as the type-I base inequality. Let $\tilde{J} = \{l, m\}$ where $l \in J \setminus J^<$ then $\beta = d(J) - s(V_1 \setminus I) + d_m - (d(J) - s(V_1 \setminus I)) = d_m$. We obtain

$$\sum_{i \in I, j \in J:(i,j) \in E} x_{ij} + \sum_{j \in J \setminus J^<} (d(J) - s(V_1 \setminus I) - d_m) z_j \quad + \textstyle\sum_{j \in J^< \setminus \{m\}} d_j z_j + (d_m - d_m) z_m \tag{42}$$
$$\geq d(J) - s(V_1 \setminus I) - d_m.$$

If we add $\sum_{j \in J \setminus J^<} d_m z_j + d_m z_m \geq d_m$ to inequality (42) we obtain (15). Hence, (15) cannot be a facet.

8. Assume that $s(V_1) = d(J)$, $d_j \geq d(J) - s(V_1 \setminus I)$ for all $j \in J$ and for contradiction $|I| \geq 2$. Because of assumption $s(V_1) = d(J)$ we have $d_j \geq d(J) - s(V_1 \setminus I) = s(V_1) - s(V_1 \setminus I) = s(I)$ for all $j \in J$. Under these assumptions inequality (15) reduces to $\sum_{i \in I, j \in J:(i,j) \in E} x_{ij} + \sum_{j \in J} s(I) z_j \geq s(I)$. Let $I' = I \setminus \{i'\}$ and $I^{''} = \{i'\}$ where $i' \in I$ ($I' \neq \emptyset$ and $I^{''} \neq \emptyset$ because $|I| \geq 2$ by assumption). Consider the following inequalities in the form of inequality (15) with set I replaced with sets I' and I'', respectively

$$\sum_{i \in I \setminus \{i'\}, j \in J:(i,j) \in E} x_{ij} + \sum_{j \in J} (s(I) - s_{i'}) z_j \geq s(I) - s_{i'}, \tag{43}$$

$$\sum_{j \in J:(i',j) \in E} x_{i'j} + \sum_{j \in J} s_{i'} z_j \geq s_{i'}. \tag{44}$$

Inequality (43) is valid because $d(J) - s(V_1 \setminus I') = d(J) - s(V_1 \setminus I) - s_{i'} = s(I) - s_{i'} > 0$. Furthermore, the coefficient of z_j is $\min\{d_j, s(I) - s_{i'}\} = s(I) - s_{i'}$ because of the assumption $d_j \geq d(J) - s(V_1 \setminus I) = s(I)$ for all $j \in J$. Inequality (44) is valid because $d(J) - s(V_1 \setminus I^{''}) = s(V_1) - s(V_1 \setminus I^{''}) = s(I^{''}) = s_{i'} > 0$ and similarly the coefficient of z_j is $\min\{s_{i'}, d_j\} = s_{i'}$, because $d_j \geq s(I) \geq s_{i'}$ for all $j \in J$ by assumption. By adding inequalities (43) and (44) we obtain inequality (15) with set I. Hence, (15) cannot be a facet.

Sufficiency. We use the technique in §I.4.3 Theorem 3.6 [19]. We show that inequality (15), plus any linear combination of the demand constraints $\sum_{i\in V_1:(i,j)\in E} x_{ij} + d_j z_j = d_j$ for all $j \in V_2$ is the only inequality that is satisfied at equality by all points (x, z) feasible to TPMC that are tight at (15), i.e., we show that if all points of TPMC at which (15) is tight satisfy

$$\sum_{(i,j)\in E} \alpha_{ij} x_{ij} + \sum_{j\in V_2} \psi_j z_j = \hat{\alpha}, \tag{45}$$

then

1. $\alpha_{ij} = u_j$, $j \in V_2 \setminus J$, $i \in V_1$, $(i,j) \in E$,
2. $\alpha_{ij} = u_j$, $j \in J$, $i \in V_1 \setminus I$, $(i,j) \in E$,
3. $\alpha_{ij} = \bar{\alpha} + u_j$, $j \in J$, $i \in I$, $(i,j) \in E$,
4. $\psi_j = u_j d_j$, $j \in V_2 \setminus J$
5. $\psi_j = \bar{\alpha}\left(\min\left\{d(J) - s(V_1 \setminus I), d_j\right\}\right) + u_j d_j$, $j \in J$,
6. $\hat{\alpha} = \bar{\alpha}\left(d(J) - s(V_1 \setminus I)\right) + \sum_{j\in V_2} u_j d_j$.

In the proof we consider three different types of points at which (15) is tight. These points are solutions to TPMC but are subject to additional systems of constraints. Throughout, let ϵ be a very small number greater than zero unless noted otherwise.

1. Suppose that $d_l > d(J) - s(V_1 \setminus I)$ for $l = \arg\max_{j\in J}\{d_j\}$. Consider a point where only markets $j \in \{r\} \cup J \setminus \{l\}$ are satisfied for some $r \in V_2 \setminus J$ and constraints

$$\begin{aligned}
\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} &= 0 && \\
\sum_{i\in V_1\setminus I, j\in J:(i,j)\in E} x_{ij} &= d(J) - d_l && \\
\sum_{i\in V_1:(i,r)\in E} x_{ir} &= d_r && \\
x_{ij} &= 0, && i \in V_1, j \in \{l\} \cup V_2 \setminus (J \cup \{r\}) \\
x_{ij} &\geq \epsilon, && i \in V_1 \setminus I, j \in J \setminus \{l\} \\
x_{ir} &\geq \epsilon, && i \in V_1 \\
\sum_{j\in V_2:(i,j)\in E} x_{ij} &\leq s_i - \epsilon, && i \in V_1 \\
z_j &= 1, && j \in \{l\} \cup V_2 \setminus (J \cup \{r\}) \\
z_j &= 0, && j \in \{r\} \cup J \setminus \{l\}
\end{aligned}$$

in addition to the original constraints are satisfied, which we refer to as System 1. We know that a solution to System 1 exists from facet conditions 9 and 11. For a solution to be feasible to System 1 the demand of markets $j \in \{r\} \cup J \setminus \{l\}$ have to be met, i.e., $s(V_1) \geq d(J) - \max_{j\in J}\{d_j\} + \max_{j\in V_2\setminus J}\{d_j\}$. Additionally, we would like to change a given solution by increasing and decreasing the x values by ϵ hence the need for $>$ relationship in facet condition 9.

2. Suppose that $d_l > d(J) - s(V_1 \setminus I)$ for some $l \in J$. Consider a point where only markets $j \in J \setminus \{l\}$ are satisfied and constraints

$$\begin{aligned}
\sum_{i \in I, j \in J:(i,j) \in E} x_{ij} &= 0 \\
\sum_{i \in V_1 \setminus I, j \in J:(i,j) \in E} x_{ij} &= d(J) - d_l \\
x_{ij} &= 0, && i \in V_1, j \in \{l\} \cup V_2 \setminus J \\
x_{ij} &\geq \epsilon, && i \in V_1 \setminus I, j \in J \setminus \{l\} \\
\sum_{j \in V_2:(i,j) \in E} x_{ij} &\leq s_i - \epsilon, && i \in V_1 \setminus I \\
z_j &= 1, && j \in \{l\} \cup V_2 \setminus J \\
z_j &= 0, && j \in J \setminus \{l\}
\end{aligned}$$

in addition to the original constraints are satisfied, which we refer to as System 2. We know that a solution to System 2 exists from facet condition 2 since there exists at least one $j \in J$ such that $s(V_1) \geq s(V_1 \setminus I) > d(J) - d_j$, and from facet condition 11.

3. We define $\hat{J} \subset J$ such that $d(J \setminus \hat{J}) > s(V_1 \setminus I)$. Due to the choice of $\hat{J}$ we have $d_j < d(J) - s(V_1 \setminus I)$ for all $j \in \hat{J}$ so $\hat{J} \subseteq J^<$ (if $d_{j'} \geq d(J) - s(V_1 \setminus I)$ and $j' \in \hat{J}$ then we cannot have $d(J \setminus \hat{J}) > s(V_1 \setminus I)$). In this point, markets in set $\hat{J} \cup V_2 \setminus J$ are rejected and constraints

$$\begin{aligned}
\sum_{i \in I, j \in J:(i,j) \in E} x_{ij} &= d(J \setminus \hat{J}) - s(V_1 \setminus I) \\
\sum_{i \in V_1 \setminus I, j \in J:(i,j) \in E} x_{ij} &= s(V_1 \setminus I) \\
x_{ij} &= 0, && i \in V_1, j \in \hat{J} \cup V_2 \setminus J \\
x_{ij} &\geq \epsilon, && i \in V_1, j \in J \setminus \hat{J} \\
\sum_{j \in J \setminus \hat{J}:(i,j) \in E} x_{ij} &\leq s_i - \epsilon, && i \in I \\
z_j &= 1, && j \in \hat{J} \cup V_2 \setminus J \\
z_j &= 0, && j \in J \setminus \hat{J}
\end{aligned}$$

in addition to the original constraints are satisfied, which we refer to as System 3. We consider a set $\hat{J}$ such that all demand in set $J \setminus J^<$ is satisfied and $\sum_{i \in I, j \in J:(i,j) \in E} x_{ij} > 0$. This is possible due to facet conditions 7, 11, and non-negativity of x variables.

In order to establish the values of the coefficients α_{ij}, ψ_j and $\hat{\alpha}$, we construct a feasible solution to the given systems 1, 2 and 3. Then a small change in the solution is made. By evaluating (45) at both solutions, which are on the face defined by (15) and comparing the resulting expressions, the possible values of a set of coefficients are obtained.

We start by showing that

1. $\alpha_{ij} = u_j$, $j \in V_2 \setminus J$, $i \in V_1$, $(i,j) \in E$.

   Consider any solution to system 1 with any market $r \in V_2 \setminus J$ that is satisfied. Choose arbitrary suppliers $i, i' \in V_1$ such that $(i,r), (i',r) \in E$. Construct a new point by decreasing the flow on edge (i,r) by ϵ and increasing the flow on edge (i',r) by ϵ. Note that this point is also on the face defined by inequality (15). Thus,

$$\alpha_{ij} = u_j, j \in V_2 \setminus J, i \in V_1, (i,j) \in E.$$

2. $\alpha_{ij} = u_j$, $j \in J$, $i \in V_1 \setminus I$, $(i,j) \in E$. Note that if $|V_1 \setminus I| = 1$, then $\alpha_{ij} = u_j$, $j \in J$ trivially holds. We condition on the number of markets in set J.

   (a) $J = \{k\}$. Note that, from facet condition 6, we have $|V_1 \setminus I| = 1$, so the result holds.

   (b) $|J| \geq 2$. By assumption, $|V_1 \setminus I| > 1$. Due to facet condition 2 there exists $k \in J$ such that $d_k > d(J) - s(V_1 \setminus I)$. We consider a solution to system 2 with $l = k$. Choose any market $j \in J \setminus \{k\}$, any suppliers $i, i' \in V_1 \setminus I$ such that $(i,j), (i',j) \in E$. Make an ϵ-change of flow between the two suppliers i, i' and market j. Thus,

$$\alpha_{ij} = u_j, j \in J \setminus \{k\}, i \in V_1 \setminus I, (i,j) \in E.$$

   Next we show that $\alpha_{ik} = u_k$ for all $i \in V_1 \setminus I$. If there exists another j^* such that $d_{j^*} > d(J) - s(V_1 \setminus I)$, $j^* \neq k$ then we consider a point satisfying System 2 with $l = j^*$, and use the same argument as before to show that $\alpha_{ik} = u_k$ for all $i \in V_1 \setminus I$. If no such j^* exists then $d_j \leq d(J) - s(V_1 \setminus I)$ for all $j \in J \setminus \{k\}$. In this case k is the only market in J with $d_k > d(J) - s(V_1 \setminus I)$. Then from facet condition 7 we know that there exists a solution to a variant of System 3 with $\hat{J} \subseteq J^< \setminus \{j\}$ for some $j \in J \setminus \{k\}$ (in which we set $\epsilon = 0$ in case facet condition 7 is satisfied at equality), where along with market k we can satisfy at least one more market, j. Choose suppliers $i, i' \in V_1 \setminus I$ such that $(i,k), (i',k), (i,j), (i',j) \in E$. Decrease flow on edges (i,j), (i',k) by ϵ and increase flow on edges (i,k), (i',j) by ϵ. Note that since we are using a solution to a variant of system 3 in which we set $\epsilon = 0$ inequality (15) is still tight. Thus,

$$\alpha_{ik} - \alpha_{ij} - \alpha_{i'k} + \alpha_{i'j} = \alpha_{ik} - u_j - \alpha_{i'k} + u_j = \alpha_{ik} - \alpha_{i'k} = 0.$$

   Therefore, $\alpha_{ik} = u_k$ for all $i \in V_1 \setminus I$.

3. $\alpha_{ij} = \bar{\alpha} + u_j$, $j \in J$, $i \in I$, $(i,j) \in E$.

   Consider a solution to system 3 with $\hat{J} \subseteq J^<$. Choose any market $j \in J \setminus \hat{J}$, any two suppliers $i, i' \in I$ such that $(i,j), (i',j) \in E$. Make an ϵ-change of flow between the two suppliers i, i' and market j. Thus,

$$\alpha_{ij} = \alpha^1_j, j \in J \setminus \hat{J}, i \in I, (i,j) \in E.$$

   Let $\alpha^1_j = \bar{\alpha}_j + u_j$, $j \in J \setminus \hat{J}$. Facet condition 10 and definition of $\hat{J}$ (i.e. $\hat{J} \subseteq J^<$) implies that for any $k_1 \in J^<$ we can redefine $\hat{J}$ to either include k_1 or not. More specifically, if $k_1 \in \hat{J}$ then market k_1 is rejected. To show that $\alpha_{ik_1} = \alpha^1_{k_1}$ for all $i \in I$, $(i,k_1) \in E$ we choose another $\hat{J}$ such that $k_1 \notin \hat{J}$. Using the same argument as before we obtain $\alpha_{ik_1} = \alpha^1_{k_1}$ for all $i \in I$, $(i,k_1) \in E$. As a result, we have shown that $\alpha_{ij} = \alpha^1_j, j \in \hat{J}, i \in I, (i,j) \in E$. Next we show that $\bar{\alpha}_j = \bar{\alpha}$, $j \in J \setminus \hat{J}$. Choose any markets $j, j' \in J \setminus \hat{J}$, any suppliers $i \in V_1 \setminus I$, $i' \in I$ such that $(i,j), (i',j), (i,j'), (i',j') \in E$. Decrease flow on edges (i,j'), (i',j) by ϵ and increase flow on edges (i,j), (i',j') by ϵ. Thus,

$$\alpha_{ij} - \alpha_{ij'} - \alpha_{i'j} + \alpha_{i'j'} = u_j - u_{j'} - \alpha^1_j + \alpha^1_{j'} = 0.$$

   By again using $\alpha^1_j = \bar{\alpha}_j + u_j$ and $\alpha^1_{j'} = \bar{\alpha}_{j'} + u_{j'}$, we obtain

$$\bar{\alpha}_j = \bar{\alpha}_{j'}.$$

   Since j and j' can be chosen as any market in $J \setminus \hat{J}$ we conclude that $\bar{\alpha}_j = \bar{\alpha}$, $j \in J \setminus \hat{J}$. Furthermore, since as before we can rearrange set $\hat{J}$ to include or not include any $k_1 \in J^<$ we get $\bar{\alpha}_j = \bar{\alpha}$, $j \in \hat{J}$.

4. $\psi_j = u_j d_j$, $j \in V_2 \setminus J$. We rewrite (45) using the information obtained until now and get

$$\bar{\alpha} \sum_{i \in I, j \in J : (i,j) \in E} x_{ij} + \sum_{(i,j) \in E} u_j x_{ij} + \sum_{j \in V_2} \psi_j z_j = \hat{\alpha}. \tag{46}$$

Consider any solution to system 1 with any market $r \in V_2 \setminus J$ that is satisfied. Then we construct a new solution based on this solution where we set $z_r = 1$ and $x_{ir} = 0$ for all $i \in V_1$, $(i,r) \in E$ and all other variables remain the same. Note that this solution is also on the face defined by (15) since $r \in V_2 \setminus J$ and the new solution is a solution to system 2. We compare face (45) evaluated at these two solutions. Thus,

$$u_r \sum_{i \in V_1 : (i,r) \in E} x_{ir} - \psi_r = 0.$$

Because $\sum_{i \in V_1 : (i,r) \in E} x_{ir} = d_r$ we have $\psi_r = u_r d_r$.

5. $\psi_j = \bar{\alpha} \left(\min \{ d(J) - s(V_1 \setminus I), d_j \} \right) + u_j d_j$, $j \in J$.

   We consider 2 cases.

   (a) $d_{j'} < d(J) - s(V_1 \setminus I)$ for some $j' \in J$.

   We consider a solution to system 3 with $\hat{J}$ such that $d(\hat{J}) + d_{j'} \le d(J) - s(V_1 \setminus I)$. This is a feasible solution due to facet condition 10 where $k_1 = j'$. We evaluate (46) at this solution and obtain

$$\bar{\alpha}(d(J \setminus \hat{J}) - s(V_1 \setminus I)) + \sum_{i \in V_1, j \in J \setminus \hat{J} : (i,j) \in E} u_j x_{ij} + \sum_{j \in \hat{J} \cup V_2 \setminus J} \psi_j = \hat{\alpha}.$$

   Then we use the same solution except now we set $z_{j'} = 1$, $x_{ij'} = 0$, $i \in V_1$, $(i, j') \in E$ (so we redefine $\hat{J}$ as $\hat{J}' = \hat{J} \cup \{j'\}$) and $\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I) - d_{j'}$ and evaluate (46) again. Note that this solution is also on the face defined by (15) because we had $z_{j'} = 0$, $\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I)$ and we changed it with $z_{j'} = 1$, $\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I) - d_{j'}$ and the coefficient of $z_{j'}$ is $d_{j'}$ in inequality (15). Thus,

$$\begin{aligned} \bar{\alpha}(d(J \setminus \hat{J}) - s(V_1 \setminus I) - d_{j'}) &+ \textstyle\sum_{i \in V_1, j \in J \setminus \hat{J}' : (i,j) \in E} u_j x_{ij} \\ &+ \textstyle\sum_{j \in \hat{J} \cup V_2 \setminus J} \psi_j + \psi_{j'} = \hat{\alpha}. \end{aligned}$$

   Taking the difference between (46) evaluated at these two solutions, we obtain

$$\psi_{j'} = \bar{\alpha} d_{j'} + u_{j'} \sum_{i \in V_1 : (i,j') \in E} x_{ij'} = \bar{\alpha} d_{j'} + u_{j'} d_{j'}.$$

   (b) $d_{j'} \ge d(J) - s(V_1 \setminus I)$ for some $j' \in J$.

   We consider a solution to system 3 with any feasible $\hat{J}$ such that the right hand side of inequality $\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I)$ is nonnegative and market j' is satisfied. In the solution we can set $\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} = \sum_{i \in I : (i,j') \in E} x_{ij'}$. This is a feasible solution since $d_{j'} \ge d(J) - s(V_1 \setminus I)$ by assumption and we know that for inequality (15) to be tight we cannot have $\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} > d(J) - s(V_1 \setminus I)$. Hence, $\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} \le d(J) - s(V_1 \setminus I)$ and we can choose a solution in which a part (or all) of the demand of market j' is met by suppliers in set I. We use $\psi_j = \bar{\alpha} d_j + u_j d_j$ for all $j \in J^<$ and recall that markets in set $\hat{J} \subseteq J^<$ are rejected. We evaluate (46) at this solution and obtain

$$\begin{aligned} \bar{\alpha}(d(J \setminus \hat{J}) - s(V_1 \setminus I) + d(\hat{J})) + u_{j'} \sum_{i \in I : (i,j') \in E} x_{ij'} &+ \sum_{i \in V_1 \setminus I, j \in J \setminus \hat{J} : (i,j) \in E} u_j x_{ij} \\ &+ \sum_{j \in \hat{J} \cup V_2 \setminus J} u_j d_j = \hat{\alpha}. \end{aligned}$$

Then we use the same solution except now we set $z_{j'} = 1$, $z_q = 0$, $q \in \hat{J}$ (this is still a feasible solution since $s(V_1) \geq s(V_1 \setminus I) \geq d(J) - d_{j'}$ by assumption, i.e., once market j' is rejected all other markets in set J can be satisfied) and $\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} = 0$ (implying that $\sum_{i\in I:(i,j')\in E} x_{ij'} = 0$) and reevaluate (46). Note that this solution is also on the face defined by (15) because we had $z_{j'} = 0$, $z_q = 1$, $q \in \hat{J}$, $\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} = d(J \setminus \hat{J}) - s(V_1 \setminus I)$ and we changed it with $z_{j'} = 1$, $z_q = 0$, $q \in \hat{J}$, $\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} = 0$ and the coefficient of $z_{j'}$ is $d(J) - s(V_1 \setminus I)$. Thus,

$$\bar{\alpha}(0) + 0 + \sum_{i\in V_1\setminus I, j\in J\setminus\{j'\}:(i,j)\in E} u_j x_{ij} + \sum_{j\in V_2\setminus J} u_j d_j + \psi_{j'} = \hat{\alpha}.$$

Taking the difference between (46) evaluated at these two solutions, we get $\bar{\alpha}(d(J) - s(V_1 \setminus I)) + u_{j'} \sum_{i\in V_1:(i,j')\in E} x_{ij'} - \sum_{i\in V_1, j\in \hat{J}} u_j x_{ij} + \sum_{j\in\hat{J}} u_j d_j - \psi_{j'} = 0$. Because $\sum_{i\in V_1:(i,j')\in E} x_{ij} = d_{j'}$ and $\sum_{i\in V_1, j\in\hat{J}} u_j x_{ij} = \sum_{j\in\hat{J}} u_j d_j$ we have $\psi_{j'} = \bar{\alpha}(d(J) - s(V_1 \setminus I)) + u_{j'} d_{j'}$.

6. $\hat{\alpha} = \bar{\alpha}\left(d(J) - s(V_1 \setminus I)\right) + \sum_{j\in V_2} u_j d_j$.
   Rewriting equality (45), we get

$$\begin{aligned} &\bar{\alpha}\left(\textstyle\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} + \sum_{j\in J} \min\left\{d(J) - s(V_1 \setminus I), d_j\right\} z_j\right) \\ &\quad + \textstyle\sum_{(i,j)\in E: j\in V_2} u_j x_{ij} + \sum_{j\in V_2} u_j d_j z_j = \hat{\alpha}. \end{aligned} \tag{47}$$

   Evaluating (47) at any point (x, z) feasible to TPMC that is tight at inequality (15) gives

$$\bar{\alpha}\left(d(J) - s(V_1 \setminus I)\right) + \sum_{j\in V_2} u_j \left(\sum_{i\in V_1:(i,j)\in E} x_{ij} + d_j z_j \right) = \hat{\alpha}.$$

   From equality (1b) in the definition of TPMC we have $\sum_{i\in V_1:(i,j)\in E} x_{ij} + d_j z_j = d_j$ for all $j \in V_2$. Thus, $\hat{\alpha} = \bar{\alpha}\left(d(J) - s(V_1 \setminus I)\right) + \sum_{j\in V_2} u_j d_j$.

□

Our next result shows that the coefficient update scheme in Proposition 6 is neither lifting nor coefficient strengthening. We show that both a type-I base inequality (16) and the corresponding inequality (19) can be facets of $\mathcal{T}_1$ under certain conditions.

Proposition 8. *If the following conditions hold, then type-I base inequality* (16) *and the corresponding inequality* (19) *are facets of* $\mathcal{T}_1$.

1. *If there exists* $j \in J^* \setminus \tilde{J}$ *with* $\beta_j < \beta_0$ *then* $\beta_j - \beta < \beta_0'$ *and* $\beta(\tilde{J} \setminus \{j_p, j_{p-1}\}) + \beta_j \leq \beta_0$ *where* $\tilde{J} = \{j_1, j_2, \ldots, j_p\}$ *and* $\beta_{j_1} \leq \beta_{j_2} \leq \cdots \leq \beta_{j_p}$.

2. *For all* $j \in J \setminus J^*$, $\beta_j < \beta_0'$ *and* $\beta(\tilde{J} \setminus \{j_p\}) + \beta_j \leq \beta_0$.

Proof. We first show that there exists $dim(\mathcal{T}_1) = |J| + 1$ many affinely independent points that satisfy inequality (19) at equality. Consider the following points:

- Let $t = 0$, $z_j = 1$ for all $j \in \tilde{J}$, $z_j = 0$ for all $j \in J \setminus \tilde{J}$. In this case, the left-hand side of inequality (19) is $\beta(\tilde{J}) - p\beta = \beta_0 + \beta - p\beta = \beta_0 - (p-1)\beta = \beta_0'$.

- For each $j' \in \tilde{J}$, $t = \beta_{j'} - \beta$, $z_{j'} = 0$, $z_j = 1$ for all $j \in \tilde{J} \setminus \{j'\}$, $z_j = 0$ for all $j \in J \setminus \tilde{J}$. In this case, the left-hand side of inequality (19) is $\beta_{j'} - \beta + \beta(\tilde{J} \setminus \{j'\}) - (p-1)\beta = \beta(\tilde{J}) - p\beta = \beta_0 + \beta - p\beta = \beta_0 - (p-1)\beta = \beta_0'$. This point also satisfies type-I base inequality (16) at equality.

- For each $j' \in J^* \setminus \tilde{J}$ we consider two cases:

1. $\beta_{j'} = \beta_0$.
   Let $t = 0$, $z_{j'} = 1$, $z_j = 0$ for all $j \in J \setminus \{j'\}$. The left-hand side of inequality (19) is $\min\{(\beta_{j'} - \beta), \beta_0^{'}\} = \min\{(\beta_0 - \beta), \beta_0 - (p-1)\beta\} = \beta_0 - (p-1)\beta$ since p is the number of elements in set $\tilde{J}$ and $p \geq 2$, for $\tilde{J}$ to be a minimal cover. This point also satisfies type-I base inequality (16) at equality.

2. $\beta_{j'} < \beta_0$.
   Let $t = \beta_0 - \beta(\tilde{J} \setminus \{j_p, j_{p-1}\}) - \beta_{j'}$, $z_j = 1$, for all $j \in \tilde{J} \setminus \{j_p, j_{p-1}\}$, $z_{j_p} = 0$, $z_{j_{p-1}} = 0$, $z_{j'} = 1$, $z_j = 0$ for all $J \setminus (\tilde{J} \cup \{j'\})$. From facet condition 1 we have $\beta_{j'} - \beta < \beta_0^{'}$ hence the left-hand side of inequality (19) is $\beta_0 - \beta(\tilde{J} \setminus \{j_p, j_{p-1}\}) - \beta_{j'} + \beta(\tilde{J} \setminus \{j_p, j_{p-1}\}) - (p-2)\beta + \beta_{j'} - \beta = \beta_0 - (p-1)\beta = \beta_0^{'}$. Note that due to facet condition 1, $t \geq 0$. Furthermore, this point also satisfies type-I base inequality (16) at equality.

- For each $j' \in J \setminus J^*$ first observe that $\beta_{j'} < \beta_0$ since by definition of J^*, $\beta_{j'} < \beta_{j_p} \leq \beta_0$. Let $t = \beta_0 - \beta(\tilde{J} \setminus \{j_p\}) - \beta_{j'}$, $z_j = 1$, for all $j \in \tilde{J} \setminus \{j_p\}$, $z_{j_p} = 0$, $z_{j'} = 1$, $z_j = 0$ for all $J \setminus (\tilde{J} \cup \{j'\})$. From facet condition 2 we have $\beta_{j'} < \beta_0^{'}$ hence the left-hand side of inequality (19) is $\beta_0 - \beta(\tilde{J} \setminus \{j_p\}) - \beta_{j'} + \beta(\tilde{J} \setminus \{j_p\}) - (p-1)\beta + \beta_{j'} = \beta_0 - (p-1)\beta = \beta_0^{'}$. Note that due to facet condition 2, $t \geq 0$. Furthermore, this point also satisfies type-I base inequality (16) at equality.

In total we have described $1 + |\tilde{J}| + |J^* \setminus \tilde{J}| + |J \setminus J^*| = |J| + 1$ many points. It is easy to see that these points are affinely independent. Furthermore, except for the first described point ($t = 0$, $z_j = 1$ for all $j \in \tilde{J}$, $z_j = 0$ for all $j \in J \setminus \tilde{J}$) all the other $|J|$ many points also satisfy type-I base inequality (16) at equality. If we replace the first point with the point $t = \beta_0$, $z_j = 0$ for all $j \in J$, which satisfies the type-I base inequality at equality, then we still get $|J| + 1$ many affinely independent points. Hence, both the type-I base inequality (16) and the corresponding inequality (19) are facets of $\mathcal{T}_1$ under conditions 1 and 2. □

Suppose that inequality $\sum_{j \in J} t_j \leq \alpha_0$ is given as a type-II base inequality in the form of (24) for set $\mathcal{S}_2$, where $\alpha_j = 0$ for all $j \in J$. Assume that there exists $\tilde{J}$ and m such that $\alpha_0 > d(J \setminus \tilde{J})$ and $\alpha_0 - d(J \setminus \tilde{J}) < \max_{j \in \tilde{J}}\{d_j\}$. These conditions imply that $m = 0$ and $\alpha = \alpha_0 - d(J \setminus \tilde{J})$. Then we obtain the corresponding inequality (27)

$$\sum_{j \in J} t_j + \sum_{j \in \tilde{J}} \alpha z_j \leq \alpha_0 + (|\tilde{J}| - 1)\alpha, \tag{48}$$

which is valid for $\mathcal{S}_2$, under these assumptions.

Proposition 9. *Inequality* (48)*, valid for* $\mathcal{S}_2$*, defines a facet of* $\mathcal{T}_2$ *only if*

1. $\tilde{J} \neq \emptyset$.

In addition, if the following conditions hold then (48) *is a facet of* $\mathcal{T}_2$*:*

2. $\alpha_0 < d(J \setminus \tilde{J}) + \min_{j \in \tilde{J}}\{d_j\}$,

3. $\alpha_0 < d(J \setminus \tilde{J}) + \max_{j \in \tilde{J}}\{d_j\} - \max_{j \in J \setminus \tilde{J}}\{d_j\}$,

4. $|J \setminus \tilde{J}| \geq 2$.

Proof. Necessity.
1. Assume that $\tilde{J} = \emptyset$. Then inequality (48) reduces to

$$\sum_{j \in J} t_j \leq \alpha_0 - \alpha. \tag{49}$$

This case implies that $\alpha = \alpha_0 - d(J \setminus \tilde{J}) = \alpha_0 - d(J)$. Thus, inequality (49) becomes $\sum_{j \in J} t_j \le d(J)$ which is dominated by $t_j + d_j z_j \le d_j$ for all $j \in J$.

Sufficiency. We show that there exists $dim(\mathcal{T}_2) = 2|J|$ many affinely independent points that satisfy inequality (48) at equality. Let $\epsilon > 0$ be a very small number and $j^* = \arg\max_{j \in \tilde{J}}\{d_j\}$ (j^* exists due to facet condition 1). Consider the following points:

- For each $j' \in \tilde{J}$, let $z_{j'} = 0$, $t_{j'} = \alpha_0 - d(J \setminus \tilde{J})$, $z_j = 1$, $j \in \tilde{J} \setminus \{j'\}$, $t_j = 0$, $j \in \tilde{J} \setminus \{j'\}$, $z_j = 0$, $j \in J \setminus \tilde{J}$, $t_j = d_j$, $j \in J \setminus \tilde{J}$. Note that this is a feasible solution due to the assumption that $\alpha_0 > d(J \setminus \tilde{J})$ and facet condition 2. Furthermore, for each such point we construct another point by increasing $t_{j'}$ by ϵ and decreasing any t_j, $j \in J \setminus \tilde{J}$ by ϵ (j exists due to facet condition 4). This gives $2|\tilde{J}|$ many points.

- For each $j'' \in J \setminus \tilde{J}$, let $z_{j''} = 1$, $t_{j''} = 0$, $z_j = 0$, $j \in (J \setminus (\tilde{J} \cup \{j''\})) \cup \{j^*\}$, $t_j = d_j$, $j \in J \setminus (\tilde{J} \cup \{j''\})$, $t_{j^*} = \alpha_0 - d(J \setminus \tilde{J}) + d_{j''}$, $z_j = 1$, $j \in \tilde{J} \setminus \{j^*\}$, $t_j = 0$, $j \in \tilde{J} \setminus \{j^*\}$. Note that this is feasible due to facet condition 3. For each such point we construct another point by increasing t_{j^*} by ϵ and decreasing any t_j, $j \in J \setminus (\tilde{J} \cup \{j''\})$ by ϵ (j exists due to facet condition 4). This gives $2|J \setminus \tilde{J}|$ many points.

It is easy to see that these points are affinely independent.

□

Now, suppose that we start with a type-II base inequality $\sum_{i \in I, j \in J:(i,j) \in E} x_{ij} \le s(I)$ in Proposition 7. Note that inequality $\sum_{i \in I, j \in J:(i,j) \in E} x_{ij} \le s(I)$ is a relaxation of the supply constraints (1c). Let $t_j = \sum_{i \in I:(i,j) \in E} x_{ij}$ and $\alpha_j = 0$ for all $j \in J$ in inequality (24). Suppose that there exists $\tilde{J}$ and m such that $s(I) > d(J \setminus \tilde{J})$ and $s(I) - d(J \setminus \tilde{J}) < \max_{j \in \tilde{J}}\{d_j\}$. These conditions imply that $m = 0$ and $\alpha = s(I) - d(J \setminus \tilde{J})$. Then we obtain the inequality

$$\sum_{i \in I, j \in J:(i,j) \in E} x_{ij} + \sum_{j \in \tilde{J}} \alpha z_j \le s(I) + (|\tilde{J}| - 1)\alpha, \tag{50}$$

which is valid for X.

Proposition 10. *Inequality* (50)*, valid for X, defines a facet of conv(X) only if*

1. $\tilde{J} \neq \emptyset$.

In addition, if the following conditions hold then (50) *is a facet of conv*(X)*:*

2. $s(V_1) > d(J \setminus \tilde{J}) + \max_{j \in (V_2 \setminus J) \cup \tilde{J}}\{d_j\}$,

3. $s(I) < d(J \setminus \tilde{J}) + \min_{j \in \tilde{J}}\{d_j\}$,

4. $s(I) \le d(J \setminus \tilde{J}) + \max_{j \in \tilde{J}}\{d_j\} - \max_{j \in J \setminus \tilde{J}}\{d_j\}$.

Proof. Necessity.

1. If we replace t_j by $\sum_{i \in I:(i,j) \in E} x_{ij}$ for all $j \in J$ and α_0 by $s(I)$ we can use the same argument as in the necessity of facet condition 1 in Proposition 9.

Sufficiency. For the proof we use §I.4.3 Theorem 3.6 [19]. We show that inequality (50), plus any linear combination of the demand constraints $\sum_{i \in V_1:(i,j) \in E} x_{ij} + d_j z_j = d_j$ for all $j \in V_2$ is the only inequality that is satisfied at equality by all points (x, z) feasible to TPMC that are tight at (50), i.e., we show that if all points of TPMC at which (50) is tight satisfy

$$\sum_{(i,j) \in E} \lambda_{ij} x_{ij} + \sum_{j \in V_2} \omega_j z_j = \hat{\lambda}, \tag{51}$$

then

1. $\lambda_{ij} = u_j$, $j \in V_2 \setminus J$, $i \in V_1$, $(i,j) \in E$,
2. $\lambda_{ij} = u_j$, $j \in J$, $i \in V_1 \setminus I$, $(i,j) \in E$,
3. $\lambda_{ij} = \bar{\lambda} + u_j$, $j \in J$, $i \in I$, $(i,j) \in E$,
4. $\omega_j = u_j d_j$, $j \in V_2 \setminus \tilde{J}$,
5. $\omega_j = \bar{\lambda}\alpha + u_j d_j$, $j \in \tilde{J}$,
6. $\hat{\lambda} = \bar{\lambda}\left(s(I) + (|\tilde{J}| - 1)\alpha\right) + \sum_{j \in V_2} u_j d_j$.

In the proof we consider four different types of points at which (50) is tight that make use of the facet conditions. Throughout, let ϵ be a very small number greater than zero unless noted otherwise.

1. Consider a point where only markets $j \in J \setminus \tilde{J} \cup \{r\}$ are satisfied for some $r \in V_2 \setminus J$, and constraints

$$\begin{aligned}
\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} &= d(J \setminus \tilde{J}) && \\
\sum_{i \in V_1 : (i,r) \in E} x_{ir} &= d_r && \\
x_{ij} &= 0, && i \in V_1, j \in \tilde{J} \cup V_2 \setminus (J \cup \{r\}) \\
x_{ij} &\geq \epsilon, && i \in I, j \in J \setminus \tilde{J} \\
x_{ir} &\geq \epsilon, && i \in V_1 \\
\sum_{j \in V_2 : (i,j) \in E} x_{ij} &\leq s_i - \epsilon, && i \in V_1 \\
z_j &= 1, && j \in \tilde{J} \cup V_2 \setminus (J \cup \{r\}) \\
z_j &= 0, && j \in \{r\} \cup J \setminus \tilde{J}
\end{aligned}$$

   in addition to the original constraints are satisfied, which we refer to as System 1. We know that a solution to System 1 exists from assumption $s(I) > d(J \setminus \tilde{J})$ and facet condition 2.

2. Consider a point where only markets $j \in J \setminus \tilde{J}$ are satisfied, and constraints

$$\begin{aligned}
\sum_{i \in I, j \in J : (i,j) \in E} x_{ij} &= d(J \setminus \tilde{J}) && \\
x_{ij} &= 0, && i \in V_1, j \in \tilde{J} \cup V_2 \setminus J \\
x_{ij} &\geq \epsilon, && i \in I, j \in J \setminus \tilde{J} \\
\sum_{j \in V_2 : (i,j) \in E} x_{ij} &\leq s_i - \epsilon, && i \in I \\
z_j &= 1, && j \in \tilde{J} \cup V_2 \setminus J \\
z_j &= 0, && j \in J \setminus \tilde{J}
\end{aligned}$$

   in addition to the original constraints are satisfied, which we refer to as System 2. We know that a solution to System 2 exists from assumption $s(I) > d(J \setminus \tilde{J})$.

3. Consider a point where only markets $j \in J \setminus \tilde{J} \cup \{l\}$ are satisfied for some $l \in \tilde{J}$, and constraints

$$\begin{aligned}
\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} &= s(I) \\
\sum_{i\in V_1\setminus I, j\in J:(i,j)\in E} x_{ij} &= d(J\setminus\tilde{J}) + d_l - s(I) \\
x_{ij} &= 0, & i \in V_1, j \in \tilde{J}\setminus\{l\} \cup V_2 \setminus J \\
x_{ij} &\geq \epsilon, & i \in V_1, j \in J \setminus \tilde{J} \cup \{l\} \\
\sum_{j\in V_2:(i,j)\in E} x_{ij} &\leq s_i - \epsilon, & i \in V_1 \setminus I \\
z_j &= 1, & j \in \tilde{J}\setminus\{l\} \cup V_2 \setminus J \\
z_j &= 0, & j \in J \setminus \tilde{J} \cup \{l\}
\end{aligned}$$

in addition to the original constraints are satisfied, which we refer to as System 3. We know that a solution to System 3 exists from facet conditions 2 and 3.

4. Consider a point where only markets $j \in J \setminus (\tilde{J} \cup \{j'\}) \cup \{l^*\}$ are satisfied for $l^* = \arg\max_{j\in\tilde{J}}\{d_j\}$ and some $j' \in J \setminus \tilde{J}$, and constraints

$$\begin{aligned}
\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} &= s(I) \\
\sum_{i\in V_1\setminus I, j\in J:(i,j)\in E} x_{ij} &= d(J\setminus\tilde{J}) + d_{l^*} - d_{j'} - s(I) \\
x_{ij} &= 0, & i \in V_1, j \in \{j'\} \cup \tilde{J}\setminus\{l^*\} \cup V_2 \setminus J \\
z_j &= 1, & j \in \{j'\} \cup \tilde{J}\setminus\{l^*\} \cup V_2 \setminus J \\
z_j &= 0, & j \in J \setminus (\tilde{J} \cup \{j'\}) \cup \{l^*\}
\end{aligned}$$

in addition to the original constraints are satisfied, which we refer to as System 4. We know that a solution to system 4 exists from facet conditions 2 and 4.

1. $\lambda_{ij} = u_j$, $j \in V_2 \setminus J$, $i \in V_1$, $(i,j) \in E$.

   Consider any solution to system 1 with any market $j = r \in V_2 \setminus J$ that is satisfied. Choose arbitrary suppliers $i, i' \in V_1$ such that $(i,j), (i',j) \in E$. Construct a new point by decreasing the flow on edge (i,j) by ϵ and increasing the flow on edge (i',j) by ϵ. Note that this point is also on the face defined by inequality (50). Thus,

$$\lambda_{ij} = u_j, j \in V_2 \setminus J, i \in V_1, (i,j) \in E.$$

2. $\lambda_{ij} = u_j$, $j \in J$, $i \in V_1 \setminus I$, $(i,j) \in E$.

   Consider any solution to system 3 with market $j \in J \setminus \tilde{J} \cup \{l\}$ satisfied for some $l \in \tilde{J}$. Choose arbitrary suppliers $i, i' \in V_1 \setminus I$ such that $(i,j), (i',j) \in E$. Construct a new point by decreasing the flow on edge (i,j) by ϵ and increasing the flow on edge (i',j) by ϵ. Note that this point is also on the face defined by inequality (50) since $i, i' \in V_1 \setminus I$. Thus,

$$\lambda_{ij} = u_j, j \in J \setminus \tilde{J} \cup \{l\}, i \in V_1 \setminus I, (i,j) \in E.$$

   Note that since we can use the above argument for any $l \in \tilde{J}$, we have $\lambda_{il} = u_l$ for all $l \in \tilde{J}$, $i \in V_1 \setminus I$, $(i,l) \in E$.

3. $\lambda_{ij} = \bar{\lambda} + u_j$, $j \in J$, $i \in I$, $(i,j) \in E$.

   Consider any solution to system 2. Choose arbitrary suppliers $i, i' \in I$ such that $(i,j), (i',j) \in E$ for $j \in J \setminus \tilde{J}$. Construct a new point by decreasing the flow on edge (i,j) by ϵ and increasing the flow on edge (i',j) by ϵ. Note that this point is also on the face defined by inequality (50). Thus,

   $$\lambda_{ij} = \lambda_j^1, j \in J \setminus \tilde{J}, i \in I, (i,j) \in E.$$

   Next we consider a solution to system 3 with $\epsilon = 0$. Choose arbitrary suppliers $i, i' \in I$ and market $j \in J \setminus \tilde{J}$ such that $(i,j), (i',j), (i,l), (i',l) \in E$. Construct a new point by decreasing the flow on edges (i,j), (i',l) by ϵ and increasing the flow on edges (i',j), (i,l) by ϵ. Note that this point is also on the face defined by inequality (50). Thus,

   $$-\lambda_{ij} + \lambda_{il} + \lambda_{i'j} - \lambda_{i'l} = -\lambda_j^1 + \lambda_{il} + \lambda_j^1 - \lambda_{i'l} = \lambda_{il} - \lambda_{i'l} = 0.$$

   Because l is any market in set $\tilde{J}$, $\lambda_{ij} = \lambda_j^1$, $j \in \tilde{J}$, $i \in I$, $(i,j) \in E$.

   Let $\lambda_j^1 = \bar{\lambda}_j + u_j$, $j \in J$. Next we show that $\bar{\lambda}_j = \bar{\lambda}$, $j \in J$. We consider a solution to system 3 with $\epsilon = 0$. Choose any markets $j, j' \in J$, any suppliers $i \in V_1 \setminus I$, $i' \in I$ such that $(i,j), (i',j), (i,j'), (i',j') \in E$. Decrease flow on edges (i,j'), (i',j) by ϵ and increase flow on edges (i,j), (i',j') by ϵ. Thus,

   $$\lambda_{ij} - \lambda_{ij'} - \lambda_{i'j} + \lambda_{i'j'} = u_j - u_{j'} - \lambda_j^1 + \lambda_{j'}^1 = 0.$$

   By again using $\lambda_j^1 = \bar{\lambda}_j + u_j$ and $\lambda_{j'}^1 = \bar{\lambda}_{j'} + u_{j'}$, we obtain

   $$\bar{\lambda}_j = \bar{\lambda}_{j'} = \bar{\lambda}.$$

4. $\omega_j = u_j d_j$, $j \in V_2 \setminus \tilde{J}$.

   We rewrite (51) using the information obtained until now, and get

   $$\bar{\lambda} \sum_{i \in I, j \in J : (i,j) \in E} x_{ij} + \sum_{(i,j) \in E} u_j x_{ij} + \sum_{j \in V_2} \omega_j z_j = \hat{\lambda}. \tag{52}$$

   Consider any solution to system 1 with market $r \in V_2 \setminus J$ that is satisfied. Then we construct a new solution based on this solution where we set $z_r = 1$ and $x_{ir} = 0$ for all $i \in V_1$, $(i,r) \in E$ and all other variables remain the same. This is a solution to System 2. Thus this solution is also on the face defined by (50). We compare inequality (51) evaluated at these two solutions. Thus,

   $$u_r \sum_{i \in V_1 : (i,r) \in E} x_{ir} - \omega_r = 0.$$

   Because $\sum_{i \in V_1 : (i,r) \in E} x_{ir} = d_r$ we have $\omega_r = u_r d_r$, $r \in V_2 \setminus J$.

   Next we show that $\omega_j = u_j d_j$, $j \in J \setminus \tilde{J}$. First we consider a solution to system 3 where we choose $l = l^* = \arg\max_{j \in \tilde{J}} \{d_j\}$. This is a feasible choice due to facet condition 2. We evaluate (52) at this solution, and get

   $$\bar{\lambda}(s(I)) + \sum_{i \in V_1, j \in J \setminus \tilde{J} \cup \{l^*\} : (i,j) \in E} u_j x_{ij} + \sum_{j \in V_2 \setminus J \cup \tilde{J} \setminus \{l^*\}} \omega_j = \hat{\lambda}. \tag{53}$$

   Next we consider a solution to system 4 where some market $j' \in J \setminus \tilde{J}$ is rejected. We evaluate (52) at this solution, and obtain

   $$\bar{\lambda}(s(I)) + \sum_{i \in V_1, j \in J \setminus (\tilde{J} \cup \{j'\}) \cup \{l^*\} : (i,j) \in E} u_j x_{ij} + \sum_{j \in V_2 \setminus J \cup \tilde{J} \setminus \{l^*\}} \omega_j + w_{j'} = \hat{\lambda}. \tag{54}$$

   We subtract (54) from (53) and obtain $u_{j'} \sum_{i \in V_1 : (i,j') \in E} x_{ij'} - \omega_{j'} = 0$. Because $\sum_{i \in V_1 : (i,j') \in E} x_{ij'} = d_{j'}$ we have $\omega_{j'} = u_{j'} d_{j'}$, $j' \in J \setminus \tilde{J}$.

5. $\omega_j = \bar{\lambda}\alpha + u_j d_j$, $j \in \tilde{J}$.

Consider any solution to system 3 with any market $l \in \tilde{J}$ that is satisfied. Then (51) reduces to

$$\bar{\lambda}(s(I)) + \sum_{i\in V_1, j\in J\setminus \tilde{J}\cup\{l\}:(i,j)\in E} u_j x_{ij} + \sum_{j\in V_2\setminus J\cup\tilde{J}\setminus\{l\}} \omega_j = \hat{\lambda}. \tag{55}$$

We also consider a solution to system 2 where market $l \in \tilde{J}$ is rejected. Then (51) reduces to

$$\bar{\lambda}(d(J\setminus\tilde{J}) + \sum_{i\in V_1, j\in J\setminus\tilde{J}:(i,j)\in E} u_j x_{ij} + \sum_{j\in V_2\setminus J\cup\tilde{J}} \omega_j = \hat{\lambda}. \tag{56}$$

We subtract (56) from (55) and obtain, $\bar{\lambda}(s(I) - d(J\setminus\tilde{J})) + u_l \sum_{i\in V_1:(i,l)\in E} x_{il} - \omega_l = 0$. Since $s(I) - d(J\setminus\tilde{J}) = \alpha$ and $\sum_{i\in V_1:(i,l)\in E} x_{il} = d_l$ we conclude that $\omega_l = \bar{\lambda}\alpha + u_l d_l$ for $l \in \tilde{J}$.

6. $\hat{\lambda} = \bar{\lambda}\left(s(I) + (|\tilde{J}| - 1)\alpha\right) + \sum_{j\in V_2} u_j d_j$.

We rewrite (51), and get

$$\bar{\lambda}\left(\sum_{i\in I, j\in J:(i,j)\in E} x_{ij} + \sum_{j\in\tilde{J}} \alpha z_j\right) + \sum_{(i,j)\in E} u_j x_{ij} + \sum_{j\in V_2} u_j d_j z_j = \hat{\lambda}. \tag{57}$$

Evaluating (57) at any point (x, z) feasible to $TPMC$ that is tight at inequality (50) gives

$$\bar{\lambda}\left(s(I) + (|\tilde{J}| - 1)\alpha\right) + \sum_{j\in V_2} u_j \left(\sum_{i\in V_1:(i,j)\in E} x_{ij} + d_j z_j\right) = \hat{\lambda}.$$

From the definition of $TPMC$ we have $\sum_{i\in V_1:(i,j)\in E} x_{ij} + d_j z_j = d_j$ for all $j \in V_2$. Thus, $\hat{\lambda} = \bar{\lambda}\left(s(I) + (|\tilde{J}| - 1)\alpha\right) + \sum_{j\in V_2} u_j d_j$.

□

Even though Propositions 6 and 7 are general results for mixed-integer cover and knapsack sets $\mathcal{S}_1$ and $\mathcal{S}_2$, we observed that many of the facets for TPMC can be derived from the recursive application of these results.

Example 4. *(Continued.) Observe that inequalities* (28), (29) *and* (30) *satisfy all the conditions given in Proposition 8 and inequality* (32) *satisfies all the conditions given in Proposition 10, and hence they are facets of conv(X).*

Finally, while the blossom inequalities (3) are strong for the case that $d_j \leq 2$ for all $j \in V_2$, they are not facet-defining for the general case of TPMC based on our experience with PORTA [7].

5 Computational Results

In this section we present our computational results for the TPMC problem. We conduct the experiments on an Intel Xeon x5650 Processor at 2.67GHz with 4GB RAM. We use IBM ILOG CPLEX 12.4 as the MIP solver. We test the TPMC problem for various settings of V_1 and V_2. There are 12 combinations of V_1 and V_2 as shown in Tables 1 and 2, in the first column. For each combination, we create 3 instances and report the averages. We observed that most instances of the TPMC problem are solved under a minute for each setting of V_1 and V_2. Therefore, we found "hard" instances by continually generating and solving instances until we were able to find 3 that were solved in at least 15 minutes under default CPLEX settings. Problem parameters are generated using a discrete uniform distribution with supply values $s_i \in [10, 20]$, demand values $d_j \in [10, 20]$, weights $w_{ij} \in [20, 50]$ and lost revenues $r_j \in [5000, 6000]$. In our computations, we impose a time limit of half an hour, and consider the following four algorithms:

(1) BB (Branch and Bound): TPMC formulation, (1a)-(1e) with no cuts,

(2) UC (User Cuts): TPMC formulation, (1a)-(1e) with only user cuts,

(3) CD (CPLEX Default Settings): TPMC formulation, (1a)-(1e) with default CPLEX cuts,

(4) UCD (User Cuts and CPLEX Default Settings): TPMC formulation, (1a)-(1e) with user cuts and default CPLEX cuts.

Table 1: Comparison of Algorithms BB and UC

$\lvert V_1\rvert, \lvert V_2\rvert$	**RGap**		**RCuts**		**EGap**		**ECuts**		**Time (unslvd)**		**B&C Nodes**	
	BB	UC	BB	UC	BB	UC	BB	UC	BB	UC	BB	UC
200,230	86.7%	86.7%	-	u8	0.8%	1.0%	-	u189	1800(3)	1800(3)	176305.3	107412
200,240	1.7%	1.7%	-	u4.3	0.7%	0.8%	-	u186	1800(3)	1800(3)	168384.7	91004.7
200,250	28.1%	28.1%	-	u4.7	0.2%	0.2%	-	u103.3	1800(3)	1800(3)	141624.3	89966.7
300,330	54.3%	54.3%	-	u4	0.5%	0.5%	-	u125.7	1800(3)	1800(3)	75466.3	56085.3
300,340	1.6%	1.6%	-	u6	0.9%	0.9%	-	u123.3	1800(3)	1800(3)	60125.7	50122.7
300,350	58%	29.5%	-	u4.3	0.4%	0.4%	-	u98.3	1800(3)	1800(3)	51141.3	44166.7
400,430	0.8%	0.7%	-	u4	0.4%	0.5%	-	u74.7	1800(3)	1800(3)	29565	32466.7
400,440	81%	54.3%	-	u4.7	0.3%	0.3%	-	u72.7	1800(3)	1800(3)	23177.7	26600
400,450	0.4%	0.4%	-	u4.7	0.2%	0.2%	-	u57.7	1800(3)	1800(3)	22593	23733.3
500,530	83.2%	83.2%	-	u7	0.5%	0.2%	-	u52	1800(3)	1690.5(2)	18152.3	16380
500,540	81.4%	81.4%	-	u4	0.4%	0.4%	-	u61.3	1800(3)	1800(3)	16115	17573.3
500,550	0.6%	0.6%	-	u6.3	0.4%	0.4%	-	u21.7	1800(3)	1800(3)	14767.7	18100
Average	43.5%	35.2%	-	u5.2	0.5%	0.5%	-	u96.4	1800(3)	1790.9(2.9)	66451.5	47801

Table 2: Comparison of Algorithms CD and UCD

$\lvert V_1\rvert, \lvert V_2\rvert$	**RGap**		**RCuts**		**EGap**		**ECuts**		**Time (unslvd)**		**B&C Nodes**	
	CD	UCD	CD	UCD	CD	UCD	CD	UCD	CD	UCD	CD	UCD
200,230	58.4%	58.1%	10.7	6,u2.3	0.2%	0.2%	568.7	569,u37.3	1342.3(1)	1249.8(1)	64767	57478.7
200,240	1.6%	1.6%	10	9.3,u2.7	0.4%	0.6%	307.3	219.7,u101.7	1420.9(2)	1333.9(2)	93360.7	74344.3
200,250	28.1%	1%	7.3	4,u3.3	0.1%	0.1%	573.3	412,u17.3	1265.4(1)	815.6(1)	53962.7	39750
300,330	0.8%	0.7%	12	8,u2	0.3%	0.3%	164.7	178.3,u50.3	1800(3)	1227.1(2)	72057.7	33551.3
300,340	1.6%	1.5%	13.3	7.3,u1.3	0.2%	0%	334.7	239,u16.3	1678.3(1)	1067.6(1)	49950	31914.3
300,350	29.5%	29.5%	4.7	5.7,u2	0.2%	0.2%	161	139.3,u53.3	1025.9(1)	901.7(1)	29653.7	23473.7
400,430	0.7%	0.7%	10.7	8.3,u2.7	0.2%	0.3%	114.7	105.7,u41	1800(3)	1234.9(2)	34852	21395.7
400,440	27.5%	27.8%	12	8,u3.3	0.2%	0.1%	128.7	167.7,u25	1800(3)	1216.6(2)	24729.3	16442.7
400,450	0.4%	0.4%	6	6.3,u3.7	0.1%	0.1%	133	173.3,u23	1800(3)	1800(3)	19398.7	22166.7
500,530	42.3%	42.3%	7.7	5.3,u3	0.2%	0.2%	76.3	86.3,u55	1800(3)	1661.7(2)	34852	21089.7
500,540	27.9%	27.9%	7	8.7,u1	0.4%	0.2%	58.3	151.3,u32.7	1800(3)	1800(3)	20709	18757.3
500,550	0.6%	0.6%	7	8.7,u3.7	0.4%	0.3%	26.3	68,u34.7	1800(3)	1800(3)	17482	18707.3
Average	18.3%	16%	9	7.1,u2.6	0.2%	0.2%	220.6	209.1,u40.6	1611.1(2.3)	1342.4(1.9)	42981.2	31589.3

In Tables 1 and 2, column **RGap** reports the average percentage integrality gap at the root node just before branching, which is $100 \times (zub - zrb)/zub$, where zub is the objective function value of the best integer solution obtained within time limit and zrb is the best lower bound obtained at the root node. Column **RCuts** reports the average number of cuts added at the root node. In column **EGap**, we report the average percentage end gap at termination output by CPLEX, which is $100 \times (zub - zbest)/zub$, where $zbest$ is the best lower bound available within time limit. Column **ECuts** reports the average number of cuts added after the problem is solved to optimality within the time limit. Column **Time (unslvd)** reports the average solution time in seconds and the number of unsolved instances in parentheses in cases where not all three instances are solved to optimality within time limit. We denote the user cuts by **u** and for the other cuts, i.e., cuts added by CPLEX we do not use a prefix. In column **B&C Nodes** we report the average number of branch-and-cut tree nodes explored. At the end of Tables 1 and 2 we give the averages of **RGap**, **RCuts**, **EGap**, **ECuts**, **Time (unslvd)** and **B&C Nodes**, respectively. For each value in the tables we report the numbers rounded to the first decimal place.

User cuts are generated every 10000 B&C nodes. For the variable upper bound inequalities (14) we add a violated inequality if $s_i < d_j$, $i \in V_1$, $j \in V_2$, $(i, j) \in E$ and $\bar{x}_{ij} > s_i(1 - \bar{z}_j)$. Recall that inequalities (19) are related to the weight inequalities for 0/1 knapsack problems. The exact separation of weight inequalities involves solving 0/1 knapsack problems. Weismantel, Kaparis and Letchford give exact pseudo-polynomial separation algorithms for weight inequalities [15, 26]. The optimization problems for finding the most violated inequalities (15) and (27) involve nonlinear objectives and constraints that resemble knapsack constraints. Thus, we use a heuristic separation for inequalities (15), (19) and (27). Let $(\bar{x}, \bar{z})$ be a fractional point. The heuristic for finding a violated inequality (15) takes $(\bar{x}, \bar{z})$ and selects sets I and J simultaneously. Set J includes a market with fractional $\bar{z}$ value, and other markets that receive demand from the same suppliers as the market with fractional $\bar{z}$. All the suppliers that do not send demand to markets in set J are placed in set I. More details for this heuristic can be found in Algorithm 2. The heuristic for finding a violated inequality (19) uses the type-I base inequalities (15), and adds the smallest p coefficients of the z variables that exceed the right-hand side, β_0 to obtain the cover $\tilde{J}$. For all the instances in Tables 1 and 2 the violated inequality (15) (i.e. type-I base inequality) found by the heuristic separation has the coefficients of all the z variables equal to the right-hand side, β_0. It is easy to see that if at least two coefficients of z variables are not strictly less than the right-hand side, β_0 in a given type-I base inequality, the new inequality of type (19) cannot be a facet of $\text{conv}(X)$. Therefore, for the given instances no violated inequality of type (19) is generated. Note that our separation heuristic for inequality (19) is different than that of [14, 15, 26] because our choice of set J also impacts the continuous term $t = \sum_{i \in I, j \in J:(i,j) \in E} x_{ij}$, which is not present in their setting. We have three heuristics for finding a violated inequality (27). Two of them uses the supply constraints as a base inequality for a certain choice of J (i.e. $\sum_{j \in J:(i,j) \in E} x_{ij} \leq s_i$ for $i \in V_1$ and $J \subseteq V_2$), one of which finds an inequality with $|\tilde{J}| = 1$ and the other finds an inequality with $|\tilde{J}| = |J| - 1$. The details for these heuristics are given by Algorithms 3 and 4, respectively. The third heuristic uses $\sum_{i \in V_1, j \in V_2:(i,j) \in E} x_{ij} \leq s(V_1)$ as a base inequality and finds a violated inequality with $\tilde{J}$ that includes the rejected markets and markets that have fractional $\bar{z}$ values. More details on this heuristic is given in Algorithm 5.

Table 1 compares the performance of algorithms BB and UC, to isolate the reduction in the root gap (8.3%) using our inequalities. Similarly, Table 2 compares the performance of the algorithms CD and UCD, to illustrate the marginal benefit of incorporating our inequalities into default CPLEX, where we observe a reduction in the root gap of 2.3%. Due to the reduction in the integrality gap the number of branch-and-cut nodes is almost always lower for UC and UCD compared to BB and CD, respectively. The solution times and the number of unsolved instances are slightly lower for algorithms that include our proposed inequalities. However, the end gap is not lower for algorithms UC and UCD compared to BB and CD, respectively. In conclusion, our preliminary computational results show that our proposed inequalities does have some positive effects, but the separation heuristics need to be significantly improved.

Algorithm 2 Heuristic separation for inequalities (15)

Input: $(\bar{x}, \bar{z})$

Output: Sets I and J and the corresponding cut for each fractional $\bar{z}$

```
I ← V_1
s(V_1 \ I) = 0
d(J) = 0
tempSupplies ← ∅
tempDemand ← ∅
switch = 0
for all the fractional variables z̄_j do
    tempDemand = {j}
    J = {j}
    while |tempDemand| ≥ 1 or |tempSupplies| ≥ 1 do
        if switch = 0 then
            for all the supplies i that have an edge to all nodes j in tempDemand do
                if x̄_ij > 0 then
                    I ← I \ {i}
                    s(V_1 \ I) ← s(V_1 \ I) + s_i
                    tempSupplies ← tempSupplies ∪ {i}
                end if
            end for
            switch = 1
            tempDemand ← ∅
        end if
        if switch = 1 then
            for all demand j that have an edge to all nodes i in tempSupplies do
                if x̄_ij > 0 then
                    J ← J ∪ {j}
                    d(J) ← d(J) + d_j
                    tempDemand ← tempDemand ∪ {j}
                end if
            end for
            switch = 0
            tempSupplies ← ∅
        end if
    end while
    if d(J) > s(V_1 \ I) and |J| ≥ 2 and max_{j∈J}{d_j} > d(J) − s(V_1 \ I) then
        if Σ_{i∈I, j∈J:(i,j)∈E} x̄_ij + Σ_{j∈J}(min{d(J) − s(V_1 \ I), d_j}) z̄_j < d(J) − s(V_1 \ I) then
            add inequality (15) with I and J
        end if
    end if
    I ← V_1
    s(V_1 \ I) = 0
    d(J) = 0
    switch = 0
end for
```

Algorithm 3 Heuristic separation for inequalities (27) that finds $|\tilde{J}| = 1$

Input: $(\bar{x}, \bar{z})$

Output: Sets I, J, $\tilde{J}$ and the corresponding cut for each fractional $\bar{z}$

$I, J, \tilde{J} \leftarrow \emptyset$
$d(J \setminus \tilde{J}) = 0$
$\alpha = 0$
for all the fractional variables $\bar{z}_j$ **do**
  $J \leftarrow \{j\}$, $\tilde{J} \leftarrow \{j\}$
  for all i such that $\bar{x}_{ij} > 0$ **do**
    $I \leftarrow \{i\}$
    for all $j^* \neq j$ **do**
      if $\bar{x}_{ij^*} = d_{j^*}$ **then**
        $J \leftarrow J \cup \{j^*\}$
        $d(J \setminus \tilde{J}) = d(J \setminus \tilde{J}) + d_{j^*}$
      end if
    end for
    $\alpha = s_i - d(J \setminus \tilde{J})$
    if $|J| \geq 2$ and $\sum_{j \in J:(i,j) \in E} \bar{x}_{ij} + \alpha \bar{z}_j > s_i$ **then**
      add inequality (27) with I, J, $\tilde{J}$ and α
    end if
    $I \leftarrow \emptyset$, $J \leftarrow \{j\}$, $d(J \setminus \tilde{J}) = 0$
  end for
end for

Algorithm 4 Heuristic separation for inequalities (27) that finds $|\tilde{J}| = |J| - 1$

Input: $(\bar{x}, \bar{z})$

Output: Sets I, J, $\tilde{J}$ and the corresponding cut for each fractional $\bar{z}$

```
J_0 ← {j ∈ V_2 : z̄_j = 0}
J_1 ← {j ∈ V_2 : z̄_j = 1}
I ← ∅
α = 0
maxdjJ̃ = max_{j∈J_1} {d_j}
for all the fractional variables z̄_j do
    J̃ ← J_1 ∪ {j}
    if maxdjJ̃ < d_j then
        maxdjJ̃ = d_j
    end if
    for all i ∈ V_1 do
        for all j' ∈ J_0 do
            if x̄_{ij'} > 0 and s_i > d_{j'} and s_i − d_{j'} < maxdjJ̃ then
                α = s_i − d_{j'}
                I ← {i}, J ← J̃ ∪ {j'}
                if ∑_{j∈J:(i,j)∈E} x̄_{ij} + α ∑_{j∈J̃} z̄_j > s_i + (|J̃| − 1)α then
                    add inequality (27) with I, J, J̃ and α
                end if
            end if
        end for
    end for
end for
```

Algorithm 5 Heuristic separation for inequalities (27) that finds general $\tilde{J}$

Input: $(\bar{x}, \bar{z})$

Output: Sets I, J, $\tilde{J}$ and the corresponding cut

```
J_f ← {j ∈ V_2 : 0 < z̄_j < 1}
J_1 ← {j ∈ V_2 : z̄_j = 1}
I ← V_1
J ← V_2
J̃ ← J_f ∪ J_1
α = 0
maxdjJ̃ = max_{j∈J_1∪J_f} {d_j}
if s(V_1) − d(V_2 \ J̃) > 0 and s(V_1) − d(V_2 \ J̃) < maxdjJ̃ then
    α = s(V_1) − d(V_2 \ J̃)
    if ∑_{i∈V_1, j∈V_2:(i,j)∈E} x̄_{ij} + α ∑_{j∈J̃} z̄_j > s(V_1) + (|J̃| − 1)α then
        add inequality (27) with I, J, J̃ and α
    end if
end if
```

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A Proofs of Section 2

In this section, we assume that all data are integral.

Proposition 1. *The decision version of TPMC is NP-complete even when:*

1. $s_i = 1$ *for all* $i \in V_1$, $d_j = d \geq 3$ *for all* $j \in V_2$, $w_{ij} = 0$ *for all* $(i,j) \in E$ *and* $r_j = 1$ *for all* $j \in V_2$.
2. $|V_1| = 1$ *and* $w_{ij} = 0$ *for all* $(i,j) \in E$.

Proof. Since TPMC is a mixed integer linear problem with rational data, it is in NP. We present two reductions to verify the two parts of this result.

1. We reduce every instance of the Exact 3-Cover (E3C) problem to an instance of TPMC. An instance of E3C is given as: Let B be a base set where $|B| = 3q$ for some $q \in \mathbb{N}$. Let C be a collection of subsets of B where each subset is of cardinality 3. Does there exist $D \subseteq C$ such that $|D| = q$ and the union of sets in D covers every element of B?

   It is well-known that E3C is strongly NP-complete [10]. Given an instance of E3C, we construct an instance of TPMC as follows: For every element in B, we construct a node in V_1 and for every element in C we construct a node in V_2. For $i \in V_1$, we use the notation $B(i)$ to denote the element of B corresponding to node i. Similarly, for $j \in V_2$, we let $C(j)$ denote the element of C corresponding to node j. We add an edge between $i \in V_1$ and $j \in V_2$ if $B(i) \in C(j)$. Let $s_i = 1$ for all $i \in V_1$. Let $d_j = 3$ for all $j \in V_2$. Let $w_{ij} = 0$ for all $(i,j) \in E$. Let $r_j = 1$ for all $j \in V_2$.

   Next, we verify that there exists $D \subseteq C$ such that $|D| = q$ and D covers every element of B if and only if there exists a feasible solution to TPMC with a cost at most $|C| - q$. Note that the size of the TPMC instance is polynomially bounded by the size of the E3C instance.

   (⇒) Assume that there exists $\{D_1, \dots, D_q\} =: D \subseteq C$ such that D covers every element of B. Let $D(u)$ represent the element of D (and therefore of C) that contains $u \in B$. Now construct the solution

   $$\hat{x}_{ij} = \begin{cases} 1 & \text{if } B(i) = u \text{ and } C(j) = D(u) \\ 0 & \text{otherwise.} \end{cases}$$
   $$\hat{z}_j = \begin{cases} 1 & \text{if } C(j) \notin \{D_1, \dots, D_q\} \\ 0 & \text{otherwise.} \end{cases}$$

   It is straightforward to verify that $(\hat{x}, \hat{z})$ satisfies all the constraints of TPMC and $\sum_{(i,j)\in E} w_{ij}\hat{x}_{ij} + \sum_{j \in V_2} r_j \hat{z}_j = |C| - q$.
   (⇐) Consider a solution $(\hat{x}, \hat{z})$ of TPMC such that

   $$\sum_{(i,j)\in E} w_{ij}\hat{x}_{ij} + \sum_{j\in V_2} r_j \hat{z}_j = \sum_{j \in V_2} \hat{z}_j \leq |C| - q. \tag{58}$$

   Since there are $3q$ supply nodes, each with a capacity of 1, the demand of at most q nodes can be satisfied. Therefore, from (58), we conclude that there are exactly q nodes whose demands are satisfied. Let $D = \{C(j) \mid \sum_{i \in V_1} \hat{z}_j = 0\}$. Clearly, $|D| = q$ and D covers every element of B. As a result TPMC is strongly NP-complete.

2. We reduce every instance of the Subset Sum (SS) problem to an instance of TPMC. An instance of SS is given as: Let A be a finite set, $a_n \in \mathbb{Z}^+$ be the size of each element $n \in A$ and B be a positive integer. Does there exist a subset $A^{'} \subseteq A$ such that the sum of the sizes of the elements in $A^{'}$ is exactly B?

   It is well-known that SS is NP-complete [10]. Given an instance of SS, we construct an instance of TPMC as follows: We construct a single node $V_1 = \{1\}$ and for every element in A we construct a node in V_2. We add all the edges between the nodes in V_1 and V_2. Let the single supply be $s_1 = B$. Let demand of market j be $d_j = a_j$ for all $j \in V_2 = A$. Finally, let the unit shipping costs and lost revenues be $w_{1j} = 0$ and $r_j = d_j$, for $j \in V_2$.

   Next, we verify that there exists subset $A^{'} \subseteq A$ such that the sum of the sizes of the elements in $A^{'}$ is exactly B if and only if there exists a feasible solution to TPMC with a cost of at most $\sum_{n \in A} a_n - B$. Note that the size of the TPMC instance is polynomially bounded by the size of the SS instance.

   ($\Rightarrow$) Assume that there exists a subset $A^{'} \subseteq A$ such that the sum of the sizes of the elements in $A^{'}$ is exactly B. Now construct the solution

$$\hat{x}_{1j} = \begin{cases} a_j & \text{if } j \in A^{'} \\ 0 & \text{otherwise,} \end{cases}$$

$$\hat{z}_j = \begin{cases} 1 & \text{if } j \notin A^{'} \\ 0 & \text{otherwise.} \end{cases}$$

   It is straightforward to verify that $(\hat{x}, \hat{z})$ satisfies all the constraints of TPMC and $\sum_{(i,j)\in E} w_{ij}\hat{x}_{ij} + \sum_{j \in V_2} r_j \hat{z}_j = \sum_{n \in A} a_n - B$.
   ($\Leftarrow$) Consider a solution $(\hat{x}, \hat{z})$ of TPMC such that

$$\sum_{(i,j)\in E} w_{ij}\hat{x}_{ij} + \sum_{j \in V_2} r_j \hat{z}_j = \sum_{j \in V_2} a_j \hat{z}_j \leq \sum_{n \in A} a_n - B. \tag{59}$$

   The total demand satisfied by any feasible solution is at most B since we cannot satisfy more than the supply. Furthermore, since each edge has a cost per unit flow of 0, we have that $\sum_{(i,j)\in E} w_{ij}\hat{x}_{ij} = 0$. Therefore, from (59), the total demand satisfied must equal B. Let the set of satisfied demand nodes be $A^{'} = \{j \in A : \hat{z}_j = 0\}$, so we have $\sum_{n \in A'} a_n = B$.

□

Proposition 2. *Suppose that $d_j \leq 2$ for all $j \in V_2$. Then there exists a polynomial-time algorithm to solve TPMC.*

Proof. We can convert a given instance of TPMC with $d_j \leq 2$ for all $j \in V_2$ and arbitrary supplies into an equivalent instance with all supplies equal to 1. Observe that in any feasible solution since $d_j \leq 2$ for all $j \in V_2$, no supply can send more than $2|V_2|$ units. Therefore, if $s_i > 1$ for some $i \in V_1$, then we construct an updated instance by replacing supply node $i \in V_1$ with $\min\{s_i, 2|V_2|\}$ supply nodes with a capacity of 1 and unit shipping cost to demand node j of w_{ij} for $(i,j) \in E$. Notice that the resulting instance is polynomial in the size of the original problem. Therefore from now on we assume that $s_i = 1$ for all $i \in V_1$.

We show that TPMC with $d_j \leq 2$ for all $j \in V_2$ is equivalent to the problem of finding a minimum weight perfect matching on a suitably constructed general graph $G' = (V', E')$.

1. For each $i \in V_1$, we add a corresponding $i \in V'$ and similarly for each $j \in V_2$ we add $j \in V'$. (When we use notation $V_1 \subseteq V'$, V_1 represents the vertices of V' corresponding to the vertices V_1 of G; similary for V_2.)

2. Let $M_1 = \{j \in V_2 : d_j = 1\}$ and $M_2 = \{j \in V_2 : d_j = 2\}$.

3. For each demand node $j \in V_2$, add a node $j' \in V'$ (note that this is in addition to $j \in V'$ for $j \in V_2$ as described in 1.). Add an edge $(j, j') \in E'$ with a cost of r_j. We refer to the set of nodes $j' \in V'$ corresponding to $j \in M_1$ as M_1'. (We define M_2' similarly.)

4. For each $i \in V_1$ such that $(i, j) \in E$ and $j \in M_1$, add the edge $(i, j) \in E'$ with cost of w_{ij}.

5. For each $i \in V_1$ such that $(i, j) \in E$ and $j \in M_2$, add two nodes, $ij1, ij2 \in V'$. Add edges $(i, ij1), (ij1, ij2), (ij2, j), (ij2, j') \in E'$ with costs $\frac{w_{ij}}{2}$, 0, $\frac{w_{ij}}{2}$, $\frac{w_{ij}}{2}$ respectively.

6. If $|V_1|$ is odd, we add an additional artificial node $\{0\}$ to V'. Let $V_1' \subseteq V'$ be defined as $V_1' := V_1 \cup M_1'$ if $|V_1|$ is even and $V_1' := V_1 \cup M_1' \cup \{0\}$ if $|V_1|$ is odd.

7. For all $u, v \in V_1'$ such that $u \neq v$, add an edge $(u, v) \in E'$ with a cost of 0. Therefore, the subgraph induced by the nodes in V_1' is a complete graph/clique.

Note that the size of the resulting minimum weight perfect matching problem is polynomial in the size of the TPMC problem. Figure 2 illustrates the original graph of a TPMC instance, where the demand of market A is 2 and that of market B is 1. Figure 3 illustrates the new graph. (The clique induced by $V_1 \cup \{B'\} \cup \{0\}$ is not shown.)

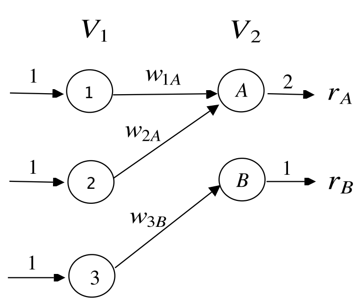

Figure 2: A TPMC instance

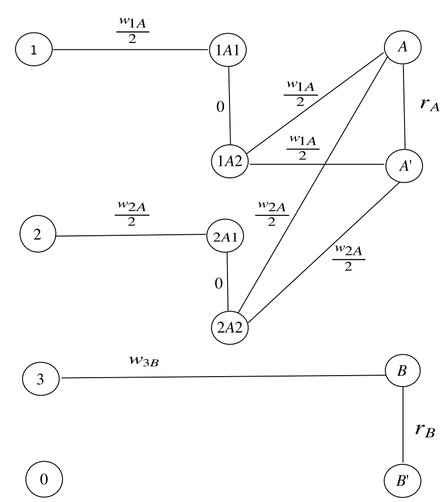

Figure 3: Construction of G'

We next show that any solution to the TPMC problem corresponds to a perfect matching in $G' = (V', E')$. Consider a feasible solution (x, z) to the TPMC problem. If $z_j = 0$ for $j \in M_1$, then there exists exactly one supply node i such that $x_{ij} = 1$. For constructing a matching in G', we choose edge (i, j), where $i \in V_1$ and $j \in M_1$, thereby covering nodes i and j in V'. If $z_j = 0$ for $j \in M_2$, then there exists two supply nodes i_1 and $i_2 \in V_1$ such that $x_{i_1 j} = x_{i_2 j} = 1$. For constructing a matching in G', without loss of generality, we choose edges $(i_1, i_1 j1)$, $(i_1 j2, j)$, $(i_2, i_2 j1)$ and $(i_2 j2, j')$, thereby covering nodes $i_1, i_2, i_1 j1, i_1 j2, i_2 j1, i_2 j2, j, j'$. If $z_j = 1$ for $j \in V_2$, then no supply node i sends demand to j and for the matching we choose edge (j, j'), hence covering nodes j and j' in V'. Moreover if $j \in M_2$, we choose edges $(ij1, ij2)$ for all $(i, j) \in E, i \in V_1$ in the matching and therefore the nodes $ij1$, $ij2$, j, j' are also covered. Hence whether $z_j = 1$ or $z_j = 0$, and whether $j \in M_1$ or $j \in M_2$, the nodes in V_2, M_2', and the nodes $ij1, ij2$ for all $(i, j) \in E, j \in M_2$ are always covered by the edges in the matching we have selected thus far. To complete the proof we show how nodes $i \in V_1'$ are also covered in all cases by extending the matching we have until now.

Let $\bar{M}_1 = \{j \in M_1 : z_j = 0\}$, $\bar{M}_2 = \{j \in M_2 : z_j = 0\}$ and $\bar{V}_1 = \{i \in V_1 : x_{ij} = 1\}$. In other words, set $\bar{M}_1$ represents the nodes $j \in M_1$ whose unit demands are satisfied, set $\bar{M}_2$ represents the nodes $j \in M_2$ whose demands, $d_j = 2$, are satisfied, and set $\bar{V}_1$ represents the set of supply nodes that send demand. Observe that the nodes in $\bar{V}_1$ are also covered in the matching constructed thus far. However, the nodes $j \in V_1 \setminus \bar{V}_1$, and $j' \in M_1'$ for $j \in \bar{M}_1$ and $\{0\}$ (if it exists) are not yet covered. Note that $|\bar{V}_1| = |\bar{M}_1| + 2|\bar{M}_2|$. We consider two cases.

1. $|V_1|$ is even. If $|\bar{V}_1|$ is even, then $|V_1| - |\bar{V}_1|$ and $|\bar{M}_1|$ are even. If $|\bar{V}_1|$ is odd, then $|V_1| - |\bar{V}_1|$ and $|\bar{M}_1|$ are odd. Therefore, $|V_1| - |\bar{V}_1| + |\bar{M}_1|$ is always even. Thus, we can cover all nodes $i \in V_1 \setminus \bar{V}_1$ and $j' \in M_1'$ for $j \in \bar{M}_1$ using $\frac{|V_1| - |\bar{V}_1| + |\bar{M}_1|}{2}$ many disjoint edges that exist between them (recall that the subgraph induced by the nodes $i \in V_1'$ form a complete graph).

2. $|V_1|$ is odd. If $|\bar{V}_1|$ is even, then $|V_1| - |\bar{V}_1|$ is odd and $|\bar{M}_1|$ is even. If $|\bar{V}_1|$ is odd, then $|V_1| - |\bar{V}_1|$ is even and $|\bar{M}_1|$ is odd. Therefore, $|V_1| - |\bar{V}_1| + |\bar{M}_1|$ is always odd. Recall that when $|V_1|$ is odd we have an additional dummy node $\{0\}$ that forms a fully connected graph with nodes $i \in V_1$ and $j \in M_1'$. Therefore, we obtain an even number of nodes that need to be covered by choosing $\frac{|V_1| - |\bar{V}_1| + |\bar{M}_1| + 1}{2}$ disjoint edges.

So we have verified that given any solution to the TPMC problem we can find a perfect matching in $G' = (V', E')$. Moreover, it is straightforward to check that the cost of this matching is equal to the cost of the given solution to TPMC.

Next we show that any solution to the perfect matching in $G' = (V', E')$ corresponds to a feasible solution of the TPMC problem. Let P be the set of edges that are in the perfect matching. If edge $(j', j) \in P$ for $j' \in M_1'$, $j \in M_1$ (or $j \in M_2$, $j' \in M_2'$), then set $z_j = 1$. Set all remaining $z_j = 0$. If edge $(i, j) \in P$ for $j \in M_1$, then we set $x_{ij} = 1$. If edge $(i, ij1) \in P$, then set $x_{ij} = 1$. Set all remaining $x_{ij} = 0$. Note that due to the construction of graph G', a supply node $i \in V_1$ can send at most 1 unit of demand. Similarly for $j \in M_1$ a single edge that has j as one of its endpoints will be selected. For $j' \in M_2'$, $j \in M_2$ if edge $(j, j') \in P$, then for any $i \in V_1$ edges $(ij2, j), (ij2, j') \notin P$. However, if edge $(j, j') \notin P$ then for a perfect matching there must exist exactly two $i_1, i_2 \in V_1$ such that $(i_1j2, j), (i_2j2, j') \in P$. Therefore, for any $j \in M_2$ either the demand is fully satisfied or it is rejected altogether. Finally, it is easy to see that the cost of the solution to the TPMC problem is equivalent to the cost of the corresponding perfect matching in G', completing the proof.

□

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