On obtaining the convex hull of quadratic inequalities via aggregations

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Outline

1
Introduction

1.1
QCQP: Need for convexification

Quadratically Constrained Quadratic Program

QCQP

Quadratic objective, quadratic constraints:

$$\begin{aligned} \max \quad & x^\top Q_0 x + b_0^\top x \\ \text{s.t.} \quad & x^\top Q_i x + b_i^\top x \leq d_i \ \forall i \in [m] \end{aligned}$$

Quadratically Constrained Quadratic Program

QCQP

May be equivalently written as:

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & x^\top Q_i x + b_i^\top x \leq d_i \ \forall i \in [m] \end{array}$$

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1. So, we care about finding:

$$\text{conv}\left\{ x \;\middle|\; x^\top Q_i x + b_i^\top x \leq d_i \ \forall i \in [m] \right\}$$

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2. This is challenging to compute! So we can consider convexification of relaxations (similar to integer programming)

1.2
Two row relaxation

Two row relaxation

- We can select two rows and try and find the convex hull of their interesection:

$$\mathcal{C}2 = \left\{ x \in \mathbb{R}^n \;\middle|\; x^\top Q_i x + b_i^\top x \le d_i \;\forall i \in [2] \right\}$$

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- (For some technical reasons), let us consider the “open version" of the above set:

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- It turns out convex hull of $\mathcal{O}2$ is well understood!

Lets first talk about aggregation

- Given $\lambda \in \mathbb{R}^m_+$ and

$$S := \left\{ x \;\middle|\; x^\top Q_i x + b_i^\top x \spadesuit d_i \; \forall i \in [m] \right\},$$

where $\spadesuit \in \{\leq, <\}$.

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where $\spadesuit \in \{\leq, <\}$.

Then:

$$S^\lambda := \left\{ x \;\middle|\; x^\top \left(\sum_{i=1}^m \lambda_i Q_i\right) x + \left(\sum_{i=1}^m \lambda_i b_i\right)^\top x \;\; \spadesuit \;\; \left(\sum_{i=1}^m \lambda_i d_i\right) \; \forall i \in [m] \right\}$$

is a relaxation of S.

- Basically, we are multiplying i^{th} constraint by λ_i and then add them together.

Convex hull of $\mathcal{O}2$

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Theorem ([Yildiran (2009)])

Given a set $\mathcal{O}2$*, such that* $\operatorname{conv}(\mathcal{O}2) \neq \mathbb{R}^n$*, there exists* $\lambda^1, \lambda^2 \in \mathbb{R}^2_+$ *such that:*

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- The paper [Yildiran (2009)] gives algorithm to compute λ_1 and λ_2.
- The quadratic constraints $(\mathcal{O}2)^{\lambda^i}$ $i \in \{1, 2\}$ has very nice properties:
  - $\sum_{j=1}^{2} \lambda_j^i Q_j$ has at most one negative eigenvalue for both $i \in \{1, 2\}$!

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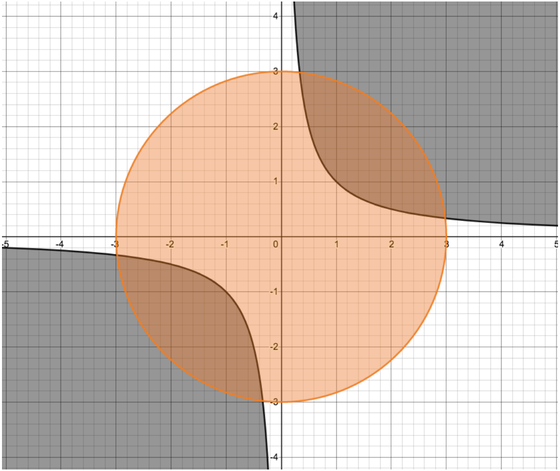
  - Basically, the sets $(\mathcal{O}2)^{\lambda_i}$ $i \in \{1, 2\}$ are either ellipsoid (may be degenarate) or hyperboloid which is union of two convex sets.
  - Henceforth, we call quadratic constraints with the “quadratic part" having at most one negative eigenvalue as a *good* constraint.

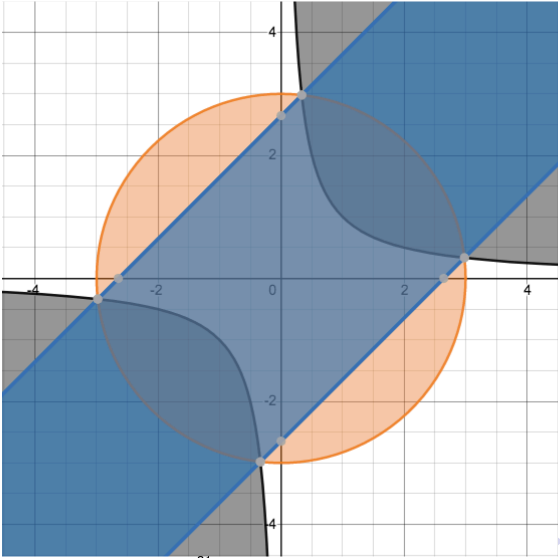
Example

$$S := \left\{ x, y \;\middle|\; \begin{array}{rcl} -xy & < & -1 \\ x^2 + y^2 & < & 9 \end{array} \right\}$$



Example - contd 1

$$\mathrm{conv}(S) := \left\{ x, y \;\middle|\; \begin{array}{lcl} (x-y)^2 & < & 7 \\ x^2 + y^2 & < & 9 \end{array} \right\}$$



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- Understanding the blue quadratic: $\lambda^1 = (2, 1)$

$$\begin{array}{lcccc} & -xy & < & -1 & \times 2 \\ + & x^2 + y^2 & < & 9 & \times 1 \\ \hline \end{array}$$

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- $\lambda^2 = (0, 1)$, so the second aggregated constraints is $x^2 + y^2 < 9$.

Literature survey (incomplete!)

Related results:

- [Yildiran (2009)]
- [Burer and Kılınc-Karzan (2017)] (second order cone intersection with a nonconvex quadratic)
- [Modaresi and Vielma (2017)] (closed version of results)

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Other related papers:

- [Tawarmalani, Richard, Chung (2010)] (Covering bilinear knapsack)
- [Santana and Dey (2020)] (polytope and one quadratic constraint)
- [Ye and Zhang (2003)], [Burer and Anstreicher (2013)], [Beinstock (2014)] [Burer (2015)], [Burer and Yang (2015)], [Anstreicher (2017)] (extended trust-region problem)
- [Burer and Ye (2019)], [Wang and Kılınc-Karzan (2020, 2021)], [Argue, Kılınc-Karzan, and Wang (2020)] (general conditions for the SDP relaxation being tight)
- [Bienstock, Chen, and Muñoz (2020)], [Muñoz and Serrano (2020)] (Cut for QCQP using intersction approach)
- ...

Questions we consider...

The main goal of this study is to understand the power of aggregation.

- What happens for $m > 2$, i.e. for the case of two rows.
- If we cannot obtain the convex hull via aggregation, then can be identify explicit examples.
- etc.

2 Main results

Results described in a high level

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- The above result represents the limit of aggregations. Basically, aggregations do not lead to convex hull even when the technical suffcient condition does not hold for $m = 3$ or when $m > 3$.

2.1 Three rows

Three rows: main result

Theorem

Let $n \geq 3$ and

$$\mathcal{O}3 = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0,\ i \in [3] \right\}.$$

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Assume

- *(Positive definite linear combination, (PDLC)) There exists $\theta \in \mathbb{R}^3$ such that*
$$\sum_{i=1}^{3} \theta_i \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix} \succ 0.$$
- *(Non-trivial convex hull)* $\text{conv}(\mathcal{O}3) \neq \mathbb{R}^n$.

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Then

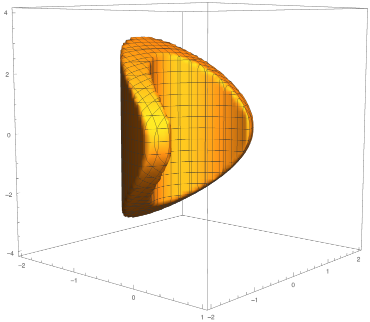
$$\mathrm{conv}(\mathcal{O}3) = \bigcap_{\lambda \in \Omega} (\mathcal{O}3)^\lambda.$$

Example

- $$S := \left\{ (x,y,z) \;\middle|\; \begin{array}{rcl} x^2 + y^2 & < & 2 \\ -x^2 - y^2 & < & -1 \\ -x^2 + y^2 + z^2 + 6x & < & 0 \end{array} \right\}$$
- PDLC condition holds, $\text{conv}(S) \neq \mathbb{R}^3$
- $$\text{conv}(S) := \left\{ (x,y,z) \;\middle|\; \begin{array}{rclc} x^2 + y^2 & < & 2 & \\ -2x^2 + z^2 + 6x & < & -1 & \star \\ -x^2 + y^2 + z^2 + 6x & < & 0 & \end{array} \right\}$$

  $\star$: sum of second and third constraint describing S

Example -contd 1



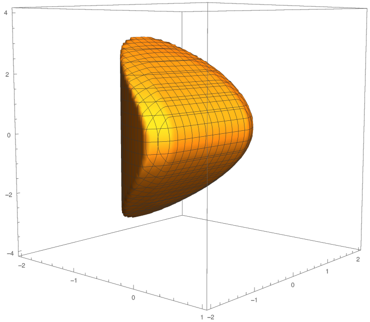


Figure: Plots of sets S (left) and $\mathrm{conv}(S)$ (right).

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How many aggregated inequalities needed to describe convex hull?	2	∞ (Conjecture!)
Structure of aggregated inequalities	Polynomial-time algorithm exists to find them	Even checking if $\lambda \in \Omega$ is not clear.

The closed case

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- *(Non-trivial convex hull)* $\mathrm{conv}(\mathcal{C}3) \neq \mathbb{R}^n$.
- *(No low-dimensional components)* $\mathcal{C}3 \subseteq \overline{\mathrm{int}(\mathcal{C}3)}$.

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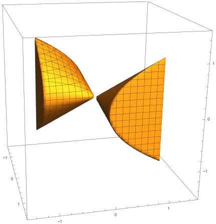
$$\overline{\mathrm{conv}(\mathcal{C}3)} = \bigcap_{\lambda \in \Omega'} (\mathcal{C}3)^\lambda.$$

2.2
Counter examples

$m = 3$ but not satisfying PDLC condition

$$S := \left\{ (x, y, z) \;\middle|\; \begin{array}{rcl} x^2 & < & 1 \\ y^2 & < & 1 \\ -xy + z^2 & < & 0 \end{array} \right\}$$



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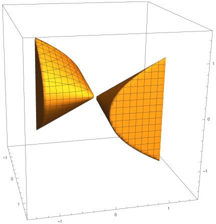

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$$\boxed{\blacktriangleright\ \text{conv}(S) \neq \cap_{\lambda \in \Omega} S^{\lambda}}$$

m = 4 (and satisfying PDLC)

$$S := \left\{ (x,y,z) \;\middle|\; \begin{array}{rcl} x^2 + y^2 + z^2 + 2.2(xy + yz + xz) & < & 1 \\ -2.1x^2 + y^2 + z^2 & < & 0 \\ x^2 - 2.1y^2 + z^2 & < & 0 \\ x^2 + y^2 - 2.1z^2 & < & 0 \end{array} \right\}$$

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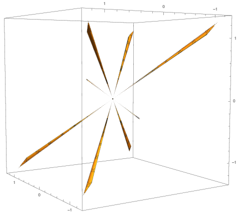

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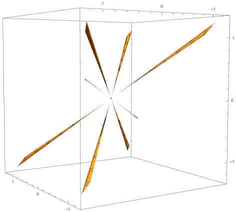
$$S := \left\{ (x,y,z) \;\middle|\; \begin{array}{rcl} x^2+y^2+z^2+2.2(xy+yz+xz) & < & 1 \\ -2.1x^2+y^2+z^2 & < & 0 \\ x^2-2.1y^2+z^2 & < & 0 \\ x^2+y^2-2.1z^2 & < & 0 \end{array} \right\}$$

- PDLC condition holds, $\operatorname{conv}(S) \neq \mathbb{R}^3$



- $\boxed{\operatorname{conv}(S) \neq \cap_{\lambda\in\Omega} S^{\lambda}}$

Do we need a finite number of aggregations?

$$S := \{x, y \mid x^2 \leq 1,\ y^2 \leq 1,\ (x-1)^2 + (y-1)^2 \geq 1\},$$

- PDLC does not hold
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- Let $\Omega^+ := \{\lambda \in \mathbb{R}^3_+ \mid S^\lambda \supseteq \text{conv}(S)\}$
- $\text{conv}(S) = \bigcap_{\lambda \in \Omega^+} S^\lambda$.
- $\text{conv}(S) \subsetneq \bigcap_{\lambda \in \tilde{\Omega}^+} S^\lambda$ for any $\tilde{\Omega}^+ \subseteq \Omega^+$ which is finite.

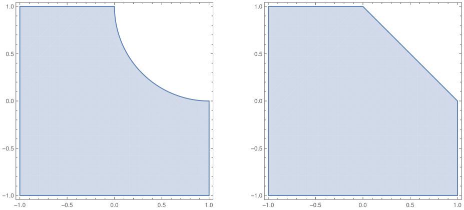

Figure: Plots of sets S (left) and $\text{conv}(S)$ (right).

3
Proof outlines

3.1
A new S-lemma

A new S-Lemma for 3 quadratic constraints

Lemma

Let $n \geq 3$ and let $g_1, g_2, g_3 : \mathbb{R}^n \to \mathbb{R}$ be homogeneous quadratic functions:

$$g_i(x) = x^\top Q_i x.$$

Assuming there is a linear combination of Q_1, Q_2, Q_3 that is positive definite, the following equivalence holds

$$\{x \in \mathbb{R}^n : g_i(x) < 0,\ i \in [3]\} = \emptyset \iff \exists \lambda \in \mathbb{R}^3_+ \setminus \{0\}, \sum_{i=1}^{3} \lambda_i Q_i \succeq 0.$$

Main ingredients for proving new S-lemma

- SDP strong duality (under staler condition)
-

Theorem ([Barvinok (2001)])

If $\mathcal{A} \subseteq \mathbb{S}^n$ is an affine subspace such that the intersection $\mathbb{S}^n_+ \cap \mathcal{A}$ is non-empty, bounded and $\dim(\mathcal{A}) \geq \binom{n+1}{2} - \binom{r+2}{2}$ then there is a matrix $X \in \mathbb{S}^n_+ \cap \mathcal{A}$ such that $\text{rank}(X) \leq r$.

3.2 Rest of proof

Lets try to prove $\text{conv}(S) = \cap_{\lambda \in \Omega} S^{\lambda}$

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- (Homogenization) The above together with $\mathrm{conv}(S) \neq \mathbb{R}^n$ can be shown to imply: $\{x | \alpha^\top x = \beta x_{n+1}\}$ (call it H) does not intersect homogenization of S:

$$H \cap \left\{ (x, x_{n+1} \,|\, [x \quad x_{n+1}] \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} < 0,\ i \in [3] \right\} = \emptyset.$$

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- (Apply S-lemma, assuming PDLC) We obtain $\lambda \in \Omega$ such that

$$H \cap \left\{ (x, x_{n+1} \,|\, [x \quad x_{n+1}] \left(\sum_{i=1}^{3} \lambda_i \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix} \right) \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} < 0, \right\} = \emptyset.$$

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- It turns, the above is sufficient to show that $x^* \neq S^{\lambda}$.

Thank You