Constrained Infinite Group Relaxations of MIPs

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Outline

Introduction

Sublinear Functions and S-free Convex Sets

Constructing Minimal Inequalities

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Extreme Inequalities for two Rows

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Introduction

A relaxation of MIPs

Simplex Tableau

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$a_{1,1}x_1$	$\cdots +$	$a_{1,k}x_k$	$+$	$a_{1,k+1}y_{k+1}$	$\cdots +$	$a_{1,n}y_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
x_{B_m}	$=$	f_m	$+$	$a_{m,1}x_1$	$\cdots +$	$a_{m,k}x_k$	$+$	$a_{m,k+1}y_{k+1}$	$\cdots +$	$a_{m,n}y_n$
$y_{B_{m+1}}$	$=$	f_{m+1}	$+$	$a_{m+1,1}x_1$	$\cdots +$	$a_{m+1,k}x_k$	$+$	$a_{m+1,k+1}y_{k+1}$	$\cdots +$	$a_{m+1,n}y_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
y_{B_p}	$=$	f_p	$+$	$a_{p,1}x_1$	$\cdots +$	$a_{p,k}x_k$	$+$	$a_{p,k+1}y_{k+1}$	$\cdots +$	$a_{pn}y_n$

1. $x_{B_1}, \dots, x_{B_m} \in \mathbb{Z}_+$
2. $y_{B_{m+1}}, \dots, y_{B_p} \in \mathbb{R}_+$
3. $x_1, \dots, x_k \in \mathbb{Z}_+$
4. $y_{k+1}, \dots, y_n \in \mathbb{R}_+$

Solution is ‘fractional’, i.e. $f_1, \dots, f_m$ are not all integer.

A relaxation of MIPs

Relaxation Step 1: Drop Some Constraints

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$a_{1,1}x_1$	$\cdots +$	$a_{1,k}x_k$	$+$	$a_{1,k+1}y_{k+1}$	$\cdots +$	$a_{1,n}y_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
x_{B_m}	$=$	f_m	$+$	$a_{m,1}x_1$	$\cdots +$	$a_{m,k}x_k$	$+$	$a_{m,k+1}y_{k+1}$	$\cdots +$	$a_{m,n}y_n$
$y_{B_{m+1}}$	$=$	f_{m+1}	$+$	$a_{m+1,1}x_1$	$\cdots +$	$a_{m+1,k}x_k$	$+$	$a_{m+1,k+1}y_{k+1}$	$\cdots +$	$a_{m+1,n}y_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
y_{B_p}	$=$	f_p	$+$	$a_{p,1}x_1$	$\cdots +$	$a_{p,k}x_k$	$+$	$a_{p,k+1}y_{k+1}$	$\cdots +$	$a_{pn}y_n$

1. $x_{B_1}, \ldots, x_{B_m} \in \mathbb{Z}_+$
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A relaxation of MIPs

Relaxation of Simplex Tableau

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$a_{1,1}x_1$	$\cdots +$	$a_{1,k}x_k$	$+$	$a_{1,k+1}y_{k+1}$	$\cdots +$	$a_{1,n}y_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
x_{B_m}	$=$	f_m	$+$	$a_{m,1}x_1$	$\cdots +$	$a_{m,k}x_k$	$+$	$a_{m,k+1}y_{k+1}$	$\cdots +$	$a_{m,n}y_n$

1. $x_{B_1}, \ldots, x_{B_m} \in \mathbb{Z}_+$
2. $x_1, \ldots, x_k \in \mathbb{Z}_+$
3. $y_{k+1}, \ldots, y_n \in \mathbb{R}_+$

Solution is ‘fractional’, i.e. $f_1, \ldots, f_m$ are not all integer.

A relaxation of MIPs

Relaxation Step 2: Drop Integrality Requirement

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$a_{1,1}x_1$	$\cdots +$	$a_{1,k}x_k$	$+$	$a_{1,k+1}y_{k+1}$	$\cdots +$	$a_{1,n}y_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
x_{B_m}	$=$	f_m	$+$	$a_{m,1}x_1$	$\cdots +$	$a_{m,k}x_k$	$+$	$a_{m,k+1}y_{k+1}$	$\cdots +$	$a_{m,n}y_n$

1. $x_{B_1}, \ldots, x_{B_m} \in \mathbb{Z}_+$
2. $x_1, \ldots, x_k \in \mathbb{Z}_+ \xrightarrow{\text{Relaxation}} x_1, \ldots, x_k \in \mathbb{R}_+$
3. $y_{k+1}, \ldots, y_n \in \mathbb{R}_+$

Solution is 'fractional', i.e. $f_1, \ldots, f_m$ are all not integer.

A relaxation of MIPs

Relaxation Step 2: Drop Integrality Requirement

Basic Variable		rhs				Columns Corresponding to Continous Variables				
x_{B_1}	$=$	f_1	$+$	$a_{1,1}y_1$	$\cdots +$	$a_{1,k}y_k$	$+$	$a_{1,k+1}y_{k+1}$	$\cdots +$	$a_{1,n}y_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
x_{B_m}	$=$	f_m	$+$	$a_{m,1}y_1$	$\cdots +$	$a_{m,k}y_k$	$+$	$a_{m,k+1}y_{k+1}$	$\cdots +$	$a_{m,n}y_n$

1. $x_{B_1}, \ldots, x_{B_m} \in \mathbb{Z}_+$
2. $y_1, \ldots, y_k, y_{k+1}, \ldots, y_n \in \mathbb{R}_+$

$f_1, \ldots, f_m$ are all not integer.

Relaxation Step 3: Continuous ‘Group’ Relaxation

$$\begin{array}{ccccccccccccc}
\text{Basic Variable} & & \text{rhs} & & & \text{Columns With Continous Variables} & & & & & \\
x_{B_1} & = & f_1 & + & a_{1,1}y_1 & \cdots + & a_{1,k}y_k & + & a_{1,k+1}y_{k+1} & \cdots + & a_{1,n}y_n \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\
x_{B_m} & = & f_m & + & a_{m,1}y_1 & \cdots + & a_{m,k}y_k & + & a_{m,k+1}y_{k+1} & \cdots + & a_{m,n}y_n
\end{array}$$

1. $x_{B_1}, \ldots, x_{B_m} \in \mathbb{Z}_+ \xrightarrow{\text{Relaxation}} x_{B_1}, \ldots, x_{B_m} \in \mathbb{Z}$
2. $y_1, \ldots, y_k, y_{k+1}, \ldots, y_n \in \mathbb{R}_+$

$f_1, \ldots, f_m$ are all not integer.

Gomory and Johnson (1972), Johnson (1974)

The continuous infinite group relaxation

$$
\begin{aligned}
x = f + \sum_{i=1}^{n} a^i y_i \\
x \in \mathbb{Z}^m \\
y_i \in \mathbb{R}_+
\end{aligned}
$$

The continuous infinite group relaxation

$$x = f + \sum_{i=1}^{n} a^i y_i$$
$$x \in \mathbb{Z}^m$$
$$y_i \in \mathbb{R}_+$$

Relaxation by addition of more non-negative variables.

$$\Downarrow$$

$$x = f + \sum_{w \in W} w y(w)$$
$$x \in \mathbb{Z}^m$$
$$y(w) \in \mathbb{R}_+$$
$$y \text{ has finite support}$$

where W is some subset of $\mathbb{R}^m$.
We call this set $R(f, \mathbb{Z}^m, W)$.

- Andersen, Louveaux, Weismantel, Wolsey (2007) W is finite
- Borozan and Cornuéjols (2007) $W = \mathbb{Q}^m$
- Cornuéjols and Margot (2008) W finite and $W = \mathbb{Q}^m$
- Basu, Conforti, Cornuéjols, Zambelli (2009a) $W = \mathbb{R}^m$

Constrained continuous infinite group relaxation

Suppose we had not relaxed the constraints (such as non-negativity) on basic variables.

$$x = f + \sum_{i=1}^{n} a^i y_i$$
$$x \in S := \{u \in \mathbb{Z}^m \mid Gu \leq h\}$$
$$y_i \in \mathbb{R}_+$$

Relaxation by addition of more non-negative variables.

$$\Downarrow$$

$$x = f + \sum_{w \in W} w y(w)$$
$$x \in S := \{u \in \mathbb{Z}^m \mid Gu \leq h\}$$
$$y(w) \in \mathbb{R}_+$$
$$y \text{ has finite support}$$

where W is some subset of $\mathbb{R}^m$.
We call this set $R(f, S, W)$.

(We assume $\operatorname{conv}(S)$ is full-dimensional)

Valid inequality

$$x = f + \sum_{w \in W} w y(w)$$
$$x \in S := \{u \in \mathbb{Z}^m \mid Gu \leq h\}$$
$$y(w) \in \mathbb{R}_+$$
$$y \text{ has finite support}$$

Definition

A function $\pi : W \to \mathbb{R}$ is called a valid inequality for $R(f, S, W)$ if

$$\sum_{w \in W} \pi(w) y(w) \geq 1 \quad \forall (x, y) \in R(f, S, W). \tag{1}$$

Example: using valid inequality

$$x = f + \sum_{w \in \mathbb{R}^2} wy(w) \tag{2}$$

$$x \in S \tag{3}$$

$$y(w) \in \mathbb{R}_+ \tag{4}$$

$$\text{y has finite support.} \tag{5}$$

Let $f = (0.5, 0.5)$ and let $S = \{(x_1, x_2) \in \mathbb{Z}^2 \mid 0 \leq x_1 \leq 1,\ x_2 \geq 0\}$. It can be verified that the function

$$\pi(w) = \begin{cases} 1.5w_1 + 0.5w_2 & \text{if } w_2 \geq 0,\ 3w_1 \geq w_2 \\ -4.5w_1 + 2.5w_2 & \text{if } w_2 \geq 2w_1,\ w_2 \geq 3w_1 \\ 1.5w_1 - 0.5w_2 & \text{if } w_2 \leq 0, 2w_1 \geq w_2 \end{cases} \tag{6}$$

yields a valid inequality for $R(f, S, \mathbb{R}^2)$ of the form $\sum_{w \in \mathbb{R}^2} \pi(w)y(w) \geq 1$.

Example: using valid inequality

$$x = f + \sum_{w \in \mathbb{R}^2} w y(w) \tag{2}$$

$$x \in S \tag{3}$$

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yields a valid inequality for $R(f, S, \mathbb{R}^2)$ of the form $\sum_{w \in \mathbb{R}^2} \pi(w) y(w) \geq 1$.

π can be used to generate a valid inequality for a mixed integer set such as:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} y_1 + \begin{pmatrix} -1 \\ -2 \end{pmatrix} y_2 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} y_3 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} y_4$$

$$x_1 \in \{0, 1\},\ x_2 \in \mathbb{Z}_+, y_1, y_2, y_3, y_4 \geq 0. \tag{7}$$

Let r^i denote the column corresponding to y_i above. Then $\sum_{i=1}^{4} \pi(r^i) y_i \geq 1$ is the inequality

$$3y_1 - 0.5y_2 + 0.5y_3 + 1.5y_4 \geq 1,$$

and cuts off the solution $y = \bar{0}$ and $x = (0.5, 0.5)^T$.

2

Sublinear Functions and *S*-free Convex Sets

Sublinear valid inequality

- Basu, Conforti, Cornuéjols, Zambelli (2009a) $S = \mathbb{Z}^m$, $W = \mathbb{R}^m$

Proposition

Let $\pi : \mathbb{R}^m \to \mathbb{R}$ be a valid inequality for $R(f, S, \mathbb{R}^m)$. Then there exists a function $\pi' : \mathbb{R}^m \to \mathbb{R}$ such that

1. *π' is a valid inequality,*
2. *$\pi'(w) \leq \pi(w)\ \forall w \in \mathbb{R}^m$, [Equivalently π' is a stronger inequality than π, since $y(w)$ are non-negative variables]*
3. *$\pi'(0) = 0$,*
4. *π' is positively homogenous, i.e. $\pi'(\lambda w) = \lambda \pi'(w)\ \forall \lambda \geq 0\ \forall w \in \mathbb{R}^m$,*
5. *π' is subadditive, i.e. $\pi'(w^1) + \pi'(w^2) \geq \pi'(w^1 + w^2)\ \forall w^1, w^2 \in \mathbb{R}^m$.*

Sublinear valid inequality

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5. *π' is subadditive, i.e. $\pi'(w^1) + \pi'(w^2) \geq \pi'(w^1 + w^2)\ \forall w^1, w^2 \in \mathbb{R}^m$.*
6. *π' is a convex function.*

From now on we are interested only in sublinear valid functions.

$P(\pi)$

Definition

Let $\pi : \mathbb{R}^m \to \mathbb{R}$ be a valid inequality for $R(f, S, \mathbb{R}^m)$. Define $P(\pi) = \{u \in \mathbb{R}^m \mid \pi(u - f) \leq 1\}$.

$P(\pi)$

Definition

Let $\pi : \mathbb{R}^m \to \mathbb{R}$ be a valid inequality for $R(f, S, \mathbb{R}^m)$. Define $P(\pi) = \{u \in \mathbb{R}^m \,|\, \pi(u - f) \leq 1\}$.

Proposition

Let $\pi : \mathbb{R}^m \to \mathbb{R}$ be a valid sublinear inequality for $R(f, S, \mathbb{R}^m)$. Then

- *$P(\pi)$ is a full-dimensional closed convex set.*
- *$interior(P(\pi)) \cap S = \emptyset$.*

We say a set K is S-free, if $\text{interior}(K) \cap S = \emptyset$.

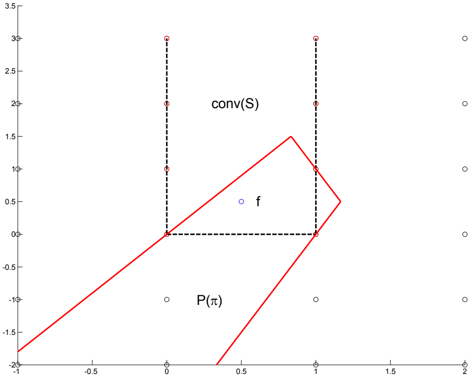
So, $P(\pi)$ is a S-free convex set.

Example π and $P(\pi)$

$f = (0.5, 0.5)$, $S = \{(x_1, x_2) \in \mathbb{Z}^2 \mid 0 \le x_1 \le 1,\ x_2 \ge 0\}$.

$$\pi(w) = \begin{cases} 1.5w_1 + 0.5w_2 & \text{if } w_2 \ge 0,\ 3w_1 \ge w_2 \\ -4.5w_1 + 2.5w_2 & \text{if } w_2 \ge 2w_1,\ w_2 \ge 3w_1 \\ 1.5w_1 - 0.5w_2 & \text{if } w_2 \le 0, 2w_1 \ge w_2 \end{cases} \tag{8}$$

$P(\pi) := \{w \in \mathbb{R}^m \mid \pi(w - f) \le 1\} =$
$\{(w_1, w_2) \in \mathbb{R}^2 \mid -9w_1 + 5w_2 \le 0, 3w_1 + w_2 \le 4, 3w_1 - w_2 \le 3\}$.



3
Constructing Minimal Inequalities

Maximal S-free convex sets and minimal inequalities

Definition (Minimal Inequalities)

A valid function π is minimal if there exists no valid function π' for $R(f, S, W)$ such that $\pi' \neq \pi$ and $\pi' \leq \pi$.

Maximal S-free convex sets and minimal inequalities

Definition (Minimal Inequalities)

A valid function π is minimal if there exists no valid function π' for $R(f, S, W)$ such that $\pi' \neq \pi$ and $\pi' \leq \pi$.

Definition (Maximal S-free convex set)

A convex set $K \subset \mathbb{R}^m$ is a maximal $\mathrm{S\text{-}free}$ convex set if $\mathrm{interior}(K) \cap S = \emptyset$ and there exists no convex set K' such that $\mathrm{interior}(K') \cap S = \emptyset$ and $K' \supsetneq K$.

Maximal S-free convex sets and minimal inequalities

Definition (Minimal Inequalities)

A valid function π is minimal if there exists no valid function π' for $R(f, S, W)$ such that $\pi' \neq \pi$ and $\pi' \leq \pi$.

Definition (Maximal S-free convex set)

A convex set $K \subset \mathbb{R}^m$ is a maximal S-free convex set if $\mathrm{interior}(K) \cap S = \emptyset$ and there exists no convex set K' such that $\mathrm{interior}(K') \cap S = \emptyset$ and $K' \supsetneq K$.

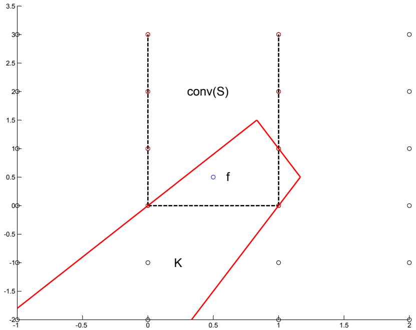
Theorem (Basu, Conforti, Cornuéjols, Zambelli)

Let K be a S-free convex set such that $K \cap conv(S)$ is full-dimensional. Then K is a maximal S-free convex set iff it is a polyhedron with at least one point belonging to S in the relative interior of each facet.

We next show how to construct minimal inequalities using maximal S-free convex sets.

Interpreting *S*-free polyhedron as a disjunction

Integer points belonging to S lie in any one of the half-spaces defined by facets of K



Interpreting S-free polyhedron as a disjunction

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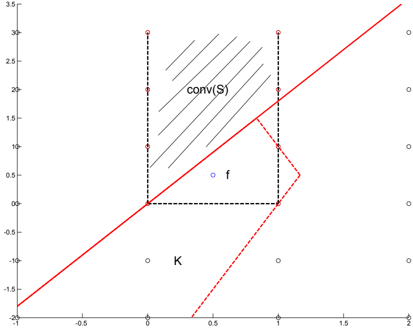

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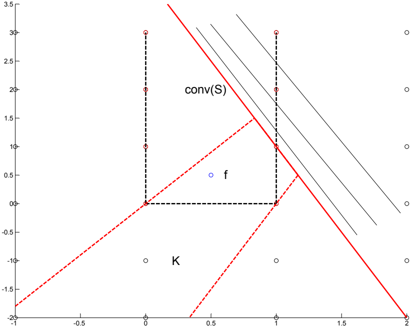

Interpreting S-free polyhedron as a disjunction

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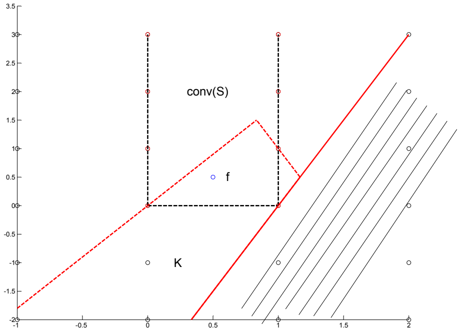

A disjunctive cut using S-free polyhedron

Let K be a maximal S-free convex set with f in it's interior.

A disjunctive cut using S-free polyhedron

Let K be a maximal S-free convex set with f in it's interior.

- Let the facets of $K - f$ be $(g^1)^T x \leq 1$, $(g^2)^T x \leq 1$, ..., $(g^l)^T x \leq 1$.

A disjunctive cut using S-free polyhedron

Let K be a maximal S-free convex set with f in it's interior.

- Let the facets of $K - f$ be $(g^1)^T x \leq 1$, $(g^2)^T x \leq 1$, ..., $(g^l)^T x \leq 1$.
- Since K is S-free, we arrive at the following disjunction:

$$\left(\begin{array}{rcl} \sum_{w \in \mathbb{R}^m} w y(w) & = & x - f \\ (g^1)^T (x - f) & \geq & 1 \end{array} \right) \bigvee \left(\begin{array}{rcl} \sum_{w \in \mathbb{R}^m} w y(w) & = & x - f \\ (g^2)^T (x - f) & \geq & 1 \end{array} \right) \dots \bigvee \dots$$

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$$\Downarrow$$

$$(g^1)^T\left(\sum_{w \in \mathbb{R}^m} w y(w)\right) \geq 1 \quad \bigvee \quad (g^2)^T\left(\sum_{w \in \mathbb{R}^m} w y(w)\right) \geq 1 \quad \ldots \bigvee \ldots$$

$$\Downarrow$$

A disjunctive cut using S-free polyhedron

Let K be a maximal S-free convex set with f in it's interior.

- Let the facets of $K - f$ be $(g^1)^T x \leq 1$, $(g^2)^T x \leq 1$, ..., $(g^l)^T x \leq 1$.
- Since K is S-free, we arrive at the following disjunction:

$$\begin{pmatrix} \sum_{w \in \mathbb{R}^m} wy(w) & = & x - f \\ (g^1)^T(x - f) & \geq & 1 \end{pmatrix} \bigvee \begin{pmatrix} \sum_{w \in \mathbb{R}^m} wy(w) & = & x - f \\ (g^2)^T(x - f) & \geq & 1 \end{pmatrix} \dots \bigvee \dots$$

$$\Downarrow$$

$$(g^1)^T(\sum_{w \in \mathbb{R}^m} wy(w)) \geq 1 \ \bigvee \ (g^2)^T(\sum_{w \in \mathbb{R}^m} wy(w)) \geq 1 \ \dots \bigvee \dots$$

$$\Downarrow$$

$$\pi(w) = \max_{1 \leq i \leq l}\{(g^i)^T w\}$$

$$\sum_{w \in \mathbb{R}^m} \pi(w)y(w) \geq 1 \tag{9}$$

Constructing minimal inequalities

Construction (Construction of Valid inequality: π^K)

1. *Let $K \subset \mathbb{R}^m$ be a maximal S-free polyhedron such that $K \cap conv(S)$ is full-dimensional and K contains f in its interior.*

Constructing minimal inequalities

Construction (Construction of Valid inequality: π^K)

1. *Let $K \subset \mathbb{R}^m$ be a maximal S-free polyhedron such that $K \cap conv(S)$ is full-dimensional and K contains f in its interior.*
2. *$K - f$ can be written as a set $\{x \mid (g^j)^T x \leq 1, j \in \{1, ..., l\}\}$ (as $\bar{0}$ belongs to the interior of $K - f$ and K is polyhedral).*

Construction (Construction of Valid inequality: π^K)

1. *Let* $K \subset \mathbb{R}^m$ *be a maximal* S-*free polyhedron such that* $K \cap conv(S)$ *is full-dimensional and* K *contains* f *in its interior.*
2. $K - f$ *can be written as a set* $\{x \mid (g^j)^T x \leq 1, j \in \{1, ..., l\}\}$ *(as* $\bar{0}$ *belongs to the interior of* $K - f$ *and* K *is polyhedral).*
3. *Define the function* $\pi^K : \mathbb{R}^m \to \mathbb{R}$ *as*

$$\pi^K(u) = max_{1 \leq j \leq l}\{(g^j)^T u\}.$$

Constructing minimal inequalities

Construction (Construction of Valid inequality: π^K)

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3. *Define the function $\pi^K : \mathbb{R}^m \to \mathbb{R}$ as*

$$\pi^K(u) = max_{1 \leq j \leq l}\{(g^j)^T u\}.$$

Proposition (π^K is Minimal)

Let $K \subset \mathbb{R}^m$ be a maximal S-free polyhedral set which contains f in its interior. Then

1. *π^K is a valid function for $R(f, S, \mathbb{R}^m)$,*
2. *$P(\pi^K) := \{x \mid \pi^K(x - f) \leq 1\} = K$,*
3. *π^K is a minimal valid function for $R(f, S, \mathbb{R}^m)$,*

Proof outline - I

1. Validity: π^K is a sublinear function and $\pi^K(x - f) \geq 1 \; \forall \; x \in S$.

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   - Let $x \in K$ and let $K - f := \{x \,|\, (g^j)^T x \leq 1, j \in \{1, ..., l\}\}$.
   - $\Rightarrow (g^j)^T(x - f) \leq 1$.
   - $\Rightarrow \pi^k(x - f) \leq 1$. Because $(\pi^K(u) = \max_{1 \leq j \leq l}\{(g^j)^T u\})$.
   - $P(\pi^K) := \{x \,|\, \pi^K(x - f) \leq 1\} \supseteq K$.

   $P(\pi^K)$ is S-free and convex. By maximality of K, $P(\pi^K) = K$.

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- By proposition before, there exists a sublinear valid function π' such that $\pi' \leq \pi'' < \pi^K$.

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Proving Minimality:

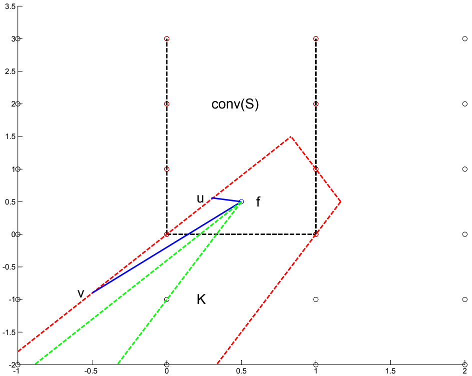
- Assume by contradiction π^K is not minimal. Then there exists $\pi'' : \mathbb{R}^m \rightarrow \mathbb{R}$ such that π'' is valid function and $\pi'' < \pi^K$.
- By proposition before, there exists a sublinear valid function π' such that $\pi' \leq \pi'' < \pi^K$.
- Now the contradiction is obtained by proving that $\pi' = \pi^K$.

Three cases:

1 If $w \in \mathbb{R}^m$ and $w \notin$ recession cone of $K - f$, then $\pi'(w) = \pi^K(w)$: This case follows from the fact that $P(\pi') := \{x \,|\, \pi'(x-f) \leq 1\} \supseteq \{x \,|\, \pi^K(x-f) \leq 1\} = P(\pi^k) = K$, or $P(\pi') = P(\pi^K)$.

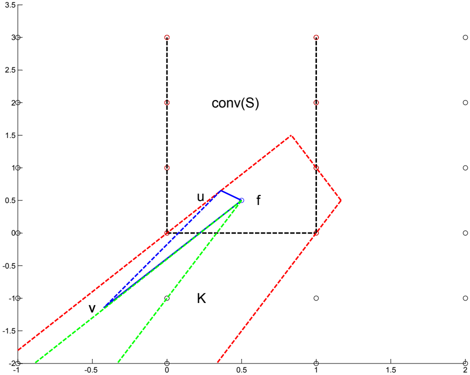
Proof outline - II

2 If w belongs to a face of the recession cone of $K - f$, then $\pi'(w) = \pi^K(w)$: Let $w = (v - f) - (u - f)$
$\pi'(w) + \pi'(u - f) \geq \pi'(v - f)$ or $\pi'(w) \geq \pi'(v - f) - \pi'(u - f) = 0 = \pi^K(w)$.



Proof outline - III

3 If w belongs to interior of the recession cone of $K - f$, then $\pi'(w) = \pi^K(w)$:
Scale w such that $\pi^K(w) = -1$. Then there exist u and v such that
$w = (v - f) - (u - f)$ and $u \in \text{boundary}(K)$ and $v \in \text{boundary}(\text{reccesion cone}(K))$.
$\pi'(w) + \pi'(u - f) \geq \pi'(v)$ or $\pi'(w) \geq \pi'(v) - \pi'(u - f) = 0 - 1 = \pi^K(w)$.



4

Relationship Between Maximal S-free convex sets and Maximal Lattice-free Sets

Formulation, Critical Subset, and Order

Definition (Formulation, Critical Subset, and Order)

Let K be an S-free polyhedron.

- A polyhedral set $P \subseteq \mathbb{R}^m$ is called a 'formulation' for S if $P \cap \mathbb{Z}^m = S$ where $P = \cap_{1 \leq j \leq c} P^j$ and $P^j = \{x \in \mathbb{R}^m \,|\, (a^j)^T x \leq b^j\}$ ($a^j \in \mathbb{Z}^{m \times 1}$ and $b^j \in \mathbb{Z}$).

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- For any subset $\mathcal{J}$ of $\{1, ..., c\}$, denote $S^{P,\mathcal{J}} = (\cap_{j \in \mathcal{J}} P^j) \cap \mathbb{Z}^m$. A subset $\mathcal{J} \subseteq \{1, ..., c\}$ is critical if
  1. K is $S^{P,\mathcal{J}}$-free, and
  2. For each $j \in \mathcal{J}$ (if $\mathcal{J}$ is nonempty), $\exists\, p \in \text{interior}(K) \cap \mathbb{Z}^m$ such that $(a^j)^T p > b^j$ and $(a^k)^T p \leq b^k \;\forall k \in \mathcal{J} \setminus \{j\}$.

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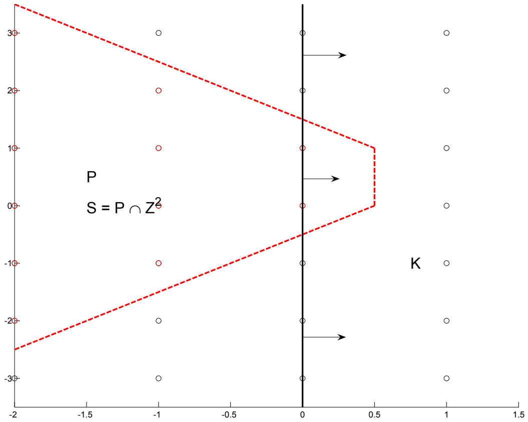
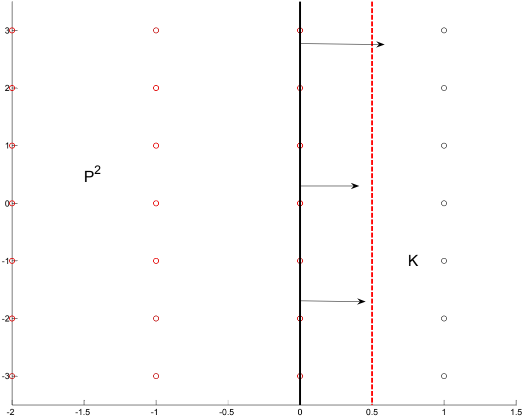
- If K is an S-free convex set, P is a formulation of S, and t is the cardinality of the largest critical subset of $\{1, ..., c\}$, then K is of order t with respect to P.

Example

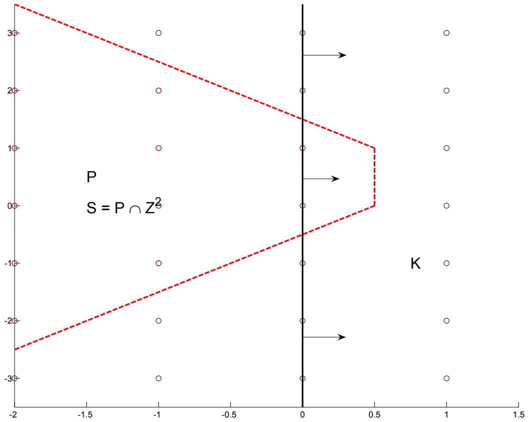


Example

Cardinality of critical set 1

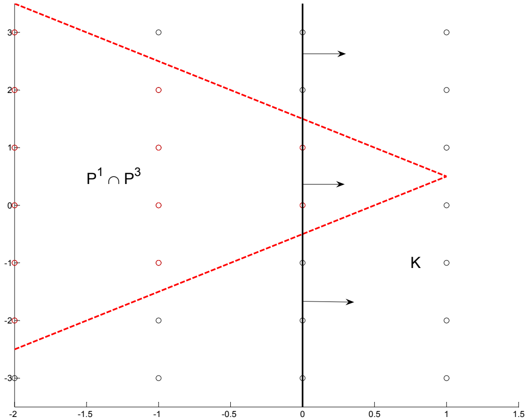


Example



Example

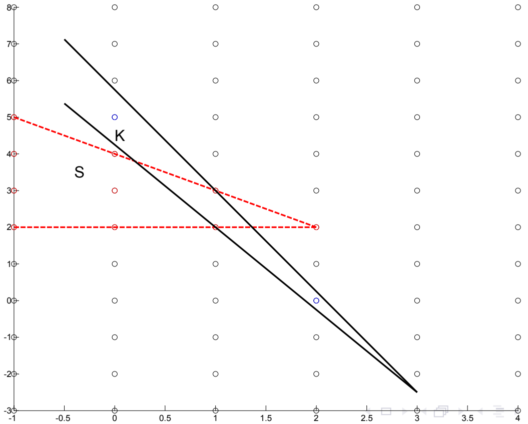
Cardinality of critical set 2, Order = 2



Example

$$P \;:=\; \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \geq 2, x_1 + x_2 \leq 4\}, \quad S = P \cap \mathbb{Z}^2$$
$$K \;:=\; \{(x_1, x_2) \in \mathbb{R}^2 | 2.25x_1 + x_2 \geq 4.25, 2.75x_1 + x_2 \leq 5.75\}$$

Then K is a maximal S-free convex set of order 2 wrt to the formulation P.



Relationship between maximal lattice-free sets and maximal S-free convex sets

Lemma

Let K be a maximal S-free polyhedron of order t wrt formulation P where $t \geq 1$. Let c inequalities describe P and let $\mathcal{J}$ be a critical subset of $\{1, ..., c\}$ of maximal cardinality. (WLOG $1 \in \mathcal{J}$). Let $Q^1 = \{q \in interior(K) \cap \mathbb{Z}^m \mid (a^1)^T q > b^1, (a^k)^T q \leq b^k \, \forall k \in \mathcal{J} \setminus \{1\}\}$ and $\hat{b}^1 = min\{(a^1)^T x \mid x \in Q^1\}$. Set $K^1 := K \cap \{x \in \mathbb{R}^m \mid (a^1)^T x \leq \hat{b}^1\}$ and $\bar{S}^1 := S^{P, \mathcal{J} \setminus \{1\}}$. Then,

1. *K^1 is a maximal m-dimensional $\bar{S}^1$-free convex set of order $t - 1$ wrt the polyhedron defined by the inequalities in the set $\mathcal{J} \setminus \{1\}$*
2. *The number of facets of K^1 is one more than the number of facets of K.*
3. *If $p \in S \cap bnd(K)$, then $p \in bnd(K^1)$.*

Illustration of lemma

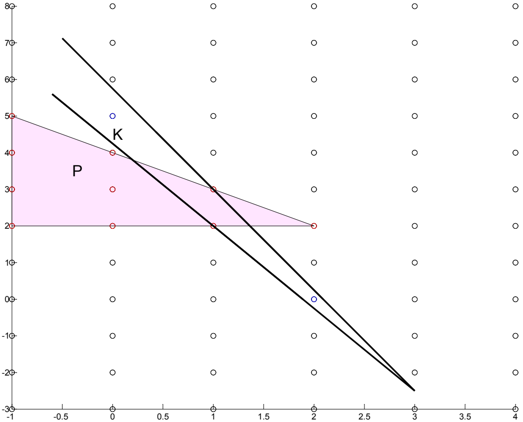


Illustration of lemma

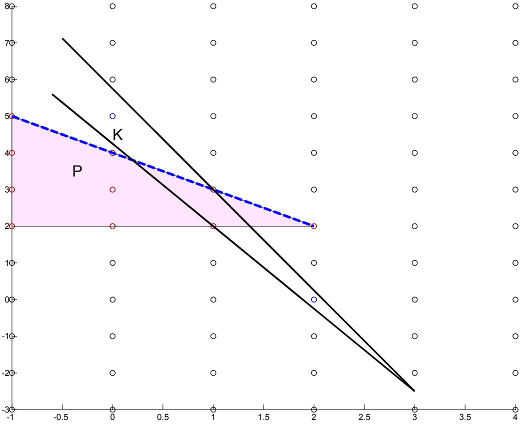


Illustration of lemma

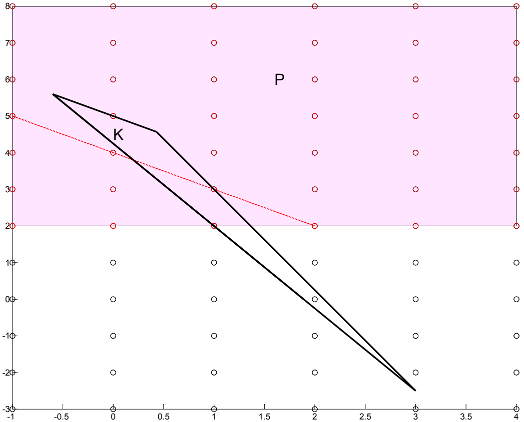


Illustration of lemma

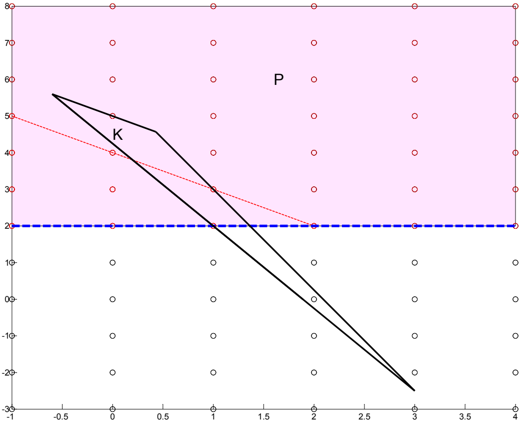
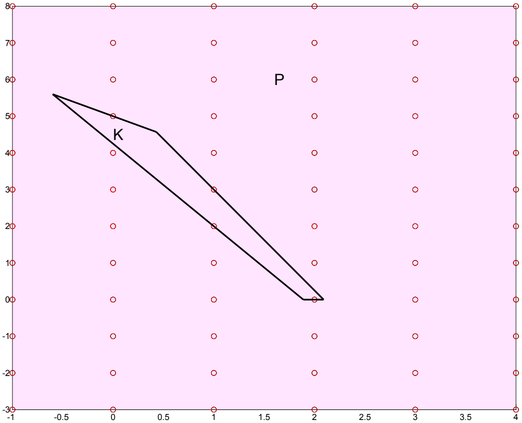


Illustration of lemma



4

Extreme inequalities for two rows

Definition

A valid function π is extreme if there do not exist two valid functions π^1 and π^2 for $R(f, S, W)$ such that $\pi^1 \neq \pi^2$ and $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$.

Proposition

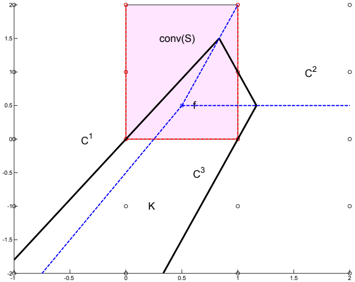
Extreme inequalities for $R(f, S, W)$ are minimal inequalities for $R(f, S, W)$.

Definition

Given $K - f = \{x \in \mathbb{R}^m \,|\, (g^j)^T x \leq 1, j \in \{1, ..., l\}\}$, define the cone $C^j = \{x \in \mathbb{R}^m \,|\, (g^j - g^k)^T x \geq 0 \,\forall k \neq j\}$ (If K is a half-space, then set $C^1 := \mathbb{R}^m$). Let V^j be a finite set of generators for C^j.

Observe that $\pi^K|_{C^j}(u) = (g^j)^T u$.

Proving inequalities are extreme



Proposition (Finite $\Leftrightarrow$ Infinite)

Let K be a full-dimensional maximal S-free polyhedron and let $W := \cup_{1\leq j\leq l} V^j$, where V^j is a finite set of generators for C^j. Then $\pi^K : \mathbb{R}^m \to \mathbb{R}$ is an extreme inequality for $R(f, S, \mathbb{R}^m)$ if and only if $\pi^K|_W : W \to \mathbb{R}$ is an extreme inequality for $R(f, S, W)$.

For two rows...

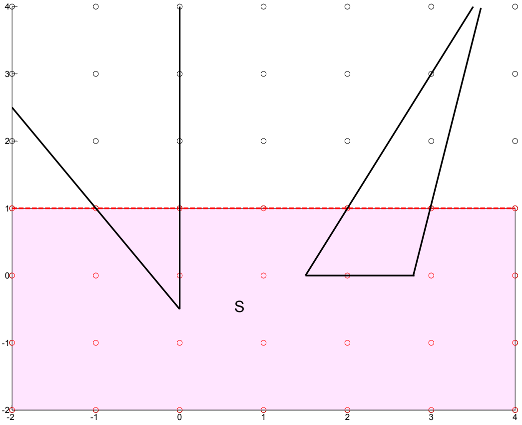
Cornuéjols and Margot (2008): $\text{interior}(K) \cap \mathbb{Z}^2 = \emptyset$.

Theorem (Classification)

Let K be a full-dimensional maximal S-free convex set in $\mathbb{R}^2$ with at least one integer point in its interior and let $K \cap conv(S)$ be bounded.

1. *Order of K:*
   1.1 *If π^K is extreme for $R(f, S, \mathbb{R}^m)$, then the order of K is at most 2.*
   1.2 *If K is not a half-space and π^K is extreme for $R(f, S, \mathbb{R}^m)$, then the order of K is at most 1.*
2. *Number of facets of K:*
   2.1 *If K is a half-space, then π^K is extreme for $R(f, S, \mathbb{R}^m)$ if and only if $bnd(K)$ contains at least two points belonging to S.*
   2.2 *If K has two facets, then π^K is extreme for $R(f, S, \mathbb{R}^m)$ if and only if one of the facets of K contains at least two points belonging to S.*
   2.3 *If K has three facets, then π^K is extreme for $R(f, S, \mathbb{R}^m)$.*

Example of extreme inequalities



Discussion

1. Maximal S-free convex polyhedrons yield minimal inequalities for $R(f, S, W)$.
2. Presented some conditions for checking extremality of valid inequalities.
3. Classification of extreme inequalities based on two rows. “Interesting disjunctions” based on two integer variables are now understood.

What about stronger relaxations...

A stronger relaxation

The relaxation $R(f, S, W)$

$$\begin{aligned} x = f + \sum_{w \in W} w y(w) \\ x \in S \\ y(w) \in \mathbb{R}_+ \\ y \text{ has finite support} \end{aligned}$$

was obtained by relaxing the integer non-basic variables to be continuous variables.

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$$
\begin{aligned}
x = f + \sum_{u \in G} u z(u) + \sum_{w \in W} w y(w) \\
x \in S \\
z(u) \in \mathbb{Z}_+ \\
y(w) \in \mathbb{R}_+ \\
x, y \text{ have finite support}
\end{aligned}
$$

Call this set $R(f, S, W, G)$

Valid inequalities for this stronger relaxation

Let $\mathcal{J}$ be a critical subset of S. Define $\phi : G \to \mathbb{R}$ as

$$\phi(u) = \inf\{\pi(w) \mid w = u + x, x \in \mathbb{Z}^m, x \in \text{rec.cone}(\text{conv}(S^{Q,\mathcal{J}}))\}. \qquad (10)$$

$(S^{Q,\mathcal{J}} = (\cap_{j\in\mathcal{J}} Q^j) \cap \mathbb{Z}^m)$. Then

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$(S^{Q,\mathcal{J}} = (\cap_{j \in \mathcal{J}} Q^j) \cap \mathbb{Z}^m)$. Then

$$\sum_{w \in \mathbb{R}^m} \pi(w)y(w) + \sum_{u \in G} \phi(u)z(u) \geq 1. \tag{11}$$

is a valid inequality for

$$\begin{aligned} x = f + \sum_{u \in G} uz(u) + \sum_{w \in W} wy(w) \\ x \in S \\ z(u) \in \mathbb{Z}_+ \\ y(w) \in \mathbb{R}_+ \\ y, z \text{ have finite support} \end{aligned}$$

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$$\sum_{w\in\mathbb{R}^m} \pi(w)y(w) + \sum_{u\in G} \phi(u)z(u) \geq 1. \qquad (11)$$

The proof of validity is the following:

1. Since $\mathcal{J}$ is a critical set, $P(\pi)$ is an $S^{Q,\mathcal{J}}$-free convex set. Therefore $\pi : \mathbb{R}^m \to \mathbb{R}$ is a valid inequality for $R(f, S^{Q,\mathcal{J}}, \mathbb{R}^m, \emptyset)$.
2. Since the set $M := \text{rec.cone}(\text{conv}(S^{Q,\mathcal{J}})) \cap \mathbb{Z}^m$ is a monoid, i.e., $0 \in M$ and M is closed under addition, by application of Theorem 1 from Balas and Jeroslow (1980), (π, ϕ) yields a valid inequality for $R(f, S^{Q,\mathcal{J}}, \mathbb{R}^m, G)$.
3. Since $S^{Q,\mathcal{J}} \supseteq S$, $R(f, S^{Q,\mathcal{J}}, \mathbb{R}^m, G)$ is a relaxation of $R(f, S, \mathbb{R}^m, G)$. Therefore (π, ϕ) yields a valid inequality for $R(f, S, \mathbb{R}^m, G)$.

Thank You.