A Family of Facets for Multiple-Constraint Infinite Group Relaxation of MIPs

Santanu S. Dey, Jean-Philippe P. Richard

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Outline

Most Successful Cutting Planes for Unstructured Problems

Diasbled Cut	Year	Mean Performance Degradation
Gomory Mixed Integer	1960	2.52X
Mixed Integer Rounding	2001	1.83X
Knapsack Cover	1983	1.40X
Flow Cover	1985	1.22X
Implied Bound	1991	1.19X
Flow path	1985	1.04X
Clique	1983	1.02X
GUB Cover	1998	1.02X
Disjunctive	1979	0.53X

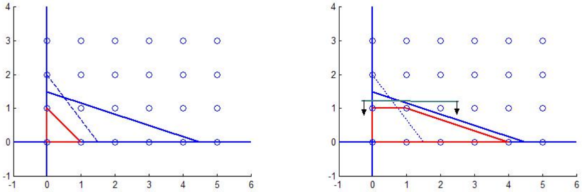
Table taken from Bixby, Rothberg [2007].

"Typical" Single Row Relaxations for Unstructured MIPs

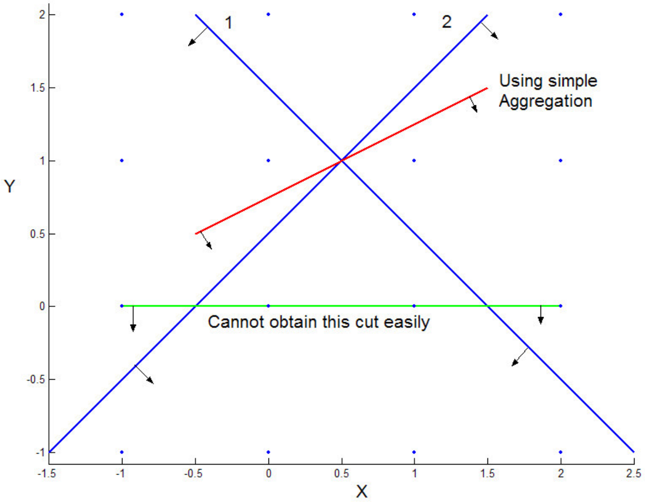
Let $P = \{x \in \mathbb{Z}^n_+ | Ax \leq b\}$. The single row relaxation approach to cut generation is:

- Drop all constraints except one: $\sum_j A_{ij}x_j \leq b_i$.
- Use a Black-box program to generate the cutting plane $\sum_j \alpha_j x_j \leq \beta$ using one constraint and bounds on variables.
- Cutting plane $\sum_j \alpha_j x_j \leq \beta$ is valid for P

Example: GMIC, Knapsack Cover.



Not All Cuts Can Be Generated Through Single-Row Relaxation - In One Step



Studies Suggests Studying High-Dimensional Group Problems

1. Fischetti and Saturni [2007]

   "... then the research on improved bounds based on mod-1 considerations should concentrate on finding row combinations different from those in the optimal tableau (a topic investigated in a recent paper by Fischetti and Lodi [9]), or has to take into account two or more tableau rows at a same time so as to better approximate the corner polyhedron."

2. Dash and Günlük [2006c]

   "GMI cuts have the smallest possible cut coefficient for continuous variables among all group cuts. Indeed, for most of the instances where no violated group cuts exist after GMI cuts are added are problems with continuous variables."

3. Gomory and Johnson [2003]

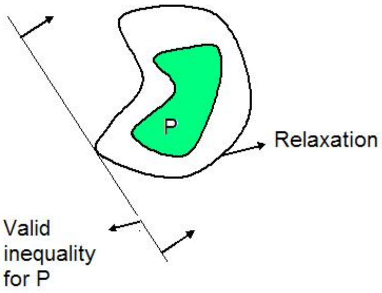
   "There are reasons to think that such inequalities would be stronger since they deal with the properties of two rows, not one. They can also much more accurately reflect the structure of the continuous variables."

Group Cutting Planes.

Group Cut: Basic Idea

1. Generate the *Group Problem* which is a relaxation of a MIP. This relaxation will consider information from multiple rows.
2. Generate a valid inequality for the Group Problem.

Since Group Problem is a relaxation of the original MIP, the valid inequality for the Group Problem is valid for the original MIP.



Group Relaxation

Feasible region of standard IP:

$$\begin{aligned} 2.3x + 0.9y + 1.1z &= 5.5 \\ 2.4x - 1.3y + 0.7z &= 3.5; \quad x, y, z \in \mathbb{Z}_+ \end{aligned}$$

Feasible Points: $(2, 1, 0)$, $(0, 0, 5)$.

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Feasible points: (2, 1, 0), (0, 0, 5), (4, 2, 5) ...

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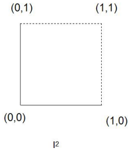
  Feasible points: $(2, 1, 0)$, $(0, 0, 5)$, $(4, 2, 5)$ ...

- Rewrite (1) in 'Group Space':

$$\begin{pmatrix} .3 \\ .4 \end{pmatrix} x + \begin{pmatrix} .9 \\ .7 \end{pmatrix} y + \begin{pmatrix} .1 \\ .7 \end{pmatrix} z = \begin{pmatrix} .5 \\ .5 \end{pmatrix}$$



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- Rewrite (1) in 'Group Space':

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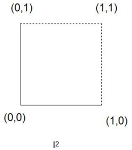


- Relaxation Step 2: Introduce additional variables.

$$\sum_{u \in I^2} u t(u) = \begin{pmatrix} .5 \\ .5 \end{pmatrix},$$

where $I^2 = \{u \in \mathbb{R}^2 | 0 \leq u_1, u_2 < 1, \}$ and $t(u) > 0$ for a finite subset of I^2.

m-Dimensional Group Problem

Definition (Infinite Group Problem, Johnson 1974)

For $r \in I^m$ and $r \neq o$, the group Problem $PI(r,m)$ is the set of functions $t : I^m \to \mathbb{R}$ such that

1. $\sum_{u \in I^m} ut(u) = r$, $r \in I^m$,
2. $t(u)$ is a non-negative integer for $u \in I^m$,
3. t has a finite support, i.e., $t(u) > 0$ for a finite subset of I^m. □

Inequalities for Infinite Group Problems

Definition (Valid Inequality, Johnson 1974)

A function $\phi : I^m \rightarrow \mathbb{R}_+$ is defined as a valid inequality for PI(r,m) if

1. $\phi(o) = 0$,
2. $\phi(r) = 1$, and
3. $\sum_{u \in I^m} \phi(u)t(u) \geq 1$, $\forall\ t \in PI(r, m)$.

$\square$

A Hierarchy of Valid Cutting Planes

Definition (Subadditive Inequality, Gomory and Johnson (1972a,b))

A function $\phi : I^m \rightarrow \mathbb{R}_+$ is defined as a valid subadditive inequality for PI(r,m) if ϕ is valid and $\phi(u) + \phi(v) \geq \phi(u + v)\ \forall u, v \in I^m$. □

Definition (Minimal Inequality, Gomory and Johnson (1972a,b))

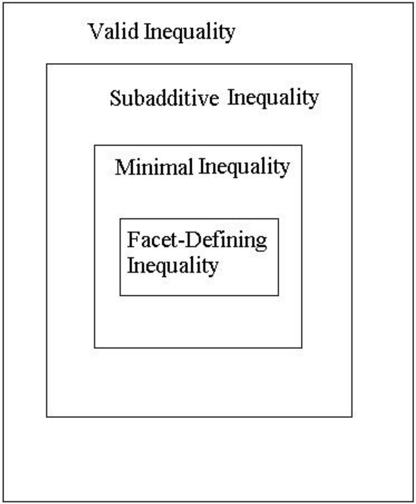
A function $\phi : I^m \rightarrow \mathbb{R}_+$ is defined as a minimal inequality for PI(r,m) if there exists no valid function $\phi' \neq \phi$ such that $\phi'(u) \leq \phi(u)\ \forall u \in I^m$. □

Let $P(\phi)$ be the set of points t that satisfy ϕ at equality, i.e., $P(\phi) = \{t \in PI(r, m) \mid \sum_{u \in I^m, t(u)>0} \phi(u)t(u) = 1\}$.

Definition (Facet-Defining Inequality, Gomory and Johnson (2003))

We say that an inequality ϕ is facet-defining for $PI(r, m)$ if there does not exist a valid function ϕ^* such that $P(\phi^*) \supsetneq P(\phi)$. □

A Hierarchy of Valid Cutting Planes - II



Technique to Prove Valid Function is Facet-Defining

Given a function $\phi : I^m \to \mathbb{R}_+$

1. Prove function is subadditive valid function.
2. Prove function is minimal: Use Gomory and Johnson's Characterization,

$$\phi(u) + \phi(r - u) = 1 \quad \forall u \in I^m. \tag{2}$$

3. Prove $E(\phi)$[1] is unique. [Facet Theorem, Gomory and Johnson (2003)]

[1]Notation $E(\phi)$: Let f be a 'variable function', i.e, for each point $u \in I^m$, we define $f(u)$ to be a non-negative variable. $E(\phi)$ is the system of equations $f(u) + f(v) = f(u + v)$ for all $u, v \in I^m$ such that $\phi(u) + \phi(v) = \phi(u + v)$.

GMIC is a Facet of One-Dimensional Group Relaxation

1. One row from a simplex tableau

$$\sum_{i=1}^{n} a_i x_i = b_1$$

2. Compute fractional part of each coefficient,

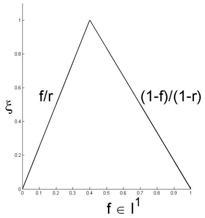
$$f_i = a_i (mod 1)$$
$$r = b(mod 1)$$

3. The GMIC is generated as

$$\sum_{i=1}^{n} \xi(f_i) x_i \geq 1$$

$$\xi(f) = \begin{cases} f/r & f < r \\ (1-f)/(1-r) & f \geq r. \end{cases}$$

Figure: The GMIC



How Will Two-Dimensional Group Cuts Work?

1. Extract two rows from a simplex tableau

$$\sum_{i=1}^{n} a_{1i} x_i = b_1$$

$$\sum_{i=1}^{n} a_{2i} x_i = b_2$$

2. Compute fractional part of each coefficient,

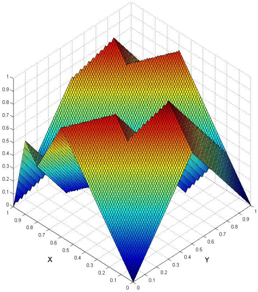
$$f_{ji} = a_{ji}(mod 1) \quad j \in \{1, 2\}$$

3. A group cut is generated as

$$\sum_{i=1}^{n} \phi(f_{1i}, f_{2i}) x_i \geq 1$$

where ϕ is a valid function for $PI(r, 2)$, where $r = (b_1(mod 1), b_2(mod 1))$.

Figure: A valid function ϕ



A Simple Family: Aggregation

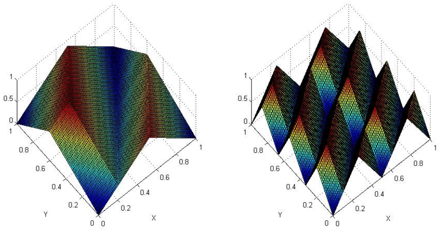
Homomorphism: Generalization of ‘K-Cuts’

Definition

The homomorphism $\lambda : I^m \rightarrow I^m$ is defined as $\lambda(x_1, ..., x_m) = (\lambda_1 x_1(mod 1), ..., \lambda_m x_m(mod 1))$, where $\lambda_1, ..., \lambda_m$ are non-zero integers.

Theorem (Homomorphism Theorem, Gomory and Johnson (1972), D. and Richard(2006))

ϕ is facet-defining for $PI(r, m)$ iff $\phi \circ \lambda$ is facet-defining for $PI(v, m)$, where $\lambda(v) = r$.



a. Three-gradient facet

b. Homomorphism of the three-gradient facet

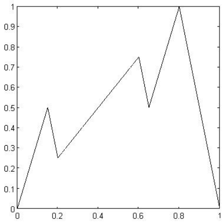
Apply One-Dimensional Facets to Aggregation of Constraints: Facet-Defining

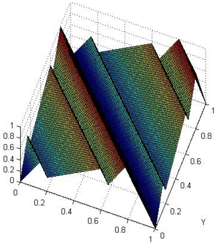
Definition

Given $\zeta : I^1 \rightarrow \mathbb{R}_+$ a piecewise linear and continuous valid inequality for $PI(c, 1)$, we construct the function τ valid for $PI((r_1, r_2), 2)$ as $\tau(x, y) = \zeta(\lambda_1 x + \lambda_2 y)(mod 1)$, where $\lambda_1 f_1 + \lambda_2 f_2 = c$, $\lambda_1, \lambda_2 \in \mathbb{Z}$, and λ_1 and λ_2 are not both zero.

Theorem (Aggregation Theorem)

κ is facet-defining for $PI(r, 2)$ iff ζ is facet-defining for $PI(c, 1)$.





All 'Two Gradient' Cuts for $PI(r, 2)$ are Aggregation-Based Cuts

Theorem (Two-Gradient Theorem)

All continuous, piecewise linear, two-gradient facet of $PI(r, 2)$ can be derived from a facet of $PI(r, 1)$ using aggregation.

Some consequences:

- Gives a complete characterization of continuous functions with only two gradients.
- All two slope functions for $PI(r, 1)$ are facet-defining. This is a two-dimensional analog for a similar result in one-dimension [Gomory and Johnson (1972b)].

Lifting-Space (Superadditive) Representation of Group Cuts.

From Group- to Lifting-Space Representation

Definition (Lifting-Space Representation)

Given a valid inequality $\phi : I^m \to \mathbb{R}_+$ for $PI(r, m)$, we define the lifting-space representation of ϕ as $[\phi]_r : \mathbb{R}^m \to \mathbb{R}$ where

$$[\phi]_r(x) = \sum_{i=1}^{m} x_i - \left(\sum_{i=1}^{m} r_i\right) \phi(\mathbb{P}(x)).$$

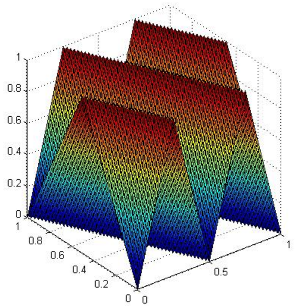
□

Proposition

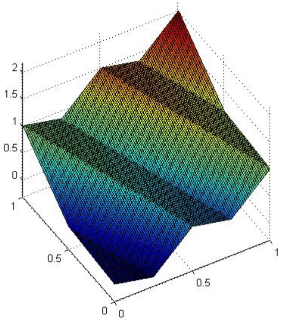
Given m rows of tableau $Ax = b$, if $\sum_i \phi(\mathbb{P}(A_i))x_i \geq 1$ is valid, then $\sum_i [\phi]_r(A_i)x_i \leq [\phi]_r b$ is valid, where $r = \mathbb{P}(b)$. □

Notation: For $a \in \mathbb{R}^m$, $\mathbb{P}(a) = (a_1(mod 1), a_2(mod 1), ...a_m(mod 1))$.

Example of a Function in Group- and Lifting-Space



Group-Space Representation



Lifting-Space Representation

Explanation of Lifting-Space Representation

For the tableau row:

$$\sum_{i=1}^{n} a_i x_i = b \tag{3}$$

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- Start with Group Cut: $\sum_{i=1}^{n} \phi(\mathbb{P}(a_i))x_i \geq 1$.

Explanation of Lifting-Space Representation

For the tableau row:

$$\sum_{i=1}^{n} a_i x_i = b \tag{3}$$

- Start with Group Cut: $\sum_{i=1}^{n} \phi(\mathbb{P}(a_i))x_i \geq 1$.
- Multiply $r = b(mod1)$ to group cut:

$$\sum_{i=1}^{n} r\phi(\mathbb{P}(a_i))x_i \geq r \tag{4}$$

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$$\sum_{i=1}^{n} r\phi(\mathbb{P}(a_i))x_i \geq r \tag{4}$$

- Subtract (4) from tableau row (3):

$$\sum_{i=1}^{n} [\phi]_r(a_i)x_i \leq [\phi]_r b \tag{5}$$

From Lifting- to Group-Space Representation

Proposition

If ϕ is valid function for $PI(r, m)$,

1. $[\phi]_r(x + e_i) = [\phi]_r(x) + 1$, *where e_i is the i^{th} unit vector of $\mathbb{R}^m$. We say that $[\phi]_r$ is pseudo-symmetric.*
2. $[\phi]_r$ *is superadditive iff ϕ is subadditive.* □

A function $\pi : D \to \mathbb{R}$ is superadditive if $\pi(u) + \pi(v) \leq \pi(u + v)\ \forall u, v \in D$.

Definition (Group-Space Representation)

Given a superadditive function $\psi : \mathbb{R}^m \to \mathbb{R}$ that is pseudo-symmetric, we define the group-space representation of ψ as $[\psi]_r^{-1} : I^m \to \mathbb{R}$ where $[\psi]_r^{-1}(x) = \frac{\sum_{i=1}^m \tilde{x}_i - \psi(x)}{\sum_{i=1}^m \tilde{r}_i}$. □

Sequential-Merge Inequalities.

Sequential-Merge Operation

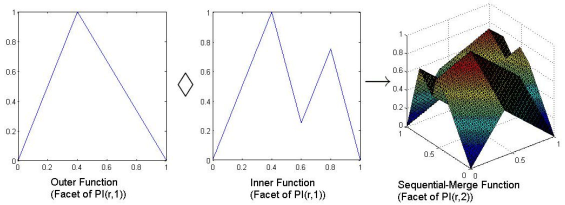
Definition

Assume that g and h are valid functions for $PI(r_1, 1)$ and $PI(r_2, m)$ respectively. We define the sequential-merge of g and h as the function $g\Diamond h : I^{m+1} \to \mathbb{R}_+$ where

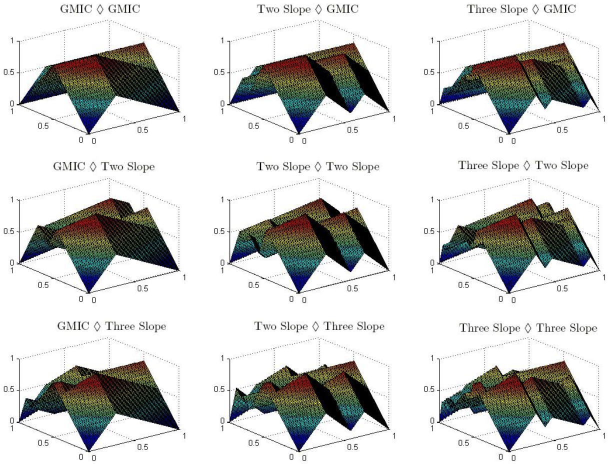
$$g\Diamond h(x_1, x_2) = [[g]_{r_1}(x_1 + [h]_{r_2}(x_2))]_r^{-1}(x_1, x_2) \tag{6}$$

and $r = (r_1, r_2)$. □

$$g\Diamond h = \frac{(\sum_{i=1}^m r_2^i)h(x_2) + r_1 g(\mathbb{P}(x_1 + \sum_{i=1}^m x_2^i - (\sum_{i=1}^m r_2^i)h(x_2)))}{r_1 + \sum_{i=1}^m r_2^i}$$



Examples of Sequential-Merge Inequalities



Explanation of Sequential-Merge Operator

Given: (1)$m+1$ Tableau Rows:

$$\sum_i a_i^1 x_i = b^1 \quad \}\textbf{... First Row}$$

$$\sum_i a_i^2 x_i = b^2 \quad \}\textbf{... Next m Rows}$$

(2) Valid group functions g and h.

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1. Generate valid inequality [h] for the last m-rows:

$$\sum_i [h](a_i^2)x_i \leq [h](b^2) \tag{7}$$

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2. Add (7) to the first row of tableau, i.e.,

$$\sum_i ([h](a_i^2) + a^1)x_i \leq [h](b^2) + b^1 \tag{8}$$

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3. Generate the valid cut [g] for (8):

$$\sum_i [g]([h](a_i^2) + a^1)x_i \leq [g]([h](b^2) + b^1) \tag{9}$$

Explanation of Sequential-Merge Operator

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4. Convert (9) to Group-Space.

Sequential-Merge Operator Generates Minimal Inequalities

Proposition (Validity)

$g \lozenge h$ is a valid subadditive function for $PI(r, m+1)$ where $r \equiv (r_1, r_2)$, if:

- *g and h are valid subadditive functions for $PI(r_1, 1)$ and $PI(r_2, m)$ respectively, and*
- *$[g]_{r_1}$ is nondecreasing.* $\square$

Proposition (Non-Dominance)

$g \lozenge h$ is a minimal function for $PI(r, m+1)$ where $r \equiv (r_1, r_2)$, if:

- *g and h are valid, minimal functions for $PI(r_1, 1)$ and $PI(r_2, m)$,*
- *$[g]_{r_1}$ is nondecreasing.* $\square$

Sequential-Merge Operator Generates Facets for High-Dimensional Group Problems

Theorem (Sequential-Merge Theorem)

$g \Diamond h$ is a facet-defining inequality for $PI((r_1, r_2), m+1)$ if:

- *g and h are continuous, piecewise linear, facet-defining inequalities of $PI(r_1, 1)$ and $PI(r_2, m)$ respectively,*
- *$E(g)$ and $E(h)$ have unique solution,*
- *$[g]_{r_1}$ and $[h]_{r_2}$ are nondecreasing.*

□

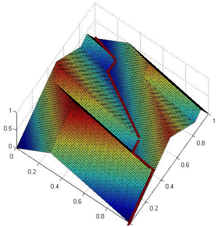
Outline of Proof

Aim: To prove $g\Diamond h$ is the unique solution to $E(g\Diamond h)$. [Facet Theorem (Gomory and Johnson (2003)]

Assume by Contradiction: $\exists$ a valid function for $PI(r, m+1)$, ψ, such that $\psi \neq g\Diamond h$ and ψ is a solution to $E(g\Diamond h)$.

Three steps:

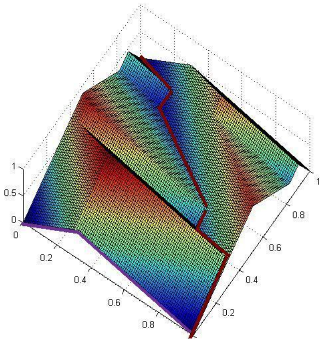
- Prove $\psi = g\Diamond h$ on the support. Support of $g\Diamond h$: $\{(x, y) \in I^{m+1} \mid x = -Y + R_2 h(y)\}$.



1. g and h are continuous, piecewise linear.
2. $[h]_r$ is non-decreasing.
3. $E(h)$ is unique.

Outline of Proof - Contd.

- Prove $\psi(x, 0) = g \Diamond h(x, 0)\ \forall 0 \leq x < 1$.
  1. $E(g)$ is unique.



- Finally prove $\psi(u) = g \Diamond h(u)\ \forall u \in I^{m+1}$.

Assumptions of the Sequential-Merge Theorem: Necessary or Not?

Under very general conditions the Sequential-Merge operator generates facets of high-dimensional group problem using two facets of lower-dimensional group problems. ‘Reminiscent’ of general results such as Homomorphism/Automorphism, Aggregation, etc.

1. g being facet-defining in $g \Diamond h$ is necessary.
2. h being facet-defining in $g \Diamond h$ is not necessary: Need to search more general conditions.
3. $[h]$ being non-decreasing may also not be necessary.

Deriving Coefficients of Continuous Variables

Johnson (1974): If ϕ is subadditive, valid coefficients for continuous variables can be found as the slope at the origin of function ϕ:

$$mu_{\phi}(w) = \lim_{h \to 0^+} \frac{\phi(\mathbb{P}(wh))}{h} \tag{10}$$

Proposition (Mixed Integer Extension)

Let $c_g^+ = lim_{\epsilon \to 0^+} \frac{g(\epsilon)}{\epsilon} = \frac{1}{r_1}$, $c_g^- = lim_{\epsilon \to 0^+} \frac{g(1-\epsilon)}{\epsilon}$, $c_h(y) = lim_{\epsilon \to 0^+} \frac{h(\epsilon y)}{\epsilon}$. The coefficients of the continuous variables for $g \diamondsuit h$ are given by

$$\mu_{g \diamondsuit h}(x, y) = \begin{cases} \frac{R_2 c_h(y) + r_1 c_g^+ (x + Y - R_2 c_h(y))}{r_1 + R_2} & \text{if } (x + Y - R_2 c_h(y)) \geq 0 \\ \frac{R_2 c_h(y) - r_1 c_g^- (x + Y - R_2 c_h(y))}{r_1 + R_2} & \text{if } (x + Y - R_2 c_h(y)) \leq 0 \end{cases}$$

□

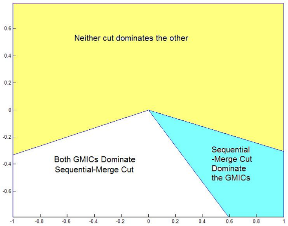
Notation: $X = \sum_{i=1}^{m} x_i$.

Sequential-Merge Facets Generate Strong Coefficients for Continuous Variables

Proposition (Non-Dominance of Continuous Variables' Coefficients)

The coefficients for continuous variables of GMIC$\Diamond$GMIC are not dominated by those of GMICs based on single constraints.

$\square$



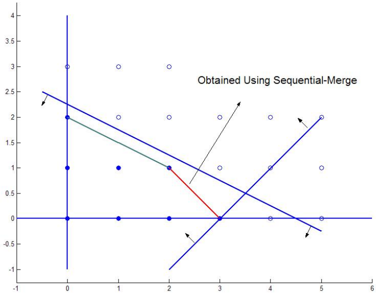
Example Where Sequential-Merge Inequality Cannot Be Derived Using Disjunction

Example IP:

$$\frac{1}{3}x - \frac{1}{3}y \leq 1$$
$$\frac{1}{3}x + \frac{2}{3}y \leq \frac{3}{2}$$
$$x, y \in \mathbb{Z}_+$$

Simplex Tableau:

$$x + 2s_1 + s_2 = \frac{7}{2}$$
$$y - s_1 + s_2 = \frac{1}{2}$$



Results on Günlük and Pochet Instances [2001]

$$\Delta_{GMIC \diamondsuit GMIC} = \frac{z^{GMIC \diamondsuit GMIC} - z^{GMIC}}{z^{GMIC} - z^{LP}} \times 100 \tag{11}$$

Integer	Continuous	Rows	$\Delta_{GMIC \diamondsuit GMIC}$
40	0	20	34.65
60	0	30	39.83
80	0	40	25.36
100	0	50	26.83
40	5	20	28.27
60	5	30	25.79
80	5	40	31.68
100	5	50	31.72

Results on Atamtürk Mixed Integer Knapsack [2003]

Integer	Continuous	Rows	$\Delta_{GMIC \Diamond GMIC}$
250	20	100	0.82
250	20	75	5.03
250	20	50	13.59
250	10	100	1.51
250	10	75	6.88
250	10	50	13.97
250	5	100	2.09
250	5	75	7.96
250	5	50	14.22
500	20	100	0.50
500	20	75	5.77
500	20	50	13.63
500	10	100	0.12
500	10	75	5.70
500	10	50	13.50
500	5	100	1.92
500	5	75	7.00
500	5	50	14.33

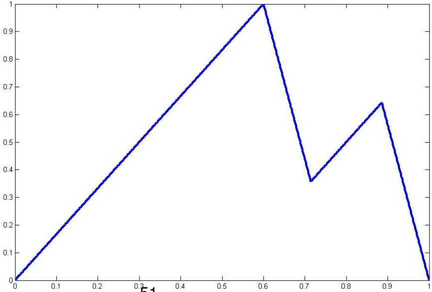
From Higher-To-Lower Dimensions

Some Two-Step MIR inequalities are Sequential-Merge Inequalities

Some two-step MIR inequalities (Dash and Günlük) can be generated in the following fashion:
$\xi\Diamond\xi(x) = [[\xi]_r(x + [\xi]_r(x)]^{-1}(x, x)$.

- Use the same row twice, instead of using two rows of the tableau.
- Use the GMIC for both g and h in $g\Diamond h$.

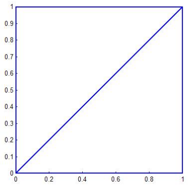
Figure: Two-Step MIR

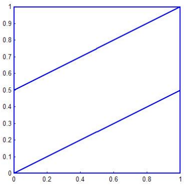


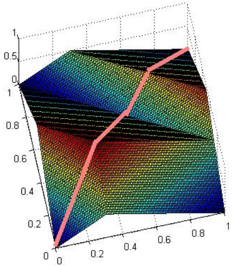
‘Projected’ Sequential-Merge Cuts

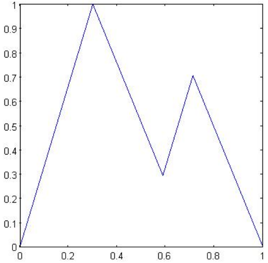
Obtain functions for lower-dimensional group problems using functions of higher-dimensional group problems.

Basic Idea: $\phi(x_1) = \tilde{\phi}(nx_1, x_1)$.

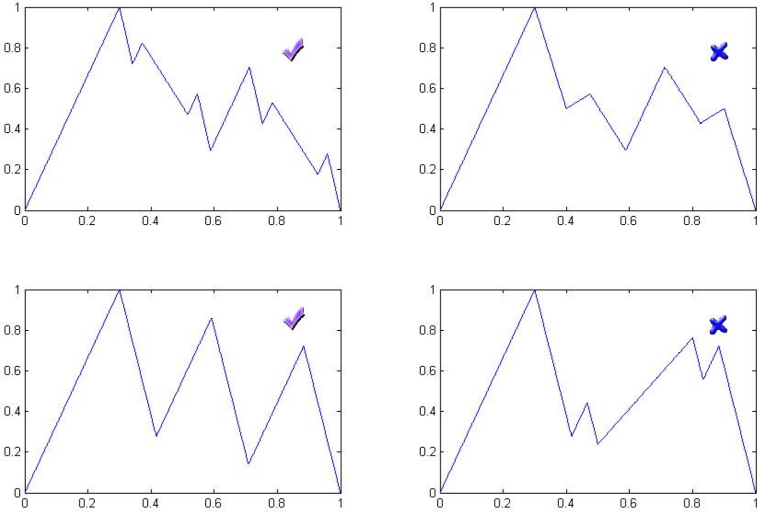








Some Other Projected Sequential-Merge Functions



Sufficient Condition For Projected Sequential-Merge Inequality To Be Facet-Defining

Notation:

1. ξ : GMIC
2. $g\Diamond^1_n h(x) = g\Diamond h(nx, x)$.

Theorem (Projected Sequential-Merge Theorem)

Let $\phi : I^{m+1} \to \mathbb{R}_+$ be a facet-defining inequality of $PI((r_1, ..., r_{m+1}), m+1)$. If ϕ is of the form $g_1\Diamond g_2\Diamond g_3 ... g_m\Diamond\xi$ where g_i is a piecewise linear, continuous, facet-defining inequality of $PI(r_i, 1)$, $[g_i]$ is nondecreasing and $E(g_i)$ is unique up to scaling then $\phi' : I^m \to \mathbb{R}_+$ defined as $g_1\Diamond g_2\Diamond g_3 ... g_m\Diamond^1_n\xi$ is facet-defining for $PI((r_1, ..., r_m), m)$. □

Pitfalls...

- 'Low Rank'
  1. Aggregation is 'rank 1'.
  2. Sequential-Merge is 'rank 2'.
- Exponential increase in number of cuts: Although group cuts are easy to derive, the number of cuts increase exponentially.

... Challenges

- Is it possible to create a 'Library of Operations' to create 'guaranteed' facet-defining inequalities? (Not all 'Operations' seem to create facets)
  1. Aggregation
  2. Homomorphism (K-Cuts...)
  3. Sequential-Merge
  4. ...
- All cuts cannot be generated using a sequence of operations on lower-dimensional cuts. [Cook et al.(1990)]
- Identify operation/parameter to be selected based on different criteria:
  1. Coefficient of continuous variables.
  2. Coefficient of integer variables.
  3. ...

Conclusions

- First known facets: The new inequalities form the first family of facets for high-dimensional group problems.
- Generic result: The sequential-merge theorem is a very general result, creating a large family of facet-defining inequalities.
- Strong continuous coefficients: These new inequalities have strong coefficients for continuous variables. (A well-known weakness of one-dimensional group cuts).
- Stronger than first split closure: These inequalities can produce cuts that are not part of the first split closure.

http://www.optimization-online.org/DB_HTML/2007/05/1671.html

Thank You.