Some cut-generating functions for second-order conic sets

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Abstract

In this paper, we study cut generating functions for conic sets. Our first main result shows that if the conic set is bounded, then cut generating functions for integer linear programs can easily be adapted to give the integer hull of the conic integer program. Then we introduce a new class of cut generating functions which are non-decreasing with respect to second-order cone. We show that, under some minor technical conditions, these functions together with integer linear programming-based functions are sufficient to yield the integer hull of intersections of conic sections in $\mathbb{R}^2$.

1 Introduction: Subadditive dual of conic integer programs

A natural generalization of linear integer programs is *conic integer programs*. Given a *regular* cone $K \subseteq \mathbb{R}^n$, that is a cone that is closed, convex, and full dimensional, we can define a conic integer program as:

$$\begin{array}{ll} \inf & c^\top x \\ \text{s.t.} & Ax - b \in K \\ & x \in \mathbb{Z}^n_+, \end{array} \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. As is standard, we will henceforth write the constraint $Ax - b \in K$ as $Ax \succeq_K b$, where we use the notation that $u \succeq_K v$ if and only if $u - v \in K$. In the case where K is the non-negative orthant, that is $K = \mathbb{R}^m_+$, the conic integer program is a standard linear integer program.

A natural way to generate cuts for conic integer programs is via the notion of cut-generating functions [8]. Consider a function $f : \mathbb{R}^m \to \mathbb{R}$ that satisfies the following:

1. f is *subadditive*, that is $f(u) + f(v) \geq f(u + v)$ for all $u, v \in \mathbb{R}^m$,
2. f is *non-decreasing with respect to* K, that is $f(u) \geq f(v)$ whenever $u \succeq_K v$,
3. $f(0) = 0$.

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Then it is straightforward to see that the inequality

$$\sum_{j=1}^{n} f(A^j)x_j \geq f(b), \tag{2}$$

is valid for the conic integer program (1), where A^j is the j-th column of A. We denote the set of functions satisfying (1.), (2.) and (3.) above as $\mathcal{F}_K$.

In the paper [27], it was shown that, assuming a technical '*discrete Slater*' condition holds, the closure of the convex hull of the set of integer feasible solutions to (1) is described by inequalities of the form (2) obtained from $\mathcal{F}_K$. This result from [27] generalizes result on subadditive duality of linear integer programs [16, 17, 18, 31], that is inequalities (2) give the convex hull of (1) when $K = R^m_+$ and the constraint matrix A is rational. Also see [20, 21] for related models and results.

In the case where $K = \mathbb{R}^m_+$ and assuming A is rational, a lot more is known about the subset of functions from $\mathcal{F}_{\mathbb{R}^m_+}$ that are sufficient to describe the convex hull of integer solutions (also called as the integer hull). For example, these functions have a constructive characterization using the Chvátal-Gomory procedure [6], it is sufficient to consider functions that consider only 2^n constraints at a time [30], or for a fixed A there is a finite list of functions independent of b that describes the integer hull [31].

The main goal of this paper is to similarly better understand structural properties of subsets of functions from $\mathcal{F}_K$ that are sufficient to produce the integer hull of the underlying conic representable set $\{x \in \mathbb{R}^n \,|\, Ax \succeq_K b\}$.

2 Main results

We will refer to the *dual cone* of a cone K as K^* which we remind the reader is the set $K^* := \{y \in \mathbb{R}^m \,|\, y^\top x \geq 0 \; \forall x \in K\}$. Given a positive integer m, we denote the set $\{1, \ldots, m\}$ by $[m]$. And given a subset X of $\mathbb{R}^n$ we denote its integer hull by X^I.

2.1 Bounded sets

Given a regular cone K we call as *linear composition* the set of functions f obtained as follows: Let the vectors $w^1, w^2, \cdots, w^p \in K^*$ and the function $f : \mathbb{R}^m \to \mathbb{R}$ be given by

$$f(v) = g((w^1)^\top v, (w^2)^\top v, \ldots, (w^p)^\top v), \tag{3}$$

where $g \in \mathcal{F}_{\mathbb{R}^p_+}$ satisfies $g(u) = -g(-u)$ for all $u \in \mathbb{R}^p$. It is straightforward to see that linear composition functions belong to $\mathcal{F}_K$ and also satisfy $f(v) = -f(-v)$ for all $v \in \mathbb{R}^m$, which implies that f generates valid inequalities of the form (2) even when the variables are not required to be non-negative. Our first result describes a class of conic sets for which linear composition functions are sufficient to produce the convex hull.

Theorem 1. *Let $K \subseteq \mathbb{R}^m$ be a regular cone. Consider the conic set $T = \{x \in \mathbb{R}^n \,|\, Ax \succeq_K b\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Assume T has nonempty interior. Let $\pi^\top x \geq \pi_0$ be a valid inequality for T^I where $\pi \in \mathbb{Z}^n$ is non-zero. Assume $B := \{x \in T \,|\, \pi^\top x \leq \pi_0\}$ is nonempty and bounded. Then, for some natural number $p \leq 2^n$, there exist vectors $y^1, y^2, \ldots, y^p \in K^*$ such that $\pi^\top x \geq \pi_0$ is a valid inequality for the integer hull of the polyhedron $Q = \{x \in \mathbb{R}^n \,|\, (y^i)^\top Ax \geq (y^i)^\top b, \; i \in [p]\}$, where $(y^i)^\top A$ is rational for all $i \in [p]$.*

We highlight here that particular care was taken in Theorem 1 to ensure that the outer approximating polyhedron has rational constraints.

Since a valid inequality for Q^I can be obtained using a subadditive function $g \in \mathcal{F}_{\mathbb{R}^p_+}$ that satisfies $g(u) = -g(-u)$ for all $u \in \mathbb{R}^p$ [26] (note that the constraints matrix defining Q is rational), Theorem 1 implies that if a cut separates a bounded set from T, then it can be obtained using exactly one function (3) with $p \leq 2^n$. Geometrically, Theorem 1 can be interpreted as the fact that if the set of points separated is bounded, then the cut can be obtained using a rational polyhedral outer approximation.

We obtain the following corollary immediately: If the set $\{x \in \mathbb{R}^n \,|\, Ax \succeq_K b\}$ is compact and has a non-empty interior, then it is sufficient to restrict attention to linear composition functions to obtain the convex hull. A proof of Theorem 1 is presented in Section 3.

2.2 New family of cut-generating functions

In the previous section we stated that any valid inequality for the integer hull of a bounded conic set can be obtained using linear composition functions. So what happens when the underlying set is not bounded? Consider the simple unbounded set $T' = \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 \geq 1,\ x_1, x_2 \geq 0\}$, which is one branch of a hyperbola[1]. This set is conic representable, that is $T' = \{x \in \mathbb{Z}^2_+ : Ax \succeq_K b\}$, where K is the second-order cone $\mathcal{L}^3$ and

$$A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}. \tag{4}$$

(We use the notation $\mathcal{L}^m := \left\{x \in \mathbb{R}^m \,|\, \sqrt{x_1^2 + x_2^2 + \cdots + x_{m-1}^2} \leq x_m\right\}$ to represent the second-order cone in $\mathbb{R}^m$.) The integer hull of T' is given by the following two inequalities:

$$x_1 \geq 1,\ x_2 \geq 1. \tag{5}$$

It is straightforward to verify that the inequalities (5) are *not valid for any polyhedral outer approximation of T'*. Indeed any polyhedral outer approximation of T' contains integer points not belonging to T'. See Proposition 3 in Section 4. Therefore, applying the cut-generating recipe (3) a finite number of times (that is considering integer hulls of a finite number of polyhedral outer approximations of T') does not yield $x_1 \geq 1$. However, we note here that we can use linear composition (3) to obtain a cut of the form $x_1 + \frac{1}{k}x_2 \geq 1$ where $k \in \mathbb{Z}_+$ and $k \geq 1$. Clearly

$$\bigcap_{k\in\mathbb{Z}_+, k\geq 1} \left\{x \in \mathbb{R}^2 \,|\, x_1 + \frac{1}{k}x_2 \geq 1\right\} = \{x \in \mathbb{R}^2 \,|\, x_1 \geq 1\}.$$

However, it would be much nicer if we could *directly obtain* $x_1 \geq 1$ without resorting to obtaining it as an implication of an infinite sequence of cuts.

Many papers [14, 15, 10, 11, 12, 29, 28, 3, 22] have explored various families of subadditive functions for linear integer programs. Our second result, in the same spirit, is a parametrized family of functions that belongs to $\mathcal{F}_K$, where K is the second-order cone $\mathcal{L}^m$. The formal result is as follows:

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[1] In this paper, we refer as the branch of a hyperbola to both the curve and the convex region delimited by this curve. Same for parabolas and ellipses.

Theorem 2. *Let* $j \in [m-1]$*. Define* $\Gamma_j := \{\gamma \in \mathbb{R}^m \mid \gamma_m \geq \sum_{i=1}^{m-1} |\gamma_i|,\ \gamma_m > |\gamma_j|\}$*. Suppose* γ *either belongs to* Γ_j *or belongs to the interior of* $\mathcal{L}^m$*. Consider the real-valued function* $f_\gamma : \mathbb{R}^m \to \mathbb{R}$ *defined as:*

$$f_\gamma(v) = \begin{cases} \gamma^\top v + 1 & \text{if } v_j \neq 0 \text{ and } \gamma^\top v \in \mathbb{Z}, \\ \lceil \gamma^\top v \rceil & \text{otherwise.} \end{cases} \tag{6}$$

Then, $f_\gamma \in \mathcal{F}_{\mathcal{L}^m}$*.*

To see an example of use of f_γ, consider $j = 1$ and $\gamma = (0, 0.5, 0.5)$. Then applying the resulting function f_γ to the columns of (4) gives us the inequality $x_1 \geq 1$.

Note that the validity of the first inequality in (5) can be explained via the disjunction $x_1 \leq 0 \vee x_1 \geq 1$. Therefore, some of the cuts generated using (6) can be viewed as split disjunctive cuts. Significant research has gone into describing split disjunctive cuts (newer implied conic constraints) for conic sections [9, 4, 24, 25, 19, 32, 7]. However, to the best of our knowledge, there is no family of subadditive functions in $\mathcal{F}_{\mathcal{L}^m}$ which have been described in closed form previously.

It is instructive to compare cuts obtained using (6) with two well-known approaches for generating cuts for the integer hull of second-order conic sets [23, 1]. Note that the CG cuts described in [23] is a special case[2] of cuts generated via linear composition (3). Therefore as discussed above, the CG cuts described in [23] cannot generate (5) directly. The conic MIR procedure described in [1] begins with first generating an extended formulation which applied to T' would be of the form:

$$\begin{aligned} t_0 &\leq x_1 + x_2 \\ t_1 &\geq 2 \\ t_2 &\geq |x_1 - x_2| \\ t_0 &\geq \|t\|_2 \\ & x_1, x_2 \in \mathbb{Z}_+, t \in \mathbb{R}^3_+. \end{aligned}$$

Then, cuts for the set $\{(x, t_2) \in \mathbb{Z}^2_+ \times \mathbb{R} \mid t_2 \geq |x_1 - x_2|\}$ are considered. However, this set is integral in this case and therefore no cuts are obtained. Thus, the conic MIR procedure does not generate the inequalities (5).

Remark 1. *The function* f_γ *defined in (6) is piecewise linear, and it is therefore tempting to think it may also belong to* $\mathcal{F}_{\mathcal{R}^m_+}$*. However it is straightforward to check that* f_γ *is not necessarily non-decreasing with respect to* $\mathbb{R}^3_+$*. Let* $j = 1$ *and* $\gamma = (0, \rho, \rho)$ *where* ρ *is a positive scalar. Then*

$$f_\gamma(v_1, v_2, v_3) = \begin{cases} \rho(v_2 + v_3) + 1 & \text{if } v_1 \neq 0 \text{ and } \rho(v_2 + v_3) \in \mathbb{Z}, \\ \lceil \rho(v_2 + v_3) \rceil & \text{otherwise.} \end{cases} \tag{7}$$

Consider the vectors $u = (0, 0, 1/\rho)$ *and* $v = (-1, 0, 1/\rho)$*. Then* $u \geq_{\mathbb{R}^3_+} v$*, but* $f_\gamma(u) = 1 < 2 = f_\gamma(v)$*.*

A proof of Theorem 2 is presented in Section 5.

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[2] More precisely, in [23] the variables are assumed to be non-negative, in which case we can drop the requirement of g satisfying $g(u) = -g(-u)$ in the definition of linear composition.

2.3 Cuts for integer conic sets in $\mathbb{R}^2$

As mentioned earlier, the family of functions (6) yields the inequalities (5). Indeed, we are able to verify a more general result in $\mathbb{R}^2$. To explain this result, we will need the following results:

Lemma 1. *Let G be one branch of a hyperbola in $\mathbb{R}^2$. Then G can be represented as $G = \{x \in \mathbb{R}^2 \mid Ax \succeq_{\mathcal{L}^3} b\}$, where $A \in \mathbb{R}^{3\times 2}$ is such that $A_{11}, A_{12} = 0$. Moreover, the asymptotes of G have equations*

$$(A_{21} + A_{31})x_1 + (A_{22} + A_{32})x_2 = b_3 + b_2 \tag{8}$$

$$(-A_{21} + A_{31})x_1 + (-A_{22} + A_{32})x_2 = b_3 - b_2. \tag{9}$$

In order to generate cuts for G in Lemma 1 using functions (6) we first require the variables to be non-negative. Therefore, let us write G as

$$A^1x_1^+ - A^1x_1^- + A^2x_1^+ - A^2x_1^- \succeq_{\mathcal{L}^3} b \tag{10}$$

$$x_1^+, x_1^-, x_2^+, x_2^- \geq 0 \tag{11}$$

$$x_j = x_j^+ - x_j^- \ j \in \{1, 2\}. \tag{12}$$

Assuming that the asymptotes of G are rational, we may assume that the coefficients in (8) and (9) are integers and then let $\tau = \gcd(A_{21} + A_{31}, A_{22} + A_{32})$. Let $j = 1$ and $\gamma = \left(0, \frac{1}{\tau}, \frac{1}{\tau}\right)$. Then we apply the function f_γ to obtain the following cut for (10), (11):

$$\frac{(A_{21} + A_{31})}{\tau}x_1^+ - \frac{(A_{21} + A_{31})}{\tau}x_1^- + \frac{(A_{22} + A_{32})}{\tau}x_2^+ - \frac{(A_{22} + A_{32})}{\tau}x_2^- \geq f_\gamma(b). \tag{13}$$

Now, using (12) and observing that the coefficient of x_j^+ is the negative of x_j^- in (13), $j = 1, 2$, we can project the inequality (13) to the space of the original x variables. The resulting cut is parallel to the asymptote (8). We can do a similar calculation to obtain a cut parallel to the other asymptote (9). We state all this concisely in the next proposition.

Proposition 1. *Let $G = \{x \in \mathbb{R}^2 \mid Ax \succeq_{\mathcal{L}^3_+} b\}$ be one branch of a hyperbola with rational asymptotes, where $A \in \mathbb{R}^{3\times 2}$ and $A_{11}, A_{12} = 0$. Then the following inequalities are valid for G^I:*

$$(u^j)^\top A^1x_1 + (u^j)^\top A^2x^2 \geq \tau^j f_{\gamma^j}(b), \tag{14}$$

where $u^1 = (0, 1, 1)$, $u^2 = (0, -1, 1)$, $\tau^j = \gcd((u^j)^\top A^1, (u^j)^\top A^2)$ and $\gamma^j := \frac{1}{\tau^j}u^j$, $j = 1, 2$.

We are now ready to state the main result of this section.

Theorem 3. *Let*

$$W = \bigcap_{i\in[m]} W^i, \tag{15}$$

where $W^i = \{x \in \mathbb{R}^2 \mid A^ix \succeq_{\mathcal{L}^{m_i}} b^i\}$, $A^i \in \mathbb{R}^{m_i\times 2}$, $b^i \in \mathbb{R}^{m_i}$ and $\mathcal{L}^{m_i}$ is the second-order cone in $\mathbb{R}^{m_i}$. Assume W has nonempty interior and each constraint $A^ix \succeq_{\mathcal{L}^{m_i}} b^i$ in the description of W is either a half-space or a single conic section, such as a parabola, an ellipse, or one branch of a hyperbola. Also assume that if W^i is a hyperbola, then it is non-degenerate and it is written as in Lemma 1, that is $A^i \in \mathbb{R}^{3\times 2}$ and $A^i_{11}, A^i_{12} = 0$. Finally, we assume that each W^i is non-redundant, that is, for all $j \in [m]$, W is strictly contained in $\cap_{i\in[m],i\neq j}W^i$. Then the following statements hold:

1. *If $W \cap \mathbb{Z}^2 = \emptyset$, then this fact can be certified with the application of at most two inequalities generated from (3) or (14);*
2. *Assume* $\text{interior}(W) \cap \mathbb{Z}^2 \neq \emptyset$. *If $\pi^\top x \geq \pi_0$ defines a face of W^I where $\pi \in \mathbb{Z}^2$ is non-zero, then this inequality can be obtained with application of exactly one function (3) or it is one of the inequalities (14).*

Proof of Lemmma 1 and Theorem 3 are presented in Section 6.

3 Cutting-planes separating bounded set of points

In this section, we prove Theorem 1. We begin by stating two well-known lemmas.

Lemma 2. *Let $K \in \mathbb{R}^n$ be a closed cone and let K^* denote its dual. Then* $\text{interior}\,(K^*) = \{y \in \mathbb{R}^n \mid y^\top x > 0 \ \forall x \in K \setminus \{0\}\}$.

Recall the recession cone of a nonempty closed convex set $C \in \mathbb{R}^n$ is the set of vectors d such that $x + td \in C$ for all $x \in C$ and for all positive scalar t. Hereafter, we will denote the recession cone of a set C and its dual by $\text{rec.cone}(C)$ and $\text{rec.cone}^*(C)$ respectively.

Lemma 3. *Let $C \in \mathbb{R}^n$ be a nonempty closed convex set. Then the following statements hold:*

(i) for every $c \in \text{interior}\,(\text{rec.cone}^*(C))$ the problem $\inf\{c^\top x \,|\, x \in C\}$ is bounded;

(ii) for every $c \notin \text{rec.cone}^*(C)$ the problem $\inf\{c^\top x \,|\, x \in C\}$ is unbounded.

Next we present two technical results that will be used to prove Proposition 2, the main result which will imply Theorem 1.

Lemma 4. *Let $K \subseteq \mathbb{R}^m$ be a regular cone. Consider the conic set $T = \{x \in \mathbb{R}^n \mid Ax \succeq_K b\}$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Assume T has nonempty interior. Consider the set $U := \{x \in T \mid \alpha \leq \pi^\top x \leq \beta\}$, where $\pi \in \mathbb{Z}^n$ is non-zero and α, β are scalars. If $\pi \in \text{interior}(\text{rec.cone}^*(T))$, then U is bounded. Moreover, there exist vectors $y^1, y^2, \ldots, y^{2n-2} \in K^*$ such that the polyhedron*

$$D := \{x \in \mathbb{R}^n \mid \alpha \leq \pi^\top x \leq \beta,\ (y^i)^\top Ax \geq (y^i)^\top b,\ i \in [2n-2]\}$$

is bounded and contains U, where $(y^i)^\top A$ is rational for all $i \in [2n-2]$.

Proof. Notice that the recession cone of the polyhedron $\{x \in \mathbb{R}^n \mid \alpha \leq \pi^\top x \leq \beta\}$ is the subspace orthogonal to π. Let $\{v^1, v^2, \ldots, v^{n-1}\}$ be a orthogonal basis for this subspace. Since $\pi \in \text{interior}\,(\text{rec.cone}^*(T))$, there exist a positive constant ε such that $w^i := \pi + \varepsilon v^i$ and $w^{i+n-1} := \pi - \varepsilon v^i$ belong to $\text{interior}\,(\text{rec.cone}^*(T))$ for all $i \in [n-1]$. Note that, since $\pi \in \mathbb{Z}^n$, we may assume that v^i's are rational. Also, we may assume that ε is rational. Thus w^i is rational for all $i \in [2n-2]$. Now consider the following primal-dual pair for each $i \in [2n-2]$:

$$\inf\{(w^i)^\top x \,|\, x \in T\}, \tag{16}$$

$$\sup\{b^\top y \mid y^\top A = (w^i)^\top,\ y \in K^*\}. \tag{17}$$

By assumption, the primal problem (16) is strictly feasible and, since $w^i \in \text{interior}\,(\text{rec.cone}^*(T))$, it follows from Lemma 3 that (16) is bounded. Therefore, strong duality for conic programming implies that the dual problem (17) is solvable [5]. Let y^i be one of its optimal solution. Then $y^i \in K^*$ and $(y^i)^\top Ax \geq (y^i)^\top b$ is a valid inequality for T, and hence for U. Therefore, we obtain the polyhedron D which contains U. Note that, since y^i is dual feasible, we have $(y^i)^\top A = (w^i)^\top$. Thus, $(y^i)^\top A$ is rational for all $i \in [2n-2]$. It remains to prove that D is bounded. Let $d \in \text{rec.cone}(D)$. Then, in particular, d belongs to the recession cone of $\{x \in \mathbb{R}^n \mid \alpha \leq \pi^\top x \leq \beta\}$. Thus, there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ such that

$$d = \sum_{i=1}^{n-1} \lambda_i v^i. \tag{18}$$

Also, d belongs to the recession cone of $\{x \in \mathbb{R}^n \mid (y^i)^\top Ax \geq (y^i)^\top b,\ i \in [2n-2]\}$. Thus d must satisfy

$$(w^i)^\top d = (y^i)^\top Ad \geq 0,\ i \in [2n-2]. \tag{19}$$

Plugging (18) into (19) and then using the definition of w^i's, the fact that $\{v^1, v^2, \dots, v^{n-1}\}$ is an orthogonal basis, and that π is orthogonal to v^i we conclude that

$$\varepsilon\lambda_i \left\|v^i\right\|^2 \geq 0, \quad -\varepsilon\lambda_i \left\|v^i\right\|^2 \geq 0,\ i \in [n-1].$$

Finally, since $\varepsilon > 0$, these inequalities imply that $\lambda_i = 0$ for all $i \in [n-1]$. Therefore, the recession cone of D only contains the zero vector which proves that D is bounded. □

The next lemma states that under some conditions it is possible to separate a point from a set using a rational separating hyperplane.

Lemma 5. *Let $C \subseteq \mathbb{R}^n$ be a closed convex set. Assume* interior (rec.cone$^*(C)$) $\neq \emptyset$*. Let $z \notin C$. Then there exist $\pi \in \mathbb{Q}^n$, $\pi \neq 0$, and $\pi_0 \in \mathbb{R}$ such that $\pi^\top z < \pi_0 \leq \pi^\top x$ for all $x \in C$.*

Proof. The standard separation theorem ensures that there exist $w \in \mathbb{R}^n$, $w \neq 0$, and $w_0 \in \mathbb{R}$ such that $w^\top z < w_0 \leq w^\top x$ for all $x \in C$. As interior (rec.cone$^*(C)$) $\neq \emptyset$ there exist $w^1, w^2, \cdots, w^{n+1} \in$ interior (rec.cone$^*(C)$) affinity independent. For every $i \in [n+1]$ let $w_0^i = \inf\{(w^i)^\top x \mid x \in C\}$. In view of Lemma 3 we have that w_0^i is finite for all $i \in [n+1]$. Since $w_0 - w^\top z > 0$ and z is fixed, we can chose $\varepsilon_i > 0$, $i \in [n+1]$, such that

$$|\sum_{i=1}^{n+1} \varepsilon_i (w^i)^\top z - \sum_{i=1}^{n+1} \varepsilon_i w_0^i| < w_0 - w^\top z. \tag{20}$$

Moreover, since $w^1, w^2, \cdots, w^{n+1}$ are affinity independent, the cone generated by these vectors is full dimensional. Thus, the scalars $\varepsilon_i > 0$, $i \in [n+1]$, can be chosen such that $\pi := w + \sum_{i=1}^{n+1} \varepsilon_i w^i \in \mathbb{Q}^n$.

Now observe that

$$\begin{aligned}\pi^\top z &= w^\top z + \sum_{i=1}^{n+1} \varepsilon_i (w^i)^\top z \\ &< w_0 + \sum_{i=1}^{n+1} \varepsilon_i w_0^i \\ &\leq \inf\{w^\top x \mid x \in C\} + \sum_{i=1}^{n+1} \inf\{(\varepsilon_i w^i)^\top x \mid x \in C\} \\ &\leq \inf\{(w^\top + \sum_{i=1}^{n+1} \varepsilon_i w^i)^\top x \mid x \in C\} \\ &\leq (w^\top + \sum_{i=1}^{n+1} \varepsilon_i w^i)^\top x \quad \forall x \in C \\ &= \pi^\top x \quad \forall x \in C,\end{aligned}$$

where the first inequality follows from (20). Therefore, $\pi^\top z < \pi_0 \leq \pi^\top x$ for all $x \in C$, where $\pi_0 := w_0 + \sum_{i=1}^{n+1} \varepsilon_i w_0^i$. ☐

The next result will imply Theorem 1.

Proposition 2. *Let $K \subseteq \mathbb{R}^m$ be a regular cone. Consider the conic set $T = \{x \in \mathbb{R}^n \mid Ax \succeq_K b\}$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Assume T has nonempty interior. Assume $B := \{x \in T \mid \pi^\top x \leq \pi_0\}$ is nonempty and bounded, where $\pi \in \mathbb{Z}^n$ is non-zero. Then, for some natural number $p \leq 2^n$, there exist vectors $y^1, y^2 \ldots, y^p \in K^*$ such that the polyhedron*

$$P = \{x \in \mathbb{R}^n \mid \pi^\top x \leq \pi_0,\ (y^i)^\top Ax \geq (y^i)^\top b,\ i \in [p]\} \tag{21}$$

contains B and $P^I = B^I$, where $(y^i)^\top A$ is rational for all $i \in [p]$.

Proof. It is sufficient to construct a polyhedron P given in (21) such that $B \subseteq P$ and $P^I = B^I$. If the number of inequalities defining P is greater than 2^n, it can be shown [30] that the result holds with at most 2^n constraints.

Note that

$$d^\top \pi > 0, \text{ for all } d \in \text{rec.cone}(T) \setminus \{0\}. \tag{22}$$

Indeed, if $d \in \text{rec.cone}(T)$ is such that $d^\top \pi \leq 0$, then $d \in \text{rec.cone}(B)$, which implies that $d = 0$ since B is bounded. Now, in view of Lemma 2, (22) implies that $\pi \in \text{interior}\,(\text{rec.cone}^*(T))$.

Let us show how to obtain the dual multipliers y_i's that will define P. First, consider the primal-dual pair

$$\inf\{\pi^\top x \mid x \in T\}, \tag{23}$$

$$\sup\{b^\top y \mid y^\top A = \pi^\top,\ y \in K^*\}. \tag{24}$$

Since $\pi \in \text{interior}\,(\text{rec.cone}^*(T))$, Lemma 3 implies that the program (23) is bounded. Moreover, by assumption, it is strictly feasible. Then it follows from strong duality for conic programming that the program (24) is solvable. Let y^0 be an optimal solution for (24). Then $y^0 \in K^*$ and

$$(y^0)^\top Ax \geq (y^0)^\top b$$

is a valid inequality for T, and hence for B. We have obtained the first multiplier. Let

$$P^0 = \{x \in \mathbb{R}^n \mid \pi^\top x \leq \pi_0,\ (y^0)^\top Ax \geq (y^0)^\top b\} = \{x \in \mathbb{R}^n \mid (y^0)^\top b \leq \pi^\top x \leq \pi_0\},$$

where the second equality comes from the fact that $(y^0)^\top A = \pi^\top$ since y^0 is dual feasible. Note that $(y^0)^\top A$ is rational since $\pi \in \mathbb{Z}^n$. Clearly, $P^0 \supseteq B$. However, unless the space has dimension one, P^0 is not bounded. In fact, the recession cone of P^0 is the subspace orthogonal to π. Nevertheless, it follows from Lemma 4 applied to $U := P^0 \cap T$ that there exist vectors $y^1, y^2, \ldots, y^{2n-2} \in K^*$ such that the polyhedron

$$P^1 = \{x \in \mathbb{R}^n \mid (y^0)^\top b \leq \pi^\top x \leq \pi_0,\ (y^i)^\top Ax \geq (y^i)^\top b,\ i \in [2n-2]\}$$

is bounded and contains U (which contains B), where $(y^i)^\top A$ is rational for all $i \in [2n-2]$. If $(P^1)^I = B^I$, then we are done by setting P to P^1. Otherwise, as P^1 is bounded, there is only a finite number of integer points in $P^1 \setminus B$. For each one of these points we construct an rational inequality defined by dual multipliers that separates it from B; and then adding these inequalities to the description of P^1 we obtain P. Now we show how to obtain such dual multipliers. Let z be an integer point in $P^1 \setminus B$. As T is closed and convex it follows from Lemma 5 that there exist a non-zero vector $w \in \mathbb{Q}^n$ and a scalar w_0 such that $w^\top z < w_0 \leq w^\top x$ for all $x \in T$. Then the conic program $\min\{w^\top x \mid x \in T\}$ is bounded, and hence it follows from strong duality for conic programming that its dual program $\max\{b^\top y \mid y^\top A = w^\top,\ y \in K^*\}$ is solvable. Let y be an optimal solution for this dual program. Then $y^\top Ax \geq y^\top b$ is a valid inequality for T and $(y)^\top A$ is rational since $w \in \mathbb{Q}^n$. Thus, we can add $y^\top Ax \geq y^\top b$ to the description of P^1. Observe that $y^\top Az = w^\top z < w_0 \leq y^\top b \leq y^\top Ax$, for all $x \in T$, showing that $y^\top Ax \geq y^\top b$ indeed separates z from T, and hence from B. $\square$

Proof. of Theorem 1 Let $\pi^\top x \geq \pi_0$ be a valid inequality for T^I, where $\pi \in \mathbb{Z}^n$ is non-zero. Suppose $B = \{x \in T \mid \pi^\top x \leq \pi_0\}$ is nonempty and bounded. Then, by Proposition 2, using dual multipliers $y^0, y^1, \ldots, y^p \in K^*$, and letting $P = \{x \in \mathbb{R}^n \mid \pi^\top x \leq \pi_0,\ (y^i)^\top Ax \geq (y^i)^\top b,\ i \in [p]\}$, we have that (i) $P \supseteq B$ and (ii) $P \cap \mathbb{Z}^n = B \cap \mathbb{Z}^n$. The only integer points in B are those that satisfy $\pi^\top x = \pi_0$ and $\text{interior}(B) \cap \mathbb{Z}^n = \emptyset$. Therefore, $\text{interior}(P) \cap \mathbb{Z}^n = \emptyset$ and hence $\pi^\top x \geq \pi_0$ is a valid inequality for the integer hull of $Q = \{x \in \mathbb{R}^n \mid (y^i)^\top Ax \geq (y^i)^\top b,\ i \in [p]\}$. In addition, $(y^i)^\top A$ is rational for all $i \in [p]$ in view of Proposition 2. $\square$

4 Cannot construct appropriate polyhedral outer approximation of hyperbola

Proposition 3. *Let* $T' := \{(x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\}$*. Every polyhedral outer approximation of* T' *contains points of the form* $(0,k)$ *(and similarly points of form* $(k,0)$*) for* k *sufficiently large natural number.*

Proof. Suppose $\{x \in \mathbb{R}^2 \,|\, \alpha_1^i x_1 + \alpha_2^i x_2 \geq \beta_i,\ i \in [q]\}$, is a polyhedral outer approximation of T' where q is some natural number. Since the recession cone of this polyhedron contains the recession cone of T', that is $\mathbb{R}^2_+$, we have that $\alpha_1^i, \alpha_2^i \geq 0$.

We will prove that there exist points of the form $(0, k)$ belonging to this outer approximation by showing that for all $i \in [q]$ there exist a k_i such that $(\alpha^i)^\top(0, t) \geq \beta_i$ for all $t \in [k_i, \infty) \cap \mathbb{Z}$. If $\alpha_2^i = 0$, then $\beta_i \leq 0$ (since $\alpha_1^i \frac{1}{k} + \alpha_2^i k \geq \beta_i$ for all $k \in \mathbb{R}_+$). Therefore $k_i = 0$. If $\alpha_2^i > 0$, then $k_i = \frac{\beta_i}{\alpha_2^i}$. ☐

5 A family of cut-generating functions in $\mathcal{F}_{\mathcal{L}^m}$ and its properties

In this section, we show that f_γ defined in (6) belongs to $\mathcal{F}_K$. Clearly f_γ satisfies property (3.) in the definition of $\mathcal{F}_K$, i.e., $f_\gamma(0) = 0$. In the next two propositions we prove that f_γ also satisfies properties (1.) and (2.).

Proposition 4. *The function f_γ defined in (6) is subadditive.*

Proof. Let $u, v \in \mathbb{R}^m$. If at least one of these vectors fits in the first clause of (6), then we have

$$f_\gamma(u+v) \leq \left\lceil \gamma^\top(u+v) \right\rceil + 1 \leq \left\lceil \gamma^\top u \right\rceil + \left\lceil \gamma^\top v \right\rceil + 1 \leq f_\gamma(u) + f_\gamma(v).$$

Now, suppose that neither u nor v satisfies the first clause. If $u + v$ does not fit in the first clause, then we are done because $\lceil \cdot \rceil$ is a subadditive function. Assume $u + v$ satisfies the first clause, i.e.,

$$u_j + v_j \neq 0, \quad \gamma^\top(u+v) = \gamma^\top u + \gamma^\top v \in \mathbb{Z}. \tag{25}$$

In this case, u_j and v_j cannot be simultaneously zero, say $u_j \neq 0$. Then

$$\gamma^\top u \notin \mathbb{Z}, \tag{26}$$

because u does not satisfies the first clause. It follows from (25) and (26) that

$$\gamma^\top v \notin \mathbb{Z}. \tag{27}$$

Finally, (25), (26), (27) together imply

$$f_\gamma(u) + f_\gamma(v) = \left\lceil \gamma^\top u \right\rceil + \left\lceil \gamma^\top v \right\rceil = \gamma^\top u + \gamma^\top v + 1 = f_\gamma(u+v),$$

where the second inequality follows from the fact that $\gamma^\top u + \gamma^\top v \in \mathbb{Z}$. ☐

Lemma 6. *Let $w \in \mathcal{L}^m$ and $j \in [m-1]$. Let Γ_j be defined as in Theorem 2. If $\gamma \in \Gamma_j \cap \text{interior}\,(\mathcal{L}^m)$, then $\gamma^\top w \geq 0$. If, in addition, $w_j \neq 0$, then $\gamma^\top w > 0$.*

Proof. We have that $\gamma \in \mathcal{L}^m$. Therefore, since $w \succeq_{\mathcal{L}^m} 0$ and $\mathcal{L}^m$ is a self-dual cone, we conclude that $\gamma^\top w \geq 0$. Now, assume $w_j \neq 0$. If either γ or w is in the interior of $\mathcal{L}^m$, then it follows directly from Lemma 2 that $\gamma^\top w > 0$. Assume $\gamma, w \notin \text{interior}\,(\mathcal{L}^m)$. Then

$$w_m = \sqrt{w_1^2 + w_2^2 + \cdots + w_{m-1}^2} \tag{28}$$

$$\gamma_m = \sqrt{\gamma_1^2 + \gamma_2^2 + \cdots + \gamma_{m-1}^2}. \tag{29}$$

Two observations follows: (i) as $w_j \neq 0$, equation (28) implies that for all $i \in [m-1]$ such that $i \neq j$ we have $w_m > |w_i|$; (ii) since $\gamma_m > |\gamma_j|$, equation (29) implies that $\gamma_i \neq 0$ for some $i \in [m-1]$ such that $i \neq j$. Now, for all $i \in [m-1]$ such that $\gamma_i \geq 0$ we multiply $w_m > -w_i$ by γ_i, and for all $i \in [m-1]$ such that $\gamma_i < 0$ we multiply $w_m > w_i$ by $-\gamma_i$. In view of observations (i) and (ii), at least one of the resulting inequalities remains strictly. Then adding them all we obtain

$$\begin{aligned} \sum_{i\in[m-1]:\ \gamma_i \geq 0} \gamma_i w_m + \sum_{i\in[m-1]:\ \gamma_i<0} -\gamma_i w_m &> \sum_{i\in[m-1]:\ \gamma_i\geq 0} \gamma_i(-w_i) + \sum_{i\in[m-1]:\ \gamma_i<0} (-\gamma_i) w_i \\ \Rightarrow \sum_{i\in[m-1]} |\gamma_i| w_m &> - \sum_{i\in[m-1]} \gamma_i w_i \\ \Rightarrow \gamma_m w_m &> - \sum_{i\in[m-1]} \gamma_i w_i, \end{aligned}$$

where the last implication follows from the fact that $\gamma_m \geq \sum_{i=1}^{m-1} |\gamma_i|$ and $w_m \geq 0$. The result follows from this last inequality. $\square$

Proposition 5. *The function f_γ defined in (6) is non-decreasing with respect to $\mathcal{L}^m$.*

Proof. Let $u, v \in \mathbb{R}^m$. Suppose $u \succeq_{\mathcal{L}^m} v$. By applying Lemma 6 to $w = u - v$ we conclude

$$\gamma^\top u \geq \gamma^\top v, \tag{30}$$

where the inequality (30) holds strictly whenever $u_j - v_j \neq 0$. Now, we use these facts to prove that $f_\gamma(v) \leq f_\gamma(u)$. If u fits in the first clause of (6), then $f_\gamma(v) \leq \gamma^\top v + 1 \leq \gamma^\top u + 1 = f_\gamma(u)$, where the second inequality follows from (30). Assume u does not satisfies the first clause. If v does not fit in the first clause, then the result follows directly from (30) and the fact that $\lceil \cdot \rceil$ is non-decreasing. Suppose v satisfies the first clause, i.e, $v_j \neq 0$ and $\gamma^\top v \in \mathbb{Z}$. In this case, if $u_j = 0$, then $u_j - v_j \neq 0$ and hence (30) holds strictly. Therefore, we conclude that $f_\gamma(v) = \gamma^\top v + 1 \leq \lceil \gamma^\top u \rceil = f_\gamma(u)$. On the other hand, if $u_j \neq 0$, then $\gamma^\top u \notin \mathbb{Z}$ (since u does not satisfy the first clause), and using (30) we obtain $\gamma^\top v < \lceil \gamma^\top u \rceil$ and hence $f_\gamma(v) = \gamma^\top v + 1 \leq \lceil \gamma^\top u \rceil = f_\gamma(u)$, which completes the proof. $\square$

6 Application of cut-generating functions in $\mathbb{R}^2$

In this section, we will prove Theorem 3. We begin with proofs of two technical lemmas.

Lemma 7. *Let $W^i = \{x \in \mathbb{R}^2 \mid A^i x \succeq_{\mathcal{L}^{m_i}} b^i\}$ be a parabola, where $A^i \in \mathbb{R}^{m_i \times 2}$, $b^i \in \mathbb{R}^{m_i}$ and $\mathcal{L}^{m_i}$ is the second-order cone in $\mathbb{R}^{m_i}$. If $\pi \in \operatorname{rec.cone}^*(W^i) \setminus \operatorname{interior}(\operatorname{rec.cone}^*(W^i))$, $\pi \neq 0$, then the problem $\inf\{\pi^\top x \mid x \in W^i\}$ is unbounded.*

Proof. Up to a rotation, any parabola in $\mathbb{R}^2$ can be written as $\{(x, y) \in \mathbb{R}^2 \mid y \geq \rho(x - x_0)^2 + y_0\}$, where $\rho > 0$. In this case, the recession cone of the parabola is a vertical line. As $\pi \in \operatorname{rec.cone}^*(W^i) \setminus \operatorname{interior}(\operatorname{rec.cone}^*(W^i))$ we must have $\pi_2 = 0$, in which case $\pi_1 \neq 0$ and the problem is clearly unbounded. $\square$

Lemma 8. *Let W be defined as in the statement of Theorem 3. Assume, in addition, that W is unbounded. Let $\pi \neq 0$ be such that $\pi \notin \operatorname{interior}(\operatorname{rec.cone}^*(W))$. If the problem*

$$\alpha := \inf\{\pi^\top x \mid x \in W\} \tag{31}$$

is bounded, then there exists $i_0 \in [m]$ *such that*

$$\alpha = \inf\{\pi^\top x \mid W^{i_0}\}. \tag{32}$$

Moreover, $W^{i_0} = \{x \in \mathbb{R}^2 \mid A^{i_0} x \succeq_{\mathcal{L}^{m_{i_0}}} b^{i_0}\}$ *is either:*
(i) a half-space defined by $\pi^\top x \geq \alpha$*; or*
(ii) one branch of a hyperbola whose one of the asymptotes is orthogonal to π*.*

Proof. Since the primal problem (31) is bounded and strictly feasible, we have that its dual

$$\sup\{\sum_{i=1}^{m} (b^i)^\top y^i \mid \sum_{i=1}^{m} (y^i)^\top A^i = \pi^\top,\ y^i \in \mathcal{L}^*_{m_i}\ \forall i \in [m]\} \tag{33}$$

is solvable [5]. We will show that (33) admits an optimal solution for which $y^i = 0$ for all $i \in [m]$ except for one particular $i_0 \in [m]$.

Since (31) is bounded, it follows from Lemma 3 that $\pi \in \text{rec.cone}^*(W)$. On the other hand, by assumption π is not in the interior of that cone. Therefore, using Lemma 2 we conclude that there exist a non-zero vector $d_0 \in \text{rec.cone}(W)$ such that $\pi^\top d_0 = 0$. Then any feasible solution $(y^1, y^2, \cdots, y^m)$ of (33) satisfies

$$0 = \pi^\top d_0 = \sum_{i=1}^{m} (y^i)^\top A^i d_0.$$

Moreover, each term in this summation is non-negative since $A^i d_0 \succeq_{\mathcal{L}_{m_i}} 0$ for all $i \in [m]$ (because $d_0 \in \text{rec.cone}(W)$) and $y^i \in \mathcal{L}^*_{m_i}$. As a result, we have $(y^i)^\top A^i d_0 = 0\ \forall i \in [m]$. As d_0 is a non-zero vector in $\mathbb{R}^2$, we conclude that for each $i \in [m]$ there must exist a scalar λ_i such that

$$(y^i)^\top A^i = \lambda_i \pi^\top. \tag{34}$$

We claim that $\lambda_i \geq 0$ for all $i \in [m]$. To prove the claim, all we need to show is that $(y^i)^\top A^i$ and π are in the same half-space. By assumption $\pi \in \text{rec.cone}^*(W)$. Since $\text{rec.cone}^*(W)$ is contained in a half-space (otherwise we would have $\text{rec.cone}(W) = \{0\}$ which contradicts the fact that W is unbounded), it is enough to prove that $(y^i)^\top A^i \in \text{rec.cone}^*(W)$. To see why this is true, note that for all $d \in \text{rec.cone}(W^i)$ we have $A^i d \succeq_{\mathcal{L}_{m_i}} 0$, which implies $(y^i)^\top A^i d \geq 0$. Thus, $(y^i)^\top A^i \in \text{rec.cone}^*(W^i) \subseteq \text{rec.cone}^*(W)$, where the last containment follows form the fact that $\text{rec.cone}(W^i) \supseteq \text{rec.cone}(W)$.

Now, suppose $(y^1, y^2, \cdots, y^m)$ is an optimal solution of the dual problem (33). If $\lambda_i = 0$, then we must have $(b^i)^\top y^i = 0$, because $(b^i)^\top y^i > 0$ would imply the dual problem to be unbounded and $(b^i)^\top y^i < 0$ would imply that the current solution is not optimal. Hence we have that if $\lambda_i = 0$, then we can set $y^i = 0$ without altering the objective value. On the other hand, since $\pi \neq 0$, (34) combined with the equality in (33) imply that the λ's add up to 1. Thus, we cannot have $\lambda_i = 0$ for all $i \in [m]$. Suppose $\lambda_i, \lambda_j > 0$ for some $i, j \in [m]$, $i \neq j$. We claim that

$$(b^i)^\top y^i = \frac{\lambda_i}{\lambda_j} (b^j)^\top y^j.$$

Without loss of generality, assume by contradiction that

$$(b^i)^\top y^i < \frac{\lambda_i}{\lambda_j} (b^j)^\top y^j.$$

Then, since $\lambda_i + \lambda_j \leq 1$ we obtain

$$(b^i)^\top y^i + (b^j)^\top y^j < \frac{\lambda_i}{\lambda_j}(b^j)^\top y^j + (b^j)^\top y^j \leq \frac{1}{\lambda_j}(b^j)^\top y^j.$$

In this case, we could set $\lambda_i = 0,\ \lambda_j = 1$ and $y^i = 0$ to obtain a new feasible solution with objective value strictly larger. But this contradicts the fact that y is an optimal solution. Thus, the claim holds and by setting $\lambda_i = 0, \lambda_j = 1$ and $y^i = 0$ we obtain a new feasible solution with the same objective value, and hence optimal. In this case, we set $i_0 = j$.

Consider now the primal-dual pair

$$\inf\{\pi^\top x \mid \ A^{i_0} x \succeq_{\mathcal{L}_{m_{i_0}}} b^{i_0}\}, \tag{35}$$

$$\sup\{(b^{i_0})^\top y^{i_0} \mid (y^{i_0})^\top A^{i_0} = \pi^\top,\ y^{i_0} \in \mathcal{L}^*_{m_{i_0}}\}. \tag{36}$$

Let x^* be an ε-optimal solution to the original primal (31). Clearly, x^* is feasible for (35). Note now that the dual solution constructed for (33) above, when restricted to the y^{i_0} component is a feasible solution to (36). Thus, we obtain an ε primal-dual feasible solution pair. Therefore we obtain (32).

To prove the second part of the lemma, we first observe that $\text{rec.cone}^*(W^{i_0}) \subseteq \text{rec.cone}^*(W)$. If $\pi \notin \text{rec.cone}^*(W^{i_0})$, then (32) would be unbounded in view of Lemma 3. Since $\pi \notin \text{interior}\,(\text{rec.cone}^*(W))$, we have that $\pi \notin \text{interior}\,(\text{rec.cone}^*(W^{i_0}))$. Therefore, $\pi \in \text{rec.cone}^*(W^{i_0}) \backslash \text{interior}\,(\text{rec.cone}^*(W^{i_0}))$.

Now, W^{i_0} cannot define an ellipse because then $W \subseteq W^{i_0}$ will be unbounded. Since $\pi \in \text{rec.cone}^*(W^{i_0}) \setminus \text{interior}(\text{rec.cone}^*(W^{i_0}))$, if W^{i_0} was a parabola, then problem (32) would be unbounded in view of Lemma 7. Therefore, only two possibilities remain:
(i) W^{i_0} is defined by a linear inequality, say $\mu^\top x \geq \mu_0$. In this case μ must be a multiple of π, otherwise problem (32) would be unbounded. Thus, we may assume $\pi = \mu$ and then $\mu_0 = \alpha$.
(ii) W^{i_0} is one branch of a hyperbola. In this case, $\text{rec.cone}(W^{i_0})$ is defined by the asymptotes of the hyperbola. As $\pi \in \text{rec.cone}^*(W^{i_0}) \setminus \text{interior}(\text{rec.cone}^*(W^{i_0}))$, π must be orthogonal to one of the asymptotes. ☐

Next we prove Lemma 1 that was stated in Section 2.3.

Proof. of Lemma 1 Any conic section (parabola, ellipse, hyperbola) in $\mathbb{R}^2$ is a curve defined by a quadratic equation of the form

$$\frac{1}{2}x^\top Qx + d^\top x + s = 0, \tag{37}$$

where s is a scalar, $d \in \mathbb{R}^2$ and $Q = VDV^\top$. In this factorization, $V \in \mathbb{R}^{2\times 2}$ is orthonormal and

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

where λ_1, λ_2 are the eigenvalues of Q. In particular, the curve defined by (37) is a hyperbola if and only if one of these eigenvalues is positive and the other is negative. After changing variables $y := V'x$ and completing squares, equation (37) can be written in exactly one of the following forms

$$[\beta_1(y_1 - \alpha_1)]^2 - [\beta_2(y_2 - \alpha_2)]^2 = \pm\eta^2, \tag{38}$$

where η and α_i, β_i, for $i = 1, 2$, are constants depending on the coefficients of (37). In what follows, we assume that the coefficient of η^2 is positive. If it was negative, then we could multiply (38) by -1 and all we will do next would be analogous. Under this assumption, one branch of the hyperbola is given by

$$\begin{aligned}
G^+ &:= \{y \in \mathbb{R}^2 \mid (\eta)^2 + [\beta_2(y_2 - \alpha_2)]^2 \leq [\beta_1(y_1 - \alpha_1)]^2,\ \beta_1(y_1 - \alpha_1) \geq 0\} \\
&= \{y \in \mathbb{R}^2 \mid \sqrt{\eta^2 + [\beta_2(y_2 - \alpha_2)]^2} \leq \beta_1(y_1 - \alpha_1)\} \\
&= \{y \in \mathbb{R}^2 \mid (\eta, \beta_2(y_2 - \alpha_2), \beta_1(y_1 - \alpha_1)) \in \mathcal{L}^3\} \\
&= \{y \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & 0 \\ 0 & \beta_2 \\ \beta_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \succeq_{\mathcal{L}^3} \begin{bmatrix} -\eta \\ \beta_2\alpha_2 \\ \beta_1\alpha_1 \end{bmatrix}\}.
\end{aligned}$$

Then, going back to the space of the original variables we obtain

$$G^+ = \{x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & 0 \\ \beta_2 v_{12} & \beta_2 v_{22} \\ \beta_1 v_{11} & \beta_1 v_{21} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \succeq_{\mathcal{L}^3} \begin{bmatrix} -\eta \\ \beta_2\alpha_2 \\ \beta_1\alpha_1 \end{bmatrix}\}, \tag{39}$$

where v_{ij} are the entries of the matrix V. The other branch of the hyperbola is given by

$$G^- := \{y \in \mathbb{R}^2 \mid (\eta)^2 + [\beta_2(y_2 - \alpha_2)]^2 \leq [\beta_1(y_1 - \alpha_1)]^2,\ \beta_1(y_1 - \alpha_1) \leq 0\}.$$

After the change of variables $\tilde{y} := -y$ we obtain

$$\begin{aligned}
G^- &= \{\tilde{y} \in \mathbb{R}^2 \mid (\eta)^2 + [\beta_2(-\tilde{y}_2 - \alpha_2)]^2 \leq [\beta_1(-\tilde{y}_1 - \alpha_1)]^2,\ \beta_1(-\tilde{y}_1 - \alpha_1) \leq 0\} \\
&= \{\tilde{y} \in \mathbb{R}^2 \mid (\eta)^2 + [\beta_2(\tilde{y}_2 + \alpha_2)]^2 \leq [\beta_1(\tilde{y}_1 + \alpha_1)]^2,\ \beta_1(\tilde{y}_1 + \alpha_1) \geq 0\} \\
&= \{\tilde{y} \in \mathbb{R}^2 \mid (\eta, \beta_2(\tilde{y}_2 + \alpha_2), \beta_1(\tilde{y}_1 + \alpha_1)) \in \mathcal{L}^3\} \\
&= \{\tilde{y} \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & 0 \\ 0 & \beta_2 \\ \beta_1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} \succeq_{\mathcal{L}^3} \begin{bmatrix} -\eta \\ -\beta_2\alpha_2 \\ -\beta_1\alpha_1 \end{bmatrix}\}.
\end{aligned}$$

Going back to the space of the original variables we obtain

$$G^- = \{x \in \mathbb{R}^2 \mid \begin{bmatrix} 0 & 0 \\ -\beta_2 v_{12} & -\beta_2 v_{22} \\ -\beta_1 v_{11} & -\beta_1 v_{21} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \succeq_{\mathcal{L}^3} \begin{bmatrix} -\eta \\ -\beta_2\alpha_2 \\ -\beta_1\alpha_1 \end{bmatrix}\}. \tag{40}$$

It follows from (38) that the asymptotes of G^+ have equations

$$\begin{aligned}
\beta_1 y_1 + \beta_2 y_2 &= \beta_1\alpha_1 + \beta_2\alpha_2, \\
\beta_1 y_1 - \beta_2 y_2 &= \beta_1\alpha_1 - \beta_2\alpha_2.
\end{aligned}$$

In the space of x variables they become

$$(\beta_1 v_{11} + \beta_2 v_{12})x_1 + (\beta_1 v_{21} + \beta_2 v_{22})x_2 = \beta_1\alpha_1 + \beta_2\alpha_2, \tag{41}$$

$$(\beta_1 v_{11} - \beta_2 v_{12})x_1 + (\beta_1 v_{21} - \beta_2 v_{22})x_2 = \beta_1\alpha_1 - \beta_2\alpha_2. \tag{42}$$

The asymptotes of G^- are obtained in a similar way. □

Lemma 9. *Let G be one branch of a non-degenrate hyperbola in $\mathbb{R}^2$. Let $\pi^\top x \geq \pi_0$ be a face of G^I such that $\pi \in \mathbb{Z}^2$ is non-zero and orthogonal to one of the asymptotes. Then this valid inequality is one of the inequalities (14).*

Proof. Using the same notation adopted in the proof of Lemma 1 above, we assume $G = G^+$. If $G = G^-$, then the proof is analogous. Note that G is contained in the set

$$\begin{aligned} H := \{x \in \mathbb{R}^2 \mid\ & (\beta_1 v_{11} + \beta_2 v_{12})x_1 + (\beta_1 v_{21} + \beta_2 v_{22})x_2 \geq \beta_1\alpha_1 + \beta_2\alpha_2, \\ & (\beta_1 v_{11} - \beta_2 v_{12})x_1 + (\beta_1 v_{21} - \beta_2 v_{22})x_2 \geq \beta_1\alpha_1 - \beta_2\alpha_2\}. \end{aligned}$$

Assume π is orthogonal to the asymptote (41). The proof of the case in which π is orthogonal to second asymptote is similar. Since $\pi \in \mathbb{Z}^2$ is non-zero, we may assume that the coefficients of x_1 and x_2 in (41) are integer. Let

$$\tau := \gcd\{\beta_1 v_{11} + \beta_2 v_{12}, \beta_1 v_{21} + \beta_2 v_{22}\}.$$

Since the hyperbola is non-degenerate, the line

$$(\beta_1 v_{11} + \beta_2 v_{12})x_1 + (\beta_1 v_{21} + \beta_2 v_{22})x_2 = \beta_1\alpha_1 + \beta_2\alpha_2$$

does not intersects G. However, for all $\varepsilon > 0$, the equation

$$\frac{\beta_1 v_{11} + \beta_2 v_{12}}{\tau}x_1 + \frac{\beta_1 v_{21} + \beta_2 v_{22}}{\tau}x_2 = \frac{\beta_1\alpha_1 + \beta_2\alpha_2}{\tau} + \varepsilon \tag{43}$$

intersects G along a ray. Moreover, (43) has integer solutions if and only if the right-hand-side is integral.

Therefore, if $(\beta_1\alpha_1 + \beta_2\alpha_2)/\tau \in \mathbb{Z}$, then the inequality

$$\frac{\beta_1 v_{11} + \beta_2 v_{12}}{\tau}x_1 + \frac{\beta_1 v_{21} + \beta_2 v_{22}}{\tau}x_2 \geq \frac{\beta_1\alpha_1 + \beta_2\alpha_2}{\tau} + 1 \tag{44}$$

is a face of G^I, and hence it is equivalent to $\pi^\top x \geq \pi_0$. On the other hand, if $(\beta_1\alpha_1 + \beta_2\alpha_2)/\tau \notin \mathbb{Z}$, then

$$\frac{\beta_1 v_{11} + \beta_2 v_{12}}{\tau}x_1 + \frac{\beta_1 v_{21} + \beta_2 v_{22}}{\tau}x_2 \geq \left\lceil \frac{\beta_1\alpha_1 + \beta_2\alpha_2}{\tau} \right\rceil \tag{45}$$

is a face of G^I, and hence it is equivalent to $\pi^\top x \geq \pi_0$.

Observe now that (44) and (45) are one of the inequalities (14) in view of Proposition 1. ☐

Next we use Lemma 8 and Lemma 9 above to proof Theorem 3.

Proof. of Theorem 3 First, we observe that if W is bounded, then the result follows directly from Theorem 1. Suppose W is unbounded. We have two cases:

Case 1: $W \cap \mathbb{Z}^2 = \emptyset$. In this case, there exist $\pi = (\pi_1, \pi_2)$ with π_1, π_2 integer relatively primes and a integer π_0 such that [13, 2]

$$W \subseteq \{x \in \mathbb{R}^2 \mid \pi_0 \leq \pi^\top x \leq \pi_0 + 1\}. \tag{46}$$

We will show that the the cut $\pi^\top x \geq \pi_0 + 1$ can be obtained using subadditive functions (3) or using one of the inequalities (14). Analogous proof holds for the cut $\pi^\top x \leq \pi_0$. A consequence of W being between these two lines is that rec.cone(W) is orthogonal to π and, therefore, $\pi \notin$ interior (rec.cone$^*(W)$) in view of Lemma 2. Then, by Lemma 8,

$$\alpha := \inf\{\pi^\top x \mid W^{i_0}\} = \inf\{\pi^\top x \mid x \in W\}, \tag{47}$$

for some $i_0 \in [m]$, where there are only two possibilities for $W^{i_0} = \{x \in \mathbb{R}^2 : \ A^{i_0}x \succeq_{\mathcal{L}^{m_{i_0}}} b^{i_0}\}$:
(i) W^{i_0} is the half-space $\pi^\top x \geq \alpha$: In this case, since $A^{i_0}x \succeq_{\mathcal{L}^{m_{i_0}}} b^{i_0}$ is non-redundant, we have that the line $\pi^\top x = \alpha$ intersects W. Note that $\pi_0 \leq \alpha$ in view of (46). Since W is unbounded and its recession cone is orthogonal to π, if $\alpha = \pi_0$, then W would contain a integer point from the line $\pi^\top x = \pi_0$. Therefore, $\alpha > \pi_0$ in which case $\pi^\top x \geq \lceil \alpha \rceil = \pi_0 + 1$ is a valid inequality for W^I and this cut can be obtained using a subadditive function (3).
(ii) W^{i_0} is a hyperbola whose one of the asymptotes is orthogonal to π: Without loss of generality, we may assume that the asymptote orthogonal to π has equation $\pi^\top x = \alpha$. Let

$$\beta = \begin{cases} \alpha + 1 \text{ if } \alpha \in \mathbb{Z} \\ \lceil \alpha \rceil \text{ if } \alpha \notin \mathbb{Z}. \end{cases} \tag{48}$$

Since the hyperbola is non-degenerate, we have that $\pi^\top x \geq \beta$ is a valid inequality for $(W^{i_0})^I$. Moreover, $\pi^\top x = \beta$ contains a ray of W^{i_0} since $\beta > \alpha$. Then, since π_1 and π_2 are relatively primes and $\beta \in \mathbb{Z}$, we have that $\pi^\top x \geq \beta$ is, in addition, a face of $(W^{i_0})^I$. Now, it follows from Lemma 9 that this face is one of the inequalities (14). Finally, note that $\pi_0 \leq \alpha < \pi_0 + 1$. Thus, we have that $\beta = \pi_0 + 1$.
Case 2: interior$(W) \cap \mathbb{Z}^2 \neq \emptyset$. By assumption, the components of π are integers and, without loss of generality, we may also assume they are relatively primes. We now have three cases.

1. $\pi \notin$ rec.cone$^*(W)$: In this case, by Lemma 3, we have that $\inf\{\pi^\top x \mid x \in W\}$ is unbounded. Since we assume that interior$(W) \cap \mathbb{Z}^2 \neq \emptyset$, we obtain that $\inf\{\pi^\top x \mid x \in W \cap \mathbb{Z}^2\}$ is unbounded [27], which contradicts the fact that $\pi^\top x \geq \pi_0$ is a valid inequality for W^I.

2. $\pi \in$ interior(rec.cone$^*(W)$) : In this case, $\alpha = \inf\{\pi^\top x \mid x \in W\}$ is bounded by Lemma 3. Let $x^0 \in W \cap \mathbb{Z}^2$. Then it follows from Lemma 4 that the set $E := \{x \in \mathbb{R}^2 \mid \alpha \leq \pi^\top x \leq \pi^\top x^0\} \cap W$ is bounded. Thus, $\{x \in \mathbb{R}^2 \mid \pi^\top x \leq \pi_0\} \cap W \subseteq E$ is also bounded. Therefore, it follows from Theorem 1 that the valid inequality $\pi^\top x \geq \pi_0$ can be obtained using functions (3).

3. $\pi \in$ rec.cone$^*(W) \setminus$ interior(rec.cone$^*(W)$): Since interior$(W) \cap \mathbb{Z}^2 \neq \emptyset$ and $\inf\{\pi^\top x \mid x \in W \cap \mathbb{Z}^2\}$ is bounded, we have that $\alpha := \inf\{\pi^\top x \mid x \in W\}$ is bounded [27]. Then, by Lemma 8, $\alpha = \inf\{\pi^\top x \mid W^{i_0}\}$, for some $i_0 \in [m]$, where there are only two possibilities for $W^{i_0} = \{x \in \mathbb{R}^2 : \ A^{i_0}x \succeq_{\mathcal{L}^{m_{i_0}}} b^{i_0}\}$:
   (i) W^{i_0} is the half-space $\pi^\top x \geq \alpha$: Since $A^{i_0}x \succeq_{\mathcal{L}^{m_{i_0}}} b^{i_0}$ is non-redundant, we have that the line $\pi^\top x = \alpha$ intersects W. Thus, $\pi^\top x \geq \lceil \alpha \rceil$ is a valid inequality for W^I and this cut can be obtained using a subadditive function (3). Now, we only need to show that $\lceil \alpha \rceil = \pi_0$. It is enough to show that the line $\pi^\top x = \lceil \alpha \rceil$ intersects $W \cap \mathbb{Z}^2$. Note that the line $\pi^\top x = \lceil \alpha \rceil$ intersects W (otherwise we would have $W \subseteq \{x \in \mathbb{R}^2 \mid \pi^\top x < \lceil \alpha \rceil\}$ which contradicts the fact that $W \cap \mathbb{Z}^2 \neq \emptyset$ since $\pi^\top x \geq \lceil \alpha \rceil$ is valid inequality for W^I). Thus, $\{x \in W \mid \pi^\top x = \lceil \alpha \rceil\} \neq \emptyset$. Moreover, since $\pi \in$ rec.cone$^*(W) \setminus$ interior(rec.cone$^*(W)$), there

exist a non-zero vector $d \in \text{rec.cone}(W)$ such that $\pi^\top d = 0$. Therefore, d is in the recession cone of $\{x \in W \mid \pi^\top x = \lceil \alpha \rceil\}$. Hence, $\pi^\top x = \lceil \alpha \rceil$ contains a ray of W. Thus, $\pi^\top x = \lceil \alpha \rceil$ contains an integer point of W since π_1 and π_2 are relatively primes.

(ii) W^{i_0} is a hyperbola whose one of the asymptotes is orthogonal to π: As in Case 1 *(ii)*, we can show that $\pi^\top x \geq \beta$ is a face of W^{i_0}, where β is defined in (48). Moreover, by Lemma 9, $\pi^\top x \geq \beta$ is one of the inequalities (14). Now, only remains to show that $\beta = \pi_0$. It is enough to show that $\pi^\top x = \beta$ intersects $W \cap \mathbb{Z}^2$. Clearly, $\pi^\top x \geq \beta$ is a valid inequality for $W^I \subseteq W^{i_0}$. Since $\alpha < \beta$, we have that the line $\pi^\top x = \beta$ intersects W (otherwise we would have $W \subseteq \{x \in \mathbb{R}^2 \mid \pi^\top x < \beta\}$ which contradicts the fact that $W \cap \mathbb{Z}^2 \neq \emptyset$). Therefore, as in the case *(i)* above, we can prove that $\pi^\top x = \beta$ contains a ray of W. Thus, $\pi^\top x = \beta$ contains an integer point of W since π_1 and π_2 are relatively primes and $\beta \in \mathbb{Z}$.

□

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References

[1] A. Atamtürk and V. Narayanan. Conic mixed-integer rounding cuts. *Mathematical Programming*, 122(1):1–20, 2008.

[2] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli. Maximal lattice-free convex sets in linear subspaces. *Mathematics of Operations Research*, 35(3):704–720, 2010.

[3] A. Basu, R. Hildebrand, and M. Köppe. Light on the infinite group relaxation. *arXiv:1410.8584*, 2015.

[4] P. Belotti, J. C. Góez, I. Pólik, T. K. Ralphs, and T. Terlaky. On families of quadratic surfaces having fixed intersections with two hyperplanes. *Discrete Applied Mathematics*, 161(16-17):2778–2793, 2013.

[5] A. Ben-Tal and A. Nemirovski. Lectures on modern convex optimization. Technical report, 2013. `http://www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf`.

[6] C. E. Blair and R. G. Jeroslow. The value function of an integer program. *Mathematical Programming*, 23(1):237–273, 1982.

[7] S. Burer and F. Kılınç-Karzan. How to convexify the intersection of a second order cone and a nonconvex quadratic. Technical report, June 2014. `http://www.andrew.cmu.edu/user/fkilinc/files/nonconvex_quadratics.pdf`.

[8] M. Conforti, G. Cornuéjols, A. Daniilidis, C. Lemaréchal, and J. Malick. Cut-generating functions and S-free sets. *Mathematics of Operations Research*, 40(2):276–391, 2015.

[9] D. Dadush, S. S. Dey, and J. P. Vielma. The split closure of a strictly convex body. *Operations Research Letters*, 39(2):121–126, 2011.

[10] S. Dash and O. Günlük. Valid inequalities based on simple mixed integer set. *Mathematical Programming*, 106:29–53, 2006.

[11] S. S. Dey and J.-P. P. Richard. Facets of the two-dimensional infinite group problems. *Mathematics of Operations Research*, 33:140–166, 2008.

[12] S. S. Dey and J.-P. P. Richard. Some relations between facets of low- and high-dimensional group problems. *Mathematical Programming*, 123:285–313, 2010.

[13] S. S. Dey and L. A. Wolsey. Two row mixed-integer cuts via lifting. *Mathematical Programming*, 124(1):143–174, 2010.

[14] R. E. Gomory and E. L. Johnson. Some continuous functions related to corner polyhedra, part I. *Mathematical Programming*, 3:23–85, 1972.

[15] R. E. Gomory and E. L. Johnson. T-space and cutting planes. *Mathematical Programming*, 96:341–375, 2003.

[16] R. G. Jeroslow. Minimal inequalities. *Mathematical Programming*, 17(1):1–15, 1979.

[17] E. L. Johnson. Cyclic groups, cutting planes, shortest paths. In T.C. Hu and Stephen M. Robinson, editors, *Mathematical Programming*, pages 185 – 211. Academic Press, 1973.

[18] E. L. Johnson. On the group problem and a subadditive approach to integer programming. In E.L. Johnson P.L. Hammer and B.H. Korte, editors, *Discrete Optimization II Proceedings of the Advanced Research Institute on Discrete Optimization and Systems Applications of the Systems Science Panel of NATO and of the Discrete Optimization Symposium co-sponsored by IBM Canada and SIAM Banff, Aha. and Vancouver*, volume 5 of *Annals of Discrete Mathematics*, pages 97 – 112. Elsevier, 1979.

[19] F. Kilinç-Karzan and S. Yildiz. Two-term disjunctions on the second-order cone. *Mathematical Programming*, 154(1-2):463–491, 2015.

[20] F. Kılınç-Karzan. On minimal valid inequalities for mixed integer conic programs. *Mathematics of Operations Research*, 41(2):477–510, 2016.

[21] F. Kılınç-Karzan and D. E. Steffy. On sublinear inequalities for mixed integer conic programs. *Mathematical Programming*, December 2015. `http://link.springer.com/article/10.1007/s10107-015-0968-0`.

[22] M. Köppe and Y. Zhou. An electronic compendium of extreme functions for the gomory–johnson infinite group problem. *arXiv:1411.5121*, 2015.

[23] M. T. Çezik and G. Iyengar. Cuts for mixed 0-1 conic programming. *Mathematical Programming*, 104(1):179–202, 2005.

[24] S. Modaresi, M. R. Kilinç, and J. P. Vielma. Split cuts and extended formulations for mixed integer conic quadratic programming. *Operations Research Letters*, 43(1):10–15, 2015.

[25] S. Modaresi and J. P. Vielma. Convex hull of two quadratic or a conic quadratic and a quadratic inequality. *http://www.optimization-online.org/DBHTML/2014/11/4641.html*, 2014.

[26] G. L. Nemhauser and L. A. Wolsey. A recursive procedure to generate all cuts for 0-1 mixed integer programs. *Mathematical Programming*, 46(3):379–390, February 1990.

[27] D. A. Morán R., S. S. Dey, and J. P. Vielma. A strong dual for conic mixed-integer programs. *SIAM Journal on Optimization*, 22(3):1136–1150, 2012.

[28] J.-P. P. Richard and S. S. Dey. The group-theoretic approach in mixed integer programming. In M. Jünger, T. M. Liebling, D. Naddef, G. L. Nemhauser, W. R. Pulleyblank, G. Reinelt, G. Rinaldi, and L. A. Wolsey, editors, *50 Years of Integer Programming 1958-2008 - From the Early Years to the State-of-the-Art*, pages 727–801. Springer, 2010.

[29] J.-P. P. Richard, Y. Li, and L. A. Miller. Valid inequalities for mips and group polyhedra from approximate liftings. *Mathematical Programming*, 118(2):253–277, 2009.

[30] A. Schrijver. *Theory of linear and integer programming.* Wiley-Interscience series in discrete mathematics and optimization. Wiley, 1999.

[31] L.A. Wolsey. The b-hull of an integer program. *Discrete Applied Mathematics*, 3(3):193 – 201, 1981.

[32] S. Yildiz and G. Cornuéjols. Disjunctive cuts for cross-sections of the second-order cone. *Operations Research Letter*, 43(4):432–437, 2015.