EXACT AUGMENTED LAGRANGIAN DUALITY FOR MIXED INTEGER QUADRATIC PROGRAMMING

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Abstract. Mixed integer quadratic programming (MIQP) is the problem of minimizing a quadratic function over mixed integer points in a rational polyhedron. This paper focuses on the augmented Lagrangian dual (ALD) for MIQP. ALD augments the usual Lagrangian dual with a weighted nonlinear penalty on the dualized constraints. We first prove that ALD will reach a zero duality gap asymptotically as the weight on the penalty goes to infinity under some mild conditions on the penalty function. We next show that a finite penalty weight is enough for a zero gap when we use any norm as the penalty function. Finally, we prove a polynomial bound on the weight on the penalty term to obtain a zero gap.

1. Introduction. We consider the following rational (mixed) integer quadratic programming (MIQP) problem with decision variable $x \in \mathbb{R}^n$:

$$\begin{align*}
\text{minimize } & c^\top x + \frac{1}{2} x^\top Q x \\
\text{subject to } & Ax = b, x \in X,
\end{align*}$$

where the parameters are defined throughout the paper as: a rational symmetric positive semi-definite matrix $Q \in \mathbb{Q}^{n \times n}$, a rational matrix $A \in \mathbb{Q}^{m \times n}$, rational vectors $c \in \mathbb{Q}^n$ and $b \in \mathbb{Q}^m$, a mixed integer linear set $X$ such that $X = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} : Ex \leq f\}$, where $E \in \mathbb{Q}^{m_2 \times n}$ is a rational matrix and $f \in \mathbb{Q}^{m_2}$ is a rational vector with $n_1 + n_2 = n$. We consider dualizing the constraints $Ax = b$.

While for continuous quadratic programming (QP) with convex objective, it is well known that even the classical Lagrangian dual (LD) will reach a zero duality gap and strong duality holds [1], it is not true for MIQP, as the integer variables introduce non-convexity. In fact, LD may have a non-zero duality gap for the problem. Therefore, to close the gap, the idea of penalizing violation of the dualized constraints with a nonlinear penalty gives rise to the well known augmented Lagrangian dual (ALD), which is

$$z_{LD+}^\rho := \sup_{\lambda \in \mathbb{R}^m} \inf_{x \in X} \{c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) + \rho \psi(b - Ax)\},$$

where $\rho > 0$ is the penalty weight, and $\psi(\cdot)$ is the penalty function which usually satisfies $\psi(0) = 0$ and $\psi(u) > 0$ if $u \neq 0$ [15].


It should be noted that an exact penalty representation usually requires a much restricted penalty function, like norm functions, see for example [15]. Norm function is

On the other hand, the size (for example, in binary coding) of the penalty weight is rarely discussed. While there are discussions for the size and computational complexity of MILP [19, 2], QP [18] and MIQP [7], we might be able to utilize their ideas to show the small size of the penalty weight.

In this paper, we significantly generalize the results of [9]. In particular, we
1. Prove that the duality gap of ALD will asymptotically reach zero under mild conditions as the penalty weight goes to infinity;
2. Prove that the duality gap will reach zero given that the penalty function is any norm, and the penalty weight is sufficiently large but still finite;
3. Prove that the size of the penalty weight which attains zero duality gap is polynomially bounded with respect to the problem data.

The paper is organized as follows. In Section 2 we provide definitions and formal statement of main results of the paper. In Section 3 we present several key lemmas useful across the paper. In Section 4 we exhibit properties of ALD as the penalty weight goes to infinity, and show the (asymptotic) zero duality gap for a large class of penalty functions. In Section 5 we use any norm function as the penalty function and show that a finite penalty weight whose size is polynomially bounded with respect to the input parameters, yields zero duality gap.

2. Main Results. In this section, we introduce some definitions and briefly present our main results.

**Assumption 1.** The MIQP (1) is feasible and the optimal value is bounded.

**Definition 2.** The augmented Lagrangian relaxation is defined as
\[
z^\text{LR+}_p(\lambda) := \inf_{x \in X} \left\{ c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) + \rho \psi(b - Ax) \right\},
\]
where \( \psi \) is a penalty function. Recall that the augmented Lagrangian dual is
\[
z^\text{LD+}_p := \sup_{\lambda} \inf_{x \in X} \left\{ c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) + \rho \psi(b - Ax) \right\}.
\]

**Definition 3.** The continuous relaxation of (1) is denoted as \( z^{\text{NLP}} \)
\[
z^{\text{NLP}} := \inf \left\{ c^\top x + \frac{1}{2} x^\top Q x : Ax = b, Ex \leq f, x \in \mathbb{R}^{n_1 + n_2} \right\}.
\]

**Remark 4.** We use \( \tilde{\lambda} \) to denote the optimal dual variables (of \( z^{\text{NLP}} \)) for the constraints \( Ax = b \) and \( \tilde{\lambda}_E \) to denote the optimal dual variables for \( Ex \leq f \). The existence of \( \tilde{\lambda} \) and \( \tilde{\lambda}_E \) is guaranteed by the boundedness of the continuous relaxation, which is given by Lemma 13.

**Remark 5.** For any \( \rho, \lambda \), we have \( z^{\text{LR+}}_p(\lambda) \leq z^{\text{LD+}}_p \leq z^{\text{IP}} \). Moreover, we have \( z^{\text{NLP}} = \inf \left\{ c^\top x + \frac{1}{2} x^\top Q x + \tilde{\lambda}^\top (b - Ax) : Ex \leq f, x \in \mathbb{R}^{n_1 + n_2} \right\} \leq z^{\text{LR+}}_p(\lambda) \leq z^{\text{LD+}}_p \leq z^{\text{IP}} \).

**Definition 6.** For a finite set of vectors \( T = \{ t_1, t_2, ..., t_k \} \), \( \text{conv}(T) \), \( \text{cone}(T) \) and \( \text{int.cone}(T) \) are the convex hull, conical hull and integral conical hull of \( T \), respectively. Here, \( \text{int.cone}(T) := \{ \sum_{i=1}^{k} \mu_i t_i : \mu_i \in \mathbb{Z}_+ \} \).
**Definition 7.** For any subset $T$ of a metric space, its diameter is defined as $\text{diam}(T) = \sup_{a,b \in T} \|a - b\|$, where $\| \cdot \|$ is the metric associated with the space.

**Definition 8.** (7). Given an input $O$ and an output $f(O)$, we say that $f(O)$ has $O$-small complexity, if the size (measured by standard binary encoding) of $f(O)$ is at most a polynomial function of the size of $O$.

**Definition 9.** We use $F$ to denote all input parameters of (1) including $E$, $f$, $c$, $Q$, $A$ and $b$. In addition, any object $q$ which is a function of $F$ is said to have small complexity, if $q$ has $F$-small complexity.

Below we present the main theorems of the paper.

**Theorem 10 (Asymptotic Zero Duality Gap).** Assume $\psi$ is proper, nonnegative, lower-semicontinuous and level-bounded, that is: $\psi(0) = 0$; $\psi(u) > 0$ for all $u \neq 0$; $\lim_{\delta \downarrow 0} \text{diam}\{u : \psi(u) \leq \delta\} = 0$; $\text{diam}\{u : \psi(u) \leq \delta\} < \infty$ for all $\delta > 0$. We have $\sup_{\rho > 0} z_{\text{LD}+}^\rho = z_{\text{IP}}$.

We provide a flowchart that depicts how the preliminary results proved in Section 3 are put together to prove Theorem 10.

**Theorem 11 (Sufficient Condition for Exact Penalty).** Under Assumption 1, if there exists $\delta$, such that

$$\inf\{\psi(b - Ax) : x \in X, Ax \neq b\} \geq \delta > 0$$

and $\psi(0) = 0$, then there exists a finite $\rho^*$ such that $z_{\rho^*}^{\text{LR}+}(\lambda) = z_{\text{IP}}$, which also gives $z_{\rho^*}^{\text{LD}+} = z_{\text{IP}}$.

**Theorem 12 (Exact Penalty Representation).** Suppose $\psi(\cdot)$ is any norm.

(a) There exists a finite $\rho^*$ of $F$-small complexity, such that $z_{\rho^*}^{\text{LD}+} = z_{\text{IP}}$.

(b) Moreover, for all $\lambda$, there exists a finite $\rho^*(\lambda)$ of $(F, \lambda)$-small complexity, such that $z_{\rho^*(\lambda)}^{\text{LR}+}(\lambda) = z_{\text{IP}}$.

The flowchart below describes the proof of Theorem 11 and Theorem 12.
3. Preliminary Results. Several useful lemmas are presented in this section.

Lemma 13 (Equivalence of Boundedness of MIQP and its Continuous Relaxation). Suppose the MIQP is feasible (i.e. \( z^{IP} < +\infty \)). Then the following three conditions are equivalent:

1. \( z^{NLP} \) is bounded.
2. \( \inf \{ c^T x | Ax = 0, Ex \leq 0, Qx = 0 \} \) is bounded.
3. \( z^{IP} \) is bounded.

Proof. 1 \( \Rightarrow \) 3 is obvious.

We first show 3 \( \Rightarrow \) 2, or equivalently \( \neg 2 \Rightarrow \neg 3 \). Note that the problem in 2 is always feasible. Assuming \( \neg 2 \), the set \( \{ x : c^T x \leq -1, Ax = 0, Ex \leq 0, Qx = 0 \} \) is now feasible and there exists a rational solution since the parameters are rational. Denote such a rational solution as \( r \) and without loss of generality, we assume that \( r \) is integral since we can scale \( r \) with a positive coefficient.

Now select any feasible solution for 3, as \( x \). Then we know that \( x + tr \) is still feasible for 3 for any \( t \in \mathbb{Z}_+ \). In addition, \( c^T (x + tr) + \frac{1}{2} (x + tr)^T Q (x + tr) = c^T x + \frac{1}{2} x^T Qx + tc^T r \to -\infty \) as \( t \to +\infty \). Therefore, we have 3 is unbounded, i.e. 3 \( \Rightarrow \) 2.

Next we show that 2 \( \Rightarrow \) 1. Suppose that 2 holds. From Farkas Lemma, we know that \( \exists \lambda_E \leq 0, \lambda_A, \lambda_Q \), such that \( \lambda_E^T E + \lambda_A^T A + \lambda_Q^T Q = c^T \). Now considering the NLP

\[
z^{NLP} = \inf c^T x + \frac{1}{2} x^T Qx
\]

s.t. \( Ax = b, \)

\( Ex \leq f \)

\[
= \inf (\lambda_E^T E + \lambda_A^T A + \lambda_Q^T Q)x + \frac{1}{2} x^T Qx
\]

s.t. \( Ax = b, \)

\( Ex \leq f \)
To show that $\lambda_Q x + \frac{1}{2} x^T Q x$ is bounded, we first calculate the derivative as $Q \lambda_Q + Q x$. Therefore, the convexity gives that $x = -\lambda_Q$ is a global minimizer and we arrive at 1, i.e. $z_{\text{NLP}}$ is bounded.

Remark 14. We note here that we are able to prove that the boundedness of the nonlinear integer problem implies boundedness of its continuous relaxation, using the fact that the data is rational. This is very similar to the Fundamental theorem of Integer Programming [11]. Note that other similar results may be proven under different assumptions such as existence of integer point in the interior of continuous relaxation, see [8, 13]. Also see [12].

Lemma 15 (Decomposition of Rational Mixed Integer Polyhedron). Given a rational positive semidefinite matrix $Q$, any rational mixed integer polyhedron $P \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) = \{ x : C x \leq d \} \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2})$ can be decomposed (with respect to $Q$) as $\cup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) + \text{int} \cdot \text{cone}(R_i))$ satisfying the following properties:

(a) Each $P_i$ is a rational polytope.
(b) Each cone($R_i$) is a rational, simple and pointed cone.
(c) For every cone($R_i$), if a face $C'$ satisfies that $\exists x \in C' \setminus \{0\}$, $x^T Q x = 0$, then there exists an extreme ray $r$ of $C'$ with $v^T Q r = 0$.
(d) Each polytope $P_i$ and each vector in $R_i$ has $P, Q$-small complexity.

Proof. This lemma is a direct consequence of [7, Proposition 1, Proposition 2, Lemma 2].

First, if $P$ is not pointed, we can decompose $P$ into at most $2^{n_1+n_2}$ pointed rational mixed integer polyhedron by separating $x_k \leq 0$ and $x_k \geq 0$ for all $k$. Therefore, we simply assume $P$ is pointed henceforth.

Next, using [7, Proposition 1], we can decompose $P$ as $P = \cup_i, k^1, k^2 \in K^1 (P_i^1 + \text{cone}(R_{k^1}^1))$, while conditions (a), (b), (d) are met.

Later, using [7, Lemma 2], we are able to decompose cone($R_k$) into a union of rational, simple and pointed cones, which satisfies condition (c) and maintains (a), (b), (d). Therefore, $P = \cup_i, k^2 \in K^2 (P_i^1 + \text{cone}(R_{k^2}^2))$.

Finally, we use [7, Proposition 2] and decompose $(P_i^1 + \text{cone}(R_{k^2}^2)) \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2})$ into a mixed integer rational polytope plus an integer cone, which completes the proof.

Lemma 16 (Bounded Region with Small Complexity [7]). Let $P \subseteq \mathbb{R}^n$ be a polytope and $R \subseteq \mathbb{R}^n$ be a finite set of vectors. Given the rational positive semidefinite matrix $Q$, suppose $P + \text{cone}(R)$ satisfies the following properties:

(a) $P$ is a rational polytope.
(b) cone($R$) is a rational, simple and pointed cone.
(c) $\forall x \in \text{cone}(R) \setminus \{0\}$, $x^T Q x > 0$.

Then, for any $\eta \in \mathbb{R}^n, \mu \in \mathbb{R}$ of $F$-small complexity, there exists $M$ of $F$-small complexity such that $\{ x \in P + \text{cone}(R) : \frac{1}{2} x^T Q x + \eta^T x \leq \mu \} \subseteq \{ x : \|x\| \leq M \}$. In addition, such $M$ exists for any norm.

Remark 17. For any rational mixed integer polyhedron, Lemma 15 provides a decomposition with respect to $Q$, while maintaining a small complexity. In addition, for any part of the decomposition, if no extreme ray $r$ has $v^T Q r = 0$ then no ray has
shows that under the conditions that we know that there is well-defined, i.e. feasible (Theorem 10), and as a first step we show that Lemma 13 ∈ Theorem and \( \hat{\lambda} \).

(2)

Therefore, feasible and bounded.

is feasible for \( c \) \( x \) \( z \) and the boundedness of \( (\hat{\lambda}) \). We also have \( \psi(b-Ax) \leq w \),

\[
c^\top x + \frac{1}{2}x^\top Qx + \hat{\lambda}^\top (b-Ax) + \rho w - z^{LR+}_\rho(\hat{\lambda}) \leq \epsilon.
\]

Then, the limit \( w^*_\rho := \lim_{\epsilon \downarrow 0} w^*_{\rho,\epsilon} \) exists and \( \lim_{\rho \to +\infty} w^*_\rho = 0 \).

**Proof.** First we need show that the problem (2) is well-defined, i.e feasible and bounded. Recall that we use \( \hat{\lambda} \) to denote the optimal dual variables of \( z^{NLP} \) (Remark 4), and as a first step we show that \( z^{LR+}_\rho(\hat{\lambda}) \) is finite. Observe that:

\[
z^{LR+}_\rho(\hat{\lambda}) \geq \inf \{ c^\top x + \frac{1}{2}x^\top Qx + \hat{\lambda}^\top (b-Ax) : x \in X \}
\]

\[
\geq \inf \{ c^\top x + \frac{1}{2}x^\top Qx + \hat{\lambda}^\top (b-Ax) : Ex \leq f \}
\]

\[
\geq \inf \{ c^\top x + \frac{1}{2}x^\top Qx + \hat{\lambda}^\top (b-Ax) + \hat{\lambda}_E(f-Ex) \}
\]

\[
= z^{NLP},
\]

and the boundedness of \( z^{NLP} \) is given by Lemma 13.

From the feasibility of the original problem (Assumption 1) we know that there exists an \( x \) feasible for \( z^{LR+}_\rho(\hat{\lambda}) \). Therefore, we are able to find \( \hat{x} \in X \) such that

\[
c^\top \hat{x} + \frac{1}{2}x^\top Q\hat{x} + \hat{\lambda}^\top (b-A\hat{x}) + \psi(b-A\hat{x}) \leq \epsilon + z^{LR+}_\rho(\hat{\lambda}),
\]

which means (\( \hat{x}, w = \psi(b-A\hat{x}) \)) is feasible for (2). We also have \( w^*_{\rho,\epsilon} \geq 0 \) from the non-negativity of \( \psi \). Thus, (2) is feasible and bounded.

In addition, we have \( z^{LR+}_\rho(\hat{\lambda}) \leq z^{IP} \) and for any \( x \) satisfying \( Ex \leq f \) we have that

\[
c^\top x + \frac{1}{2}x^\top Qx + \hat{\lambda}^\top (b-Ax) \geq \inf \{ c^\top x + \frac{1}{2}x^\top Qx + \hat{\lambda}^\top (b-Ax) + \hat{\lambda}_E(f-Ex) \} = z^{NLP}.
\]

Therefore,

\[
w^*_{\rho,\epsilon} \leq \frac{1}{\rho} \{ z^{LR+}_\rho(\hat{\lambda}) + \epsilon - [c^\top x + \frac{1}{2}x^\top Qx + \hat{\lambda}^\top (b-Ax)] \}, \text{ for some } x \in X
\]

4. Asymptotic Zero Duality Gap. In this section, we show that under mild conditions on the penalty function the ALD duality gap vanishes as the penalty weight \( \rho \) goes to infinity.

Throughout the section, we make the following mild assumption on the penalty function which is also the assumption made in Theorem 10.

**Assumption 18** (Conditions for Asymptotic Zero Duality Gap). We assume \( \psi \) is proper, nonnegative, lower-semicontinuous and level-bounded, that is: \( \psi(0) = 0; \psi(u) > 0 \) for all \( u \neq 0; \lim_{\delta \downarrow 0} \text{diam} \{ u : \psi(u) \leq \delta \} = 0; \text{diam} \{ u : \psi(u) \leq \delta \} < \infty \) for all \( \delta > 0 \).

**Proposition 19** (Approximation of the Penalty Term). For given \( \rho > 0 \) and \( \epsilon > 0 \), define \( w^*_{\rho,\epsilon} \) as

\[
w^*_{\rho,\epsilon} := \inf_{x, w} w \text{ s.t. } x \in X,
\]

\[
\psi(b-Ax) \leq w,
\]

\[
c^\top x + \frac{1}{2}x^\top Qx + \hat{\lambda}^\top (b-Ax) + \rho w - z^{LR+}_\rho(\hat{\lambda}) \leq \epsilon.
\]

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\[ \frac{1}{\rho} (z^{\text{IP}} + \epsilon - z^{\text{NLP}}). \]

By taking \( \epsilon \downarrow 0 \) we have

\[ 0 \leq w^*_\rho = \lim_{\epsilon \downarrow 0} w^*_{\rho,\epsilon} \leq \lim_{\epsilon \downarrow 0} \frac{1}{\rho} (z^{\text{IP}} + \epsilon - z^{\text{NLP}}) = \frac{1}{\rho} (z^{\text{IP}} - z^{\text{NLP}}). \]

In addition, as \( \epsilon \downarrow 0 \) the feasible region of (2) becomes smaller, which indicates that \( w^*_{\rho,\epsilon} \) is non-decreasing. Therefore \( w^*_\rho = \lim_{\epsilon \downarrow 0} w^*_{\rho,\epsilon} \) exists.

By taking \( \rho \to +\infty \) we therefore obtain \( \lim_{\rho \to +\infty} w^*_\rho = 0. \)

**Lemma 20 (Equivalent Form of \( z^*_\rho^{\text{LR+}}(\lambda) \)).** Consider \( w^*_\rho \) as in Proposition 19 and for any \( \delta \in (0,1) \) define \( \bar{z}^*_\rho^{\text{LR+}}(\lambda) \) as

\[ \bar{z}^*_\rho^{\text{LR+}}(\lambda) := \inf_{x,w} c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) + \rho w \]

s.t. \( x \in X, \psi(b - Ax) \leq w, (1 - \delta)w^*_\rho \leq w \leq (1 + \delta)w^*_\rho. \)

Then,

\[ z^*_\rho^{\text{LR+}}(\lambda) = \bar{z}^*_\rho^{\text{LR+}}(\lambda) \]

\[ \geq \inf_x c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) + \rho(1 - \delta)w^*_\rho \]

s.t. \( x \in X, \psi(b - Ax) \leq (1 + \delta)w^*_\rho, (1 - \delta)w^*_\rho \leq w \leq (1 + \delta)w^*_\rho. \)

**Proof.** Note that the definition of \( \bar{z}^*_\rho^{\text{LR+}}(\lambda) \) is the same as that of \( z^*_\rho^{\text{LR+}}(\lambda) \) except for the additional constraint \( (1 - \delta)w^*_\rho \leq w \leq (1 + \delta)w^*_\rho \), and thus \( z^*_\rho^{\text{LR+}}(\lambda) \leq \bar{z}^*_\rho^{\text{LR+}}(\lambda) \).

Suppose \( \alpha_\rho := z^*_\rho^{\text{LR+}}(\lambda) - z^*_\rho^{\text{LR+}}(\lambda) \) by contradiction. Then, for all \( (x,w) \) feasible for (4) we have

\[ c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) + \rho w \geq z^*_\rho^{\text{LR+}}(\lambda) = z^*_\rho^{\text{LR+}}(\lambda) + \alpha_\rho, \]

which implies \( (x,w) \) is infeasible for (2) if \( \epsilon < \alpha_\rho \). On the other hand, from the definition of \( w^*_{\rho,\epsilon} \) in (2), there exists \( \hat{x} \) such that \( (\hat{x},w^*_{\rho,\epsilon}) \) is feasible for (4). Hence, \( w^*_{\rho,\epsilon} \notin ((1 - \delta)w^*_\rho,(1 + \delta)w^*_\rho) \), a contradiction. Therefore \( \bar{z}^*_\rho^{\text{LR+}}(\lambda) = z^*_\rho^{\text{LR+}}(\lambda) \) and the inequalities (5) are straightforward to verify.

We are now ready to present the asymptotic zero duality gap.

**Theorem 10 (Asymptotic Zero Duality Gap).** Assume \( \psi \) is proper, nonnegative, lower-semicontinuous and level-bounded, that is: \( \psi(0) = 0; \psi(u) > 0 \) for all \( u \neq 0; \lim_{\delta \downarrow 0} \text{diam}\{u : \psi(u) \leq \delta\} = 0; \text{diam}\{u : \psi(u) \leq \delta\} < \infty \) for all \( \delta > 0 \). We have

\[ \sup_{\rho > 0} z^*_\rho^{\text{LD+}} = z^{\text{IP}}. \]
Lemma 21 comes.

is valid from for Assumption 18 Lemma 21 we have.

is guaranteed by is provided later. From and 268 265 264 263 262 261

where \( \kappa_\rho := \text{diam}(u : \psi(u) \leq 2(z^{\text{IP}} - z^{\text{NLP}})) \) which is obviously non-increasing with respect to \( \rho \). (6a) is guaranteed by Lemma 20. (6b) is valid from (3) and (6c) comes from the level-boundedness of Assumption 18.

We will need the following lemma that provides a uniform bound \( M \) on (6c) for \( x \) independent of \( \rho \).

**Lemma 21 (Adding a Uniform Bound on \( x \) without Changing the Value).** Under the assumption that \( \rho \geq 1 \), then \( \exists M > 0 \) independent of \( \rho \), such that

\[
\inf_x \{ c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax) : x \in X, \|b - Ax\|_\infty \leq \kappa_\rho \} = \min_x \{ c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax) : x \in X, \|b - Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M \}.
\]

A proof of Lemma 21 is provided later. From Lemma 21 we have \( z^{\text{LD}}_{\rho} \geq \min \{ c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax) : x \in X, \|b - Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M \} \). Recall that \( x = (x_1, x_2) \) with \( x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{Z}^{n_2} \). By taking \( \rho \to +\infty \), we get

\[
\lim_{\rho \to +\infty} z^{\text{LD}}_{\rho} \geq \lim_{\rho \to +\infty} \min_x c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax)
\]

such that \( x \in X, \|b - Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M \)

\[
= \lim_{\rho \to +\infty} \min_{\|x_2\|_\infty \leq M, \|x_1\|_\infty \leq M} c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax)
\]

such that \( x \in X, \|b - Ax\|_\infty \leq \kappa_\rho \)

\[
(7a) = \min_{\|x_2\|_\infty \leq M, \|x_1\|_\infty \leq M} \lim_{\rho \to +\infty} \min_x c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax)
\]

such that \( x \in X, \|b - Ax\|_\infty \leq \kappa_\rho \)

\[
(7b) \geq \min_{\|x_2\|_\infty \leq M, \|x_1\|_\infty \leq M} \lim_{\rho \to +\infty} \kappa_\rho\]

\[
(7c) \geq \min_{\|x\|_\infty \leq M} c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax)
\]

such that \( x \in X, \|b - Ax\|_\infty = 0 \)

\[
= \min_x c^T x + \frac{1}{2} x^T Q x
\]
Lemma 15 holds by and we get Lemma 15 (7c) follows from the finiteness of \( \lambda \). Note that for any \( \rho \), \( z_{\rho}^{LR+} \leq z_{\rho}^{LD+} \leq z_{\rho}^{IP} \), and thus \( \lim_{\rho \to +\infty} z_{\rho}^{LR+} = z^{IP} \). Note that by proving the theorem we also show that \( \lim_{\rho \to +\infty} z_{\rho}^{LR+} = z^{IP} \) from the non-decreasing of \( z_{\rho}^{LR+} \) with respect to \( \rho \).

We now complete the proof by proving Lemma 21.

Proof of Lemma 21. Note that \( \| b - Ax \|_{\infty} \leq \kappa_\rho \) can be written as linear constraints. Hence, apply Lemma 15 to the feasible region for \( \rho = 1 \) of (6b) and we get a decomposition \( \cup_i (P_i \cap (\mathbb{R}^n \times \mathbb{Z}^m) + \text{int.cone}(R_i)) \) with the properties listed in the lemma. Note that for all \( r \in R_i \) we have \( Ar = 0 \) from the constraints \( \| b - Ax \|_{\infty} \leq \kappa_1 \).

Therefore, the feasible region for any \( \rho \geq 1 \) can be written as

\[
\left( \bigcup_i (P_i \cap (\mathbb{R}^n \times \mathbb{Z}^m) + \text{int.cone}(R_i)) \right) \cap \left\{ x : \| b - Ax \|_{\infty} \leq \kappa_\rho \right\}
\]

Now consider the problem \( \inf_x \left\{ c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax) : x \in P_i \cap (\mathbb{R}^n \times \mathbb{Z}^m) \cap \{ x : \| b - Ax \|_{\infty} \leq \kappa_\rho \} + \text{int.cone}(R_i) \right\} \). If there exists \( r \in R_i \) such that \( r^T Q r = 0 \) (i.e. \( Q r = 0 \)), then the feasible region can be rewritten as \( P_i \cap (\mathbb{R}^n \times \mathbb{Z}^m) \cap \{ x : \| b - Ax \|_{\infty} \leq \kappa_\rho \} + \text{int.cone}(R_i) \cap \{ r \} \) and \( \mu \in \mathbb{Z}_+ \).

We can use \( y + \mu r \) such that \( y \in P_i \cap (\mathbb{R}^n \times \mathbb{Z}^m) \cap \{ x : \| b - Ax \|_{\infty} \leq \kappa_\rho \} + \text{int.cone}(R_i) \cap \{ r \} \) and \( \mu \in \mathbb{Z}_+ \) to represent \( x \). The problem is therefore

\[
\inf_{y, \mu} \left( c^T - \lambda^T A \right) \mu + c^T y + \frac{1}{2} y^T Q y + \lambda^T (b - Ay) \text{ s.t. } y \in P_i \cap (\mathbb{R}^n \times \mathbb{Z}^m) \cap \{ x : \| b - Ax \|_{\infty} \leq \kappa_\rho \} + \text{int.cone}(R_i) \cap \{ r \}, \mu \in \mathbb{Z}_+. \]

Optimize the problem over \( \mu \) and we get \( \mu = 0 \) an optimal solution (or the problem is unbounded, contrary to (6b)). Therefore, we can refine the feasible region by omitting all \( r \in R_i \) such that \( Q r = 0 \). Denote the set after the process as \( R_i^f \). Note that this process is independent of the value of \( \rho \), and hence we have

\[
z^{IP} \geq \inf_x c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax) \text{ s.t. } x \in X, \| b - Ax \|_{\infty} \leq \kappa_\rho
\]

\[
= \inf_x c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax) \text{ s.t. } x \in \bigcup_i (P_i \cap (\mathbb{R}^n \times \mathbb{Z}^m) \cap \{ x : \| b - Ax \|_{\infty} \leq \kappa_\rho \} + \text{int.cone}(R_i^f)).
\]

In addition, from (c) of Lemma 15, for all \( x \in \text{cone}(R_i^f) \setminus \{ 0 \} \), we have \( x^T Q x > 0 \).

Let

\[
V_i = \{ x \in (P_i + \text{cone}(R_i^f)) : c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax) - (z^{IP} + 1) \leq 0 \}.
\]
Note that the definition on $V_i$ is independent of $\rho$. Note that $(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i^f) \subseteq (P_i + \text{cone}(R_i^f))$ and we have

\[ z_{\text{IP}}^f \geq \inf_x c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) \]
\[ \text{s.t. } x \in \bigcup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i^f) \]
\[ = \inf_x c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) \]
\[ \text{s.t. } x \in \bigcup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i^f), \]
\[ c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) \leq z_{\text{IP}}^f + 1 \]
\[ \geq \inf_x c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) \]
\[ \text{s.t. } x \in \bigcup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i^f), \]
\[ x \in \bigcup_i V_i. \]

Using Lemma 16 we have that there exists $M_i > 0$ such that $V_i \in \{x : \|x\|_\infty \leq M_i\}$. Take $M = \max\{M_i\}$, which is independent of $\rho$, and we have

\[ \inf_x c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) \]
\[ \text{s.t. } x \in \bigcup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i^f), \]
\[ x \in \bigcup_i V_i \]
\[ \geq \inf_x c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) \]
\[ \text{s.t. } x \in \bigcup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \cap \{x : \|b - Ax\|_\infty \leq \kappa_\rho\}) + \text{int.cone}(R_i^f), \]
\[ \|x\|_\infty \leq M \]
\[ \geq \inf_x c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) \text{ s.t. } x \in X, \|b - Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M. \]

While $\inf_x \{c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax)\} \text{ s.t. } x \in X, \|b - Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M \geq z_{\text{NLP}}^f$ is bounded and the set of all possible values of $x_2$ here is finite, we can therefore replace $\inf$ by $\min$.

Therefore,

\[ \inf_x \{c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) : x \in X, \|b - Ax\|_\infty \leq \kappa_\rho\} \]
\[ \geq \min_x \{c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) : x \in X, \|b - Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M\}. \]

Since it is obvious that

\[ \inf_x \{c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) : x \in X, \|b - Ax\|_\infty \leq \kappa_\rho\} \]
\[ \leq \min_x \{c^\top x + \frac{1}{2} x^\top Q x + \lambda^\top (b - Ax) : x \in X, \|b - Ax\|_\infty \leq \kappa_\rho, \|x\|_\infty \leq M\}, \]

thus equality holds and the proof is completed.

5. Exact Penalty Representation. In this section, we will discuss conditions for an exact penalty representation. To begin with, a sufficient condition is given. We later prove the sufficiency of using norm as the penalty function for an exact penalty, while noting that a norm function always satisfies Assumption 18.
Theorem 11 (Sufficient Condition for Exact Penalty). Under Assumption 1, if there exists $\delta$, such that
\[
\inf \{ \psi(b-Ax) : x \in X, Ax \neq b \} \geq \delta > 0
\]
and $\psi(0) = 0$, then there exists a finite $\rho^*$ such that $z_{\rho^*}^{LR+}(\bar{\lambda}) = z^{IP}$, which also gives $z_{\rho^*}^{LD+} = z^{IP}$.

Proof. Under Assumption 1, using Lemma 13, we have $z^{NLP}$ is bounded. Thus, choose a feasible point $\tilde{x}$ for the MIQP and set
\[
\rho^* = \frac{1}{\delta}(c^\top \tilde{x} + \frac{1}{2} \tilde{x}^\top Q\tilde{x} - z^{NLP}) < \infty,
\]
we next show that $\rho^*$ satisfies our requirements.

First of all, as $z^{NLP}$ bounded and $c^\top \tilde{x} + \frac{1}{2} \tilde{x}^\top Q\tilde{x} \geq z^{IP}$, we have $\rho^* \in [0, +\infty)$.

Clearly $z_{\rho^*}^{LR+}(\bar{\lambda}) \leq z^{IP}$. We next show that $z_{\rho^*}^{LR+}(\bar{\lambda}) \geq z^{IP}$.

For any $x \in X$, if $Ax = b$, we have
\[
c^\top x + \frac{1}{2} x^\top Qx + \bar{\lambda}^\top (b-Ax) + \rho^* \psi(b-Ax) = c^\top x + \frac{1}{2} x^\top Qx \geq z^{IP}.
\]

On the other hand, if $Ax \neq b$, we have
\[
c^\top x + \frac{1}{2} x^\top Qx + \bar{\lambda}^\top (b-Ax) \geq z^{NLP}
\]
from the strong duality results for QP, Thus,
\[
c^\top x + \frac{1}{2} x^\top Qx + \bar{\lambda}^\top (b-Ax) + \rho^* \psi(b-Ax) \geq z^{NLP} + \rho^* \delta = c^\top \tilde{x} + \frac{1}{2} \tilde{x}^\top Q\tilde{x} \geq z^{IP}.
\]

Therefore, we have for any $x \in X$, $c^\top x + \frac{1}{2} x^\top Qx + \bar{\lambda}^\top (b-Ax) + \rho^* \psi(b-Ax) \geq z^{IP}$
and thus $z_{\rho^*}^{LR+}(\bar{\lambda}) = z^{IP}$.

We now present the exact penalty results for $\psi(\cdot) = \| \cdot \|_{\infty}$.

Theorem 22 (Exact Penalty Representation for $L^\infty$-Norm). Assuming $\psi(\cdot) = \| \cdot \|_{\infty}$, there exists a finite $\rho^*(\bar{\lambda})$ of small complexity, such that $z_{\rho^*}^{LR+}(\bar{\lambda}) = z^{IP}$.

Proof. It is sufficient to find a finite $\rho^*(\bar{\lambda})$ polynomially bounded, such that $z_{\rho^*}^{LR+}(\bar{\lambda}) \geq z^{IP}$. Since $z_{\rho^*}^{LR+}(\bar{\lambda})$ is non-decreasing with $\rho$ increasing, without loss of generality, we only consider $\rho \geq 1$. In addition, from [7, Theorem 4], $z^{IP}$, $z^{NLP}$ and $\bar{\lambda}$ have $F$-small complexity, while recalling that $F$ represents all input parameters.

The constraints $\|b-Ax\|_{\infty} \leq w$ can be written as $-w \leq b-Ax \leq w$. Therefore,
\[
z_{\rho^*}^{LR+}(\bar{\lambda}) = \inf_{x,w} (c^\top - \bar{\lambda}^\top A)x + \frac{1}{2} x^\top Qx + \rho w + \bar{\lambda}^\top b
\]
\[
s.t. \ Ax - w \leq b,
\]
\[
- Ax - w \leq b,
\]
\[
Ex \leq f,
\]
\[
x \in \mathbb{R}^n_+ \times \mathbb{Z}^{n_+}.
\]

The following lemma shows a uniform bound $M$ of small complexity can be put on $x$ independent of $\rho$. 

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Lemma 23 (A Uniform Bound on $x$ Independent of $\rho$). Under the assumption that $\rho \geq 1$, there exists $M > 0$ independent of $\rho$ and of small complexity, such that

$$z^{\text{LR+}}_{\rho}(\bar{\lambda}) = \inf_{x,w} (c^T - \bar{\lambda}^T A)x + \frac{1}{2} x^T Q x + \rho w + \bar{\lambda}^T b$$

s.t. $Ax - 1w \leq b$,

$$-Ax + 1w \leq b,$$

$$E x \leq f,$$

$$\|x\|_{\infty} \leq M,$$

$$x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}.$$

A proof of Lemma 23 is provided later. We next rewrite $x = (x_1, x_2)$ and separate $A, E, c$ respectively. We also rewrite

$$Q = \begin{bmatrix} Q^{(11)} & Q^{(12)} \\
Q^{(21)} & Q^{(22)} \end{bmatrix}.$$

Note that $Q^{(11)}$ is also positive semi-definite. Therefore, the problem can be rewritten as

$$z^{\text{LR+}}_{\rho}(\bar{\lambda}) = \inf_{x_1, x_2, w} (c_1^T - \bar{\lambda}^T A_1 + x_2^T Q^{(21)})x_1 + \frac{1}{2} x_1^T Q^{(11)} x_1$$

$$+ \rho w + \bar{\lambda}^T b + (c_2^T - \bar{\lambda}^T A_2)x_2 + \frac{1}{2} x_2^T Q^{(22)} x_2$$

s.t. $A_1 x_1 - 1w \leq -A_2 x_2 + b$,

$$-A_1 x_1 + 1w \leq A_2 x_2 - b,$$

$$E_1 x_1 \leq f - E_2 x_2,$$

$$x_1 \leq 1M, -x_1 \leq 1M,$$

$$\|x_2\|_{\infty} \leq M,$$

$$x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{Z}^{n_2}.$$

Denote $V = \{v \in \mathbb{Z}^{n_2} : \|v\|_{\infty} \leq M\}$. In addition, we use $z^{\text{LR+}}_{\rho}(\bar{\lambda}, x_2)$ to denote $z^{\text{LR+}}_{\rho}(\bar{\lambda})$ while fixing $x_2$. Therefore, $z^{\text{LR+}}_{\rho}(\bar{\lambda}) = \min_{x_2 \in V} z^{\text{LR+}}_{\rho}(\bar{\lambda}, x_2)$. Note that $z^{\text{LR+}}_{\rho}(\bar{\lambda}, x_2)$ is still non-decreasing with respect to $\rho$. Therefore, from Theorem 10 we have

$$z^{\text{IP}} = \lim_{\rho \to +\infty} z^{\text{LR+}}_{\rho}(\bar{\lambda}) = \lim_{\rho \to +\infty} \min_{x_2 \in V} z^{\text{LR+}}_{\rho}(\bar{\lambda}, x_2) = \min_{x_2 \in V} \lim_{\rho \to +\infty} z^{\text{LR+}}_{\rho}(\bar{\lambda}, x_2).$$

Thus, $\lim_{\rho \to +\infty} z^{\text{LR+}}_{\rho}(\bar{\lambda}, x_2) \geq z^{\text{IP}}$, for all $x_2 \in V$.

For arbitrary $x_2 \in V$, the dual problem of (10) (with respect to $x_1, w$) is therefore

$$z^{\text{DRD+}}_{\rho}(\bar{\lambda}, x_2) := \sup_{y_1, y_2, y_3, y_4, y_5 \geq 0} \inf_{x_1, w} (c_1^T - \bar{\lambda}^T A_1 + x_2^T Q^{(21)})x_1 + \frac{1}{2} x_1^T Q^{(11)} x_1$$

$$+ \rho w + \bar{\lambda}^T b + (c_2^T - \bar{\lambda}^T A_2)x_2 + \frac{1}{2} x_2^T Q^{(22)} x_2$$

$$+ y_1^T (A_1 x_1 - 1w + A_2 x_2 - b) - y_2^T (A_1 x_1 + 1w + A_2 x_2 - b)$$

$$+ y_3^T (E_1 x_1 - f + E_2 x_2) + y_4^T (x_1 - 1M) + y_5^T (-x_1 - 1M).$$

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Note that the problem $\inf_{x_1,x_2} c^T_1 - \lambda^T A_1 + x_1^T Q^{(21)} + (y_1 - y_2)^T A_1 + y_3^T E_1 + (y_4 - y_5)^T = \nu^T Q^{(11)}$. Therefore, the problem is

$$z^\text{DRD}^+_{\lambda}(x_2) = \sup_{y,\nu} - \frac{1}{2} \nu^T Q^{(11)} y + (A_2 x_2 - b)^T y_1 - (A_2 x_2 - b)^T y_2$$

$$+ (E_2 x_2 - f)^T y_3 - M^T (y_4 + y_5)$$

$$+ \lambda^T b + (c^T_2 - \lambda^T A_2) x_2 + \frac{1}{2} x_2^T Q^{(22)} x_2$$

s.t. $y_1, y_2, y_3, y_4, y_5 \geq 0$,

$$1^T (y_1 + y_2) = \rho,$$

$$c - A_1^T \lambda + Q^{(12)} x_2 + A_1^T (y_1 - y_2) + E_1^T y_3 + y_4 - y_5 = Q^{(11)} y.$$

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Now let \( \rho^* = \max_{x \in V} \rho^*(x_2) \) of small complexity, we have \( z_{\rho^*}^{DRD^+}(\tilde{\lambda}, x_2) \geq \)
\( z_{\rho^*}(\tilde{\lambda}, x_2) \geq z^{IP} \) for all \( x_2 \in V \). Hence, \( z_{\rho^*}^{LR^+}(\tilde{\lambda}) = \min_{x_2 \in V} z_{\rho^*}^{LR^+}(\tilde{\lambda}, x_2) =\)
\( \min_{x_2 \in V} z_{\rho^*}^{DRD^+}(\tilde{\lambda}, x_2) \geq z^{IP} \).

Note that for any \( \rho, \lambda, z_{\rho}^{LR^+}(\lambda) \leq z^{LD^+} \leq z^{IP} \), and thus \( z_{\rho^*}^{LR^+}(\tilde{\lambda}) = z^{LD^+} = z^{IP} \).\( \quad \Box \)

We next complete the proof by proving Lemma 23.

Proof of Lemma 23. Consider for \( \rho \geq 1 \), \( w_M := 2(z^{IP} - z^{NLP}) \geq 2w^* \), where the
inequality follows from (3), and define

\[
\bar{z}_{\rho}^{LR^+}(\tilde{\lambda}) = \inf_{x,w} (c^T - \tilde{\lambda}^T A)x + \frac{1}{2} x^T Qx + \rho w + \tilde{\lambda}^T b
\]

s.t. \( Ax - 1w \leq b, \)

\[ -Ax - 1w \leq -b, \]

\[ Ex \leq f, \]

\[ w \leq w_M, \]

\[ x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}. \]

Denote the feasible region for this problem as \( P \) and the feasible region for the original
problem (8) as \( P^0 \). Clearly, \( P \subseteq P^0 \) and thus \( \bar{z}_{\rho}^{LR^+}(\tilde{\lambda}) \geq z_{\rho}^{LR^+}(\tilde{\lambda}). \) Similarly as \( P \)
is larger than the feasible region of (4), we have \( \bar{z}_{\rho}^{LR^+}(\tilde{\lambda}) \leq z_{\rho}^{LR^+}(\tilde{\lambda}) \) and
thus \( \bar{z}_{\rho}^{LR^+}(\tilde{\lambda}) = z_{\rho}^{LR^+}(\tilde{\lambda}). \)

Now apply Lemma 15 to \( P \) and we get a decomposition \( P = \cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}) \times \mathbb{R})) + \text{int.cone}(R_i) \) with the properties listed in the lemma. Note that the
decomposition applies to all \( \rho \).

Note that the problem is bounded due to the boundedness of \( z^{LR^+} \) and for all
\( r \in R_i \) the \( w \)-component is 0 (from the constraints \( w \leq w_M \)), so we can omit the
\( w \)-component for any vector in \( R_i \), or simply denote it as \( R_i \times \{0\} \).

Similar to the proof of Lemma 21, when we solve the problem \( \inf_{x,w} \{ (c^T - \tilde{\lambda}^T A)x + \frac{1}{2} x^T Qx + \rho w + \tilde{\lambda}^T b : (x, w) \in P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \text{int.cone}(R_i) \times \{0\} \} \).

If there exists \( r \in R_i \) such that \( r^T Qr = 0 \) (i.e. \( Qr = 0 \)), the feasible region can be
decomposed as \( P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \text{int.cone}(R_i \backslash \{r\}) \times \{0\} + \{ \mu r : \mu \in \mathbb{Z}_+ \} \times \{0\} \).

Optimize the problem over \( \mu \) and we get \( \mu = 0 \) an optimal solution (otherwise the
problem will be unbounded). Therefore, we can refine the feasible region by omitting
all \( r \in R_i \) such that \( Qr = 0 \). Denote the set after the process as \( R_i^i \). Note that this
process is independent of \( \rho \), and hence we have

\[
\bar{z}_{\rho}^{LR^+}(\tilde{\lambda}) = \inf_{x,w} (c^T - \tilde{\lambda}^T A)x + \frac{1}{2} x^T Qx + \rho w + \tilde{\lambda}^T b
\]

s.t. \( (x, w) \in \cup_i(P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \text{int.cone}(R_i^i) \times \{0\})). \)

Now, from (c) of Lemma 15, for all \( x \in \text{cone}(R_i^i) \backslash \{0\} \), we have \( x^T Qx > 0 \).Let

\[ V_i = \{(x,w) \in (P_i + \text{cone}(R_i^i) \times \{0\}) : \]

\[ (c^T - \tilde{\lambda}^T A)x + \frac{1}{2} x^T Qx + w + \tilde{\lambda}^T b \leq \bar{z}^{IP} + 1 \} \).

Note that the definition of \( V_i \) is independent of \( \rho \). Therefore, as \( \rho \geq 1 \) we have

\[
\bar{z}_{\rho}^{LR^+}(\tilde{\lambda}) = \inf_{x,w} (c^T - \tilde{\lambda}^T A)x + \frac{1}{2} x^T Qx + \rho w + \tilde{\lambda}^T b
\]
From Lemma 16, there exists $M_i > 0$, such that $V_i \subseteq \{(x, w) : \|(x, w)\|_\infty \leq M_i\}$ and $M_i$ has small complexity. Therefore, $V_i \subseteq \{(x, w) : \|x\|_\infty \leq M_i\}$. Let $M = \max\{M_i\}$ (which is again independent of $\rho$ and has small complexity) and we have

$$z_{\rho}^{LR^+}(\lambda) \geq \inf_{x, w} (c^\top - \lambda^\top A)x + \frac{1}{2}x^\top Qx + \rho w + \bar{\lambda}^\top b$$

s.t. $(x, w) \in \cup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \text{int.cone}(R'_i) \times \{0\})$, $(x, w) \in \cup_i V_i$

$$\geq \inf_{x, w} (c^\top - \lambda^\top A)x + \frac{1}{2}x^\top Qx + \rho w + \bar{\lambda}^\top b$$

s.t. $(x, w) \in \cup_i (P_i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}) + \text{int.cone}(R'_i) \times \{0\})$, \[\|x\|_\infty \leq M\]

$$\geq \inf_{x, w} (c^\top - \lambda^\top A)x + \frac{1}{2}x^\top Qx + \rho w + \bar{\lambda}^\top b$$

s.t. $(x, w) \in P^\rho$, \[\|x\|_\infty \leq M\]

Since

$$z_{\rho}^{LR^+}(\lambda) = \inf_{x, w} (c^\top - \lambda^\top A)x + \frac{1}{2}x^\top Qx + \rho w + \bar{\lambda}^\top b$$

s.t. $(x, w) \in P^\rho$

$$\leq \inf_{x, w} (c^\top - \lambda^\top A)x + \frac{1}{2}x^\top Qx + \rho w + \bar{\lambda}^\top b$$

s.t. $(x, w) \in P^\rho$, \[\|x\|_\infty \leq M\]

equality holds and the proof is completed. \qed

Next, we will generalize the result to any norm penalty and any dual variable.

**Theorem 12** (Exact Penalty Representation). Suppose $\psi(\cdot)$ is any norm.

(a) There exists a finite $\rho^*$ of $\mathcal{F}$-small complexity, such that $z_{\rho^*}^{LD^+} = z_{IP}$.

(b) Moreover, for all $\lambda$, there exists a finite $\rho^*(\lambda)$ of $(\mathcal{F}, \lambda)$-small complexity, such that $z_{\rho^*(\lambda)}^{LR^+}(\lambda) = z_{IP}$.

Proof. Denote the $\rho^*(\lambda)$ in Theorem 22 as $\rho^*_\infty(\lambda)$ to represent the case for infinity norm.

As $\psi(\cdot)$ is a norm function, there exists $\gamma \in [1, +\infty)$ such that $\gamma \cdot \|\cdot\|_\infty \geq \psi(\cdot) \geq \|\cdot\|_\infty / \gamma$. Without loss of generality, we round up $\gamma$ to the closest integer, which is still a constant decided only by $\|\cdot\|_\infty$ and $\psi(\cdot)$. Therefore, by letting $\rho^*(\lambda) = \gamma \rho^*_\infty(\lambda)$,
which still has small complexity, we have

\[ z_{\rho^*}(\lambda) = \inf_{x \in \mathcal{X}} \{ c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax) + \rho^*(\lambda) \psi(b - Ax) \} \]

\[ \geq \inf_{x \in \mathcal{X}} \{ c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax) + \rho^*_\infty(\lambda) \| b - Ax \|_\infty \} \]

\[ = z_{\text{IP}}, \]

where the last equation comes from Theorem 22. Along with \( z_{\rho^*}(\lambda) \leq z_{\rho^*} \leq z_{\text{IP}}, \) we have \( z_{\rho^*}(\lambda) = z_{\text{IP}} \) and (a) is proven. Now it only remains to show that we can replace \( \lambda \) by any dual vector \( \hat{\lambda} \in \mathbb{R}^m. \)

From Cauchy-Schwarz inequality, we have

\[-\|\lambda\|_2 \|b - Ax\|_2 \leq \lambda^T (b - Ax) \leq \|\lambda\|_2 \|b - Ax\|_2.\]

Again, applying the property of the norm, there exists \( \eta \in [1, +\infty) \cap \mathbb{Z}_+ \) decided only by \( \| \cdot \|_2 \) and \( \psi(\cdot) \), such that \( \eta \| \cdot \|_2 \geq \psi(\cdot) \geq \| \cdot \|_2 / \eta, \) and we have

\[ \hat{\lambda}(b - Ax) - \lambda^T (b - Ax) \geq -\eta \| \hat{\lambda} - \lambda \|_2 \psi(b - Ax). \]

By setting \( \rho^*(\lambda) = [\rho^*(\hat{\lambda}) + \eta \| \hat{\lambda} - \hat{\lambda} \|_2], \) which has \( (\mathcal{F}, \hat{\lambda}) \)-small complexity, we have

\[ z_{\rho^*}(\lambda) = \inf_{x \in \mathcal{X}} \{ c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax) + \rho^*(\hat{\lambda}) \psi(b - Ax) \} \]

\[ \geq \inf_{x \in \mathcal{X}} \{ c^T x + \frac{1}{2} x^T Q x + \lambda^T (b - Ax) + \rho^*(\hat{\lambda}) \psi(b - Ax) \} \]

\[ = z_{\rho^*}(\lambda) = z_{\text{IP}}. \]

Therefore, along with \( z_{\rho^*}(\lambda) \leq z_{\text{IP}}, \) we have \( z_{\rho^*}(\lambda) = z_{\text{IP}}. \) 

\[ \square \]

**Remark 24.** The results also apply to MILP, which yield that exact penalty weight \( \rho^* \) (which is detailedly discussed in [9]) also has \( \mathcal{F} \)-small complexity.

6. **Conclusions.** In this paper, we investigate ALD for MIQP. We prove that an asymptotic zero duality gap is reachable as the penalty weight goes to infinity, under some mild conditions (Assumption 18) on the penalty function. We also show that a finite penalty weight is enough for an exact penalty when we use any norm as the penalty function. Moreover, we prove that a penalty weight of polynomial size is enough to give an exact penalty representative.

By dualizing and penalizing the difficult constraints using ALD, we can convert the problem to one with only easy constraints, while maintaining the optimality of the original optimal solutions. However, after introducing a penalty term, the new objective function is certainly more complicated in comparison to the original objective function. In addition, as ALD does not deal with integer constraints, the new problem is still non-convex in general.

A special case where the easy constraints are separable, leads us to consider the alternating direction method of multipliers (ADMM) [3] and relative update schemes, which are proposed to solve convex problems separably. However, for mixed integer problems, such methods are mainly heuristic, like [17] for MIQP based on ADMM. Future development of separable exact algorithms utilizing the strong duality results and solving general non-convex problems is an important direction of research.
REFERENCES


