Convexification of substructures in quadratically constrained quadratic program

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Mixed-integer Nonlinear Optimization: Theory and Computation
Centre De Recherches Mathamatiques

Oct, 2019

Outline

1. The convex hull of quadratic constraints over polytopes

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2. A study of rank-one sets with linear side constraints and application to the pooling problem

   Santanu S. Dey, Burak Kocuk, Asteroide Santana.

Section 1

Motivation

A QCQP for Structural engineering application

Updating FEM model for bridges

$$\begin{aligned} \min \quad & \sum_{k=1}^{m} z_k \\ \text{s.t.} \quad & |x^\top Q_k y + a_k^\top x + b_k^\top y + c_k| = z_k, \;\; k \in \{1, \dots, m\} \\ & x \in [0,1]^{n_1}, y \in [0,1]^{n_2}, \end{aligned} \tag{1}$$

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QCQP (Bilinear Program):

$$\begin{aligned} \min \quad & \sum_{k=1}^{m} z'_k + z''_k \\ \text{s.t.} \quad & x^\top Q_k y + a_k^\top x + b_k^\top y + c_k = z'_k - z''_k, \;\; k \in \{1, \dots, m\} \\ & x \in [0,1]^{n_1}, y \in [0,1]^{n_2}. \\ & 0 \le z'_k, z''_k \le u, \;\; k \in \{1, \dots, m\}. \end{aligned} \tag{2}$$

Need to obtain good dual bounds

We tried various approached.

1. McCormick relaxation + Semidefinite programming relaxation
2. Commercial solvers (for hours)

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Motivation from Integer programming: Convexification of substructures like knapsack set, flow cover sets, mixing sets, etc.

A very successful approach in integer linear programing is to generate cutting-planes implied by single constraint relaxations.

Convex hull of one-constraint relaxation

Theorem (Santana, D., Wang)

Consider the set (S) *defined as:*

$$\left.\begin{array}{l} \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} q_{ij}w_{ij} + \sum_{i=1}^{n_1} a_i x_i + \sum_{j=1}^{n_2} b_j y_j + c = 0, \\ w_{ij} = x_i y_j,\ \ i \in [n_1], j \in [n_2], \\ (x, y, w) \in [0,1]^{n_1+n_2+n_1 n_2} \end{array}\right\} (S)$$

Then, conv(S) is SOC-representable (SOCr).

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We give a constructive proof.

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Then, $\text{conv}(S)$ is SOC-representable (SOCr).

- M. Tawarmalani, N.V. Sahinidis [2001]
- M. Tawarmalani, J.-P. P. Richard, K. Chung [2010]
- K. M. Anstreicher, S. Burer [2010]
- T. T. Nguyen, J.-P. P. Richard, M. Tawarmalani [2012]
- A. Gupte Thesis [2012]
- B. Kocuk, SSD, and X. A. Sun [2018]
- And many others...(especially on convex envelopes of bilinear functions)

This substructure convexification worked well

1. We used a new branching rule.
2. Used a polyhedral outer approximation of the SOCr convex hull.

Table: Comparison with BARON

	newBB, row-wise SOCP			BARON		
Inst	Dual	Primal	% Gap	Dual	Primal	% Gap
1	2.5074	3.4785	27.91	0.3312	3.4789	90.47
2	2.8644	3.4998	18.15	0.5245	3.4993	85.01
3	3.1308	3.6810	14.94	0.4760	3.6831	87.07
4	3.1115	3.7522	17.07	0.7863	3.7530	79.04
5	3.7896	4.1328	8.30	0.3840	4.1354	90.71
6	4.6399	5.6609	18.03	2.2657	5.6605	59.97

Running-time: 10 hours.

$\%\text{gap} = \frac{\text{Primal}-\text{Dual}}{\textit{Primal}}$

Questions

- What about more general quadratic constraints?
- What about more general linear constraints?

Section 2

The convex hull of a quadratic constraint over a polytope

Our result

Theorem (Santana, D.)

Let

$$S := \{x \in \mathbb{R}^n \mid \; x^\top Q x + c^\top x = d, \; x \in P\},$$

where P is a polytope. Then, conv(S) *is SOCr. Moreover, we provide a constructve proof to generate* conv(S).

Proof of Thm (sketch)
Underlying Proof Structure

If S is a compact set, then

$$\text{conv}\,(S) = \text{conv}\,(\text{extr}(S)).$$

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As a consequence:

1. If $\text{extr}(S) \subseteq \bigcup_{k=1}^{m} T_k \subseteq S$, then

$$\boxed{\text{conv}\,(S) = \text{conv}\left(\bigcup_{k=1}^{m} \text{conv}\,(T_k)\right)}$$

2. Finally, if $\text{conv}\,(T_k)$ is SOCr, then $\text{conv}\,(S)$ is SOCr.

Proof of Thm (sketch)

Structure Lemma on Quadratic functions

Lemma

Consider the set defined by a quadratic equation. Then exactly one of the following occurs:

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2. *It is the union of boundary of two disjoint SOCP representable convex set; or*

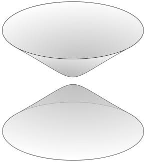
Proof of Thm (sketch)

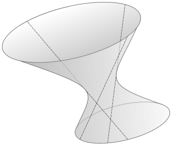
Structure Lemma on Quadratic functions

Lemma

Consider the set defined by a quadratic equation. Then exactly one of the following occurs:

1. *It is the boundary of a SOCP representable convex set,*
2. *It is the union of boundary of two disjoint SOCP representable convex set; or*
3. *It has the property that, through every point, there exists a straight line that is entirely contained in the surface.*





https://www.math.arizona.edu/~models/Ruled_Surfaces/source/1.html

Ruled surface are beautiful!

Proof of Thm (sketch)

Using the Structure Lemma

1. If in Case 1 or Case 2: (i.e., the boundry of SOCP representable convex set or union of boundary of two SOCP representable sets), then done!

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   2.2 Intersect the quadratic with each facet of the polytope;
   2.3 Each intersection yields a new quadratic set of the same form, but in lower dimension;
3. Repeat above argument for each facet.

Proof of Thm (sketch)

Proof of Structure Lemma: Reduction

Lemma

If F is an affine bijective map, then:

1. $\text{conv}(F(S)) = F(\text{conv}(S))$;
2. $\text{conv}(S)$ *is SOCr iff* $\text{conv}(F(S))$ *is SOCr.*

Then, we rewrite

$$S := \{x \in \mathbb{R}^n \mid x^\top Qx + c^\top x = d,\ x \in P\},$$

as

$$S = \Big\{(u, v, w) \in \mathbb{R}^{n_{q+}} \times \mathbb{R}^{n_{q-}} \times \mathbb{R}^{n_l} \mid \sum_{i=1}^{n_{q+}} u_i^2 - \sum_{j=1}^{n_{q-}} v_j^2 + \sum_{k=1}^{n_l} w_k = d,\ (u, v, w) \in P\Big\},$$

where we may assume $d \geq 0$.

Proof of Thm (sketch)

Proof of Observation 2: Classification

$$S = \Big\{(u,v,w) \in \mathbb{R}^{n_{q+}} \times \mathbb{R}^{n_{q-}} \times \mathbb{R}^{n_l} \mid \sum_{i=1}^{n_{q+}} u_i^2 - \sum_{j=1}^{n_{q-}} v_j^2 + \sum_{k=1}^{n_l} w_k = d,\ (u,v,w) \in P\Big\}$$

Lemma

Case	*Classification*
1) $n_l \geq 2$	*straight line*
2) $n_{q+} \leq 1,\ n_l = 0$	*Case 1 or Case 2*
3) $n_{q+}n_{q-} = 0,\ n_l \leq 1$	*Case 1 or Case 2*
4) $n_{q+},\ n_{q-} \geq 1,\ n_l = 1$	*straight line*
5) $n_{q+} \geq 2,\ n_{q-} \geq 1,\ n_l = 0$	*straight line*

Discussion

Classify: conv.hull of QCQP substructure is SOCr?

Is SOCP representable:

1. One quadratic equality constratraint ∩ polytope
2. Two quadratic inequalities (Burer, Klinc-Karzan [2017], Modaresi, Vielma [2017])

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Classify: conv.hull of QCQP substructure is SOCr?

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2. Two quadratic inequalities (Burer, Klinc-Karzan [2017], Modaresi, Vielma [2017])

Is not SOCP representable:

1. Already in 10 variables, 5 quadratic equalities, 4 quadratic inequalities, 3 linear inequalities (Fawzi [2018])

Section 3

Rank-one sets and applications to the pooling problem

3.1
Setting and Theoretical results

Rank-one sets with linear side constraints

Notation: $[m] = \{1, \ldots, m\}$

Consider again the QCQP:

$$\begin{aligned} \min \quad & x^\top Q^0 x + (a^0)^\top x \\ \text{s.t.} \quad & x^\top Q^k x + (a^k)^\top x = b_k \qquad \forall k \in [m] \\ & x \in [0,\ 1]^n. \end{aligned}$$

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Which can be rewritten as:

$$\begin{aligned} \min \quad & \langle Q^0, X \rangle + (a^0)^\top x \\ \text{s.t.} \quad & \langle Q^k, X \rangle + (a^k)^\top x \leq b_k \qquad \forall k \in [m] \\ & \text{rank}\left(\begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \right) = 1 \\ & x \in [0,\ 1]^n, \end{aligned}$$

where $\langle U, V \rangle := \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} U_{ij} V_{ij}$.

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where $\langle U, V\rangle := \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} U_{ij} V_{ij}$.

$$\mathcal{U} := \left\{ W \in \mathbb{R}_+^{n_1 \times n_2} \mid \langle A^k, W\rangle \leq b_k, \forall k \in [m],\ \operatorname{rank}(W) \leq 1 \right\}$$

Some choices of linear side constraints: a well-known result

$$\mathcal{U} := \left\{ W \in \mathbb{R}_{+}^{n_1 \times n_2} \,|\, \langle A^k, W \rangle \leq b_k, \forall k \in [m], \, \text{rank}(W) \leq 1 \right\}.$$

Known result

- All individual variable bounds:

$$\bar{\mathcal{U}} := \left\{ W \in \mathbb{R}_{+}^{n_1 \times n_2} \,|\, W_{ij} \leq 1, \forall i \in [n_1], j \in [n_2], \, \text{rank}(W) \leq 1 \right\}.$$

Then, $\text{conv}\,(\bar{\mathcal{U}})$ is the Boolean Quadric Polytope.

(Burer, Letchford [2009])

One of two arbitrary side constraints

- Single linear constraint with non-negative coefficients:

$$\mathcal{U}^1 := \left\{ W \in \mathbb{R}_+^{n_1 \times n_2} \mid \langle A^1, W \rangle \leq b_1, \text{ rank}(W) \leq 1 \right\}.$$

Then, $\text{conv}\,(\mathcal{U}^1)$ is a polyhedron.

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- Two linear constraint (assuming set is bounded):

$$\mathcal{U}^2 := \left\{ W \in \mathbb{R}_+^{n_1 \times n_2} \quad \mid \langle A^k, W \rangle \leq b_k, \ \ k = 1, 2, \ \mathrm{rank}(W) \leq 1 \right\}.$$

Then, $\mathrm{conv}\,(\mathcal{U}^2)$ is SOCr.

Arbitrary number of constraints - I

Theorem (D., Kocuk, Santana)

$$\mathcal{U} := \left\{ W \in \mathbb{R}_{+}^{n_1 \times n_2} \mid \langle A^k, W \rangle \le b_k, \forall k \in [m], \ \text{rank}(W) \le 1 \right\}.$$

Suppose the constraint matrices are of the form

$$A^k := \alpha^k \beta^\top \qquad k \in [m],$$

where $\alpha^k \in \mathbb{R}^{n_1}$ *for* $k \in [m]$ *and* $\beta \in \mathbb{R}^{n_2}_{++}$*. Moreover, let the* α^k*'s be such that* $\{u \in \mathbb{R}^{n_1}_{+} \mid (\alpha^k)^\top u \le 0, \forall\ k \in [m]\} = \{0\}$*. Then:*

1. *The* conv $(\mathcal{U})$ *is a polyhedron.*
2. *A compact extended formulation of* conv$(\mathcal{U})$ *is given by:*

$$\sum_{j=1}^{n_2} t_j = 1, \ \ t_j \ge 0 \qquad \forall j \in [n_2]$$

$$\sum_{i=1}^{n_1} \alpha_i^k \beta_j W_{ij} \le b_k t_j \qquad \forall k \in [m], \ j \in [n_2].$$

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$$\sum_{j=1}^{n_2} t_j = 1,\ \ t_j \geq 0 \qquad \forall j \in [n_2]$$

$$\sum_{i=1}^{n_1} \alpha_i^k \beta_j W_{ij} \leq b_k t_j \qquad \forall k \in [m],\ j \in [n_2].$$

Equivalent to result in (Sherali, Alameddine [1992])

Arbitrary number of constraints - II

Theorem (D., Kocuk, Santana)

Consider the set

$$\mathcal{U} := \left\{ W \in \mathbb{R}_{+}^{n_1 \times n_2} \,|\, \langle A^k, W \rangle \leq b_k, \forall k \in [m], \ \text{rank}(W) \leq 1 \right\},$$

with $n_1 = n_2 \geq 3$. *Suppose that* $\mathcal{U}$ *is bounded and the constraint matrices are of the form*

$$A^k := \alpha^k \beta \beta^\top + \gamma^k \delta \delta^\top \qquad k \in [m],$$

where $\alpha^k, \gamma^k \in \mathbb{R}_+$ *for* $k \in [m]$ *and* $\beta, \delta \in \mathbb{R}^{n_1}_{++}$.
Then, conv $(\mathcal{U})$ *is SOCr. Moreover, for a fixed m, a linear function can be optimized in polytime over* $\mathcal{U}$.

Some corollaries

- Row sum bounds:

$$\mathcal{U}^{\text{row}} := \left\{ W \in \mathbb{R}_+^{n_1 \times n_2} \mid l_i \leq \sum_{j=1}^{n_2} W_{ij} \leq u_i, \forall i \in [n_1], \right.$$
$$\left. \text{rank}(W) \leq 1 \right\}.$$

Then, $\text{conv}\left(\mathcal{U}^{\text{row}}\right)$ is a polyhedron.

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$$\left. \text{rank}(W) \leq 1 \right\}.$$

Then, $\text{conv}\left(\mathcal{U}^{\text{row}}\right)$ is a polyhedron.

- Row sum and overall sum bounds:

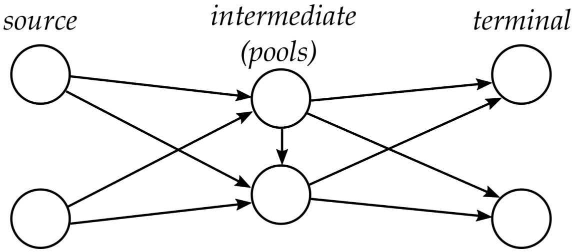
$$\mathcal{U}^{\text{row}+} := \left\{ W \in \mathbb{R}_+^{n_1 \times n_2} \mid l_i \leq \sum_{j=1}^{n_2} W_{ij} \leq u_i, \forall i \in [n_1], \right.$$
$$\left. L \leq \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} W_{ij} \leq U, \text{rank}(W) \leq 1 \right\}.$$

Then, $\text{conv}\left(\mathcal{U}^{\text{row}+}\right)$ is a polyhedron.

3.2
Application: To the Generalized Pooling Problem

Pooling problem

Haverly [1978].

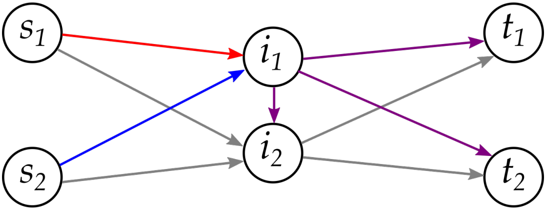


Called *generalized pooling problem* (GPP) if there are pool-pool arcs, otherwise, *standard pooling problem* (SPP).

Pooling problem

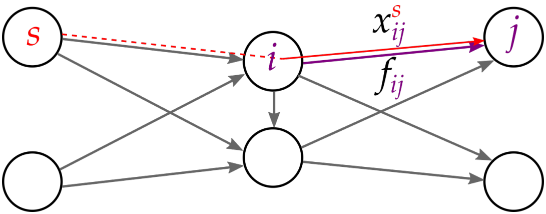
Haverly [1978].

Flow consistency requirement must be met at each pool.



pq-Formulation

SPP: Tawarmalani and Sahinidis [2002]
GPP: Alfaki and Haugland [2012]



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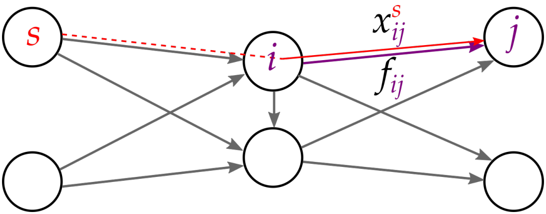
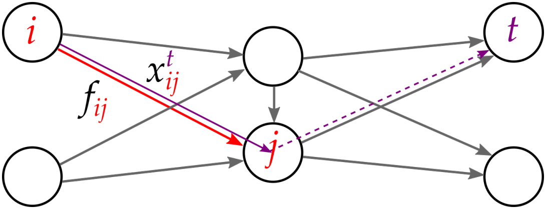


For a fixed pool i and any source s:

$$q_i^s = \frac{x_{ij}^s}{f_{ij}} \ \forall j \quad \Rightarrow \quad x_{ij}^s = q_i^s f_{ij} \ \forall j.$$

tp-Formulation

SPP: Alfaki and Haugland [2013]
GPP: Boland et al. [2016]



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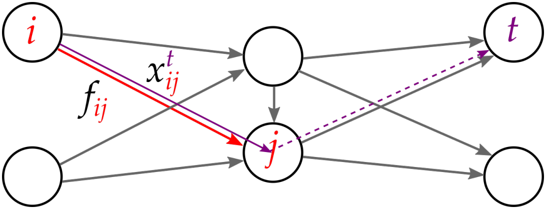


For a fixed pool j and any terminal t:

$$q_j^t = \frac{x_{ij}^t}{f_{ij}} \; \forall i \quad \Rightarrow \quad x_{ij}^t = q_j^t f_{ij} \; \forall i.$$

From pq-formulation to rank-1 (Part I)

For a fixed pool i, consider the matrix

$$[x^s_{ij}]_{(s,j)} = \begin{bmatrix} x^1_{i1} & \cdots & x^1_{ij} & \cdots & x^1_{in} \\ & & \vdots & & \\ x^s_{i1} & \cdots & x^s_{ij} & \cdots & x^s_{in} \\ & & \vdots & & \\ x^m_{i1} & \cdots & x^m_{ij} & \cdots & x^m_{in} \end{bmatrix}_{(s,j)}$$

Then,

$$\begin{aligned} x^s_{ij} = q^s_i f_{ij}, \ \forall s, j \quad & \Leftrightarrow \quad [x^s_{ij}]_{(s,j)} = [q^s_i]_s [f_{ij}]_j^\top \\ & \Leftrightarrow \quad \text{rank}([x^s_{ij}]_{(s,j)}) = 1. \end{aligned}$$

From pq-formulation to rank-1 (Part II)

Consider again the matrix $[x_{ij}^s]_{sj}, \forall s, j$, for a fixed pool i. Then,

$$q_i^s = \frac{x_{ij}^s}{f_{ij}}, \ \forall s, j \quad \Leftrightarrow \quad q_i^s = \frac{x_{ij}^s}{\sum_{s'} x_{ij}^{s'}}, \ \forall s, j \quad \Leftrightarrow \quad \text{rank}([x_{ij}^s]_{(s,j)}) = 1.$$

$$\begin{bmatrix} x_{i1}^1 & \cdots & x_{ij}^1 & \cdots & x_{in}^1 \\ & & \vdots & & \\ x_{i1}^s & \cdots & x_{ij}^s & \cdots & x_{in}^s \\ & & \vdots & & \\ x_{i1}^m & \cdots & x_{ij}^m & \cdots & x_{in}^m \end{bmatrix}_{(s,j)}$$

From pq-formulation to rank-1 (Part II)

Consider again the matrix $[x_{ij}^s]_{sj}, \forall s, j$, for a fixed pool i. Then,

$$q_i^s = \frac{x_{ij}^s}{f_{ij}}, \ \forall s, j \quad \Leftrightarrow \quad q_i^s = \frac{x_{ij}^s}{\sum_{s'} x_{ij}^{s'}}, \ \forall s, j \quad \Leftrightarrow \quad \text{rank}([x_{ij}^s]_{(s,j)}) = 1.$$

$$\begin{bmatrix} x_{i1}^1 & \cdots & x_{ij}^1 & \cdots & x_{in}^1 \\ & & \vdots & & \\ x_{i1}^s & \cdots & x_{ij}^s & \cdots & x_{in}^s \\ & & \vdots & & \\ x_{i1}^m & \cdots & x_{ij}^m & \cdots & x_{in}^m \end{bmatrix}_{(s,j)}$$

By applying our convex hull result to $\mathcal{U}^{col+}$ to each matrix $[x_{ij}^s]_{sj}, \forall s, j$, we recover the pq-relaxation!

From pq-formulation to rank-1 (Part II)

Consider again the matrix $[x_{ij}^s]_{sj}, \forall s, j$, for a fixed pool i. Then,

$$q_i^s = \frac{x_{ij}^s}{f_{ij}}, \ \forall s, j \ \Leftrightarrow \ q_i^s = \frac{x_{ij}^s}{\sum_{s'} x_{ij}^{s'}}, \ \forall s, j \ \Leftrightarrow \ \text{rank}([x_{ij}^s]_{(s,j)}) = 1.$$

$$\begin{bmatrix} x_{i1}^1 & \cdots & x_{ij}^1 & \cdots & x_{in}^1 \\ & & \vdots & & \\ x_{i1}^s & \cdots & x_{ij}^s & \cdots & x_{in}^s \\ & & \vdots & & \\ x_{i1}^m & \cdots & x_{ij}^m & \cdots & x_{in}^m \end{bmatrix}_{(s,j)}$$

By applying our convex hull result to $\mathcal{U}^{col+}$ to each matrix $[x_{ij}^s]_{sj}, \forall s, j$, we recover the pq-relaxation!

How about $\mathcal{U}^{row+}$?

From pq-formulation to rank-1 (Part II)

Consider again the matrix $[x_{ij}^s]_{sj}, \forall s, j$, for a fixed pool i. Then,

$$q_i^s = \frac{x_{ij}^s}{f_{ij}}, \ \forall s,j \quad \Leftrightarrow \quad q_i^s = \frac{x_{ij}^s}{\sum_{s'} x_{ij}^{s'}}, \ \forall s,j \quad \Leftrightarrow \quad \text{rank}([x_{ij}^s]_{(s,j)}) = 1.$$

$$\begin{bmatrix} x_{i1}^1 & \cdots & x_{ij}^1 & \cdots & x_{in}^1 \\ & & \vdots & & \\ x_{i1}^s & \cdots & x_{ij}^s & \cdots & x_{in}^s \\ & & \vdots & & \\ x_{i1}^m & \cdots & x_{ij}^m & \cdots & x_{in}^m \end{bmatrix}_{(s,j)} \qquad \begin{bmatrix} x_{i1}^1 & \cdots & x_{ij}^1 & \cdots & x_{in}^1 \\ & & \vdots & & \\ x_{i1}^s & \cdots & x_{ij}^s & \cdots & x_{in}^s \\ & & \vdots & & \\ x_{i1}^m & \cdots & x_{ij}^m & \cdots & x_{in}^m \end{bmatrix}_{(s,j)}$$

$$q'_{ij} = \frac{x_{ij}^s}{\sum_{j'} x_{ij'}^s}, \ \forall s,j \quad \Leftrightarrow \quad \text{rank}([x_{ij}^s]_{(s,j)}) = 1.$$

From pq-formulation to rank-1 (Part II)

Consider again the matrix $[x_{ij}^s]_{sj}, \forall s, j$, for a fixed pool i. Then,

$$q_i^s = \frac{x_{ij}^s}{f_{ij}}, \ \forall s, j \quad \Leftrightarrow \quad q_i^s = \frac{x_{ij}^s}{\sum_{s'} x_{ij}^{s'}}, \ \forall s, j \quad \Leftrightarrow \quad \text{rank}([x_{ij}^s]_{(s,j)}) = 1.$$

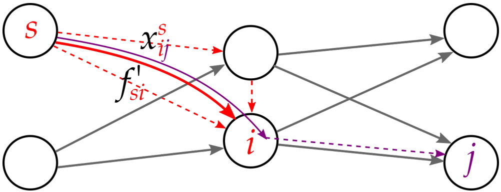
$$\begin{bmatrix} x_{i1}^1 & \cdots & x_{ij}^1 & \cdots & x_{in}^1 \\ & & \vdots & & \\ x_{i1}^s & \cdots & x_{ij}^s & \cdots & x_{in}^s \\ & & \vdots & & \\ x_{i1}^m & \cdots & x_{ij}^m & \cdots & x_{in}^m \end{bmatrix}_{(s,j)} \qquad \begin{bmatrix} x_{i1}^1 & \cdots & x_{ij}^1 & \cdots & x_{in}^1 \\ & & \vdots & & \\ x_{i1}^s & \cdots & x_{ij}^s & \cdots & x_{in}^s \\ & & \vdots & & \\ x_{i1}^m & \cdots & x_{ij}^m & \cdots & x_{in}^m \end{bmatrix}_{(s,j)}$$

$$\boxed{q'_{ij} = \frac{x_{ij}^s}{f'_{si}}, \ \forall s, j} \quad \Leftrightarrow \quad q'_{ij} = \frac{x_{ij}^s}{\sum_{j'} x_{ij'}^s}, \ \forall s, j \quad \Leftrightarrow \quad \text{rank}([x_{ij}^s]_{(s,j)}) = 1.$$

Rank-1 on pq-formulation

These new non-linear equations enforce flow consistency requirements on the source side:

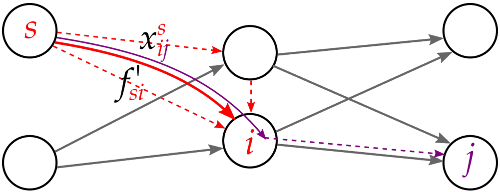
$$q'_{ij} = \frac{x^s_{ij}}{f'_{si}}, \ \forall s, j \quad \Leftrightarrow \quad x^s_{ij} = q'_{ij} f'_{si}, \ \forall s, j$$



Rank-1 on pq-formulation

These new non-linear equations enforce flow consistency requirements on the source side:

$$q'_{ij} = \frac{x^s_{ij}}{f'_{si}}, \ \forall s, j \quad \Leftrightarrow \quad x^s_{ij} = q'_{ij} f'_{si}, \ \forall s, j$$



By applying our convex hull result for $\mathcal{U}^{row+}$ to each $[x^s_{ij}]_{sj}, \forall s, j$:

- For SPP, we recover all the strength of tp-relaxation.
- For GPP, we partially recover the strength of tp-relaxation.

Most importantly, without having to define the x^t_{ij} variables of the tp-formulation.

New vs. known relaxations

Formulation	Standard Pooling (SPP)	Generalized Pooling (GPP)
pq	Known [1]	Known [2]
tp	Known [2]	Known [3]
pq×tp	Known [3]	Known [3]
rank-1-pq	$=$ pq $\cap$ tp	New
rank-1-tp	$=$ pq $\cap$ tp	New

[1]: Tawarmalani and Sahinidis [2002]

[2]: Alfaki and Haugland [2013]

[3]: Boland et al. [2016]

Computational results

Reporting the best average duality gap for each instance set:

Light-LP methods: pq, tp, rank-1-pq, rank-1-tp

Instance Set	Gap	Time	Method
Mining	4.14	1.76	rank-1-qp
Literature	18.11	0.02	rank-1-tp
Random	12.47	5.25	rank-1-qp

Medium-LP methods: pq $\cap$ tp, pq $\times$ tp, rank-1-pq $\cap$ tp, pq $\cap$ rank-1-tp, rank-1-pq $\cap$ rank-1-tp

Instance Set	Gap	Time	Method
Mining	3.82	13.73	rank-1-qp $\cap$ rank-1-tp
Literature	18.10	0.11	rank-1-qp $\cap$ rank-1-tp
Random	12.28	27.34	rank-1-qp $\cap$ rank-1-tp

Computational results

Reporting the best average duality gap for each instance set:

MILP methods: pq(q^s), rank-1-pq(q^s), rank-1-pq(q'), tp(q^t), rank-1-tp(q^t), rank-1-tp(q')

Instance Set	Gap	Time	Method
Mining	1.46	758.30	rank-1-qp(q')
Literature	0.49	150.29	rank-1-tp(q')
Random	7.37	927.50	rank-1-qp(q')

Thank you!

Section 4

Appendix

Computational results

Duality gaps via LP for Mining instances

	Paper [3]		pq		rank-1-pq		tp		rank-1-tp		pq ∩ tp		rank-1-pq ∩ tp		pq ∩ rank-1-tp		rank-1-pq ∩ rank-1-tp		pq × tp	
Inst	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time
2009H2	37.00	0.05	7.13	0.53	4.06	1.12	4.58	1.23	4.58	9.28	4.27	5.13	**3.59**	5.89	4.27	7.16	**3.59**	6.64	4.02	1697.27
2009Q3	29.36	0.03	3.84	0.06	1.59	0.23	2.16	0.08	2.16	0.17	2.01	0.34	**1.45**	0.69	2.01	0.61	**1.45**	0.47	1.96	1.23
2009Q4	41.86	0.02	24.93	0.13	14.49	0.57	17.11	0.30	17.11	0.78	16.94	1.20	13.77	1.75	16.94	1.36	13.77	1.98	**10.84**	22.59
2010Y	26.01	0.25	10.37	5.16	8.58	10.99	8.91	33.62	8.91	61.63	8.87	101.08	**8.14**	86.91	8.87	141.66	**8.14**	92.30	(**8.14**)	86.91
2010H1	32.42	0.08	14.64	1.02	13.42	1.95	13.38	8.97	13.38	6.23	13.32	10.09	**12.77**	17.47	13.32	18.48	**12.77**	15.73	12.78	285.80
2010H2	14.83	0.09	3.94	0.81	1.84	2.09	2.45	3.27	2.45	5.34	2.45	10.60	**1.80**	16.55	2.45	15.61	**1.80**	14.89	2.26	480.52
2010Q1	19.74	0.03	4.35	0.20	3.90	0.75	4.31	0.55	4.31	0.66	4.14	2.03	**3.88**	1.81	4.14	2.50	**3.88**	2.36	3.94	13.33
2010Q2	35.38	0.03	21.41	0.11	18.59	0.61	18.58	0.27	18.58	0.61	18.55	1.28	**17.90**	1.39	18.55	1.38	**17.90**	0.98	18.26	5.66
2010Q3	20.33	0.02	3.84	0.09	1.54	4.05	2.56	0.13	2.56	0.39	2.56	0.61	**1.49**	0.94	2.56	1.11	**1.49**	0.98	2.23	4.20
2010Q4	28.62	0.05	11.43	0.12	9.95	0.36	9.43	0.58	9.43	0.94	9.14	1.00	**8.83**	1.37	9.14	1.48	**8.83**	1.56	8.75	47.97
2011Y	19.30	0.14	2.45	2.83	1.50	7.64	1.43	7.95	1.43	16.39	1.41	57.67	**1.25**	322.28	1.41	60.89	**1.25**	88.55	1.28	1161.44
2011H1	9.07	0.06	2.30	0.48	1.40	0.91	1.24	0.83	1.24	1.91	1.23	2.36	1.13	1.95	1.23	2.77	1.13	5.39	**1.10**	14.53
2011H2	22.21	0.06	2.17	0.77	1.43	1.56	1.37	1.53	1.37	2.38	1.34	7.75	1.26	10.12	1.34	12.62	1.26	21.31	**1.24**	172.06
2011Q1	10.51	0.02	1.78	0.04	1.03	0.15	0.83	0.11	0.83	0.34	0.82	0.52	**0.72**	0.53	0.82	0.47	**0.72**	0.77	**0.72**	1.16
2011Q2	4.04	0.02	3.19	0.05	2.29	0.16	2.53	0.12	2.53	0.40	2.53	0.34	**2.10**	0.45	2.53	0.63	**2.10**	0.98	2.24	3.75
2011Q3	10.04	0.00	0.39	0.02	0.07	0.03	0.15	0.03	0.15	0.06	0.15	0.14	**0.07**	0.08	0.15	0.11	**0.07**	0.14	0.15	0.25
2011Q4	16.09	0.02	1.01	0.06	0.57	0.10	0.99	0.09	0.99	0.30	0.90	0.53	**0.56**	0.73	0.90	0.63	**0.56**	0.97	0.83	3.08
2012Y	8.20	0.11	2.79	1.78	1.72	5.61	2.37	7.56	2.37	16.38	2.19	21.64	**1.57**	30.39	2.19	29.50	**1.57**	41.34	(**1.57**)	30.39
2012H1	4.99	0.08	3.35	1.27	2.20	2.50	3.00	2.34	3.00	5.17	2.77	10.77	**2.02**	16.66	2.77	13.19	**2.02**	28.83	(**2.02**)	16.66
2012H2	10.21	0.02	0.97	0.11	0.42	0.47	0.74	0.19	0.74	0.33	0.66	1.02	**0.42**	0.72	0.66	0.91	**0.42**	1.33	0.64	4.75
2012Q1	13.93	0.01	9.31	0.03	4.43	0.08	8.66	0.09	8.66	0.22	8.25	0.41	**4.16**	0.33	8.25	0.31	**4.16**	0.75	8.13	2.37
2012Q2	1.68	0.02	0.70	0.03	0.30	0.09	0.49	0.08	0.49	0.23	0.49	0.34	**0.21**	0.28	0.49	0.53	**0.21**	0.41	0.49	0.95
2012Q3	5.92	0.02	1.13	0.05	0.56	0.13	0.99	0.06	0.99	0.20	0.92	0.38	**0.56**	0.33	0.92	0.42	**0.56**	0.86	0.89	1.52
2012Q4	25.81	0.00	4.91	0.02	3.50	0.02	1.91	0.03	1.91	0.03	1.91	0.05	**1.91**	0.06	1.91	0.08	**1.91**	0.09	1.91	0.22
Ave.	18.65	0.05	5.93	0.66	4.14	1.76	4.59	2.92	4.59	5.43	4.49	9.89	**3.82**	21.65	4.49	13.10	**3.82**	13.73	4.02	169.11

Computational results

Duality gaps via LP for Mining instances

	Paper [3]		rank-1-pq ∩ tp		pq ∩ rank-1-tp		rank-1-pq ∩ rank-1-tp		pq × tp	
Inst	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time
2009H2	37.00	0.05	**3.59**	5.89	4.27	7.16	**3.59**	6.64	4.02	1697.27
2009Q3	29.36	0.03	**1.45**	0.69	2.01	0.61	**1.45**	0.47	1.96	1.23
2009Q4	41.86	0.02	13.77	1.75	16.94	1.36	13.77	1.98	**10.84**	22.59
2010Y	26.01	0.25	**8.14**	86.91	8.87	141.66	**8.14**	92.30	(**8.14**)	86.91
2010H1	32.42	0.08	**12.77**	17.47	13.32	18.48	**12.77**	15.73	12.78	285.80
2010H2	14.83	0.09	**1.80**	16.55	2.45	15.61	**1.80**	14.89	2.26	480.52
2010Q1	19.74	0.03	**3.88**	1.81	4.14	2.50	**3.88**	2.36	3.94	13.33
2010Q2	35.38	0.03	**17.90**	1.39	18.55	1.38	**17.90**	0.98	18.26	5.66
2010Q3	20.33	0.02	**1.49**	0.94	2.56	1.11	**1.49**	0.98	2.23	4.20
2010Q4	28.62	0.05	**8.83**	1.37	9.14	1.48	**8.83**	1.56	8.75	47.97
2011Y	19.30	0.14	**1.25**	322.28	1.41	60.89	**1.25**	88.55	1.28	1161.44
2011H1	9.07	0.06	1.13	1.95	1.23	2.77	1.13	5.39	**1.10**	14.53
2011H2	22.21	0.06	1.26	10.12	1.34	12.62	1.26	21.31	**1.24**	172.06
2011Q1	10.51	0.02	**0.72**	0.53	0.82	0.47	**0.72**	0.77	**0.72**	1.16
2011Q2	4.04	0.02	**2.10**	0.45	2.53	0.63	**2.10**	0.98	2.24	3.75
2011Q3	10.04	0.00	**0.07**	0.08	0.15	0.11	**0.07**	0.14	0.15	0.25
2011Q4	16.09	0.02	**0.56**	0.73	0.90	0.63	**0.56**	0.97	0.83	3.08
2012Y	8.20	0.11	**1.57**	30.39	2.19	29.50	**1.57**	41.34	(**1.57**)	30.39
2012H1	4.99	0.08	**2.02**	16.66	2.77	13.19	**2.02**	28.83	(**2.02**)	16.66
2012H2	10.21	0.02	**0.42**	0.72	0.66	0.91	**0.42**	1.33	0.64	4.75
2012Q1	13.93	0.01	**4.16**	0.33	8.25	0.31	**4.16**	0.75	8.13	2.37
2012Q2	1.68	0.02	**0.21**	0.28	0.49	0.53	**0.21**	0.41	0.49	0.95
2012Q3	5.92	0.02	**0.56**	0.33	0.92	0.42	**0.56**	0.86	0.89	1.52
2012Q4	25.81	0.00	**1.91**	0.06	1.91	0.08	**1.91**	0.09	1.91	0.22
Ave.	18.65	0.05	**3.82**	21.65	4.49	13.10	**3.82**	13.73	4.02	169.11

Computational results

Duality gaps via LP for Literature instances

	pq		rank-1-pq		tp		rank-1-tp		pq ∩ tp		rank-1-pq ∩ tp		pq ∩ rank-1-tp		rank-1-pq ∩ rank-1-tp		pq × tp	
Inst	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time
L1	1.01	0.00	**0.86**	0.00	1.01	0.00	1.01	0.00	1.01	0.02	**0.86**	0.00	1.01	0.00	**0.86**	0.00	1.01	0.00
L2	55.24	0.00	55.24	0.02	55.74	0.00	**52.83**	0.00	55.24	0.00	55.24	0.02	**52.83**	0.02	**52.83**	0.00	55.24	0.02
L3	**4.55**	0.00	**4.55**	0.00	**4.55**	0.00	**4.55**	0.00	**4.55**	0.00	**4.55**	0.02	**4.55**	0.00	**4.55**	0.00	**4.55**	0.03
L4	**2.45**	0.02	**2.45**	0.06	**2.45**	0.00	**2.45**	0.00	**2.45**	0.02	**2.45**	0.05	**2.45**	0.03	**2.45**	0.09	**2.45**	0.16
L5	10.80	0.02	10.80	0.00	11.26	0.00	**9.60**	0.02	10.80	0.02	10.80	0.02	**9.60**	0.02	**9.60**	0.03	10.80	0.03
L6	22.22	0.00	22.06	0.00	**20.37**	0.00	**20.37**	0.00	**20.37**	0.00	**20.37**	0.00	**20.37**	0.00	**20.37**	0.00	**20.37**	0.00
L12	**25.00**	0.02	**25.00**	0.00	**25.00**	0.00	**25.00**	0.00	**25.00**	0.00	**25.00**	0.00	**25.00**	0.00	**25.00**	0.00	**25.00**	0.00
L13	**66.67**	0.00	**66.67**	0.00	**66.67**	0.00	**66.67**	0.00	**66.67**	0.00	**66.67**	0.00	**66.67**	0.00	**66.67**	0.00	**66.67**	0.00
L14	16.67	0.00	16.67	0.00	16.67	0.00	**6.67**	0.00	16.67	0.00	16.67	0.00	**6.67**	0.00	**6.67**	0.00	16.67	0.00
L15	37.41	0.00	**25.88**	0.00	**25.88**	0.00	**25.88**	0.00	**25.88**	0.02	**25.88**	0.00	**25.88**	0.00	**25.88**	0.02	**25.88**	0.00
C2	**1.23**	0.03	**1.23**	0.02	**1.23**	0.03	**1.23**	0.03	**1.23**	0.08	**1.23**	0.08	**1.23**	0.09	**1.23**	0.06	**1.23**	0.33
D1	**1.06**	0.09	**1.06**	0.25	**1.06**	0.12	**1.06**	0.22	**1.06**	0.55	**1.06**	0.94	**1.06**	0.94	**1.06**	1.17	**1.06**	3.92
Ave.	20.36	0.01	19.37	0.03	19.32	0.01	18.11	0.02	19.24	0.06	19.23	0.09	18.11	0.09	**18.10**	0.11	19.24	0.37

Computational results

Duality gaps via LP for Random instances

	pq		rank-1-pq		tp		rank-1-tp		pq ∩ tp		rank-1-pq ∩ tp		pq ∩ rank-1-tp		rank-1-pq ∩ rank-1-tp		pq × tp	
Inst	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time
F1	**3.79**	0.02	**3.79**	0.02	5.02	0.03	4.09	0.06	**3.79**	0.05	**3.79**	0.12	**3.79**	0.09	**3.79**	0.09	**3.79**	0.22
F2	4.81	0.39	4.60	1.02	5.15	0.41	4.50	0.89	4.30	1.27	4.29	3.43	**4.28**	3.06	**4.28**	5.98	4.29	13.70
F3	13.52	0.27	**13.19**	1.27	14.57	0.70	14.57	1.41	**13.19**	1.55	**13.19**	2.82	**13.19**	2.00	**13.19**	6.44	13.19	22.39
F4	**0.00**	0.14	**0.00**	2.48	**0.00**	1.03	**0.00**	1.72	**0.00**	1.31	**0.00**	4.80	**0.00**	3.37	**0.00**	6.25	**0.00**	6.61
F5	14.58	0.42	11.02	1.81	13.06	0.34	11.96	1.06	11.71	1.19	**10.97**	3.59	11.66	3.03	**10.97**	6.19	11.65	22.69
F6	10.69	0.31	10.03	1.73	12.25	0.27	10.45	1.17	10.15	3.19	9.92	8.11	9.97	6.56	**9.91**	12.42	9.94	60.92
F7	33.72	1.05	33.09	3.25	34.91	1.20	34.04	5.08	33.09	5.31	33.09	12.50	**33.05**	12.45	**33.05**	22.45	33.09	86.13
F8	17.15	0.58	**16.90**	1.77	17.46	0.53	17.06	1.81	**16.90**	4.22	**16.90**	8.47	**16.90**	7.89	**16.90**	14.05	**16.90**	22.58
F9	18.13	0.50	17.17	1.03	20.75	0.38	17.78	1.15	17.24	3.48	**17.15**	8.91	17.16	5.77	**17.15**	16.30	17.16	19.64
F10	7.84	0.31	**7.81**	1.30	8.08	0.61	7.86	1.41	**7.81**	2.66	**7.81**	7.27	**7.81**	4.25	**7.81**	9.17	**7.81**	12.98
F11	5.43	0.02	5.32	0.05	5.71	0.03	5.20	0.08	5.28	0.06	5.28	0.16	**5.19**	0.16	**5.19**	0.23	5.28	0.39
F12	4.57	0.14	4.54	0.30	5.14	0.19	4.78	0.34	4.48	1.06	4.48	1.69	**4.47**	1.64	**4.47**	2.31	4.48	8.73
F13	7.20	0.67	7.15	2.11	8.12	0.69	7.92	1.81	7.17	1.86	7.14	4.45	7.16	3.84	**7.13**	7.48	7.16	32.11
F14	**0.00**	0.08	**0.00**	0.48	**0.00**	0.30	**0.00**	0.80	**0.00**	0.95	**0.00**	2.00	**0.00**	2.02	**0.00**	1.75	**0.00**	3.83
F15	17.95	0.23	17.88	0.78	18.82	0.39	18.33	1.61	16.36	1.84	**16.33**	2.77	16.36	3.03	**16.33**	6.06	16.36	14.53
F16	4.64	0.06	4.29	0.09	4.41	0.05	4.31	0.11	4.28	0.28	**4.26**	0.42	4.28	0.38	**4.26**	0.64	4.27	2.31
F17	10.24	0.02	10.24	0.05	9.63	0.06	**9.62**	0.11	**9.62**	0.12	**9.62**	0.16	**9.62**	0.17	**9.62**	0.27	**9.62**	0.64
F18	**16.63**	0.09	**16.63**	0.34	**16.63**	0.19	**16.63**	0.56	**16.63**	0.80	**16.63**	0.73	**16.63**	1.08	**16.63**	2.53	**16.63**	1.87
F19	**1.06**	0.02	**1.06**	0.02	2.73	0.03	**1.06**	0.03	**1.06**	0.03	**1.06**	0.03	**1.06**	0.05	**1.06**	0.09	**1.06**	0.08
F20	12.74	0.11	12.11	0.16	13.38	0.14	12.28	0.31	11.52	0.91	11.48	0.88	11.50	1.00	**11.46**	1.34	11.52	2.63
F21	26.02	12.38	25.57	55.88	26.49	8.72	25.71	50.89	25.42	70.72	25.40	165.03	25.37	133.67	**25.36**	310.98	(**25.36**)	310.98
F22	22.64	0.23	22.17	0.55	23.43	0.34	22.61	0.66	21.91	1.42	**21.91**	3.19	**21.91**	2.86	**21.91**	4.34	**21.91**	10.83
F23	43.92	10.64	43.58	49.09	47.04	12.15	45.05	49.55	43.39	53.61	43.35	493.14	43.16	224.12	**43.13**	207.30	(**43.13**)	207.30
F24	11.19	0.22	11.19	0.50	11.19	1.02	**11.18**	2.23	11.19	2.23	11.19	5.81	**11.18**	7.45	**11.18**	11.38	11.19	5.44
Ave.	12.85	1.20	12.47	5.25	13.50	1.24	12.79	5.20	12.35	6.67	12.30	30.85	12.32	17.91	**12.28**	27.34	12.32	36.23

Computational results

Duality gaps via discretization for Mining instances

Inst	$\text{pq}(q^s)$ Gap	$\text{pq}(q^s)$ Time	$\text{rank-1-pq}(q^s)$ Gap	$\text{rank-1-pq}(q^s)$ Time	$\text{rank-1-pq}(q')$ Gap	$\text{rank-1-pq}(q')$ Time	$\text{tp}(q^t)$ Gap	$\text{tp}(q^t)$ Time	$\text{rank-1-tp}(q^t)$ Gap	$\text{rank-1-tp}(q^t)$ Time	$\text{rank-1-tp}(q')$ Gap	$\text{rank-1-tp}(q')$ Time
2009H2	3.64	1800.17	2.69	1800.18	**1.61**	1800.09	2.08	1800.37	2.08	1800.27	2.65	1800.14
2009Q3	0.80	23.67	0.54	16.45	**0.20**	10.64	0.21	12.56	0.22	18.12	0.53	8.56
2009Q4	8.82	1800.05	8.41	1800.05	**3.45**	1558.98	4.93	1800.06	4.74	1800.06	4.69	1800.02
2010Y	9.13	1800.27	8.27	1801.63	**8.09**	1800.63	8.06	1800.33	8.24	1800.25	8.44	1800.14
2010H1	13.31	1800.07	11.71	1800.15	**6.92**	1800.23	9.05	1800.09	8.55	1800.23	8.41	1800.09
2010H2	1.22	1800.11	**0.84**	1800.11	1.03	1800.33	1.00	1800.19	0.93	1800.11	2.19	1800.05
2010Q1	2.89	232.87	2.75	379.08	**1.64**	134.11	2.25	665.41	2.25	1279.26	2.32	199.84
2010Q2	12.37	1028.67	11.64	1594.03	**2.95**	232.52	4.89	1800.03	4.85	1800.05	4.23	422.70
2010Q3	0.94	39.75	0.70	40.09	**0.38**	14.31	0.70	33.23	0.70	37.08	0.80	48.44
2010Q4	6.38	132.55	5.28	160.81	**1.97**	772.50	3.84	231.61	3.84	239.50	4.95	701.95
2011Y	1.56	1800.09	1.28	1800.19	1.20	1800.11	1.17	1800.16	**1.13**	1800.20	1.28	1800.84
2011H1	0.64	1800.09	0.54	1800.08	**0.18**	871.16	0.41	1800.05	0.58	1800.14	0.31	287.62
2011H2	1.37	1800.06	1.26	1800.09	**0.37**	1800.09	1.07	1800.09	1.05	1800.12	0.66	1800.05
2011Q1	0.31	18.62	0.26	36.94	**0.12**	57.71	0.22	20.05	0.22	24.11	0.22	29.84
2011Q2	1.08	90.28	1.01	38.64	**0.49**	36.70	0.89	62.84	0.89	111.09	0.69	58.31
2011Q3	0.05	0.66	0.03	0.77	**0.01**	1.13	0.04	1.33	0.04	1.81	0.06	0.83
2011Q4	0.18	26.39	0.15	31.48	**0.10**	8.00	0.24	45.20	0.24	39.56	0.17	49.69
2012Y	1.85	1800.06	1.43	1800.09	**1.20**	1802.70	1.37	1800.08	1.37	1800.06	1.95	1800.13
2012H1	2.22	1800.03	1.72	1800.08	1.34	1800.04	**1.14**	1800.06	1.39	1800.08	2.48	1800.11
2012H2	0.19	39.77	0.15	52.55	**0.07**	52.66	0.18	55.59	0.18	60.58	0.19	72.69
2012Q1	3.35	22.00	2.15	10.64	**0.91**	11.08	1.32	9.00	1.32	23.47	1.78	23.52
2012Q2	0.16	12.02	0.12	10.39	0.12	3.97	0.16	18.55	0.16	19.77	**0.11**	22.23
2012Q3	0.29	19.80	0.23	28.28	**0.07**	28.56	0.18	20.72	0.18	48.72	0.21	33.38
2012Q4	1.77	4.73	1.65	4.16	0.70	1.03	0.44	2.31	0.44	2.22	**0.33**	8.56
Ave.	3.10	820.53	2.70	850.29	**1.46**	758.30	1.91	874.16	1.90	904.45	2.07	757.07

Computational results

Duality gaps via discretization for Literature instances

	$pq(q^s)$		rank-1-$pq(q^s)$		rank-1-$pq(q')$		$tp(q^t)$		rank-1-$tp(q^t)$		rank-1-$tp(q')$	
Inst	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time
L1	0.33	0.03	0.33	0.05	**0.26**	0.05	0.36	0.05	0.36	0.05	0.59	0.03
L2	2.96	1.12	2.96	1.70	15.66	1.38	2.70	0.20	2.70	0.09	**1.55**	0.28
L3	2.96	0.50	2.96	0.86	3.58	0.37	1.97	0.08	1.97	0.09	**1.55**	0.17
L4	1.96	16.64	1.96	19.78	2.45	1.36	0.47	0.49	0.47	0.28	**0.26**	2.14
L5	2.88	1.17	2.88	1.58	4.19	12.78	0.96	0.81	0.96	0.68	**0.05**	0.14
L6	**0.00**	0.14	**0.00**	0.17	**0.00**	0.11	**0.00**	0.04	**0.00**	0.02	**0.00**	0.06
L12	**0.00**	0.09	**0.00**	0.11	**0.00**	0.12	**0.00**	0.02	**0.00**	0.02	**0.00**	0.03
L13	**0.00**	0.08	**0.00**	0.13	**0.00**	0.20	**0.00**	0.02	**0.00**	0.03	**0.00**	0.03
L14	**0.00**	0.10	**0.00**	0.11	4.76	0.17	1.91	0.02	1.91	0.04	**0.00**	0.03
L15	0.77	0.14	**0.68**	0.12	1.20	0.08	1.20	0.03	1.20	0.06	**0.68**	0.09
C2	0.14	0.64	0.14	0.72	**0.10**	0.59	**0.10**	0.55	**0.10**	0.69	0.14	0.47
D1	1.06	1800.03	1.06	1800.00	1.04	1800.02	**1.02**	1800.02	1.03	1800.03	1.04	1800.02
Ave.	1.09	151.72	1.08	152.11	2.77	151.44	0.89	150.19	0.89	150.17	**0.49**	150.29

Computational results

Duality gaps via discretization for Random instances

	pq(q^s)		rank-1-pq(q^s)		rank-1-pq(q')		tp(q^t)		rank-1-tp(q^t)		rank-1-tp(q')	
Inst	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time
F1	**2.04**	7.06	**2.04**	9.52	2.30	4.14	2.30	6.31	2.30	4.13	2.26	10.23
F2	3.01	1800.03	2.99	1800.02	**2.39**	1800.08	2.63	1800.08	2.59	1800.08	2.63	1800.03
F3	3.42	1800.05	4.36	1800.05	**2.01**	526.69	7.50	1800.05	8.88	1800.03	5.42	1800.09
F4	**0.00**	2.44	**0.00**	3.19	**0.00**	3.91	**0.00**	3.20	**0.00**	3.78	**0.00**	4.02
F5	7.26	1800.06	7.90	1800.03	6.52	1800.03	6.78	1800.06	7.55	1800.05	**6.27**	1800.06
F6	8.25	1800.02	8.38	1800.03	**8.12**	1800.05	9.11	1800.05	8.29	1800.05	8.19	1800.06
F7	24.44	1800.05	25.67	1800.06	25.96	1800.05	24.79	1800.06	26.83	1800.04	**22.87**	1800.08
F8	10.40	1800.06	10.81	1800.05	9.53	1800.05	**7.66**	1800.08	8.91	1800.06	10.55	1800.03
F9	8.09	1800.06	8.29	1800.06	6.78	1800.05	8.25	1800.06	7.88	1800.06	**6.37**	1800.12
F10	3.08	1800.14	3.65	1800.14	4.23	726.17	4.69	764.77	4.69	653.10	**2.47**	1110.62
F11	1.25	23.02	1.25	26.14	1.30	63.05	1.47	57.05	1.47	71.55	**0.84**	9.50
F12	1.75	1800.05	1.62	1800.07	**1.11**	431.17	1.57	1800.14	1.61	1800.12	1.49	1800.16
F13	4.54	1800.05	4.22	1800.05	**1.67**	1751.19	3.81	1800.10	3.79	1800.06	4.40	1800.05
F14	**0.00**	1.09	**0.00**	0.69	**0.00**	0.88	**0.00**	1.33	**0.00**	2.36	**0.00**	3.08
F15	13.15	1800.03	12.73	1800.03	**12.22**	1800.03	12.97	1800.03	13.52	1800.05	12.43	1800.05
F16	1.37	240.83	1.35	347.73	1.16	161.31	1.15	573.34	**1.09**	1460.55	1.48	39.92
F17	1.94	10.59	1.94	19.34	**1.82**	6.59	1.83	6.95	1.83	10.22	2.50	10.50
F18	1.79	61.87	1.79	42.75	**1.16**	36.03	1.17	21.73	1.17	27.42	2.64	16.83
F19	**0.00**	0.75	**0.00**	0.56	**0.00**	0.38	**0.00**	0.39	**0.00**	0.39	**0.00**	0.44
F20	3.47	1800.11	3.50	1800.05	2.49	535.50	2.29	631.09	2.29	1337.64	**2.23**	308.12
F21	26.01	1800.05	**25.46**	1800.08	25.55	1800.06	26.20	1800.05	25.69	1800.09	25.46	1800.11
F22	10.40	1800.03	11.68	1800.03	11.11	1800.05	12.60	1800.04	12.95	1800.08	7.26	1800.13
F23	43.65	1800.06	**43.58**	1800.11	**43.58**	1800.09	46.52	1800.05	45.03	1800.09	44.69	1800.09
F24	5.96	46.45	5.96	39.28	**5.88**	12.58	5.99	12.37	5.99	28.78	5.92	20.03
Ave.	7.72	1141.46	7.88	1145.42	**7.37**	927.50	7.97	1061.64	8.10	1125.03	7.43	1038.93

Proof of Thm 1 (sketch)
Proof of Case 5

For simplicity, $n_{q+} = 2$, $n_{q-} = 1$, $n_l = 0$: $w_1^2 + w_2^2 - x_1^2 = d$.

Let (a_1, a_2, b_1) be a point in the surface, i.e.,

$$a_1^2 + a_2^2 - b_1^2 = d.$$

We show that there exist u_1, u_2, v_1 such that the line

$$\{(a_1, a_2, b_1) + \lambda(u_1, u_2, v_1) \mid \lambda \in \mathbb{R}\}$$

is entirely contained in the surface:

$$(a_1 + \lambda u_1)^2 + (a_2 + \lambda u_2)^2 - (b_1 + \lambda v_1)^2 = d \;\forall \lambda \in \mathbb{R}.$$

Proof of Thm 1 (sketch)

Proof of Case 5

Then, $\forall \lambda \in \mathbb{R}$,

$$d = (a_1 + \lambda u_1)^2 + (a_2 + \lambda u_2)^2 - (b_1 + \lambda v_1)^2$$
$$d = a_1^2 + a_2^2 - b_1^2 + \lambda^2(u_1^2 + u_2^2 - v_1^2) + 2\lambda(a_1 u_1 + a_2 u_2 - b_1 v_1)$$
$$0 = \lambda^2(u_1^2 + u_2^2 - v_1^2) + 2\lambda(a_1 u_1 + a_2 u_2 - b_1 v_1)$$
$$0 = \lambda(u_1^2 + u_2^2 - v_1^2) + 2(a_1 u_1 + a_2 u_2 - b_1 v_1).$$

Therefore,

$$\begin{cases} u_1^2 + u_2^2 - v_1^2 = 0 \\ a_1 u_1 + a_2 u_2 - b_1 v_1 = 0. \end{cases}$$

Setting $v_1 = 1$:

$$\begin{cases} u_1^2 + u_2^2 = 1 \\ a_1 u_1 + a_2 u_2 = b_1. \end{cases}$$

$$a_1^2 + a_2^2 - b_1^2 = d \;\Rightarrow\; a_1^2 + a_2^2 = d + b_1^2 \geq b_1^2 \;\Rightarrow\; \frac{|b_1|}{||a||_2} \leq 1.$$