A Lower Bound on the Split Rank of Intersection Cuts

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Outline

Introduction: split rank, intersection cuts

A lower bound on split rank of intersection cuts

Outline of proof

Application: mixing inequality

Discussion

Split cut

- Let $\mathcal{M} := \{(x, y) \in \mathbb{Z}^m \times \mathbb{R}^n \mid Gx + Hy \leq b\}$ where $G \in \mathbb{Q}^{p \times m}$, $H \in \mathbb{Q}^{p \times n}$, and $b \in \mathbb{Q}^{p \times 1}$.
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- Given a vector $a \in \mathbb{Z}^m$ and $c \in \mathbb{Z}$, any vector $x \in \mathbb{Z}^m$ satisfies the split disjunction defined as $(a^T x \leq c) \vee (a^T x \geq c + 1)$.

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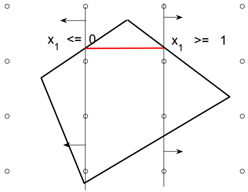
- Let $L^0_{a,c} := M^0 \cap \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \mid a^T x \leq c\}$ and $R^0_{a,c} := M^0 \cap \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \mid a^T x \geq c + 1\}$

$$M^0_{a,c} := \mathrm{conv}(L^0_{a,c} \cup R^0_{a,c})$$



Split rank: A tool for comparing cuts with split cuts

Definition (Split closure)

The first split closure M^1 is defined as $\cap_{a \in \mathbb{Z}^p, c \in \mathbb{Z}} M^0_{a,c}$. □

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Definition (Split rank)

The split rank of a valid inequality $\alpha^T x + \beta^T y \leq \gamma$ for $\mathcal{M}$ is defined as the smallest integer k such that $\alpha^T x + \beta^T y \leq \gamma$ is valid for M^k. □

Some previous results

1. (Upper bound) Split rank of any inequality for a 0-1 MIP is at most n, when there are n binary variables: Balas (1979), Nemhauser and Wolsey (1990), Balas et. al. (1993).
2. (Lower Bound) Split rank of certain class of cutting planes is not finite: Cook et. al. (1988), Li and Richard (2008).
3. (Lower Bound) The upper bound on the split rank for 0-1 MIP is achieved: Cornuéjols and Li (2002).
4. (Upper Bound) Split rank of m row mixing inequalities is at most m: Dash and Günlük (2008).
5. (Upper Bound) Upper Bound on split rank for triangle and quadrilateral inequalities: D. and Louveaux (2009).

Basic MIP model for analysis: a relaxation of simplex tableau

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\begin{aligned}
x &= f + \sum_{i=1}^{n} r^i y_i \\
x &\in \mathbb{Z}^m \\
y_i &\geq 0 \quad \forall i \in \{1, ..., n\}
\end{aligned}
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where

1. $r^i \in \mathbb{Q}^m$
2. $f \in \mathbb{Q}^m \setminus \mathbb{Z}^m$.

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Another Example: Mixing Set

$$\begin{aligned} x_i + y_0 \geq f_i \quad \forall i \in \{1, ..., m\} \\ x \in \mathbb{Z}^m, \quad y_0 \geq 0 \end{aligned}$$

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Introducing non-negative slack variable:

$$\begin{aligned} x_i = f_i - y_0 + y_i \quad \forall i \in \{1, ..., m\} \\ x \in \mathbb{Z}^m, \quad y_0 \geq 0, y_i \geq 0 \;\; \forall i \in \{1, ..., m\} \end{aligned}$$

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x &= f + \sum_{i=1}^{n} r^i y_i \\
x &\in \mathbb{Z}^m, Ax \leq b \\
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Intersection cut: lattice-free convex sets to valid cuts

Let $P \subseteq \mathbb{R}^m$ be a convex set and

- Lattice-free, i.e., $\mathrm{int}(P) \cap \mathbb{Z}^m = \emptyset$ and
- $f \in \mathrm{int}(P)$.

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Intersection Cut: [Balas (1971)]

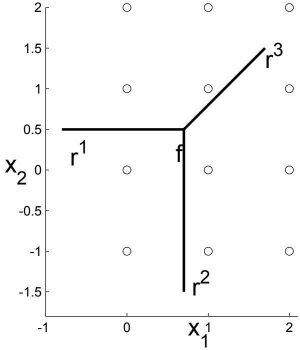
$$\sum_{i=1}^{n} \pi^P(r^i) y_i \geq 1, \tag{1}$$

where

$$\pi^P(r^i) = \begin{cases} \lambda & \text{if } \exists \lambda \geq 0, \text{ s.t. } f + \frac{r^i}{\lambda} \in \text{boundary}(P) \\ 0 & \text{if } r^i \text{ is a ray for } P \end{cases} . \tag{2}$$

Example of intersection cut

Projection on x-space



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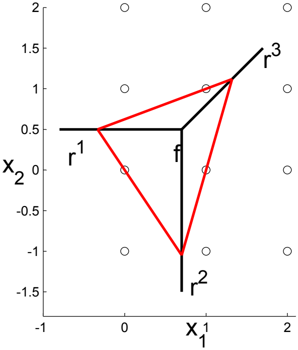

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- $(1.32, 1.12)$
- $(-0.3333, 0.5)$
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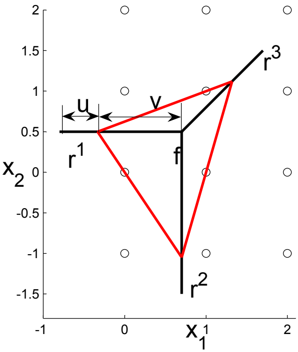

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$f + \frac{r^1}{(u+v)/v} \in \mathrm{Bnd}(P) \Rightarrow$ Coefficient of y_1 in cut is $\frac{u+v}{v}$.

$$1.4516y_1 + 1.2903y_2 + 2.38y_3 \geq 1$$

Problem statement

To obtain an insight into the (lower bounds on) split rank of intersection cuts.

Main result: split rank depends on orientation of integer points satisfied at equality

$\backsim$ "As the number of integer points lying on distinct facets of P increases, the split rank increases."

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More precisely:

- Step 1: Under a assumption on the columns of our basic relaxation of simplex tableau, construct a polyhedral subset of P (P is the lattice-free convex set used to generate the intersection cut) : *restricted lattice-free set*.

Main result: split rank depends on orientation of integer points satisfied at equality

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More precisely:

- Step 1: Under a assumption on the columns of our basic relaxation of simplex tableau, construct a polyhedral subset of *P* (*P* is the lattice-free convex set used to generate the intersection cut) : *restricted lattice-free set*.
- Step 2: Then the following result is proven: $\lceil \log_2(t) \rceil$ is a lower bound on the split rank of intersection cut where $\{x^1, x^2, ..., x^t\}$ is a subset of integer points on the boundary of the restricted lattice-free set such that no two points lie on the same facet of the restricted lattice-free set.

Assumption on columns

Some notation:

- The lattice-free set $P \subseteq \mathbb{R}^m$ that is used to generate the intersection cut may not be a bounded set.
- If P is unbounded, some of the columns r^i may be rays of P.

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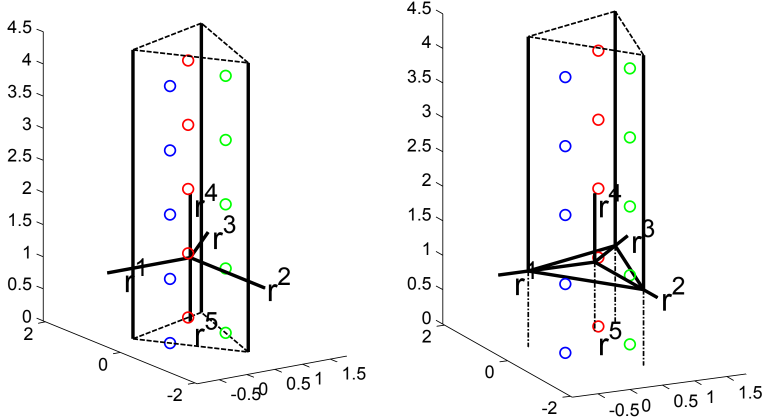
Assumption: There exist subsets $S_v \subseteq \{1, ..., n_1\}$ and $S_r \subseteq \{n_1 + 1, ..., n\}$ of the columns such that

1. $\tilde{Q} := \mathrm{conv}\left(\cup_{i \in S_v} \left\{ \underbrace{f + \frac{r^i}{\pi^P(r^i)}}_{\in \mathrm{Bnd}(P)} \right\}\right) + \mathrm{cone}\left(\cup_{i \in S_r} \{ \underbrace{r^i}_{\text{ray of } P} \}\right)$ has a dimensions of $|S_v| + |S_r| - 1$
2. $f \in \mathrm{affine.hull}(\tilde{Q})$. □

$\tilde{Q}$: Restricted lattice-free set.

Construction of $\tilde{Q}$



The lower bound result

Theorem

Let $\{x^1, x^2, ..., x^t\}$ *be a subset of integer points on the boundary of the restricted lattice-free set* $\tilde{Q}$ *such that no two points lie on the same facet of* $\tilde{Q}$*. Then a lower bound on the split rank of the intersection cut is* $\lceil log_2(t) \rceil$*.*

Outline of proof.

How to prove a lower bound on split rank?

$$\text{Set: } x = f + \sum_{i=1}^{n} r^i y_i \quad x \in \mathbb{Z}^m, y \in \mathbb{R}^n_+$$

$$\text{Intersection Cut: } \sum_{i=1}^{n} \pi^P(r^i) y_i \geq 1$$

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2. We will show that points of the form $(x^I, y^I - \delta\lambda^{S_v,S_r}) \in M^{\lceil log_2(|I|) \rceil - 1}$, but does not satisfy intersection cut; where
   - (x^I, y^I) belongs to convex hull of $\mathcal{M}$, where $I \subseteq \{1, ..., t\}$.
   - $\lambda^{S_v,S_r} \in \mathbb{R}^n$ is a fixed vector.

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   - If $\delta > 0$, then $(x^I, y^I - \delta\lambda^{S_v,S_r})$ does not satisfy the intersection cut.

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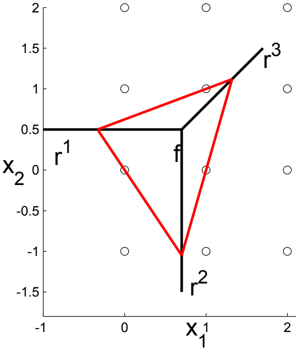
   - If $\delta > 0$, then $(x^I, y^I - \delta\lambda^{S_v,S_r})$ does not satisfy the intersection cut.

We do the following next:

1. Construction of $\lambda^{S_v,S_r} \in \mathbb{R}^n$ and $(x^I, y^I) \in \mathbb{R}^m \times \mathbb{R}^n_+$.
2. Show that $(x^I, y^I - \delta\lambda^{S_v,S_r}) \in M^{\lceil log_2(|I|)\rceil - 1}$.
3. Show that $(x^I, y^I - \delta\lambda^{S_v,S_r})$ is cut off by the intersection cut for $\delta > 0$.

Construction of λ^{S_v, S_r}



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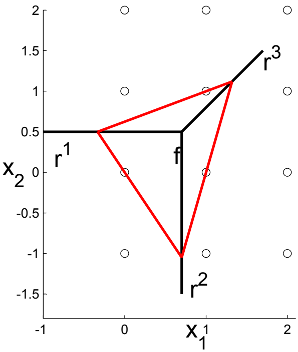

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Consider $\lambda \in \mathbb{R}^3$: $(1, 1, 2)$. Then

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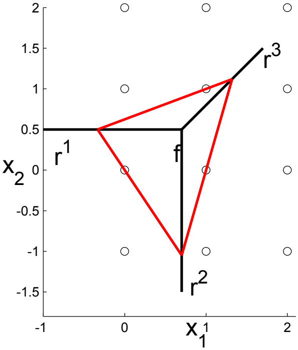

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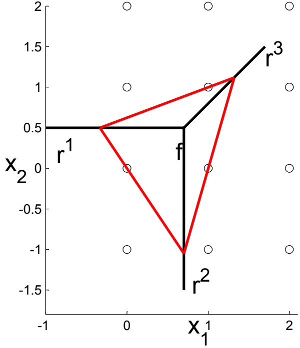

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Proposition

If Assumption 1 holds, then $\exists\, \lambda^{S_v,S_r} \in \mathbb{R}^n$ $(\lambda_i^{S_v,S_r} = 0\; \forall i \notin S_v \cup S_r)$ *such that,*

1. $\sum_{i=1}^n \lambda_i^{S_v,S_r} r^i = \bar{0}$ *($\bar{0}$ is the origin in $\mathbb{R}^m$) and*
2. $\sum_{i=1}^n \pi^P(r^i)\lambda_i^{S_v,S_r} > 0$. □

Construction of (x^l, y^l)

$$x^1 = (1,0) = f + 0r^1 + 0.4r^2 + 0.3r^3$$
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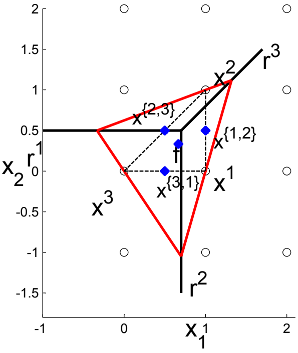

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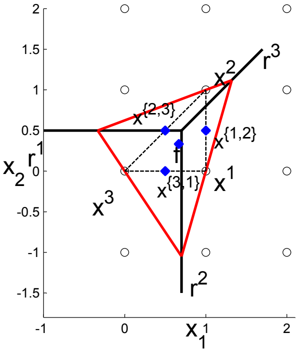

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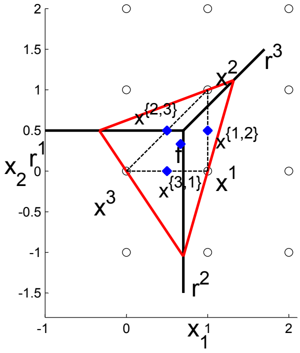

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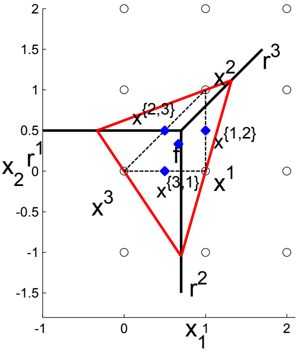
Note (x^1, y^1), (x^2, y^2), (x^3, y^3) belong to convex hull.

Construction of (x^I, y^I)



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Note (x^1, y^1), (x^2, y^2), (x^3, y^3) belong to convex hull.

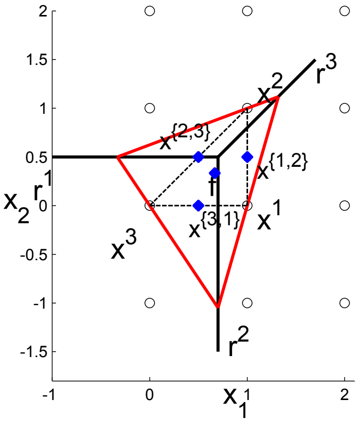
For $I \subseteq \{1, ..., t\}$, let $\quad x^I = \frac{1}{|I|}\sum_{i \in I} x^i.$

Similarly $\quad y^I = \frac{1}{|I|}\sum_{i \in I} y^i$

The points (x^I, y^I) are satisfied at equality by π^P,

$$\sum_{i=1}^{n} \pi^P(r^i) y^I_i = 1$$

A key lemma



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$$y^{\{1,2\}} := (0.05, 0.2, 0.4)$$
$$y^{\{2,3\}} := (0.2, 0.125, 0.25)$$
$$y^{\{3,1\}} := (0.15, 0.3125, 0.15)$$
$$y^{\{1,2,3\}} := (0.1, 0.210, 0.1)$$

Lemma

If $|I| > 1$*, then* $y_i^I > 0 \; \forall i \in (S_v \cup S_r)$.

Points belonging to the LP relaxation

- Remember (x^l, y^l) belongs to the LP relaxation for all l, i.e.,

$$x^l = f + \sum_{i=1}^{n} y_i^l r^i. \tag{3}$$

- There exists a vector λ^{S_v,S_r} such that

$$\sum_{i=1}^{n} \lambda_i^{S_v,S_r} r^i = \bar{0} \tag{4}$$

- If $|l| > 1$, then $y_i^l > 0 \ \forall i(S_v \cup S_r) \Rightarrow \exists \delta > 0 \ \text{s.t.} y^l - \delta\lambda^{S_v,S_r} \geq 0$

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$$\exists \delta > 0 \text{ s.t.} (x^l, y^l - \delta \lambda^{S_v, S_r}) \text{ belongs to the LP relaxation.} \tag{5}$$

Also,

$$\sum_{i=1}^{n} \pi^P (r^i) \lambda_i^{S_v, S_r} > 0 \text{ and } \sum_{i=1}^{n} \pi^P (r^i) y_i^l = 1$$

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$$\Rightarrow \sum_{i=1}^{n} \pi^P(r^i)(y^I - \delta\lambda^{S_v,S_r})_i < 1.$$

If $|I| > 2^0$, then the point $(x^I, y^I - \delta\lambda^{S_v,S_r}) \in M^0$ and is cut off by the intersection cut.

Proof by induction

- Let $\gamma^{I,k} = \max\{\delta \,|\, (x^I, y^I - \delta\lambda^{S_v,S_r}) \in M^k\}$.
- We just proved: If $|I| > 1$, then $\gamma^{I,0} > 0$.

If $I \subseteq \{1, ..., t\}$ and $|I| > 2^z$, then $\gamma^{I,z} > 0$

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If $I \subseteq \{1, ..., t\}$ and $|I| > 2^z$, then $\gamma^{I,z} > 0 \Rightarrow (x^I, y^I - \delta\lambda^{S_v,S_r}) \in M^{\lceil log_2(|I|) \rceil - 1}$.

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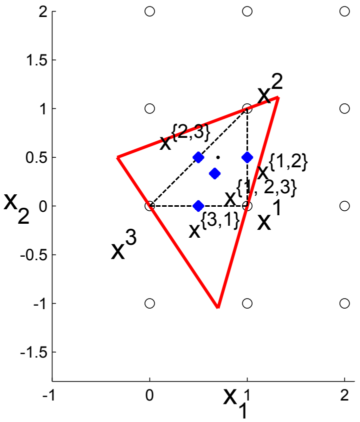

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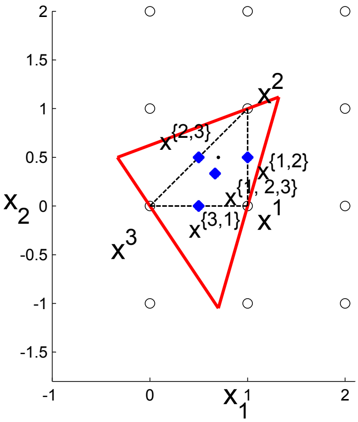

1. Any disjunction $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$ contains at least two of the integer points 'on one side', i.e., $\pi^T x^1 \leq \pi_0$ and $\pi^T x^2 \leq \pi_0$.
2. So the point $(x^{\{1,2\}}, y^{\{1,2\}} - \gamma^{\{1,2\},0}\lambda^{S_v,S_r}) \in M^0 \cap \{(x,y) | \pi^T x \leq \pi_0\}$.

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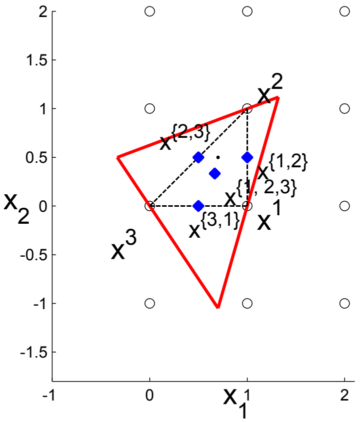

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3. So a point of the form $(x^{\{1,2,3\}}, y^{\{1,2,3\}} - \delta\lambda^{S_v,S_r}) \in \mathrm{conv}(M^0 \cap \{(x,y)|\pi^T x \leq \pi_0\}) \cup (M^0 \cap \{(x,y)|\pi^T x \geq \pi_0 + 1\})$ where $\delta > 0$.

An useful extension

Corollary

Let

$$M^0 := \begin{cases} x = f + \sum_{i=1}^{n} r^i y_i, & x \in \mathbb{R}^m, y \in \mathbb{R}^n_+, \\ Ax \leq b. \end{cases}$$

and $\mathcal{M} = M^0 \cap x \in \mathbb{Z}^m$*. Let* $P \subset \mathbb{R}^m$ *be a lattice-free convex set containing* f *in its interior. Let the inequality*

$$\sum_{i=1}^{n} \pi^P(r^i) y_i \geq 1 \tag{6}$$

be an intersection cut. Let $\{x^1, x^2, ..., x^t\}$ *be a subset of integer points on the boundary of the restricted lattice-free set such that no two points lie on the same facet of* $\tilde{Q}$ *and* $Ax^j \leq b \; \forall 1 \leq j \leq l$*. Then a lower bound on the split rank of intersection cut is* $\lceil \log_2(t) \rceil$*.* □

Application

Mixing inequalities

The mixing set:

$$x_i = f_i - y_0 + y_i \quad i \in \{1, ..., m\}$$
$$x_i \in \mathbb{Z}\, i \in \{1, ..., m\}, \quad y_0 \geq 0, y_i \geq 0\, i \in \{1, ..., m\}$$

The mixing inequality (type 1)

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- The mixing inequality is an intersection cut generated using a maximal lattice-free simplex.
- If $0 < f_1 < f_2 < ... < f_m < 1$, then each facet of the mixing lattice-free set contains at least one integer point in its relative interior of the form: $(0, 0, 0, ..0)$, $(0, 0, 0, ..., 1)$, $(0, 0, 0, ..., 1, 1)$, ..., $(1, 1, 1, ..., 1)$.

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- If $0 < f_1 < f_2 < ... < f_m < 1$, then the split rank of the mixing inequality is at least $\lceil log_2(m + 1) \rceil$.

(Note: Dash and Günlük (2008) prove an upper bound on the split rank of mixing inequalities is m.)

Discrete lot-sizing problem with initial stock

A formulation for single item discrete lot-sizing problem with initial stock s_0, with binary variables v_u representing the decision to produce in the period u, and with constant capacity C is

$$\frac{s_0}{C} + \sum_{u=1}^{t} v_u \geq \frac{d_{1t}}{C} \quad \forall 1 \leq t \leq n \tag{7}$$

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where d_{1t} is the sum of the demands from periods 1 to t.

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where d_{1t} is the sum of the demands from periods 1 to t.
Setting $y_0 = \frac{s_0}{C}$ and $f_t = \frac{d_{1t}}{C}$, we can re-write (7) as

$$y_0 + x_t \geq f_t \quad \forall 1 \leq t \leq n \tag{8}$$

$$0 \leq x_t - x_{t-1} \leq 1 \, \forall 1 \leq t \leq n \tag{9}$$

$$x_t = \sum_{u=1}^{t} v_u \, \forall 1 \leq t \leq n \tag{10}$$

$$y_0 \geq 0, \; x_t \in \mathbb{Z}_+ \, \forall 1 \leq t \leq n, \; v_u \in \{0,1\} \, \forall 1 \leq u \leq n. \tag{11}$$

If $0 < f_1 < f_2 < ... < f_n < 1$, then the split rank of facet-defining inequalities is at least $\lceil log_2(n+1) \rceil$.

Discussion

Upshot:

- This is a non-trivial lower bound on the split rank of intersection cuts.
- The main insight: the effect of the orientation of integer feasible points satisfied at equality by intersection cuts on the split rank of the inequality.
- We also used this result to derive a lower bound on the split rank of mixing inequalities.

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Possibilities of strengthening the result:

- It is however not clear whether the split rank of an inequality is a function of the structure of $\tilde{Q}$ (for the best choice of S_v and S_r) alone.
- This lower bound seems to be weak when the vertices of the lattice-free set $\tilde{Q}$ are all integral and each facet contains an integer point in its relative interior.

Theorem (Li and Richard (2008))

If $\tilde{Q} \subset \mathbb{R}^m$ is the lattice-free simplex defined by the following vertices: $(0, 0, 0, ...0)$, $(m, 0, 0, ...0)$, $(0, m, 0, ...0)$, ..., $(0, 0, 0..., m)$, then a lower bound to the split rank of the inequality is infinite.

A more general result that combines the result of above Theorem?

Thank You.