Lifting Convex Inequalities for Bilinear Programs

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July, 2021

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Introduction

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Lifting

- derives or strengthens classes of cutting planes
- first introduced for mixed integer linear programming (MILP)

Richard, J.P.P. (2010) *Lifting techniques for mixed integer programming*

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Procedure

- *Fixing* some variables and generating a *seed inequality*;
- *Lifting* the seed inequality to make it feasible for the entire set.

Existing works on lifting for nonlinear nonconvex programs

Significantly fewer articles on lifting for nonlinear programs and mixed integer nonlinear programs

- Richard, J.P.P., Tawarmalani, M. (2010)
  Lifting inequalities: a framework for generating strong cuts for nonlinear programs
- Nguyen, T.T., Richard, J.P.P. (2018)
  Deriving convex hulls through lifting and projection
- Gupte, A. (2012)
  Mixed integer bilinear programming with applications to the pooling problem
- Atamtürk, A., Narayanan, V. (2011)
  Lifting for conic mixed-integer programming
- Chung, K., Richard, J.P.P., Tawarmalani, M. (2014)
  Lifted inequalities for 0-1 mixed-integer bilinear covering sets

Lifting: an example

Example

- $$S := \left\{ (x_1, x_2, x_3) \in [0,1]^3 \,|\, x_1 x_2 + 2x_1 x_3 \geq 1 \right\}.$$
- *Fixing*: fix $x_3 = 0$ and get restricted region
$$S|_{x_3=0} := \{(x_1, x_2, x_3) \in S | x_3 = 0\}$$
- The seed inequality $\sqrt{x_1 x_2} \geq 1$ is a valid convex inequality for $S|_{x_3=0}$.

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- The seed inequality $\sqrt{x_1x_2} \geq 1$ is a valid convex inequality for $S|_{x_3=0}$.
- *Lifting*: Find $\alpha \in \mathbb{R}$ so that $\sqrt{x_1x_2} + \alpha x_3 \geq 1$ is valid for S, by solving
$$\sup\left\{\frac{1-\sqrt{x_1x_2}}{x_3}\,\middle|\, x_1x_2 + 2x_1x_3 \geq 1, x_3 \in (0,1],\ x_1, x_2 \in [0,1]\right\}.$$

Lifting: an example

Example

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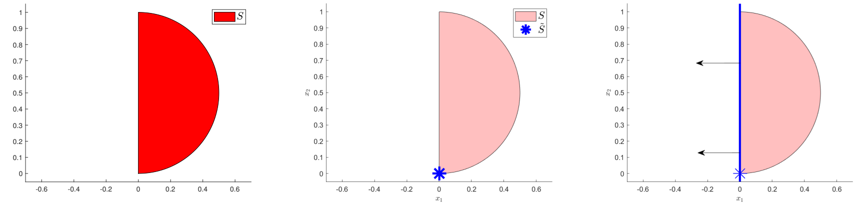
$$\sup \left\{ \frac{1 - \sqrt{x_1 x_2}}{x_3} \middle| x_1 x_2 + 2x_1 x_3 \geq 1, x_3 \in (0,1], \; x_1, x_2 \in [0,1] \right\}.$$
- Lifted inequality valid for S: $\sqrt{x_1 x_2} + 2x_3 \geq 1$.

Lifting can Fail

Example

$$S = \left\{(x_1, x_2) \in [0,1]^2 \mid -x_1^2 - (x_2 - 0.5)^2 \geq -0.5^2\right\}.$$

- $-x_1 \geq 0$ is valid for $S|_{x_2=0}$.
- No $\alpha \in \mathbb{R}$ for which $-x_1 + \alpha x_2 \geq 0$ is valid for S.



Main contributions

- *Can we always lift?*

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  Yes! For bilinear programming.

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For separable bilinear programming:

- *Bilinear covering set and bilinear cover inequality.*
- *Sequence-independent lifting for the bilinear cover inequality.*

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Existence of lifted inequality for bilinear program

Problem setup

Consider a set described by one bilinear constraint and bounds on variables:

$$S = \{(x, y) \in [0,1]^m \times [0,1]^n \mid x^\intercal Q y + a^\intercal x + b^\intercal y \geq c\},$$

where $Q \in \mathbb{R}^{m \times n}$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

Seed inequality

Definition (Seed Inequality)

Given $C \times D \subset [m] \times [n]$ and $\tilde{x}_i, \tilde{y}_j \in \{0, 1\}$ for $i \in [m] \backslash C$, $j \in [n] \backslash D$, for

$$\tilde{S} := \{(x, y) \in S \mid x_{[m] \backslash C} = \tilde{x}_{[m] \backslash C}, y_{[n] \backslash D} = \tilde{y}_{[n] \backslash D}\} \neq \emptyset,$$

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if the inequality

$$h(x_C, y_D) \geq r$$

is valid on $\tilde{S}$, it is a seed inequality on $\tilde{S}$.

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Assume h is concave on $[0,1]^{|C|+|D|}$, i.e., $h(x_C, y_D) \geq r$ is convex.

Existence of lifted inequality

Theorem

For any $k \in [m] \setminus C$, *there exists a finite* $f_k \in (-\infty, \infty)$ *for which*

$$h(x_C, y_D) + f_k x_k \geq r + f_k \tilde{x}_k$$

is valid for $\{(x, y) \in S \mid x_{([m]\setminus C)\setminus\{k\}} = \tilde{x}_{([m]\setminus C)\setminus\{k\}}, y_{[n]\setminus D} = \tilde{y}_{[n]\setminus D}\}$.

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- Applied iteratively to fixed variables, yields a valid inequality for S.
- Re-scale variables if bounds not $[0, 1]$

Lifting "from the middle" can fail

Example

Inequality

$$x \geq \frac{3}{4}$$

is valid for $S|_{\hat{x}=\frac{1}{2}}$ where

$$S = \left\{ (x, y, \hat{x}) \in [0, 1]^3 \;\middle|\; \left(x - \frac{1}{4}\right)\left(y - \frac{1}{2}\right) \geq \frac{\hat{x}}{4} + \frac{1}{8} \right\}.$$

Further, there is no $\alpha \in \mathbb{R}$ such that $x + \alpha(\hat{x} - \frac{1}{2}) \geq \frac{3}{4}$ is valid for S.

Challenges with sequential lifting

- Sequential lifting is expensive: We have to solve one nonlinear noncovex problem to lift one variable.

- How about sequence-independent lifting?

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Sequence-independent lifting: basic theory

Separable bilinear set

Definition (Separable bilinear set)

A set Q is said to be a separable bilinear set if it is of the form

$$Q := \left\{ (x, y) \in [0, 1]^n \times [0, 1]^n \;\middle|\; \sum_{i=1}^{n} a_i x_i y_i \geq d \right\},$$

where d and $a_i \in \mathbb{R}$ for $i \in [n]$, i.e., each variable x_i or y_i, for $i \in [n]$, appears in only one term.

Lifting function

Consider:

- Q: a separable bilinear set.
- $\Lambda = \{I, J_0, J_1\}$: a partition of $[n]$.
- $\tilde{Q} = \{(x, y) \in Q \,|\, x_{J_0} = y_{J_0} = 0, x_{J_1} = y_{J_1} = 1\}$.
- $h(x_I, y_I) \geq r$: a valid convex inequality for $\tilde{Q}$ (i.e., h concave).

Definition (Lifting Function)

For $\delta \in \mathbb{R}$,

$$\phi(\delta) := \max \left\{ r - h(x_I, y_I) \,\middle|\, \sum_{i \in I} a_i x_i y_i \geq \left(d - \sum_{i \in J_1} a_i \right) - \delta, x, y \in [0, 1]^n \right\}.$$

Sequence Independence

Lemma

Let $\psi : \mathbb{R} \mapsto \mathbb{R}$ *and* $\gamma_i : \mathbb{R}^2 \mapsto \mathbb{R}$ *be such that*

- $\psi(\delta) \geq \phi(\delta)$, $\forall \delta \in \mathbb{R}$;

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- *for* $i \in J_0$, $\gamma_i(x, y)$ *is concave and* $\gamma_i(x, y) \geq \psi(a_i xy), \forall x, y \in [0, 1]$,
- *for* $i \in J_1$, $\gamma_i(x, y)$ *is concave and* $\gamma_i(x, y) \geq \psi(a_i xy - a_i), \forall x, y \in [0, 1]$.

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Lemma

Let $\psi : \mathbb{R} \mapsto \mathbb{R}$ *and* $\gamma_i : \mathbb{R}^2 \mapsto \mathbb{R}$ *be such that*

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- *for* $i \in J_1$, $\gamma_i(x, y)$ *is concave and* $\gamma_i(x, y) \geq \psi(a_i xy - a_i), \forall x, y \in [0, 1]$.

Then, the following lifted inequality is a valid convex inequality for Q*:*

$$h(x_I, y_I) + \sum_{i \in J_0 \cup J_1} \gamma_i(x_i, y_i) \geq r.$$

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Bilinear cover inequality

Minimal cover

Definition (Minimal Cover)

Let $k \in \mathbb{Z}_+$. We say $a_i \in \mathbb{R}$, $i \in [k]$ form a minimal cover of $d \in \mathbb{R}$, if

1. $a_i > 0$ for all $i \in [k]$, $d > 0$,
2. $\sum_{i=1}^{k} a_i > d$
3. $\sum_{i \in K \subsetneq [k]} a_i \leq d$.

Definition (Minimal Cover Partition)

For a separable bilinear set Q, a partition $\Lambda = \{I, J_0, J_1\}$ of $[n]$ with $I \neq \emptyset$, is a minimal cover partition if: a_i, $i \in I$ form a minimal cover of $d^{\Lambda} := d - \sum_{i \in J_1} a_i$.

Example of minimal cover

Example

- $$Q = \left\{ (x, y) \in [0,1]^7 \times [0,1]^7 \;\middle|\; \sum_{i=1}^{7} i x_i y_i \geq 22 \right\}$$
- In this set, $a_i = i$ for $i \in [7]$ and $d = 22$.

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- In this set, $a_i = i$ for $i \in [7]$ and $d = 22$.
- Fix $x_1, y_1, x_4, y_4 = 0$, $x_2, y_2, x_3, y_3 = 1$ ($J_0 = \{1, 4\}$, $J_1 = \{2, 3\}$)
- $\tilde{Q}$: $5x_5y_5 + 6x_6y_6 + 7x_7y_7 \geq 17$, ($I = \{5, 6, 7\}$)

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- In this set, $a_i = i$ for $i \in [7]$ and $d = 22$.
- Fix $x_1, y_1, x_4, y_4 = 0$, $x_2, y_2, x_3, y_3 = 1$ ($J_0 = \{1, 4\}$, $J_1 = \{2, 3\}$)
- $\tilde{Q} : 5x_5y_5 + 6x_6y_6 + 7x_7y_7 \geq 17$, ($I = \{5, 6, 7\}$)
- $5, 6, 7$ is a minimal cover of 17
- $\{I, J_0, J_1\}$ is a minimal cover partition of Q

Existence of minimal cover

Recall the separable bilinear set

$$Q = \left\{ (x, y) \in [0,1]^n \times [0,1]^n \;\middle|\; \sum_{i=1}^{n} a_i x_i y_i \geq d \right\}.$$

Theorem

For a separable bilinear set Q, either there exists at least one minimal cover partition or we have that $Q = \emptyset$ or that $\mathrm{conv}(Q)$ is polyhedral.

Bilinear cover inequality

Theorem

Consider the bilinear minimal covering set

$$\tilde{Q} := \left\{ (x,y) \in [0,1]^n \times [0,1]^n \;\middle|\; \sum_{i=1}^{n} a_i x_i y_i \geq d \right\},$$

where the a_i, $i \in [n]$ form a minimal cover of d. Then the inequality

$$\sum_{i=1}^{n} \frac{\sqrt{a_i}}{\sqrt{a_i} - \sqrt{d_i}} \left(\sqrt{x_i y_i} - 1\right) \geq -1,$$

which we refer to as bilinear cover inequality is valid for $\tilde{Q}$, where $d_i = d - \sum_{j \in [n] \setminus \{i\}} a_j$.

Example of bilinear cover inequality

Bilinear Cover Inequality for $\tilde{Q}: 5x_5y_5 + 6x_6y_6 + 7x_7y_7 \geq 17$:

$$
\begin{aligned}
&\frac{\sqrt{5}}{\sqrt{5}-\sqrt{4}}(\sqrt{x_5y_5}-1)\\
&+\frac{\sqrt{6}}{\sqrt{6}-\sqrt{5}}(\sqrt{x_6y_6}-1)\\
&+\frac{\sqrt{7}}{\sqrt{7}-\sqrt{6}}(\sqrt{x_7y_7}-1)\\
&\geq -1
\end{aligned}
$$

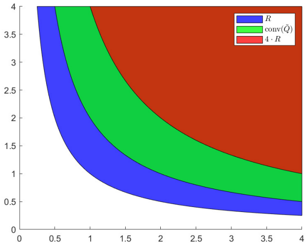
Strength of bilinear cover inequality

Theorem

$$R := \left\{ (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^{n} \frac{\sqrt{a_i}}{\sqrt{a_i} - \sqrt{d_i}} \left(\sqrt{x_i y_i} - 1 \right) \geq -1 \right\}.$$

Then

$$(4 \cdot R) \cap [0,\ 1]^n \subseteq \text{conv}(\tilde{Q}) \subseteq R \cap [0,\ 1]^n.$$



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Lifting bilinear cover inequality

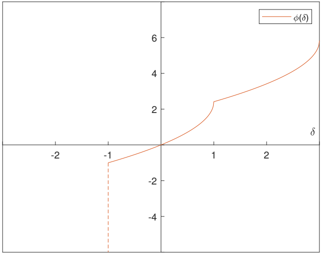
Lifting Function

Consider the lifting function for the bilinear cover inequality:

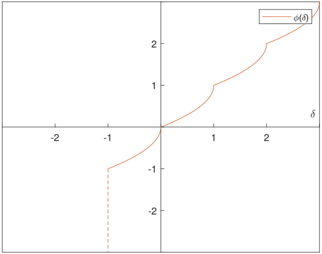
$$\phi(\delta) := \max \sum_{i=1}^{n} \frac{\sqrt{a_i}}{\sqrt{a_i} - \sqrt{d_i}} (1 - \sqrt{x_i y_i}) - 1$$
$$\text{s.t.} \sum_{i=1}^{n} a_i x_i y_i \geq d - \delta, \; x, y \in [0, 1]^n.$$

Let $\Delta := \sum_{i=1}^{n} a_i - d$ and let $a_{i_0} = \min\{a_i | a_i > \Delta\}$ if it exists.

Example



(a) $a_i = 2, \Delta = 1$



(b) $a_i = 1, \Delta = 1$

Figure: Lifting function $\phi(\delta)$ in red

Subadditive Upper Bound

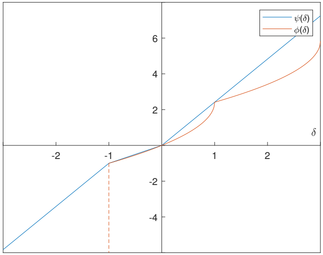
Theorem

$$\psi(\delta) := \begin{cases} l_+\delta & 0 \leq \delta \\ l_-\delta & -\Delta \leq \delta \leq 0 \\ l_+(\delta+\Delta) - 1 & \delta \leq -\Delta, \end{cases}$$

where $l_+ = \frac{\sqrt{a_{i_0}}+\sqrt{d_{i_0}}}{\Delta\sqrt{d_{i_0}}}$ *if* a_{i_0} *exists and* $l_+ = \frac{1}{\Delta}$ *otherwise, and where* $l_- = \frac{1}{\Delta}$*. Then*

- $\psi(\delta)$ *is subadditive over* $\mathbb{R}$ *with* $\psi(0) = 0$,
- $\phi(\delta) \leq \psi(\delta)$ *for* $\delta \in \mathbb{R}$.

Example



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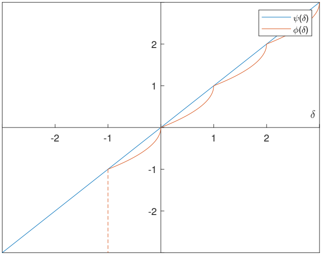

(b) $a_i = 1$, $\Delta = 1$

Figure: Lifting function $\phi(\delta)$ in red and subadditive upper bound $\psi(\delta)$ in blue

Lifted Bilinear Cover Inequality

Theorem

For the separable bilinear set
$Q = \left\{(x, y) \in [0,1]^n \times [0,1]^n \;\middle|\; \sum_{i=1}^n a_i x_i y_i \geq d\right\}$*, and a minimal cover partition* $\{I, J_0, J_1\}$,

$$\sum_{i \in I} \frac{\sqrt{a_i}}{\sqrt{a_i} - \sqrt{d_i}} \left(\sqrt{x_i y_i} - 1\right) + \sum_{i \notin I} \gamma_i(x_i, y_i) \geq -1,$$

is valid for Q *where* $\gamma_i : \mathbb{R}^2 \to \mathbb{R}$ *for* $i \in [n] \setminus I$ *are the concave functions:*

- $\gamma_i(x, y) = l_+ a_i \min\{x, y\}$ *for* $i \in J_0^+$;
- $\gamma_i(x, y) = -l_+ a_i \min\{2 - x - y, 1\}$ *for* $i \in J_1^-$;
- $\gamma_i(x, y) = \min\{-l_- a_i(1 - x - y), -l_+ a_i(1 - x - y) + l_+ \Delta - 1, 0\}$ *for* $i \in J_0^-$;

Lifted Bilinear Cover Inequality (Cont'd)

Theorem (Cont'd)

- $\gamma_i(x,y) = \min\{\tilde{g}_i(x,y), \tilde{h}_i(x,y), g_i(x,y), h_i(x,y)\}$, *for* $i \in J_1^+$ *with* $a_i > a_{i_0}$ *when* a_{i_0} *exists, and* $\gamma_i(x,y) = \min\{\tilde{g}_i(x,y), \tilde{h}_i(x,y)\}$ *in all other cases where* $i \in J_1^+$, *with*

$$\tilde{g}_i(x,y) = l_+ a_i(\min\{x,y\} - 1) + l_+\Delta - 1$$
$$\tilde{h}_i(x,y) = l_- a_i(\min\{x,y\} - 1)$$
$$g_i(x,y) = \sqrt{a_i - \Delta}\sqrt{a_i}\, l_+\sqrt{xy} - l_+(a_i - \Delta) - 1$$
$$h_i(x,y) = \frac{\sqrt{a_i}}{\sqrt{a_i} - \sqrt{d_i}}(\sqrt{xy} - 1).$$

The lifted bilinear cover inequality is SOC representable.

Future Directions

- *Computational and Theoretical Experiments*:
  How good is the bilinear cover inequality?

- *Separation Problem*:
  How to find the minimal cover and bilinear cover inequality which cuts out a current solution?

Thank you!