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Two Row Mixed-Integer Cuts Via Lifting

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Abstract Recently Andersen et al. [1], Borozan and Cornuéjols [6] and Cornuéjols and Margot [10] have characterized the extreme valid inequalities of a mixed integer set consisting of two equations with two free integer variables and non-negative continuous variables. These inequalities are either split cuts or intersection cuts derived using maximal lattice-free convex sets. In order to use these inequalities to obtain cuts from two rows of a general simplex tableau, one approach is to extend the system to include all possible non-negative integer variables (giving the two row mixed-integer infinite-group problem), and to develop lifting functions giving the coefficients of the integer variables in the corresponding inequalities. In this paper, we study the characteristics of these lifting functions.

We show that there exists a unique lifting function that yields extreme inequalities when starting from a maximal lattice-free triangle with multiple integer points in the relative interior of one of its sides, or a maximal lattice-free triangle with integral vertices and one integer point in the relative interior of each side. In the other cases (maximal lattice-free triangles with one integer point in the relative interior of each side and non-integral vertices, and maximal lattice-free quadrilaterals), non-unique lifting functions may yield distinct extreme inequalities. For the latter family of triangles, we present sufficient conditions to yield an extreme inequality for the two row mixed-integer infinite-group problem.

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1 Introduction

Research in the area of general-purpose cutting planes for mixed-integer programs (MIP) has received much attention recently. Despite some theoretical breakthroughs, two of the most effective classes of cutting planes remain the Gomory mixed-integer cuts (GMIC) (Gomory [18]) and the mixed-integer rounding (MIR) inequalities (Nemhauser and Wolsey [29]). While these cuts along with the split cuts (Cook, Kannan and Schrijver [7]) have been proven to be equivalent (see Cornuéjols and Li [8]), their original derivations differ significantly. The GMI cuts are facet-defining inequalities of the group relaxation of MIPs, the split cuts are based on the theory of disjunctive programming (Balas [4]) and the MIR inequalities are based on a simple mixed-integer set (Nemhauser and Wolsey [29]). In this paper we pursue the group relaxation approach. The GMIC and valid inequalities based on single row mixed-integer group relaxations have been studied both theoretically and computationally, whereas understanding of valid inequalities from two and multiple rows is at an early stage. Our goal is to generate valid inequalities from any two rows of an optimal simplex tableau, that are strong in a well-defined sense and that have similar properties to the GMIC.

Below we briefly discuss earlier work and the motivation for the group-relaxation approach that is taken in this paper. Group cutting planes based on relaxations of a single row of a mixed-integer program, of which GMIC is a special case, were presented by Gomory [19] and Gomory and Johnson [20,21] in the 70's, and more recently in Gomory, Johnson, and Evans [23], Gomory and Johnson [22], Aráoz et al. [2], Miller, Li and Richard [27], Richard, Li, and Miller [30] and Dash and Günlük [12]. This has led to computational work to test whether other inequalities based on single row group relaxation are effective computationally; see Cornúejols, Li and Vandenbussche [9], Fischetti and Saturni [17] and Dash and Günlük [11]. In general the results have been disappointing and the GMIC seems to be the most effective single row mixed-integer group inequality. One possible explanation for this is the fact that the GMIC has the strongest coefficients for the continuous variables among all single row group inequalities.

In Johnson [24] multiple row group inequalities were studied and in Gomory and Johnson [22] the potential advantages of valid inequalities based on multiple constraints were discussed. In particular, one weakness of the single row inequalities is that the continuous variables are modeled by aggregating them into two continuous variables, based on the signs of the coefficients. Group cuts based on multiple rows overcome this limitation and can more accurately represent the structure of the columns corresponding to continuous variables. Some extreme inequalities for two row mixed-integer group problems are presented in Dey and Richard [14,13].

A slightly different viewpoint has been taken recently by Andersen et al. [1], Borozan and Cornuéjols [6] and Cornuéjols and Margot [10]. See also Johnson [25] for related results. They have analyzed a system of two rows with two free integer variables and non-negative continuous variables. They show that extreme inequalities of the system

$$f + \sum_{w \in \mathbb{Q}^2} w y(w) \in \mathbb{Z}^2, \qquad (1)$$
$$y(w) \geq 0 \ \forall w \in \mathbb{Q}^2, \quad y \text{ has finite support}, \quad f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$$

are either split cuts or intersection cuts (Balas [3]) that can be derived using maximal lattice-free convex sets.

Our approach builds on this work. Given that the GMIC is one of the most effective single row group inequalities and has the strongest coefficients on the continuous variables, we attempt to keep similar properties when generating inequalities from two rows. Thus we view the construction of the GMIC in the following way:

1. Starting from a simplex tableau of a MIP, create a single-row mixed-integer group relaxation.
2. Fix the non-negative integer variables of the mixed-integer group relaxation to zero and generate an extreme (facet-defining) inequality with respect to the continuous variables.
3. Lift the non-negative integer variables to obtain an inequality that is extreme for the one-row mixed-integer infinite-group problem (see Nemhauser and Wolsey [28] for an overview on lifting).

We apply the same approach to the two row case. The recent results in [1,6,10] cited above tell us how to approach step 2. Our contribution is to accomplish the two row counterpart of step 3, i.e., to lift integer variables into the extreme inequalities for (1) in order to obtain new extreme inequalities for the two row mixed-integer infinite-group problem. The new inequalities derived in this way may thus be considered as the two row counterparts to the GMIC; they are both extreme inequalities for the mixed-integer infinite-group problem and have strong coefficients for the continuous variables. An extended abstract of some of the results in this paper is presented in Dey and Wolsey [15].

The rest of the paper is organized as follows. In Section 2, we present some preliminaries about the mixed-integer infinite-group problem, the continuous infinite-group problem, and a classification of maximal lattice-free convex sets in $\mathbb{R}^2$. We present the principal results of this paper in Section 3, ending with a description of a scheme to generate cutting planes based on two rows of a simplex tableau. Sections 4 - 7 present the proofs of the results presented in Section 3. Some directions of future research are discussed in Section 8.

2 Mixed-integer Group Relaxations, Valid Inequalities, and Lattice-free Convex Sets

Here we motivate and derive the mixed-integer group relaxations. We discuss valid inequalities for such sets and different measures of the strength of valid inequalities for such sets along with recent results linking valid inequalities and lattice-free convex sets. We present a classification of such sets in two dimensions - the case that is relevant for the rest of the paper. We terminate the section with a more formal description of the problem considered in this paper.

2.1 Mixed-integer Infinite Group Relaxations

Given a mixed-integer program and an associated basis representation in which the basic variables are constrained to be integer, one may wish to study the set X^{MIP} of feasible solutions of the form,

$$x_{B_u} + \sum_{j=1}^{n} a_u^j x_j + \sum_{j=1}^{p} g_u^j y_j = a_u^0 \quad u \in \{1, ..., m\}$$
$$x_B \in \mathbb{Z}_+^m, \quad x \in \mathbb{Z}_+^n, \quad y \in \mathbb{R}_+^p.$$

A first so-called group relaxation is obtained by dropping the non-negativity constraints on x_{B_u} (and possibly aggregating variables that have the same columns). The resulting set X_1^{MIP} can be written as

$$\sum_{j=1}^{n} \mathbb{P}(a^j) x_j + \mathbb{P}(\sum_{j=1}^{p} g^j y_j) \equiv \mathbb{P}(a^0)$$
$$x \in \mathbb{Z}_+^n, \quad y \in \mathbb{R}_+^p,$$

where $\mathbb{P}(u)$ denotes $u (\mathrm{mod}\, \bar{1})$, i.e., $(\mathbb{P}(u))_i = u_i (\mathrm{mod}\, 1)$ for $i \in \{1, ..., m\}$ and $\equiv$ denotes equivalence $(\mathrm{mod}\, \bar{1})$.

A second relaxation is obtained by the addition of more variables corresponding to all other possible columns. The resulting relaxation is known as the mixed-integer infinite group problem.

Specifically, let $I^m = \{(u_1, u_2, ..., u_m) \,|\, 0 \leq u_i < 1, \forall i \in \{1, ..., m\}\}$ (addition for elements in I^m is defined as modulo 1 componentwise), U be a subgroup of I^m containing $\mathbb{P}(a^j)$ $\forall j \in \{1, ..., n\}$, and W be a subset of $\mathbb{R}^m$ containing g^j $\forall j \in \{1, ..., p\}$. Since it is typically clear from context, we use the symbols $+$ and $-$ to denote addition and subtraction respectively in both $\mathbb{R}^m$ and I^m. We will also use the symbol $=$ in place of the symbol $\equiv (\mathrm{mod}\, \bar{1})$ for equations involving elements of I^m.

Definition 1 ([21],[24]) Let U be a subgroup of I^m and W be any subset of $\mathbb{R}^m$. Then the mixed-integer infinite-group problem, denoted $MI(U, W, r)$, is defined as the set of pairs of functions $x : U \rightarrow \mathbb{Z}_+$ and $y : W \rightarrow \mathbb{R}_+$ that satisfy

1. $\sum_{u \in U} u x(u) + \mathbb{P}(\sum_{w \in W} w y(w)) = r$, $r \in I^m \setminus \{\bar{0}\}$,
2. x and y have finite support. □

The key observation connecting $MI(I^m, \mathbb{R}^m, r)$ to (1) is the following: If all the $x(u)$'s are fixed to zero in $MI(I^m, \mathbb{R}^m, r)$, then the resulting set $MI(\{\bar{0}\}, \mathbb{R}^m, r)$:

$$\mathbb{P}(\sum_{w \in \mathbb{R}^m} w y(w)) = r, \quad r \in I^m \setminus \{\bar{0}\}$$
$$y(w) \in \mathbb{R}_+, y \text{ has a finite support}$$

(known as the *continuous infinite group relaxation*[1]) is essentially the set presented in (1) with $r = -\mathbb{P}(f)^2$ and $m = 2$.

Our principal goal is to find strong valid inequalities for the mixed-integer infinite group relaxation.

Definition 2 ([21],[24]) A valid inequality for $MI(U, W, r)$ is defined as a pair of functions, $\phi : U \to \mathbb{R}_+$ and $\mu_\phi : W \to \mathbb{R}_+$, such that $\sum_{u \in U} \phi(u)x(u) + \sum_{w \in W} \mu_\phi(w)y(w) \geq 1$, $\forall (x, y) \in MI(U, W, r)$, where $\phi(\bar{0}) = 0$. □

Since valid inequalities for the group problem are functions defined over subsets of I^m and $\mathbb{R}^m$, we will use the terms valid inequality and valid function interchangeably.

The interest of $MI(U, W, r)$ is that any valid inequality (ϕ, μ_ϕ) provides a valid inequality for the finite group relaxation X_1^{MIP}. Thus the inequality

$$\sum_{j=1}^{n} \phi(\mathbb{P}(a^j))x_j + \sum_{j=1}^{p} \mu_\phi(g^j)y_j \geq 1, \tag{2}$$

is valid for X_1^{MIP} and every valid inequality for X_1^{MIP} can be obtained in this way.

To obtain strong valid inequalities/functions, two important properties are now defined.

Definition 3 ([21],[24]) A valid function (ϕ, π) is minimal for $MI(U, W, r)$ if there do not exist valid functions (ϕ^*, π^*) for $MI(U, W, r)$ different from (ϕ, π) such that $\phi^*(u) \leq \phi(u)$ $\forall u \in U$ and $\pi^*(w) \leq \pi(w)$ $\forall w \in W$. □

Definition 4 ([21],[24]) A valid function (ϕ, π) is extreme for $MI(U, W, r)$ if there do not exist valid functions (ϕ_1, π_1) and (ϕ_2, π_2) for $MI(U, W, r)$ such that $(\phi_1, \pi_1) \neq (\phi_2, \pi_2)$ and $(\phi, \pi) = \frac{1}{2}(\phi_1, \pi_1) + \frac{1}{2}(\phi_2, \pi_2)$. □

Minimality of inequalities for finite and infinite variable sets are similar. Extreme inequalities correspond to facet-defining inequalities for finite (full-dimensional) sets. A third important property of valid functions is subadditivity that will be defined later. Our goal will be to find extreme valid inequalities for $MI(I^2, \mathbb{R}^2, r)$ and we will regularly use the important property (Gomory and Johnson [20]) that *all extreme inequalities are minimal and all minimal inequalities are subadditive.*

[1] Some authors have used the term continuous group problem to imply the infinite-group problem, as the underlying group is a 'continuous' set. However, we use the term to imply the problem whose variables are all non-negative continuous (except for the free integer variables of the group problem).

[2] Note here that columns corresponding to the continuous variables are assumed to be rational in (1). However, we will assume that $W = \mathbb{R}^2$ which allows the use of results from Johnson [24]. This is only a minor technical assumption as we will show that results obtained using only rational columns for (1) apply to the case when columns are irrationals.

2.2 Continuous Infinite Group Problems and Lattice-Free Convex Sets

From now on we consider only the cases in which $m = 1$ and $m = 2$. Minimal (and extreme) valid functions for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$ have been studied by several researchers. We begin with the definition of a maximal lattice-free convex set, that is the key component in the description of minimal inequalities for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$.

Definition 5 ([26]) A set S is called a maximal lattice-free convex set in $\mathbb{R}^2$ if it is convex,

1. interior$(S) \cap \mathbb{Z}^2 = \emptyset$, and
2. There exists no convex set S' satisfying (1.) with $S \subsetneq S'$. □

We state the following theorem, modified from Borozan and Cornuéjols [6]; see also Theorem 1 in Andersen et al. [1].

Theorem 1 ([6]) *An inequality of the form* $\sum_{w \in \mathbb{Q}^2} \tilde{\pi}(w) y(w) \geq 1$ *is minimal for (1) if the closure of*

$$P(\tilde{\pi}) = \{w \in \mathbb{Q}^2 | \tilde{\pi}(w - f) \leq 1\} \tag{3}$$

in $\mathbb{R}^2$ *is a maximal lattice-free convex set. Moreover, given a maximal lattice-free convex set* P *such that* $f \in$ interior(P)*, the function* $\tilde{\pi} : \mathbb{Q}^2 \rightarrow \mathbb{R}_+$ *defined as*

$$\tilde{\pi}(w) = \begin{cases} 0 & \text{if } w \in \text{recession cone of } P \\ \lambda & \text{if } f + \frac{1}{\lambda} w \in \text{boundary}(P) \end{cases} \tag{4}$$

is a minimal valid inequality for (1). □

It is possible to analyze the case when $f \in$ boundary$(P(\pi))$. However, in this paper we focus on the case when $f \in$ interior$(P(\pi))$.

It can be verified that if a function $\tilde{\pi} : \mathbb{Q}^2 \rightarrow \mathbb{R}_+$ corresponding to a maximal lattice-free set P is minimal (extreme resp.) for (1) and $f \in$ interior(P), then $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ defined as

$$\pi(w) = \begin{cases} 0 & \text{if } w \in \text{recession cone of } P \\ \lambda & \text{if } f + \frac{1}{\lambda} w \in \text{boundary}(P) \end{cases} \tag{5}$$

is minimal (extreme resp.) for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$. This is a technical verification and we relegate the proof to Appendix 1. (Note that since f is rational in (1), we assume that r (i.e., $\mathbb{P}(-f)$) is rational in the rest of the paper.) For any minimal valid function π, we denote the corresponding lattice-free maximal set by $P(\pi)$.

Next we present a classification of maximal lattice-free convex sets. The classification is from Lovász [26], Andersen et al. [1], Dey and Wolsey [16], and Cornuéjols and Margot [10].

Proposition 1 (Classification) *Let* P *be a maximal lattice-free convex set with a non-empty interior in* $\mathbb{R}^2$*. Then* P *is any one of the following:*

1. *(Split Set) The set* $\{(x_1, x_2) \in \mathbb{R}^2 \,|\, b \leq a_1 x_1 + a_2 x_2 \leq b + 1\}$ *where* $a_1, a_2, b \in \mathbb{Z}$ *and* a_1*,* a_2 *are coprime.*

2. A maximal lattice-free triangle in $\mathbb{R}^2$. In this case exactly one of the following is true:
   (a) (Type 1 triangle) All the vertices are integral and each side contains exactly one integer point in its relative interior.
   (b) (Type 2 triangle) One side of P contains more than one integer point in its relative interior.
   (c) (Type 3 triangle) The vertices are non-integral and each side contains exactly one integer point in its relative interior.

3. A lattice-free quadrilateral and each of its sides contains exactly one integer point in its relative interior. □

2.3 Problem Description: Extreme Inequalities for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$ and $MI(I^2, \mathbb{R}^2, r)$

As noted in the introduction, the approach we take is motivated by the one row case. To make this more precise, we consider an example from the one row case. Figure 1 shows two extreme inequalities for $MI(I^1, \mathbb{R}^1, 0.5)$. The pair of functions (ϕ_1, π_1), plotted in bold, was shown to be extreme for $MI(I^1, \mathbb{R}^1, 0.5)$ in Gomory and Johnson [22]. The functions $(\phi_{GMIC}, \pi_{GMIC})$ plotted in dashed lines correspond to the GMIC which is also extreme for $MI(I^1, \mathbb{R}^1, 0.5)$. Therefore, from the perspective of the mixed-integer problem, both inequalities are strong. However, if we just compare the functions π_1 and π_{GMIC}, we observe that π_{GMIC} dominates π_1. Therefore, while π_{GMIC} is extreme for $MI(\{\bar{0}\}, \mathbb{R}^1, 0.5)$, π_1 is not even minimal for $MI(\{\bar{0}\}, \mathbb{R}^1, 0.5)$.

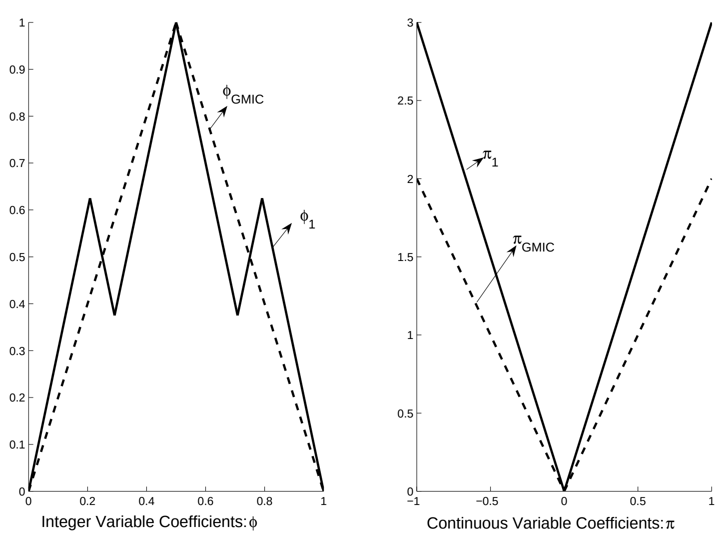

Fig. 1 Extreme functions for $MI(I^1, \mathbb{R}^1, 0.5)$

As GMIC is known to be an effective single row general cut, this suggests trying to mimic the above property in the construction of inequalities for two row group problems.

Thus the approach below for the two row case is to take extreme inequalities for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$ and lift them into strong and potentially minimal and extreme inequalities for $MI(I^2, \mathbb{R}^2, r)$.

3 The Principal Results

From now on we will assume that $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ is a minimal valid inequality for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$ corresponding to a maximal lattice-free convex set. Given π, there is a natural candidate for an approximate lifting function.

Proposition 2 (Trivial Fill-in Function) *Let $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ be a minimal function for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$ corresponding to a maximal lattice-free convex set $P(\pi)$ and let $\phi^{\bar{0}} : I^2 \to \mathbb{R}_+$, the trivial fill-in function, be defined as*

$$\phi^{\bar{0}}(u) = \inf\{\pi(w) \,|\, \mathbb{P}(w) = u, w \in \mathbb{R}^2\}. \tag{6}$$

Then $(\phi^{\bar{0}}, \pi)$ is a valid function for $MI(I^2, \mathbb{R}^2, r)$. □

Using a characterization of minimal inequalities for mixed-integer group problems from Gomory and Johnson [20] and Johnson [24], it can be verified that the valid function $(\phi^{\bar{0}}, \pi)$ is minimal if and only if $\phi^{\bar{0}}$ satisfies the following property: $\phi^{\bar{0}}(u) + \phi^{\bar{0}}(r - u) = 1 \ \forall u \in I^2$. Therefore, 'in theory' to verify that the trivial fill-in function is minimal we need to evaluate whether

$$\inf_{n^1, n^2 \in \mathbb{Z}^2}\{\pi(u + n^1) + \pi(r - u + n^2)\} = 1, \tag{7}$$

for every $u \in I^2$. We design a simpler test of minimality for the trivial fill-in function. We define a set $D(\pi)$ such that $D(\pi) + \{f\} \subseteq P(\pi)$ (see Definition 6) and prove the following result in Section 4.

Theorem 2 *$(\phi^{\bar{0}}, \pi)$ is a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$ if and only if $\mathbb{P}(D(\pi)) := \{\mathbb{P}(u) \,|\, u \in D(\pi)\} = I^2$. Moreover if $(\phi^{\bar{0}}, \pi)$ is minimal, then it is the unique minimal function for $MI(I^2, \mathbb{R}^2, r)$, i.e., if $(\phi^{\bar{0}}, \pi)$ and (ϕ', π) are minimal for $MI(I^2, \mathbb{R}^2, r)$, then $\phi' = \phi^{\bar{0}}$.* □

The set $D(\pi) + \{f\}$ consists of points $w \in P(\pi)$ such that $\phi^{\bar{0}}(\mathbb{P}(w)) + \phi^{\bar{0}}(r - \mathbb{P}(w)) = 1$. Therefore, the main burden of the proof of Theorem 2 lies in the reverse direction, i.e., to show that if $\mathbb{P}(w) \notin \mathbb{P}(D(\pi))$, then $\phi^{\bar{0}}(\mathbb{P}(w)) + \phi^{\bar{0}}(r - \mathbb{P}(w)) > 1$. Theorem 2 leads to the following corollary.

Corollary 1 *If π is extreme for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$ and $\mathbb{P}(D(\pi)) = I^2$, then $(\phi^{\bar{0}}, \pi)$ is the unique extreme function for $MI(I^2, \mathbb{R}^2, r)$.* □

Starting with a split set $P(\pi)$, taking the corresponding π and using the trivial fill-in function leads to the GMI inequalities (after application of a unimodular transformation, see [16]). Next we evaluate whether $\mathbb{P}(D(\pi)) = I^2$ for other maximal lattice-free convex sets $P(\pi)$. Corollary 1 and a characterization of extreme inequalities for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$ (Cornuéjols and Margot [10]) lead to Theorems 3 and 4.

Theorem 3 *If $P(\pi)$ is a maximal lattice-free triangle of Type 1 or 2, then $(\phi^{\bar{0}}, \pi)$ is an extreme function for $MI(I^2, \mathbb{R}^2, r)$.* □

See Section 5 for a proof of Theorem 3. Besides providing a characterization of extreme inequalities for $MI(I^2, \mathbb{R}^2, r)$ when $P(\pi)$ is a maximal lattice-free triangle of Type 1 or 2, Theorem 3 is of practical interest as it presents a simple method to generate strong inequalities for two rows of a simplex tableau. Specifically, since $D(\pi) \subseteq P(\pi)$, if $P(\pi)$ is a maximal lattice-free triangle of Type 1 or 2, then we can generate coefficients of non-basic integer variables $x(u)$ by solving $\min\{\pi(w) \,|\, \mathbb{P}(w) = u, w \in P(\pi)\}$.

Theorem 4 *If $P(\pi)$ is a maximal lattice-free triangle of Type 3, or a maximal lattice-free quadrilateral corresponding to an extreme function for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$, then $(\phi^{\bar{0}}, \pi)$ is not an extreme function for $MI(I^2, \mathbb{R}^2, r)$.* □

See Section 6 for a proof of Theorem 4. When $P(\pi)$ is a maximal lattice-free triangle of Type 3 or a maximal lattice-free quadrilateral, it should be possible to strengthen the trivial fill-in function. In these cases, it is possible that there exist different functions $\phi^1, \phi^2 : I^2 \rightarrow \mathbb{R}_+$ such that both (ϕ^1, π) and (ϕ^2, π) are extreme for $MI(I^2, \mathbb{R}^2, r)$.

It is possible to define a two step process to construct a function that generalizes the trivial fill-in function:

1. Let G be a subset of I^2. We assume that we have a valid inequality

$$\sum_{u \in G} V(u)x(u) + \sum_{w \in \mathbb{R}^2} \pi(w)y(w) \geq 1 \tag{8}$$

for the set

$$\begin{array}{c} \sum_{u \in G} ux(u) + \mathbb{P}(\sum_{w \in \mathbb{R}^2} wy(w)) = r \\ x(u) \in \mathbb{Z}_+, y(w) \in \mathbb{R}_+, x \text{ and } y \text{ have finite support.} \end{array} \tag{9}$$

One possible way of obtaining a valid inequality (8) is by standard lifting. For example, assume for simplicity that $G = \{u^1, u^2\}$. Then we may obtain the coefficients of $x(u^1)$ and $x(u^2)$ by sequential lifting as follows:

$$V(u^1) = \max_{n \in \mathbb{Z}_+, n \geq 1}\{\frac{1 - \pi(w)}{n} \,|\, \mathbb{P}(w) = r - nu^1\}, \tag{10}$$

$$V(u^2) = \max_{n_1, n_2 \in \mathbb{Z}_+, n_2 \geq 1}\{\frac{1 - \pi(w) - n_1 V(u^1)}{n_2} \,| \; \mathbb{P}(w) = r - n_1 u^1 - n_2 u^2\}. \tag{11}$$

2. Next for all integer variables $x(v)$ apply the fill-in procedure in the following fashion:

$$\phi^{G,V}(v) = \inf_{n(u) \in \mathbb{Z}_+, u \in G}\{\pi(w) + \sum_{u \in G} n(u)V(u) \,| \; \mathbb{P}(w) = v - \sum_{u \in G} n(u)u\}. \tag{12}$$

This step is essentially an update of the value function of the MIP whose objective function corresponds to the left-hand-side of (8) and whose constraints set is (9) with right-hand-side value of v.

It is easily verified that $(\phi^{G,V}, \pi)$ is a valid inequality for $MI(I^2, \mathbb{R}^2, r)$.

Next we concentrate on the case in which $|G| = 1$ to generate one extreme inequality when $P(\pi)$ is a maximal lattice-free triangle of Type 3. In particular, let $G = \{u^*\}$ and let $\phi^{u^*} : I^2 \rightarrow \mathbb{R}_+$ denote the inequality $\phi^{\{u^*\},V(u^*)}$ where $V(u^*)$ is obtained using (10) and the coefficients of the other integer variables are obtained using (12). Notice that if $u^* = \bar{0}$, then ϕ^{u^*} is the trivial fill-in function. We prove the following sufficient condition to generate an extreme valid inequality for $MI(I^2, \mathbb{R}^2, r)$ in Section 7.

Theorem 5 *Given f and $P(\pi)$, one can define two points $\bar{v}_0$ and $\bar{u}_0$ in $\mathbb{R}^2$ (see after Definition 9). If $\phi^{\mathbb{P}(\bar{v}_0)}(\mathbb{P}(\bar{v}_0)) = 1 - \pi(\bar{u}_0)$, then $(\phi^{\mathbb{P}(\bar{v}_0)}, \pi)$ is an extreme function for $MI(I^2, \mathbb{R}^2, r)$.* □

We end this section with a sketch of a procedure to generate cutting planes for two rows of a simplex tableau using the results presented above. Consider the set X_1^{MIP} with two rows,

$$x_{B_u} + \sum_{j=1}^{n} a_u^j x_j + \sum_{j=1}^{p} g_u^j y_j = r_u \quad u \in \{1, 2\}$$
$$x_B \in \mathbb{Z}^2, \quad x \in \mathbb{Z}_+^n, \quad y \in \mathbb{R}_+^p.$$

Apply the following steps:

1. Fix the non-basic integer variables to zero.
2. Select three (four for a quadrilateral inequality) continuous variables y_{j_1}, y_{j_2}, and y_{j_3} such that the positive combination of g^{j_1}, g^{j_2} and g^{j_3} spans $\mathbb{R}^2$.
3. Find a maximal lattice-free triangle $P(\pi)$ such that the inequality π is extreme for
$$x_B + g^{j_1} y_{j_1} + g^{j_2} y_{j_2} + g^{j_3} y_{j_3} = r, \quad x_B \in \mathbb{Z}^2, y_{j_1}, y_{j_2}, y_{j_3} \geq 0.$$
4. Lift the other continuous variables, i.e., use the function π as described in (5) to generate the coefficients for the other continuous variables.
5. Lift the non-basic integer variables into this inequality.
   - If $P(\pi)$ is a triangle of Type 1 or Type 2, then use the trivial fill-in function to lift all the integer variables. The coefficient of the integer variable corresponding to the column a^j can be generated as $\min\{\pi(w) \,|\, \mathbb{P}(w) = a^j, w \in P(\pi)\}$.
   - If $P(\pi)$ is a Type 3 triangle or a quadrilateral, select an integer variable x_j corresponding to column a^j with $u^j = \mathbb{P}(a^j)$. Calculate $\phi^{\bar{0}}(u^j) + \phi^{\bar{0}}(r - u^j)$. If $\phi^{\bar{0}}(u^j) + \phi^{\bar{0}}(r - u^j) = 1$, then $u^j \in \mathbb{P}(D(\pi))$. (This is a consequence of Propositions 5 and 7, see Section 4). In this case, take $\phi^{\bar{0}}(u^j)$ as the coefficient in the inequality for x_j. Denote the set of variables such that $\phi^{\bar{0}}(u^j) + \phi^{\bar{0}}(r - u^j) = 1$ by N_T. For the other variables with $\phi^{\bar{0}}(u^j) + \phi^{\bar{0}}(r - u^j) > 1$, denoted as N_L, try to improve

upon the coefficient $\phi^{\bar{0}}(u^j)$. Lift a subset $N_I \subseteq N_L$ of variables (as in (10), (11), etc.) giving an inequality of the form

$$\sum_{j\in N_T} \phi^{\bar{0}}(u^j)x_j + \sum_{j\in N_I} V_j x_j + \sum_{j=1}^{p} \pi(g^j)y_j \geq 1. \tag{13}$$

(V_i's are the coefficients obtained using lifting). Finally, apply the general fill-in function (12) to obtain the coefficients for the integer variables in the set $N_L \setminus N_I$.

It is shown in Proposition 3 that the value of the trivial (and similarly the general) fill-in function can be found by evaluating the value of the function π at a finite number of points. However, since the fill-in function is calculated via a minimization problem and due to the form of the inequalities, it is not necessary to solve the fill-in coefficient calculation (12) to optimality to obtain a valid inequality (in contrast to the traditional lifting process).

We next present an example illustrating some of the steps outlined above.

Example 1 Consider the following instance:

$$\begin{aligned} &\begin{pmatrix}1\\0\end{pmatrix} y_1 + \begin{pmatrix}0\\1\end{pmatrix} y_2 + \begin{pmatrix}-1\\-2\end{pmatrix} y_3 + \begin{pmatrix}-3\\-7\end{pmatrix} y_4 + \begin{pmatrix}-4/5\\6/5\end{pmatrix} x_1 \\ &+ \begin{pmatrix}19/10\\23/10\end{pmatrix} x_2 + \begin{pmatrix}3/10\\-7/5\end{pmatrix} x_3 + \begin{pmatrix}-2/3\\11/6\end{pmatrix} x_4 + \begin{pmatrix}1/2\\1/2\end{pmatrix} \\ &= \begin{pmatrix}1\\0\end{pmatrix} x_{B_1} + \begin{pmatrix}0\\1\end{pmatrix} x_{B_2} \end{aligned} \tag{14}$$

$$x_B \in \mathbb{Z}^2, \quad x \in \mathbb{Z}^4_+, \quad y \in \mathbb{R}^4_+.$$

- Choose three continuous variables: y_1, y_2, y_3. A maximal lattice-free triangle generating a facet for

$$\begin{pmatrix}1\\0\end{pmatrix} y_1 + \begin{pmatrix}0\\1\end{pmatrix} y_2 + \begin{pmatrix}-1\\-2\end{pmatrix} y_3 + \begin{pmatrix}1/2\\1/2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} x_{B_1} + \begin{pmatrix}0\\1\end{pmatrix} x_{B_2} \tag{15}$$
$$x_B \in \mathbb{Z}^2, \quad y_1, y_2, y_3 \in \mathbb{R}_+,$$

  is the triangle with vertices: $(3/2, 1/2)$, $(1/2, 3/2)$, and $(-1/2, -3/2)$. This triangle $P(\pi)$ is illustrated in Figure 2. Given $P(\pi)$ one can calculate $\pi(-3, -7) = 4$ using (5). Note now that $P(\pi)$ is a Type 2 triangle. Therefore it is enough to use the trivial fill-in procedure to lift the integer variables. We illustrate the computation of $\phi^{\bar{0}}(\mathbb{P}(-4/5, 6/5))$. Since in the case $D(\pi) \subset P(\pi) \subset \{(w_1, w_2) \,|\, -1/2 \leq w_1 \leq 3/2, -3/2 \leq w_2 \leq 3/2\}$ we obtain,

$$\phi^{\bar{0}}(1/5, 1/5) = \min \begin{cases} \pi(1/5, -4/5) = 1 \\ \pi(1/5, 1/5) = 2/5 \\ \pi(1/5, 6/5) = 7/5 \\ \pi(6/5, -4/5) = 2 \\ \pi(6/5, 1/5) = 7/5 \\ \pi(6/5, 6/5) = 12/5. \end{cases} \tag{16}$$

By computing the trivial fill-in function for the other integer variables, we obtain the inequality $y_1 + y_2 + y_3 + 4y_4 + (2/5)x_1 + (3/5)x_2 + (7/10)x_3 + (1/2)x_4 \geq 1$.

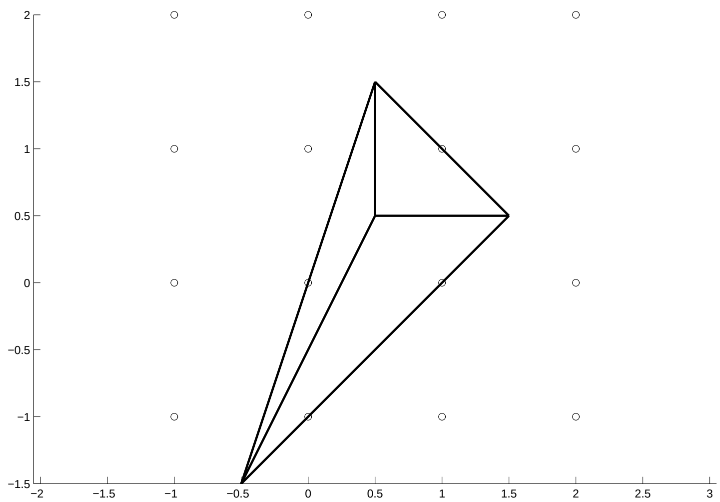

Fig. 2 $P(\pi)$ to generate facet for (15)

– Now suppose instead that the three continuous variables: y_1, y_2, y_4 are chosen. A maximal lattice-free triangle generating a facet for

$$\begin{pmatrix}1\\0\end{pmatrix} y_1 + \begin{pmatrix}0\\1\end{pmatrix} y_2 + \begin{pmatrix}-3\\-7\end{pmatrix} y_4 + \begin{pmatrix}1/2\\1/2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} x_{B_1} + \begin{pmatrix}0\\1\end{pmatrix} x_{B_2} \quad (17)$$
$$x_B \in \mathbb{Z}^2, \quad y_1, y_2, y_4 \in \mathbb{R}_+,$$

is given by the triangle with vertices: $\frac{7}{10}(1,0) + (1/2,1/2)$, $\frac{7}{4}(0,1) + (1/2,1/2)$, and $\frac{7}{26}(-3,-7) + (1/2,1/2)$. This triangle is illustrated in Figure 3.

Given $P(\pi)$, one can check that $\pi(-1,-2) = 10/7$. Note now that as $P(\pi)$ is a Type 3 triangle, we need to check whether the trivial fill-in function is sufficient to obtain strong coefficients. (See Proposition 3 for a method to compute the trivial fill-in function in this case).

1. x_1: $u^1 = \mathbb{P}((-4/5, 6/5)) = (1/5, 1/5)$. Now it can be verified that $\phi^{\bar{0}}(1/5,1/5) + \phi^{\bar{0}}(3/10,3/10) = 1$ implying that $\phi^{\bar{0}}(1/5,1/5) = 2/5$ is the coefficient in the inequality.
2. x_2: $u^2 = \mathbb{P}(19/10, 23/10) = (9/10, 3/10)$. Now it can be verified that $\phi^{\bar{0}}(9/10,3/10) + \phi^{\bar{0}}(3/5,1/5) = 1$ implying that $\phi^{\bar{0}}(9/10,3/10) = 3/7$ is the coefficient in the inequality.

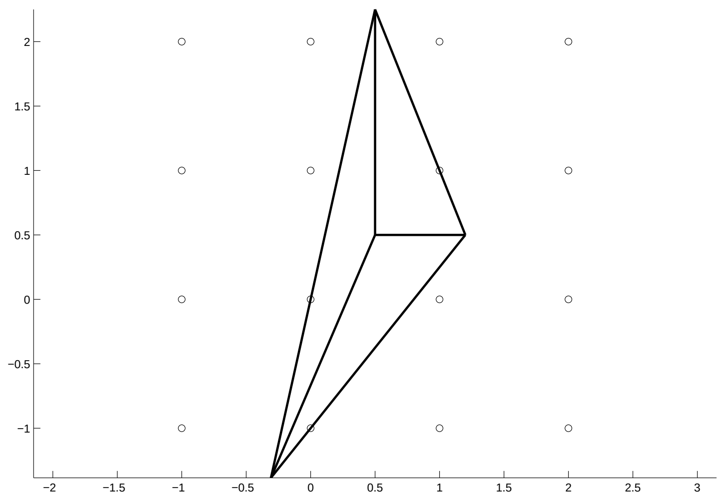

Fig. 3 $P(\pi)$ to generate facet for (17)

3. x_3: $u^3 = \mathbb{P}((3/10, -7/5)) = (3/10, 6/10)$. Now it can be verified that $\phi^{\bar{0}}(3/10, 6/10) + \phi^{\bar{0}}(2/10, 9/10) > 1$. Therefore the value of coefficient $\phi^{\bar{0}}(3/10, 6/10) = 27/35$ can possibly be improved.
4. x_4: $u^4 = \mathbb{P}((-2/3, 11/6)) = (1/3, 5/6)$. Now it can be verified that $\phi^{\bar{0}}(1/3, 5/6) + \phi^{\bar{0}}(1/6, 2/3) > 1$. Therefore the value of coefficient $\phi^{\bar{0}}(1/3, 5/6) = 2/3$ can possibly be improved.

Arbitrarily select x_4 for exact lifting. Then solve the problem:

$$\max_{n\in\mathbb{Z}, n\geq 1}\left\{\frac{1-\pi(w)}{n} | w \equiv (1/2, 1/2) - n(1/3, 5/6)\right\}$$
$$= 8/21 < 2/3 = \phi^{\bar{0}}(1/3, 1/6).$$

Now the generalized fill-in function coefficient for u^3 is given by $\phi^{u^4}(u^3) = \inf_{n\in\mathbb{Z}_+}\{(8/21)n + \pi(w) \,|\, \mathbb{P}(w) = u^3 - nu^4\} = 3/5 < \phi^{\bar{0}}(3/10, 6/10)$. Thus, the coefficients of both x_3 and x_4 have been decreased and the resulting inequality is: $(10/7)y_1 + (4/7)y_2 + (10/7)y_3 + (26/7)y_4 + (2/5)x_1 + (3/7)x_2 + (3/5)x_3 + (8/21)x_4 \geq 1$. □

4 Trivial Fill-in Function

4.1 Validity and Computation of the Trivial Fill-in Function

We begin with a lemma that motivates the definition of the trivial fill-in function.

Lemma 1 *Let $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ be a valid function for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$. Consider the function $\phi : I^2 \to \mathbb{R}_+$ defined as $\phi(u) = \pi(\hat{u})$ where $\hat{u} \in \mathbb{R}^2$ is any point such that $\mathbb{P}(\hat{u}) = u$. Then (ϕ, π) is a valid inequality for $MI(I^2, \mathbb{R}^2, r)$.*

Proof: Consider any $(\bar{x}, \bar{y}) \in MI(I^2, \mathbb{R}^2, r)$. Now consider the point $\tilde{y}$ defined as follows:

1. First set $\tilde{y}(w) = \bar{y}(w)\ \forall w \in \mathbb{R}^2$.
2. For every $u \in I^2$, such that $\bar{x}(u) > 0$ (note there is a finite number of such $u \in I^2$), update $\tilde{y}(w) := \tilde{y}(w) + \bar{x}(u)$, where $w = \hat{u}$.

Observe that $\mathbb{P}(\sum_{w\in\mathbb{R}^2} w\tilde{y}(w)) = \sum_{u\in I^2} u\bar{x}(u) + \mathbb{P}(\sum_{w\in\mathbb{R}^2} w\bar{y}(w)) = r$. Moreover the support of $\tilde{y}$ is finite as the supports of $\bar{x}$ and $\bar{y}$ are finite. Therefore $\tilde{y} \in MI(\{\bar{0}\}, \mathbb{R}^2, r)$. Also observe that $\sum_{\bar{x}(u)>0} \phi(u)\bar{x}(u) + \sum_{\bar{y}(w)>0} \pi(w)\bar{y}(w) = \sum_{w\in\mathbb{R}^2} \pi(w)\tilde{y}(w) \geq 1$ as π is valid for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$. Thus (ϕ, π) is a valid inequality for $MI(I^2, \mathbb{R}^2, r)$. □

Therefore, if we set the value of $\phi(u)$ to be that of $\pi(\hat{u})$ for any $\hat{u}$ such that $\mathbb{P}(\hat{u}) = u$, then $\phi(u)$ is a valid coefficient for $x(u)$. To obtain the best possible coefficient for the integer variables, we choose $\hat{u}$ so as to obtain the smallest possible coefficient for $\phi(u)$, thus obtaining the trivial fill-in function.

Fill-in functions were first introduced in Gomory and Johnson [21]. One interpretation of the trivial fill-in function is that its computation is equivalent to applying the procedure for strengthening coefficients of integer variables presented in Balas and Jeroslow [5]. The focus here is to prove that the trivial fill-in function provides the strongest possible coefficients for the integer variables in certain cases. Before presenting this in the next section, we show that the trivial fill-in function can be evaluated in finite time.

Proposition 3 *Let $P(\pi)$ be a maximal lattice-free polytope. Then there exist non-negative integers N_1 and N_2 such that*

$$\phi^{\bar{0}}(u) = \min_{k_1,k_2\in\mathbb{Z}} \pi(u_1 + k_1, u_2 + k_2) \quad \forall u \in I^2 \tag{18}$$

with $|k_1| \leq N_1$, $|k_2| \leq N_2$.

Proof. Since $P(\pi)$ is bounded, $\pi(w_1, w_2) > 0\ \forall (w_1, w_2) \in \mathbb{R}^2 \setminus \{\bar{0}\}$. Let $d := (d_1, d_2)$ be the vector in the direction of minimum slope with length 1. By assumption this minimum slope is positive. Since the value of π is bounded over the set $[0, 1) \times [0, 1)$, let $k = \sup\{\pi(w_1, w_2) \mid (w_1, w_2) \in [0, 1) \times [0, 1)\}$. Let l be the real such that $\pi(ld) = k$. Set $N_1 = N_2 = \lceil l \rceil$. Now for any (w_1, w_2) and $n_1, n_2 \in \mathbb{Z}$ such that either $|n_1| > N_1$ or $|n_2| > N_2$ (or both), we have $\pi(w_1 + n_1, w_2 + n_2) \geq \pi(\|(w_1 + n_1, w_2 + n_2)\|d) \geq \pi(ld) = k \geq \pi(w_1, w_2)$. □

4.2 Minimality and Extremality of the Trivial Fill-in Function

We use the following characterization of minimal inequalities for the mixed-integer infinite group relaxation by Johnson [24].

Theorem 6 (Theorem 6.1, [24]) *The pair of functions $\phi : I^2 \to \mathbb{R}_+$ and $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ is a minimal valid inequality for the convex hull of $MI(I^2, \mathbb{R}^2, r)$ with $r \neq \bar{0}$ if and only if*

1. *ϕ is subadditive, i.e., $\phi(u^1) + \phi(u^2) \geq \phi(u^1 + u^2)$ $\forall u^1, u^2 \in I^2$*
2. *$\phi(u) + \phi(r - u) = \phi(r) = 1$ for all $u \in I^2$,*
3. *$\pi(w) = \lim_{h \to 0^+} \frac{\phi(\mathbb{P}(hw))}{h}$ $\forall w \in \mathbb{R}^2$.* □

As discussed in Section 3, we define a set $D(\pi)$ that simplifies the verification of minimality of the fill-in function.

Definition 6 Let $P(\pi)$ be a maximal lattice-free triangle or maximal lattice-free quadrilateral.

1. Let d^1 , d^2, ..., d^k with $k \in \{3, 4\}$ be vectors such that $d^i + f$ are the vertices of $P(\pi)$.
2. $D_{ij}(\pi)$: Denote the j^{th} integer point in the relative interior of the edge joining d^i+f and $d^{i+1}+f$ (where $d^4 := d^1$, $d^5 := d^1$ when $P(\pi)$ is triangle, quadrilateral respectively) by X^{ij}. Each $X^{ij} = \delta^{ij} d^i + (1 - \delta^{ij}) d^{i+1} + f$ where $0 < \delta^{ij} < 1$. Let $D_{ij}(\pi) = \{\rho d^i + \gamma d^{i+1} | 0 \leq \rho \leq \delta^{ij}, 0 \leq \gamma \leq 1 - \delta^{ij}\}$.
3. $D(\pi) = \cup_{i,j} D_{ij}(\pi)$.
4. Let C^i be the cone formed by the extreme rays d^i and d^{i+1}, i.e., $C^i = \{\lambda_1 d^i + \lambda_2 d^{i+1} \mid \lambda_1, \lambda_2 \geq 0\}$. □

Geometrically, the set $D_{ij}(\pi)$ is obtained as follows: Take X^{ij}, the j^{th} integer point in the relative interior of the i^{th} edge of $P(\pi)$, and construct a parallelogram such that two of its vertices are X^{ij} and f and the other two vertices lie on the rays $f + \lambda d^i$ and $f + \mu d^{i+1}$ where $\lambda, \mu \geq 0$. If Q is this parallelogram, then $D_{ij}(\pi) = \{w - f \mid w \in Q\}$. See Figure 5 for an illustration of $D(\pi)$.

We prove Theorem 2 in the rest of the section, i.e., we prove that $(\phi^{\bar{0}}, \pi)$ is a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$ if and only if $\mathbb{P}(D(\pi)) := \{\mathbb{P}(u) \mid u \in D(\pi)\} = I^2$. Moreover we show that if $(\phi^{\bar{0}}, \pi)$ and (ϕ', π) are minimal for $MI(I^2, \mathbb{R}^2, r)$, then $\phi' = \phi^{\bar{0}}$.

We begin by establishing one direction of Theorem 2, namely if $\mathbb{P}(D(\pi)) = I^2$, then $(\phi^{\bar{0}}, \pi)$ is a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$.

The following result is standard. (See Johnson [24]).

Proposition 4 *$\phi^{\bar{0}}$ is subadditive, i.e., $\phi^{\bar{0}}(u^1) + \phi^{\bar{0}}(u^2) \geq \phi^{\bar{0}}(u^1 + u^2)$ $\forall u^1, u^2 \in I^2$.*

Note that it is easily verified that when there exists at least one point $\bar{y} \in MI(\{\bar{0}\}, \mathbb{R}^2, r)$ for which $\sum_{w \in \mathbb{R}^2} \pi(w) \bar{y}(w) = 1$, then $\phi^{\bar{0}}(r) = 1$.

The next proposition establishes some of the crucial properties of $D(\pi)$.

Proposition 5 *Let $P(\pi)$ be a bounded maximal lattice-free convex set. For any $v \in D(\pi)$ the following are true:*

1. *There exists a point $(\bar{x}, \bar{y}) \in MI(I^2, \mathbb{R}^2, r)$ with $\bar{x}(\mathbb{P}(v)) > 0$ which satisfies the inequality $(\phi^{\bar{0}}, \pi)$ at equality.*
2. *$\phi^{\bar{0}}(\mathbb{P}(v)) = \pi(v)$.*
3. *$\phi^{\bar{0}}(\mathbb{P}(v)) + \phi^{\bar{0}}(r - \mathbb{P}(v)) = 1$.*
4. *If $(\bar{\phi}, \pi)$ is any valid inequality for $MI(I^2, \mathbb{R}^2, r)$, then $\bar{\phi}(\mathbb{P}(v)) \geq \phi^{\bar{0}}(\mathbb{P}(v))$.*

Proof.

1. Since $v \in D_{ij}(\pi)$, let $v = \rho d^i + \gamma d^{i+1}, \rho \leq \delta^{ij}, \gamma \leq 1 - \delta^{ij}$. Consider the point $v' = (\delta^{ij} - \rho)d^i + (1 - \delta^{ij} - \gamma)d^{i+1}$. Since $\rho \leq \delta^{ij}, \gamma \leq 1 - \delta^{ij}$, $v' \in C^i$. Now consider the solution:
$$\bar{x}(u) = \begin{cases} 1 & \text{if } u = \mathbb{P}(v) \\ 0 & \text{otherwise} \end{cases} \tag{19}$$
$$\bar{y}(w) = \begin{cases} 1 & \text{if } w = v' \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$
Then
$$\begin{aligned} \sum_{u \in I^2} u\bar{x}(u) + \sum_{w \in \mathbb{R}^2} w\bar{y}(w) + f &= \mathbb{P}(v) + v' + f \\ &\equiv (v + v' + f)(\text{mod } \bar{1}) \\ &= \rho d^i + \gamma d^{i+1} + (\delta^{ij} - \rho)d^i \\ &\quad +(1 - \delta^{ij} - \gamma)d^{i+1} + f \\ &= X^{ij} \in \mathbb{Z}^2. \end{aligned}$$
Also
$$\begin{aligned} \textstyle\sum_{u \in I^2} \phi^{\bar{0}}(u)\bar{x}(u) + \sum_{w \in \mathbb{R}^2} \pi(w)\bar{y}(w) &\leq \pi(v) + \pi(v') \\ = (\rho + \gamma) + (\delta^{ij} - \rho + 1 - \delta^{ij} - \gamma) &= 1. \end{aligned} \tag{21}$$
Finally (21) holds at equality because of the validity of $(\phi^{\bar{0}}, \pi)$.
2. Follows from the fact that (21) holds at equality.
3. Consider the point v' constructed in the proof of part 1. Since $\mathbb{P}(v) + v' + f \in \mathbb{Z}^2$, we have that $\mathbb{P}(v) \equiv (-f - v')(\text{mod } \bar{1})$ or $r - \mathbb{P}(v) = \mathbb{P}(v')$ since $r = \mathbb{P}(-f)$. Also note that $v' \in D(\pi)$ and therefore, $\pi(v') = \phi^{\bar{0}}(\mathbb{P}(v')) = \phi^{\bar{0}}(r - \mathbb{P}(v))$. Since (21) holds at equality, we obtain that $\phi^{\bar{0}}(\mathbb{P}(v)) + \pi(v') = 1$ or $\phi^{\bar{0}}(\mathbb{P}(v)) + \phi^{\bar{0}}(r - \mathbb{P}(v)) = 1$.
4. Since $(\bar{x}, \bar{y})$ ((19) and (20)) is a solution of $MI(I^2, \mathbb{R}^2, r)$, we obtain that $\sum_{u \in I^2} \bar{\phi}(u)\bar{x}(u) + \sum_{w \in \mathbb{R}^2} \pi(w)\bar{y}(w) \geq 1$ or $\bar{\phi}(\mathbb{P}(v)) \geq 1 - \pi(v') = \phi^{\bar{0}}(\mathbb{P}(v))$. □

We next present a Corollary of Proposition 5.

Corollary 2 *Let $P(\pi)$ be a bounded maximal lattice-free convex set. Then* $\lim_{h \to 0^+} \frac{\phi^{\bar{0}}(\mathbb{P}(wh))}{h} = \pi(w) \ \forall \ w \in \mathbb{R}^2$.

Therefore the fill-in function is subadditive (Proposition 4) and satisfies the condition $\lim_{h \to 0^+} \frac{\phi^{\bar{0}}(\mathbb{P}(wh))}{h} = \pi(w) \ \forall \ w \in \mathbb{R}^2$ (Corollary 2). Thus by Theorem 6, we obtain the following result: $(\phi^{\bar{0}}, \pi)$ is minimal for $MI(I^2, \mathbb{R}^2, r)$ if and only if $\phi^{\bar{0}}(u) + \phi^{\bar{0}}(r - u) = 1 \ \forall u \in I^2$.

Now by part (3.) of Proposition 5 we obtain the proof of Theorem 2 in one direction, i.e., if $\mathbb{P}(D(\pi)) = I^2$, then $(\phi^{\bar{0}}, \pi)$ is a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$. Also observe that if $\mathbb{P}(D(\pi)) = I^2$, then by part (4.) of

Proposition 5, $(\phi^{\bar{0}}, \pi)$ is the unique minimal function for $MI(I^2, \mathbb{R}^2, r)$, i.e., if $(\phi^{\bar{0}}, \pi)$ and (ϕ', π) are minimal for $MI(I^2, \mathbb{R}^2, r)$ then $\phi' = \phi^{\bar{0}}$.

To complete the proof of Theorem 2, we need to establish that if $\mathbb{P}(D(\pi)) \subsetneq I^2$, then $(\phi^{\bar{0}}, \pi)$ is not a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$. This is done next.

We first present a property of the function π corresponding to a maximal lattice-free bounded set $P(\pi)$.

Proposition 6 *Let $P(\pi)$ be a bounded maximal lattice-free polytope in $\mathbb{R}^2$ (i.e., either a triangle or a quadrilateral) containing f in its interior and let $w^1, w^2 \in \mathbb{R}^2$. Then $\pi(w^1) + \pi(w^2) = \pi(w^1 + w^2)$ if and only if $w^1, w^2, w^1 + w^2 \in C^i$ for some i.*

Proof: $\Leftarrow$ If $w^1, w^2, w^1 + w^2 \in C^i$, then clearly $\pi(w^1) + \pi(w^2) = \pi(w^1 + w^2)$.

$\Rightarrow$ It is sufficient to prove that if w^1 and w^2 do not belong to the same cone, then $\pi(w^1) + \pi(w^2) > \pi(w^1 + w^2)$.

Since $f \in \text{int}(P(\pi))$, the set $P(\pi) - \{f\}$ can be written as

$$(\alpha^i)^T x \leq 1 \;\; i \in \{1, ..., k\}, \tag{22}$$

where $k = 3$ for a triangle and $k = 4$ for a quadrilateral. Now it is easily verified (see for example Johnson [25]) that

$$\pi(u) = \max_{1 \leq i \leq k}\{(\alpha^i)^T u\}. \tag{23}$$

Therefore, if $d \in \text{int}(C^i)$, then $\pi(d) = (\alpha^i)^T d > (\alpha^j)^T d$ for $j \neq i$ and if $d \in C^i \cap C^{i+1}$, then $\pi(d) = (\alpha^i)^T d = (\alpha^{i+1})^T d > (\alpha^j)^T d$ for $j \neq i, i+1$. (where $i + 1 = 1$ if $i = k$).

WLOG assume $w^1 + w^2 \in C^1$. Since w^1 and w^2 do not belong to the same cone, both cannot belong to C^1. Thus we obtain that either $\pi(w^1) > (\alpha^1)^T(w^1)$ or $\pi(w^2) > (\alpha^1)^T(w^2)$. Since (23) also implies that $\pi(w^j) \geq (\alpha^1)^T(w^j)$, $j \in \{1, 2\}$, we obtain $\pi(w^1) + \pi(w^2) > (\alpha^1)^T(w^1 + w^2) = \pi(w^1 + w^2)$. □

Next we present the final step in the proof of Theorem 2.

Proposition 7 *Let $P(\pi)$ be a lattice-free bounded convex set. Suppose $u^* \notin D(\pi)$ and $\phi^{\bar{0}}(\mathbb{P}(u^*)) = \pi(u^*)$. Then,*

1. *The following system has no solution*

$$\mathbb{P}(u^*)x(\mathbb{P}(u^*)) + \sum_{w \in \mathbb{R}^2} wy(w) + f \in \mathbb{Z}^2 \tag{24}$$

$$\phi^{\bar{0}}(\mathbb{P}(u^*))x(\mathbb{P}(u^*)) + \sum_{w \in \mathbb{R}^2} \pi(w)y(w) = 1 \tag{25}$$

$$x(\mathbb{P}(u^*)) \in \mathbb{Z}, x(\mathbb{P}(u^*)) \geq 1, y \geq 0. \tag{26}$$

2. *$\phi^{\bar{0}}(\mathbb{P}(u^*)) + \phi^{\bar{0}}(\mathbb{P}(r - u^*)) > 1$.*

Proof. Let s_i be the line segment between vertices $d^i + f$ and $d^{i+1} + f$ (where $d^4 := d^1$, $d^5 := d^1$ when $P(\pi)$ is triangle, quadrilateral respectively). Let p^i be the set of integer points in the interior of edge s_i of $P(\pi)$.

1. If $u^* \notin P(\pi) - \{f\}$ the result is obvious since $\phi^{\bar{0}}(\mathbb{P}(u^*)) > 1$. Consider the case when $u^* \in (P(\pi) - \{f\}) \setminus D(\pi)$. WLOG assume that $u^* \in C^1$. Assume by contradiction that there exists $(\bar{x}, \bar{y})$ that satisfies (24), (25), and (26). Therefore,
$$X = \mathbb{P}(u^*)\bar{x}(\mathbb{P}(u^*)) + \sum_{w \in \mathbb{R}^2} w\bar{y}(w) + f, \tag{27}$$
where $X \in \mathbb{Z}^2$. As $P(\pi)$ is lattice-free, $\pi(X - f) \geq 1$. Now
$$\begin{aligned} 1 &= \phi^{\bar{0}}(\mathbb{P}(u^*))\bar{x}(\mathbb{P}(u^*)) + \sum_{w \in \mathbb{R}^2} \pi(w)\bar{y}(w) \\ &= \pi(u^*)\bar{x}(\mathbb{P}(u^*)) + \sum_{w \in \mathbb{R}^2} \pi(w)\bar{y}(w) \\ &\geq \pi(u^*)\bar{x}(\mathbb{P}(u^*)) + \pi(\sum_{w \in \mathbb{R}^2} w\bar{y}(w)) \quad (28) \\ &\geq \pi(X - f). \quad (29) \end{aligned}$$
Therefore, $\pi(X - f) = 1$ and $X \in P(\pi)$. Moreover by Proposition 6, $\pi(u) + \pi(v) = \pi(u + v)$ if and only if $u, v, u + v \in C^i$. Since $u^* \in C^1$, $\bar{x}(\mathbb{P}(u^*)) \geq 1$, and (29) is satisfied at equality, we obtain that $X - f \in C^1$ or $X - f \in p^1 - f$. We also obtain that $\sum_{w \in \mathbb{R}^2} w\bar{y}(w) \in C^1$.
It is easily verified that if $u \in C^i \setminus D(\pi)$, then there does not exist any $v \in C^i$, $n \in \mathbb{Z}_+$ such that $n \geq 1$ and $nu + v + f \in p^i$: Assume by contradiction that there exists a $v \in C^i$, $n \in \mathbb{Z}_+$ such that $n \geq 1$ and $nu + v + f \in p^i$. Let $\delta^{ij} d^i + (1 - \delta^{ij})d^{i+1} + f = X^{ij}$. Since $u \in C^i \setminus D(\pi)$, by the definition of $D(\pi)$, $u = \alpha d^i + \beta d^{i+1}$ where either $\alpha > \delta^{ij}$ or $\beta > (1 - \delta^{ij})$. Now $v = (\delta^{ij} - n\alpha)d^i + (1 - \delta^{ij} - n\beta)d^{i+1}$ which implies that $v \notin C^i$, a contradiction.
Therefore as $u^* \in C^1 \setminus D(\pi)$, we obtain that there does not exist a vector $v \in C^1$ such that $v + nu^* \in p^1 - f$ where $n \in \mathbb{Z}_+$ and $n \geq 1$, which is the required contradiction to (27).
2. This follows from the proof of part (1) since
$$\phi^{\bar{0}}(\mathbb{P}(r - u^*)) = \min\{\sum_{w \in \mathbb{R}^2} \pi(w)y(w) \,|\, \mathbb{P}(u^*) + \sum_{w \in \mathbb{R}^2} wy(w) + f \in \mathbb{Z}^2\}.$$

□

Finally, if there exists a points $u \in I^2$ such that $u \notin \mathbb{P}(D(\pi))$, then by Proposition 7, $\phi^{\bar{0}}(u) + \phi^{\bar{0}}(r - u) > 1$ and by Theorem 6, the function is not minimal. This completes the proof of Theorem 2.

Next we present a proof of Corollary 1 showing that if π is extreme for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$ and $\mathbb{P}(D(\pi)) = I^2$, then $(\phi^{\bar{0}}, \pi)$ is the unique extreme function for $MI(I^2, \mathbb{R}^2, r)$.

Proof of Corollary 1: Assume by contradiction that (ϕ, π) is not extreme. Then there exist two valid functions (ϕ_1, π_1) and (ϕ_2, π_2) such that $(\phi_1, \pi_1) \neq (\phi_2, \pi_2)$ and $(\phi, \pi) = \frac{1}{2}(\phi_1, \pi_1) + \frac{1}{2}(\phi_2, \pi_2)$. It can be easily verified that (ϕ_i, π_i) must be minimal. (Otherwise, there exists $(\phi', \pi') <$

(ϕ_1, π_1) which is valid for $MI(I^2, \mathbb{R}^2, r)$. However, this shows that there exists $(\phi'', \pi'') < (\phi, \pi)$ which is valid, a contradiction to the minimality of (ϕ, π)). Now note that $\pi_1 = \pi_2 = \pi$ since π is an extreme inequality for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$. However since $\phi : I^2 \to \mathbb{R}_+$ is the unique function such that (ϕ, π) is minimal, it implies that $\phi_1 = \phi_2 = \phi$, giving the required contradiction. □

5 Unique Lifting Function

In this section, we present the proof of Theorem 3, namely if $P(\pi)$ is a maximal lattice-free triangle of Type 1 or 2, then $(\phi^{\bar{0}}, \pi)$ is an extreme function for $MI(I^2, \mathbb{R}^2, r)$.

We use the following result from Cornuéjols and Margot [10].

Theorem 7 ([10]) *If $P(\pi)$ is a maximal lattice-free triangle, then $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ is extreme for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$.*

We now present a step-by-step outline of the proof of Theorem 3.

Step 1: Rather than verifying the result for all possible triangles of Type 1 and Type 2, we work with 'standard' triangles. This is valid because of the following simple result.

Proposition 8 ([16]) *Let $P(\pi)$ be a maximal lattice-free convex set with a point $f \in \text{interior}(P(\pi))$. Let M be a two-by-two unimodular matrix. Let $P^M(\pi)$ be the maximal lattice-free set defined as $P^M(\pi) = \{x \,|\, x = M(u - v), u \in P(\pi)\}$ where $v \in \mathbb{Z}^2$. Define the functions $\phi^M : I^2 \to \mathbb{R}_+$ and $\pi^M : \mathbb{R}^2 \to \mathbb{R}_+$ as $\phi^M(u) = \phi(\mathbb{P}(M^{-1}u))$ and $\pi^M(w) = \pi(M^{-1}w)$. Then (ϕ, π) is a minimal (extreme resp.) inequality for $MI(I^2, \mathbb{R}^2, r)$ if and only if (ϕ^M, π^M) is a minimal (extreme rep.) inequality for $MI(I^2, \mathbb{R}^2, \mathbb{P}(Mr))$.*

Step 2: The following result is easily verified.

Proposition 9 ([16]) *Let P be a maximal lattice-free triangle of Type 2. Then there exists a unimodular matrix M and $v \in \mathbb{Z}^2$ such that the set $\{x \in \mathbb{R}^2 \,|\, x = M(u - v), u \in P\}$ is a maximal lattice-free convex set with the points $(0, 0)$ and $(1, 0)$ on one side and $(0, 1)$ and $(1, 1)$ in the relative interior of the other two sides.*

A similar 'standard' triangle can be obtained for triangles of Type 1. Propositions 9 and 8 together with Corollary 1 and Theorem 7 imply that it is sufficient to verify that $\mathbb{P}(D(\pi)) = I^2$ for 'standard' triangles of Type 2 and Type 1 to prove Theorem 3.

We now present a detailed proof that $\mathbb{P}(D(\pi)) = I^2$ for 'standard' maximal lattice-free triangles of Type 2. A similar proof can be given for maximal lattice-free triangles of Type 1, see [16].

Definition 7 (Refer to Figure 4.) Any point w will be represented as $w := (w_1, w_2)$. We denote the length of a line segment pq by $|pq|$. Let $P(\pi)$ be a maximal lattice-free triangle with $(0, 0)$ and $(1, 0)$ adjacent integer points in the relative interior of one side and $(0, 1)$ and $(1, 1)$ in the relative interior of the other two sides. We use the following notation for points in this section:

1. The points a^1, a^2 and a^3 represent the vertices of the lattice-free triangle $P(\pi)$.
2. $b^1 := (1, 1)$ is the integer point in the relative interior of the side a^1a^2.
3. $b^2 := (0, 1)$ is the integer point in the relative interior of the side a^2a^3.
4. $b^3 := (0, 0)$ and $b^4 := (1, 0)$ are adjacent integer points in the relative interior of the side a^3a^1.
5. The union of quadrilaterals $fc^1b^1e^1$, $fc^2b^2e^2$, $fc^3b^3e^3$, and $fc^4b^4e^4$ represents a subset of the set $D(\pi) + \{f\}$. (In particular c^1 lies on fa^1, e^1 lies on fa^2 and $f + (c^1 - f) + (e^1 - f) = b^1$. c^2 lies on fa^2, e^2 lies on fa^3 and $f + (c^2 - f) + (e^2 - f) = b^2$. c^3 lies on fa^3, e^3 lies on fa^1 and $f + (c^3 - f) + (e^3 - f) = b^3$. c^4 lies on fa^3, e^4 lies on fa^1 and $f + (c^4 - f) + (e^4 - f) = b^4$).
6. Let g be the point where b^3e^3 and b^4c^4 intersect.
7. We will assume that $f_1 \geq a^2_1$: therefore let i be the point of intersection of b^2e^2 extended and b^3e^3.

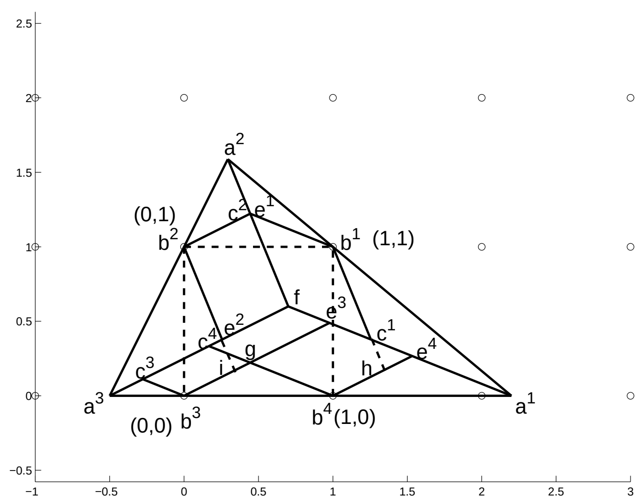

Fig. 4 A maximal lattice-free triangle with the points $(0, 0)$ and $(1, 0)$ on one side and $(0, 1)$ and $(1, 1)$ in the interior of the other two sides

Proposition 10 records some geometric properties used below.

Proposition 10 *Let $P(\pi)$ be a standard Type 2 maximal lattice-free triangle. Under the assumption $f_1 \geq a^2_1$,*

1. *e^1 and c^2 are the same point.*
2. *Triangle b^3gb^4 is symmetric to triangle $b^2c^2b^1$.*
3. *There exists a point h such that b^1c^1 extended to b^1h intersects b^4e^4.*

4. Triangle b^1hb^4 is symmetric to b^2ib^3.

Proposition 11 *If $P(\pi)$ is a standard maximal lattice-free triangle of Type 2, then $P(D(\pi)) = I^2$.*

Proof. Refer to Figure 4. We present the proof in the case in which $f_1 \geq a_1^2$. A similar proof can be presented for the case in which $f_1 \leq a_1^2$. Note that the union of the parallelograms $fc^1b^1e^1$, $fc^2b^2e^2$, $fc^3b^3e^3$ and $fc^4b^4e^4$ is a subset of $D(\pi) + \{f\}$. Using Proposition 10, we obtain that triangles b^3gb^4 and $b^2c^2b^1$ are symmetric. Since b^1, b^2, b^3, and b^4 are integer points and b^1b^2 is parallel to b^3b^4, the fractional parts of points in the triangles b^3gb^4 and $b^2c^2b^1$ are exactly the same. Similarly, b^1hb^4 is symmetric to b^2ib^3 and a similar result regarding fractional parts may be obtained. As the triangles $b^2c^2b^1$ and b^1hb^4 belong to $D(\pi) + \{f\}$, all the fractional parts within the quadrilateral $b^1b^2b^3b^4$ belong to $D(\pi) + \{f\}$, completing the proof. □

6 Non-Unique Lifting Functions

In this section we prove Theorem 4, namely if $P(\pi)$ is a maximal lattice-free triangle of Type 3 or a maximal lattice-free quadrilateral such that π is an extreme function for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$, then $(\phi^{\bar{0}}, \pi)$ is not an extreme function for $MI(I^2, \mathbb{R}^2, r)$. Observe that by Proposition 8 and Theorem 2, it is sufficient to verify $\mathbb{P}(D(\pi)) \subsetneq I^2$ for 'standard' maximal lattice-free triangles of Type 3 or 'standard' quadrilaterals.

We begin with a tool for the analysis of the area of $D(\pi)$.

Proposition 12 ([16]) *Let $P(\pi)$ be a maximal lattice-free polytope in $\mathbb{R}^2$. For any $f := (f_1, f_2) \in P(\pi)$, let $A(f) = \text{Area}(D(\pi))$. If there exists only one integer point in the interior of each side of $P(\pi)$, then A is an affine function of f, i.e., $A(f) = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2$ for some $\alpha_0, \alpha_1, \alpha_2$.*

We construct the 'standard' triangle of Type 3 in Proposition 13.

Proposition 13 ([16]) *Let P be a maximal lattice-free triangle of Type 3. Then there exists a unimodular matrix M and $v \in \mathbb{Z}^2$ such that the set $P^M := \{x \in \mathbb{R}^2 \,|\, x = M(u - v), u \in P\}$ is a maximal lattice-free convex set with the points $(0,0)$, $(1,0)$, and $(0,1)$ in the interior of its sides. Let s_1 be the side of P^M passing thorough $(1,0)$, let s_2 be the side of P^M passing through the point $(0,1)$, and let s_3 be the side of P^M passing through the point $(0,0)$. Then s_1 and s_2 intersect at a point outside the unit square. Let $-m_1$, m_2 and $-m_3$ be the slopes of s_1, s_2 and s_3 respectively. Either $1 < m_1 < \infty$, $0 < m_2 < \infty$ and $0 < m_3 < 1$ or $-\infty < m_1 < 0$, $-1 < m_2 < 0$ and $1 < m_3 < \infty$.*

We next present a proof that $\mathbb{P}(D(\pi)) \subsetneq I^2$ for 'standard' maximal lattice-free triangles of Type 3.

Definition 8 (Refer to Figure 5). Let $P(\pi)$ be a maximal lattice-free triangle with the points $(1,0)$, $(0,1)$ and $(0,0)$ in the relative interior of its sides. We use the following notation for points in this section:

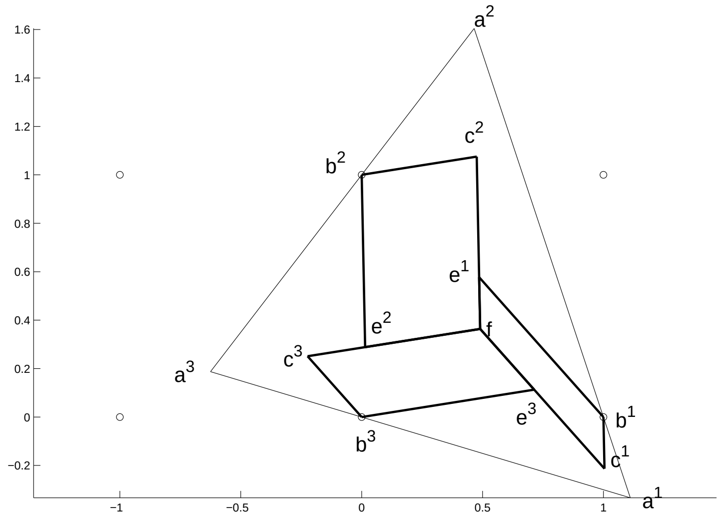

Fig. 5 An example of triangle with one integer point in the interior of each side

1. The points a^1, a^2 and a^3 represent the vertices of the lattice-free triangle $P(\pi)$.
2. $b^1 := (1, 0)$ is the integer point in the relative interior of the side a^1a^2.
3. $b^2 := (0, 1)$ is the integer point in the relative interior of the side a^2a^3.
4. $b^3 := (0, 0)$ is the integer point in the relative interior of the side a^3a^1.
5. The union of quadrilaterals $fc^1b^1e^1$, $fc^2b^2e^2$, and $fc^3b^3e^3$ represents $D(\pi) + \{f\}$. (In particular, c^1 lies on fa^1, e^1 lies on fa^2 and $f+(c^1-f)+(e^1-f) = b^1$. c^2 lies on fa^2, e^2 lies on fa^3 and $f + (c^2 - f) + (e^2 - f) = b^2$. c^3 lies on fa^3, e^3 lies on fa^1 and $f + (c^3 - f) + (e^3 - f) = b^3$).

Proposition 14 *If $P(\pi)$ is a maximal lattice-free triangle of Type 3, then $(\phi^{\bar{0}}, \pi)$ is not minimal for $MI(I^2, \mathbb{R}^2, r)$.*

Proof. It is enough to analyze the 'standard' triangle presented in Proposition 13. Let s_1, s_2 and s_3 be the sides of $P(\pi)$ passing through $(1, 0)$, $(0, 1)$, and $(0, 0)$ respectively. We assume WLOG that the slope of s_1 is negative and the slope of s_2 is positive (and s_1 is not vertical). The other case can be proven similarly. Let m_1 be the negative of the slope of s_1, m_2 be the slope of s_2 and m_3 be the negative of the slope of s_3.

We know that f is in the strict interior of the triangle $a^1a^2a^3$. By Proposition 12, the area is an affine function of the position of f. Therefore the area of $D(\pi)$ is maximized when f is the same point as either a^1, a^2 or a^3. We consider these three cases, see Figure 6.

1. f is same as a^1. The area of $D(\pi)$ is the area of the parallelogram $a^1e^2b^2c^2$. The equation of the line passing through c^2a^1 is $m_1x + y = m_1$. The

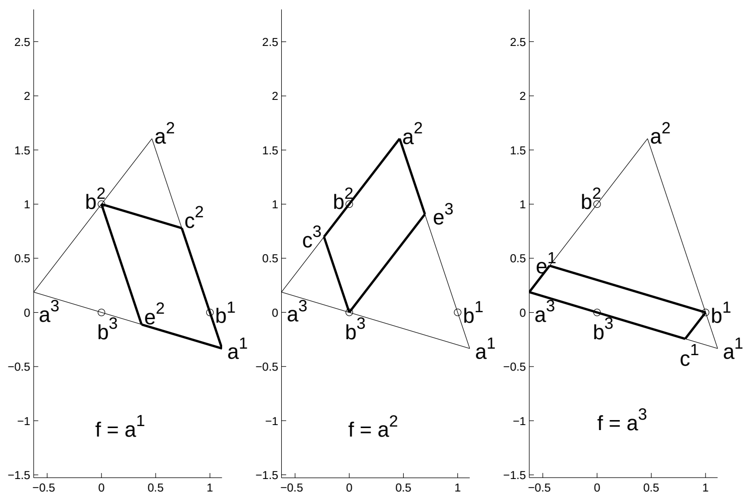

Fig. 6 f is the same as vertex of the triangle $P(\pi)$

coordinates of c^2 are $\left(\frac{m_1-1}{m_1-m_3}, \frac{m_1(1-m_3)}{m_1-m_3}\right)$. Using this, the area of $a^1e^2b^2c^2$ is $\frac{m_1-1}{m_1-m_3}$. As $m_1 > 1$ and $0 < m_3 < 1$, we obtain that Area$(a^1e^2b^2c^2) < 1$.

2. f is same as a^2. The area of $D(\pi)$ is the area of the parallelogram $a^2e^3b^3c^3$. The equation of the line passing through a^2c^3 is $-m_2x + y = 1$. The coordinates of e^3 are $\left(\frac{m_1}{m_1+m_2}, \frac{m_1m_2}{m_1+m_2}\right)$. Using this, the area of $a^2e^3b^3c^3$ is $\frac{m_1}{m_1+m_2}$. As $m_1 > 0$ and $m_2 > 0$, we obtain that Area$(a^2e^3b^3c^3) < 1$.
3. f is same as a^3. The area of $D(\pi)$ is the area of the parallelogram $a^3e^1b^1c^1$. The equation of the line passing through a^3c^1 is $m_3x + y = 0$. The coordinates of e^1 are $\left(\frac{m_3-1}{m_2+m_3}, \frac{m_3(m_2+1)}{m_2+m_3}\right)$. Using this, the area of $a^3e^1b^1c^1$ is $\frac{(1+m_2)m_3}{m_2+m_3}$. As $m_2 > 0$ and $m_3 < 1$, we obtain that Area$(a^3e^1b^1c^1) < 1$.

Thus Area$(D(\pi)) < 1$. This implies that $\mathbb{P}(D(\pi))$ is a proper subset of I^2. Therefore, it follows from Proposition 7 that $(\phi^{\bar{0}}, \pi)$ is not minimal. □

The next example illustrates that when $P(\pi)$ is a maximal lattice-free triangle of Type 3, the function $(\phi^{\bar{0}}, \pi)$ is not minimal.

Example 2 Let $P(\pi)$ be the triangle with vertices $(0.25, 1.25)$, $(-0.75, 0.25)$, and $(1.25, -5/12)$ and let $f = (0.5, 0.5)$. Then it can be verified that $P(\pi)$ is a lattice-free triangle with only one integer point in the interior of each of its sides and non-integral vertices. $\phi^{\bar{0}}(0.1, 0.2) = 1.1$ and $\phi^{\bar{0}}$ is not minimal. There are two distinct functions ϕ_1 and ϕ_2 such that both (ϕ_1, π) and (ϕ_2, π) are extreme. See Figure 7. (The proof of the extremality of these functions is similar to the proof of Theorem 7.1 in Dey and Richard [14]). □

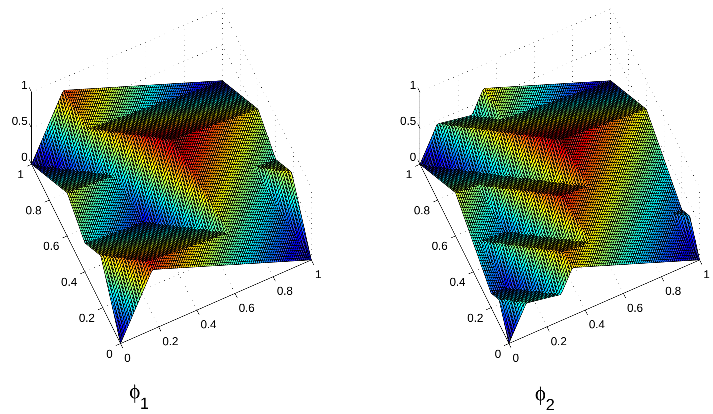

Fig. 7 There exist distinct functions ϕ_1 and ϕ_2 such that (ϕ_1, π) and (ϕ_2, π) are extreme.

In the case of maximal lattice-free quadrilaterals, the inequality π is not always extreme for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$. (see Cornuéjols and Margot [10]). When π is extreme, the proof that $\mathbb{P}(D(\pi)) \subsetneq I^2$ is similar to the proof for maximal lattice-free triangles of Type 3, see [16].

7 Some Conditions for Extremality of General Fill-in Function

As discussed in Section 3, we now analyze the properties of the function $\phi^{u^*} : I^2 \rightarrow \mathbb{R}_+$. The function ϕ^{u^*} is obtained by first lifting the integer variable $x(u^*)$ using

$$V(u^*) = \max_{n \in \mathbb{Z}_+, n \geq 1} \{ \frac{1 - \pi(w)}{n} \,|\, \mathbb{P}(w) = r - nu^* \}$$

and then applying the fill-in procedure to obtain,

$$\phi^{u^*}(u) = \inf_{n \in \mathbb{Z}_+} \{ \pi(w) + nV(u^*) \,|\, \mathbb{P}(w) = u - nu^* \} \quad \forall u \in I^2.$$

Validity of the function (ϕ^{u^*}, π) for $MI(I^2, \mathbb{R}^2, r)$ is easily verified. To establish that minimality of (ϕ^{u^*}, π) along with extremality of π for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$, implies extremality of (ϕ^{u^*}, π), we use the following result.

Theorem 8 ([24]) *If (ϕ, π) is a minimal inequality for $MI(I^2, \mathbb{R}^2, r)$, then*

1. $\phi(u) + \phi(v) \geq \phi(u + v)$ $\forall u, v \in I^2$.
2. $\phi(u) + \sum_{w \in \mathbb{R}^2} \pi(w)y(w) \geq \phi(v)$ *whenever* $u + \mathbb{P}(\sum_{w \in \mathbb{R}^2} wy(w)) = v$.

3. $\sum_{w\in\mathbb{R}^2} \pi(w)y(w) \geq \pi(w')$ *whenever* $\sum_{w\in\mathbb{R}^2} wy(w) = w'$. □

Proposition 15 *If* π *is extreme for* $MI(\{\bar{0}\}, \mathbb{R}^2, r)$ *and* (ϕ^{u^*}, π) *is minimal, then* (ϕ^{u^*}, π) *is extreme for* $MI(I^2, \mathbb{R}^2, r)$.

Proof: Assume by contradiction that (ϕ^{u^*}, π) is not extreme for $MI(I^2, \mathbb{R}^2, r)$. Then $(\phi^{u^*}, \pi) = \frac{1}{2}(\phi_1, \pi_1) + \frac{1}{2}(\phi_2, \pi_2)$ where (ϕ_i, π_i) are valid minimal functions and $(\phi_1, \pi_1) \neq (\phi_2, \pi_2)$. (Note that since (ϕ^{u^*}, π) is minimal, (ϕ_i, π_i) must be minimal).

First observe that $\pi_1 = \pi_2$ since π is extreme for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$.

Next we claim that $\phi_1(u^*) = \phi_2(u^*) = V := \max_{n\in\mathbb{Z}_+, n\geq 1} \frac{1-\pi(r-nu^*)}{n}$. Assume by contradiction that $\phi_1(u^*) \neq \phi_2(u^*)$. WLOG suppose that $\phi_1(u^*) < V$. By definition of V, there exists $\bar{n} \in \mathbb{Z}_+, \bar{w} \in \mathbb{R}^2$ such that $\bar{n}V + \pi(\bar{w}) = 1$ and $\mathbb{P}(\bar{w}) + \bar{n}u^* = r$. This implies that $\exists(\bar{x}, \bar{y}) \in MI(I^2, \mathbb{R}^2, r)$ with $\bar{x}(u^*) = \bar{n}$, $\bar{x}(v) = 0\ \forall v \in I^2, v \neq u^*$ and $y(\bar{w}) = 1$. Therefore $\phi_1(u^*)\bar{n} + \pi(\bar{w}) < 1$, a contradiction.

Finally, we claim that $\phi_1 = \phi_2 = \phi^{u^*}$. Assume by contradiction that $\phi_1 \neq \phi_2$. Since (ϕ_1, π) and (ϕ_2, π) are minimal, we obtain that $\exists v'$ such that $\phi_1(v') = \phi^{u^*}(v') + \delta$, where $\delta > 0$. By definition of ϕ^{u^*}, there exists $n^{v'} \in \mathbb{Z}_+$, $w^{v'} \in \mathbb{R}^2$ such that $\phi^{u^*}(v') \geq n^{v'}V + \pi(w^{v'}) - \frac{\delta}{2}$. Since $\phi_1(u^*) = V$, we obtain that $\phi_1(v') \geq \delta + n^{v'}\phi_1(u^*) + \pi(w^{v'}) - \frac{\delta}{2}$. Therefore, $\phi_1(v') > n^v\phi_1(u^*) + \pi(w^{v'})$. This contradicts Theorem 8 as ϕ_1 is minimal. □

More generally, it is possible to present conditions when the minimality of the general fill-in function $\phi^{G,V}$ implies the extremality of $(\phi^{G,V}, \pi)$ for $MI(I^2, \mathbb{R}^2, r)$, see [16].

The rest of this section is devoted to proving Theorem 5. We begin with some definitions.

Definition 9 (Refer to Figure 8.) Let $P(\pi)$ be a maximal lattice-free triangle with $(0, 0)$, $(1, 0)$, and $(0, 1)$ in the interior of its sides. We use the following notation for the rest of this section:

1. The points a^1, a^2 and a^3 represent the vertices of the lattice-free triangle $P(\pi)$.
2. $b^1 := (1, 0)$ is the integer point in the interior of the side a^1a^2.
3. $b^2 := (0, 1)$ is the integer point in the interior of the side a^2a^3.
4. $b^3 := (0, 0)$ is the integer point in the interior of the side a^3a^1.

Let

$$\begin{aligned}
D_{11}(\pi) &= \{\eta d^1 + \gamma d^2 \,|\, 0 \leq \eta \leq \delta^{11}, 0 \leq \gamma \leq (1 - \delta^{11})\} \\
&\qquad\qquad (\text{Quadrilateral } fc^1b^1e^1 - \{f\}) \\
D_{21}(\pi) &= \{\eta d^2 + \gamma d^3 \,|\, 0 \leq \eta \leq \delta^{21}, 0 \leq \gamma \leq (1 - \delta^{21})\} \\
&\qquad\qquad (\text{Quadrilateral } fc^2b^2e^2 - \{f\}) \\
D_{31}(\pi) &= \{\eta d^3 + \gamma d^1 \,|\, 0 \leq \eta \leq \delta^{31}, 0 \leq \gamma \leq (1 - \delta^{31})\} \\
&\qquad\qquad (\text{Quadrilateral } fc^3b^3e^3 - \{f\}).
\end{aligned}$$

Note that δ^{ij} was presented in Definition 6. (For a triangle of Type 3, $j = 1$ for all i). To describe $D(\pi) + \{f\}$, we need the following points,

1. c^1: $f + \delta^{11} d^1$.
2. e^1: $f + (1 - \delta^{11}) d^2$.
3. c^2: $f + \delta^{21} d^2$.
4. e^2: $f + (1 - \delta^{21}) d^3$.
5. c^3: $f + \delta^{31} d^3$.
6. e^3: $f + (1 - \delta^{31}) d^1$.

The set $D(\pi) + \{f\}$ is represented by the union of the quadrilaterals: $fc^1b^1e^1$, $fc^2b^2e^2$, and $fc^3b^3e^3$. Some points in $\mathbb{R}^2$ are described next:

1. g: $f + (1 - \delta^{11})d^2 + (\delta^{11} - 1 + \delta^{31})d^1$. (Note: $g = b^1 - e^3 + f$.)
2. i: $f + (1 - \delta^{11})d^2 + (\delta^{11} - 1 + \delta^{31})d^1 - (\delta^{31} - 1 + \delta^{21})d^3$. (Note: $i = g - c^3 + e^2$.)
3. j: $f + \delta^{21}d^2 + (\delta^{11} - 1 + \delta^{31})d^1 - (\delta^{31} - 1 + \delta^{21})d^3$. (Note: $j = i - e^1 + c^2$.)
4. m: Mid point of i and j.
5. k: $f + \delta^{21}d^2 - (\delta^{31} - 1 + \delta^{21})d^3$. (Note: $k = j - g + e^1$.)
6. l: midpoint of e^1 and c^2.
7. u_0: $f + \left(\frac{1-\delta^{11}+\delta^{21}}{2}\right) d^2 + (\delta^{11} + \delta^{31} - 1)d^1$. (Note: $u_0 = g - i + m$.)
8. v_0: $f + \left(\frac{1-\delta^{11}+\delta^{21}}{2}\right) d^2 + (1 - \delta^{31} - \delta^{21})d^3$. (Note: $v_0 = k - m + i$.)

It can be verified that $(u_0 - f) + (v_0 - f) + f = (1, 1)$. □

For any point $p \in \mathbb{R}^2$, we denote the point $p - f$ by $\bar{p}$. In particular, $\bar{v}_0 = v_0 - f$ and $\bar{u}_0 = u_0 - f$.

Theorem 5 provides a sufficient condition for $(\phi^{\mathbb{P}(\bar{v}_0)}, \pi)$ to be extreme for $MI(I^2, \mathbb{R}^2, r)$ for the specific point $\mathbb{P}(\bar{v}_0) \in I^2$. More precisely, we prove the following statement.

Theorem 5 If $\phi^{\mathbb{P}(\bar{v}_0)}(\mathbb{P}(\bar{v}_0)) = 1 - \pi(\bar{u}_0)$, then $(\phi^{\mathbb{P}(\bar{v}_0)}, \pi)$ is an extreme function for $MI(I^2, \mathbb{R}^2, r)$. □

Next we present some results needed in the proof of Theorem 5.

Proposition 16 *Let δ^{21}, δ^{31}, and δ^{11} be as presented in Definition 9. Then*

1. $1 - \delta^{21} < \delta^{31}$, $1 - \delta^{31} < \delta^{11}$, *and* $1 - \delta^{11} < \delta^{21}$.
2. $(1, 0) = f + \delta^{11}d^1 + (1 - \delta^{11})d^2$, $(0, 1) = f + \delta^{21}d^2 + (1 - \delta^{21})d^3$, *and* $(0, 0) = f + \delta^{31}d^3 + (1 - \delta^{31})d^1$. □

Definition 10 Let $T(\pi) \subset \mathbb{R}^2$ be the set $(D(\pi) + \{f\}) \cup H$ where $H = \text{conv}(c^2e^1gijk)$.

Lemma 2 ([16]) $\mathbb{P}(T(\pi)) = I^2$. □

We also need the following result from Gomory and Johnson [20].

Theorem 9 ([20]) *If $\phi : I^2 \to \mathbb{R}_+$ is a valid function for $MI(I^2, \emptyset, r)$ and if $\phi(u) + \phi(r - u) \leq 1 \ \forall u \in I^2$, then ϕ is subadditive.*

Proof of Theorem 5: Since π is extreme for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$, the result will follow from Proposition 15 if we show that $(\phi^{\mathbb{P}(\bar{v}_0)}, \pi)$ is minimal for $MI(I^2, \mathbb{R}^2, r)$.

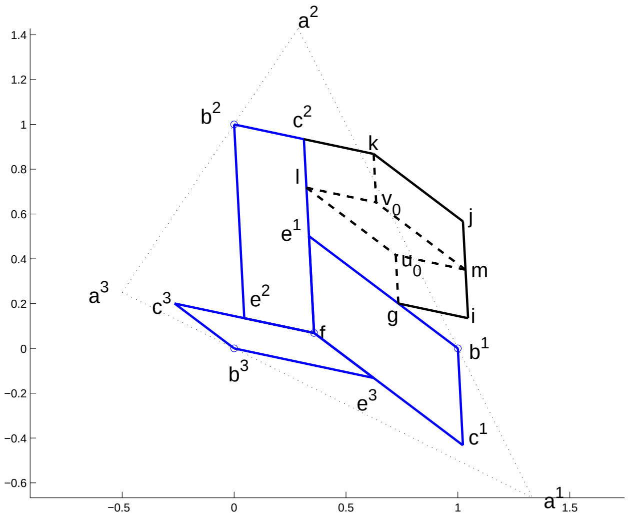

Fig. 8 $T(\pi)$

To prove that $\phi^{\mathbb{P}(\bar{v}_0)}$ is minimal, we need to verify the conditions of Theorem 6. However, using the definition of $(\phi^{\mathbb{P}(\bar{v}_0)}, \pi)$ directly is not convenient; because for any $u \in I^2$, we do not know the values of $\hat{n} \in \mathbb{Z}_+$ and $\hat{w} \in \mathbb{R}^2$ for which $\hat{n}\mathbb{P}(\bar{v}_0) + \mathbb{P}(\hat{w}) = u$ and $\phi^{\mathbb{P}(\bar{v}_0)}(u) = \hat{n}\phi^{(\mathbb{P}(\bar{v}_0))}(\mathbb{P}(\bar{v}_0)) + \pi(\hat{w})$.

Therefore, instead of working with $\phi^{\mathbb{P}(\bar{v}_0)}$, we prove the minimality of $\phi^{\mathbb{P}(\bar{v}_0)}$ by creating an upper bound $\tilde{\phi}$ on $\phi^{\mathbb{P}(\bar{v}_0)}$. We then show that $\tilde{\phi}$ satisfies the symmetry conditions, i.e., $\tilde{\phi}(u) + \tilde{\phi}(r-u) \leq 1 \ \forall u \in I^2$. Since $\phi^{\mathbb{P}(\bar{v}_0)}$ is a valid function and $\tilde{\phi}$ is an upper bound, $\tilde{\phi}$ is a valid function. Now using Theorem 9 we will show that $\tilde{\phi}$ is subadditive. It will also be verified that $\tilde{\phi}$ satisfies $\lim_{h\to 0^+} \frac{\tilde{\phi}(\mathbb{P}(wh))}{h} = \pi(w) \ \forall w \in \mathbb{R}^2$ and $\tilde{\phi}(r) = 1$. This will show that $(\tilde{\phi}, \pi)$ is minimal for $MI(I^2, \mathbb{R}^2, r)$ implying that $\tilde{\phi}$ is the same function as $\phi^{\mathbb{P}(\bar{v}_0)}$ and thus completing the proof.

The proof has two main steps. Step one involves creating the function $\tilde{\phi} : I^2 \to \mathbb{R}_+$ and showing that this function is an upper bound on the function $\phi^{\mathbb{P}(\bar{v}_0)}$. Step two involves proving that $\tilde{\phi}(u) + \tilde{\phi}(r-u) \leq 1 \ \forall u \in I^2$, $\lim_{h\to 0^+} \frac{\tilde{\phi}(\mathbb{P}(wh))}{h} = \pi(w) \ \forall w \in \mathbb{R}^2$, and $\tilde{\phi}(r) = 1$.

Step 1: To define the function $\tilde{\phi}$, we first define a function $\phi_1 : T(\pi) \to \mathbb{R}_+$. By Lemma 2, we know that $\mathbb{P}(T(\pi)) = I^2$. This allows us to define $\tilde{\phi} : I^2 \to \mathbb{R}_+$ as:

$$\tilde{\phi}(u) = \min\{\phi_1(w) | \mathbb{P}(\bar{w}) = u\}. \tag{30}$$

We next present the function ϕ_1. Refer to Figure 8. We use the symbols Q^{11}, and Q^{21} to represent the quadrilaterals $e^1 g u_0 l$ and $gimu_0$ respectively.

$$\phi_1(u) = \begin{cases} \pi(u-f) & \text{if } u \in (D(\pi)+\{f\}) \cup Q^{11} \\ \pi(u-(1,0)-f) & \text{if } u \in Q^{21} \\ V(\mathbb{P}(\bar{v}_0)) + \pi(u-v_0) & \text{otherwise.} \end{cases} \tag{31}$$

Claim: ϕ_1 is well-defined. We need to verify that if u belongs to different categories in (31), then $\phi_1(u)$ has the same value for all the categories. Since by assumption $\phi^{\mathbb{P}(\bar{v}_0)}(\mathbb{P}(\bar{v}_0)) = 1-\pi(\bar{u}_0)$ and $\pi(\bar{u}_0) = \pi((\frac{1-\delta^{11}+\delta^{21}}{2})d^2+(\delta^{11}+\delta^{31}-1)d^1) = \frac{\delta^{11}+\delta^{21}-1}{2}+\delta^{31}$, we obtain $\phi^{\mathbb{P}(\bar{v}_0)}(\mathbb{P}(\bar{v}_0)) = \frac{3}{2}-\frac{\delta^{11}}{2}-\frac{\delta^{21}}{2}-\delta^{31}$.

1. u belongs to the line segment c^2l: It is easily verified that the function is linear (both the first and third case) over this interval. Therefore it is enough to check the value of the function ϕ_1 at $u = c^2$ and $u = l$.
   (a) $u = c^2$: From the first case in (31), $\phi_1(c^2) = \pi(\bar{c^2}) = \pi(\delta^{21}d^2) = \delta^{21}$. From the third case in (31), $\phi_1(c^2) = V(\mathbb{P}(\bar{v}_0)) + \pi(c^2 - v_0)$, or

$$\begin{aligned} \phi_1(c^2) &= \frac{3}{2} - \frac{\delta^{11}}{2} - \frac{\delta^{21}}{2} - \delta^{31} \\ &\quad + \pi((\frac{-1+\delta^{11}+\delta^{21}}{2})d^2 + (\delta^{31}+\delta^{21}-1)d^3) \\ &= \delta^{21}. \end{aligned}$$

   (b) $u = l$: From the first case in (31), $\phi_1(l) = \pi(\bar{l}) = \pi(\frac{(\delta^{21}-\delta^{11}+1)}{2}d^2) = \frac{\delta^{21}-\delta^{11}+1}{2}$. From the third case in (31), $\phi_1(l) = V(\mathbb{P}(\bar{v}_0)) + \pi(l - v_0)$, or

$$\begin{aligned} \phi_1(l) &= \frac{3}{2} - \frac{\delta^{11}}{2} - \frac{\delta^{21}}{2} - \delta^{31} + \pi((\delta^{31}+\delta^{21}-1)d^3) \\ &= \frac{\delta^{21}-\delta^{11}+1}{2}. \end{aligned}$$

2. u belongs to the line segment lu_0: the proof is similar.
3. u belongs to the line segment u_0g: the proof is similar.
4. u belongs to the line segment u_0m: the proof is similar.

Finally we verify that $\tilde{\phi}$ is an upper bound on $\phi^{\mathbb{P}(\bar{v}_0)}$. This follows from the definition $\phi^{\mathbb{P}(\bar{v}_0)}(u) = \inf_{n\in\mathbb{Z}_+}\{n\phi^{(\mathbb{P}(\bar{v}_0))}(\mathbb{P}(\bar{v}_0)) + \pi(w) \,|\, \mathbb{P}(w) + n\mathbb{P}(\bar{v}_0) = u\}$. Now this claim easily follows from (31) and (30).

Step 2:

- $\lim_{h\to 0^+}\frac{\tilde{\phi}(\mathbb{P}(wh))}{h} = \pi(w)\ \forall w \in \mathbb{R}^2$: This follows from the fact that $\tilde{\phi}(u) = \phi^{\bar{0}}(u)\ \forall u \in D(\pi)$ and Corollary 2.

- $\tilde{\phi}(r) = 1$: We know that $\tilde{\phi}$ is an upper bound to $\phi^{\mathbb{P}(\bar{v}_0)}$. Therefore, $\tilde{\phi}(r) \geq 1$. Moreover we have that $\tilde{\phi}(r) \leq \phi_1(\bar{b_1}) = 1$.
- Finally we show that $\tilde{\phi}(u) + \tilde{\phi}(r-u) \leq 1\ \forall u \in I^2$. For $u \in I^2$, we call $r - u \in I^2$ the complementary point. By the definition of ϕ_1 and $\tilde{\phi}$, it is easily verified that $\tilde{\phi}(u) + \tilde{\phi}(r-u) \leq 1\ \forall u \in D(\pi)$. We now present some key complementary points:

1. Complement of $\mathbb{P}(\bar{e^1})$ is $\mathbb{P}(\bar{j})$: $\bar{e}^1 + \bar{j} + f = e^1 + j - f = e^1 + (i - e^1 + c^2) - f = i + c^2 - f = (g - c^3 + e^2) + c^2 - f = b^1 - e^3 + f - c^3 + e^2 + c^2 - f = b^1 + (-e^3 - c^3 - f) + (e^2 + c^2 + f) = (1,1)$. Therefore, $\mathbb{P}(\bar{e^1}) + \mathbb{P}(\bar{j}) = -f = r$.
2. Complement of $\mathbb{P}(\bar{g})$ is $\mathbb{P}(\bar{k})$: $\bar{g} + \bar{k} + f = \bar{e}^1 + \bar{j} + f = (1,1)$.
3. Complement of $\mathbb{P}(\bar{u_0})$ is $\mathbb{P}(\bar{v}_0)$: $\bar{u}_0 + \bar{v}_0 + f = \bar{g} + \bar{k} + f = (1,1)$.
4. Complement of $\mathbb{P}(\bar{l})$ is $\mathbb{P}(\bar{m})$: $\bar{l} + \bar{m} + f = \bar{e}^1 + \bar{j} + f = (1,1)$.
5. Complement of $\mathbb{P}(\bar{i})$ is $\mathbb{P}(\bar{c^2})$: $\bar{i} + \bar{c}^2 + f = \bar{l} + \bar{m} + f = (1,1)$.

Note that the function ϕ_1 is linear in each of the following quadrilaterals: kv_0mj, kv_0lc^2, lv_0mu_0, le^1gu_0, and u_0gim. Therefore to prove that $\tilde{\phi}(u) + \tilde{\phi}(r-u) \leq 1 \ \forall u \in I^2 \setminus \mathbb{P}(D(\pi))$, it is enough to check the following five cases:

1. $\tilde{\phi}(\mathbb{P}(\bar{e^1})) + \tilde{\phi}(\mathbb{P}(\bar{j})) \leq 1$: $\tilde{\phi}(\mathbb{P}(\bar{e^1})) \leq \phi_1(e^1) = 1 - \delta^{11}$. $\tilde{\phi}(\mathbb{P}(\bar{j})) \leq \phi_1(j) = \phi^{(\mathbb{P}(\bar{v}_0))}(\mathbb{P}(\bar{v}_0)) + \pi(j - v_0) = \frac{3}{2} - \frac{\delta^{11}}{2} - \frac{\delta^{21}}{2} - \delta^{31} + \pi((\delta^{11} + \delta^{31} - 1)d^1 + (\frac{\delta^{21}+\delta^{11}-1}{2})d^2) = \delta^{11}$. Therefore, $\tilde{\phi}(\mathbb{P}(\bar{e^1})) + \tilde{\phi}(\mathbb{P}(\bar{j})) \leq 1$.
2. $\tilde{\phi}(\mathbb{P}(\bar{g})) + \tilde{\phi}(\mathbb{P}(\bar{k})) \leq 1$: $\tilde{\phi}(\mathbb{P}(\bar{g})) \leq \phi_1(g) = \pi(\bar{g}) = \delta^{31}$. $\tilde{\phi}(\mathbb{P}(\bar{k})) \leq \phi_1(k) = \phi^{(\mathbb{P}(\bar{v}_0))}(\mathbb{P}(\bar{v}_0)) + \pi(\frac{\delta^{21}+\delta^{11}-1}{2}d^2) = 1 - \delta^{31}$.
3. $\tilde{\phi}(\mathbb{P}(\bar{u_0})) + \tilde{\phi}(\mathbb{P}(\bar{v}_0)) \leq 1$: $\tilde{\phi}(\mathbb{P}(\bar{u_0})) + \tilde{\phi}(\mathbb{P}(\bar{v}_0)) \leq \tilde{\phi}(\mathbb{P}(\bar{v}_0)) + \phi_1(u_0) \leq 1$.
4. $\tilde{\phi}(\mathbb{P}(\bar{l})) + \tilde{\phi}(\mathbb{P}(\bar{m})) \leq 1$: $\tilde{\phi}(\mathbb{P}(\bar{l})) \leq \phi_1(l) = \frac{\delta^{21}+1-\delta^{11}}{2}$. $\tilde{\phi}(\mathbb{P}(\bar{m})) \leq \phi_1(m) = \frac{1+\delta^{11}-\delta^{21}}{2}$.
5. $\tilde{\phi}(\mathbb{P}(\bar{i})) + \tilde{\phi}(\mathbb{P}(\bar{c^2})) \leq 1$: $\tilde{\phi}(\mathbb{P}(\bar{c^2})) \leq \phi_1(c^2) = \delta^{21}$. $\tilde{\phi}(\mathbb{P}(\bar{i})) \leq \phi_1(i) = \pi(i - (1,0) - f) = \pi((1 - \delta^{21})d^3) = 1 - \delta^{21}$. □

8 Concluding Remarks

Computational experiments are under way to test the effectiveness of the inequalities based on two rows of a simplex tableau. Even though our approach leads to only a very special set of inequalities, the number of these inequalities is huge - first one can select three or four continuous non-basic variables, and then for each such choice there are potentially a large number of triangles or quadrilaterals. There are also obvious questions concerning the relative effectiveness of the triangle inequalities of different types and the quadrilateral inequalities. It is also possible that data independent choices such as inequalities based on 45^o splits, or specific triangles such as those obtained from mixing sets (see [16]) may be effective complements to the GMICs.

Other than the important problem of selecting appropriate triangle or quadrilateral inequalities, various other theoretical questions remain open. This includes further analysis of lifting functions in the case of maximal lattice-free triangles of Type 3 and maximal lattice-free quadrilaterals, the incorporation of information about bounds on integer variables when lifting, the derivation of closed-form expressions for the trivial fill-in functions (see [16] for closed-form expressions of trivial fill-in function for subclasses such as sequential-merge inequalities and mixing inequalities), and a study of maximal lattice-free convex sets in higher dimensions.

References

1. Andersen, K., Louveaux, Q., Weismantel, R., Wolsey, L.: Cutting planes from two rows of a simplex tableau. In: M. Fischetti, D.P. Williamson (eds.) Proceedings 12$^{\text{th}}$ Conference on Integer Programming and Combinatorial Optimization, pp. 1–15. Springer-Verlag (2007)
2. Aráoz, J., Evans, L., Gomory, R.E., Johnson, E.L.: Cyclic group and knapsack facets. Math. Prog. **96**, 377–408 (2003)
3. Balas, E.: Intersection cuts - a new type of cutting planes for integer programming. Oper. Res. **19**, 19–39 (1971)
4. Balas, E.: Disjunctive programming. Ann. Discrete Math. **5**, 3–51 (1979)
5. Balas, E., Jeroslow, R.: Strengthening cuts for mixed integer programs. European J. Oper. Res. **4**, 224–234 (1980)
6. Borozan, V., Cornuéjols, G.: Minimal valid inequalities for integer constraints (2007). http://integer.tepper.cmu.edu
7. Cook, W., Kannan, R., Schrijver, A.: Chvátal closures for mixed integer programming problems. Math. Prog. **47**, 155–174 (1990)
8. Cornuéjols, G., Li, Y.: Elementary closures for integer programs. Oper. Res. Lett. **28**, 1–8 (2001)
9. Cornuéjols, G., Li, Y., Vandenbussche, D.: K-cuts: a variation of Gomory mixed integer cuts from the LP tableau. INFORMS J. Comput. **15**, 385–396 (2003)
10. Cornuéjols, G., Margot, F.: On the facets of mixed integer programs with two integer variables and two constraints (2008). To appear in Math. Prog.
11. Dash, S., Günlük, O.: On the strength of Gomory mixed-integer cuts as group cuts. Tech. Rep. RC23967, IBM Research report (2006)
12. Dash, S., Günlük, O.: Valid inequalities based on simple mixed-integer sets. Math. Prog. **105**, 29–53 (2006)
13. Dey, S.S., Richard, J.-P.P.: Sequential-merge facets for two-dimensional group problems. In: M. Fischetti, D.P. Williamson (eds.) Proceedings 12$^{\text{th}}$ Conference on Integer Programming and Combinatorial Optimization, pp. 30–42. Springer-Verlag (2007)
14. Dey, S.S., Richard, J.-P.P.: Facets of two-dimensional infinite group problems. Math. Oper. Res. **33**, 140–166 (2008)
15. Dey, S.S., Wolsey, L.A.: Lifting integer variables in minimal inequalities corresponding to lattice-free triangles. In: A. Lodi, A. Panconesci, G. Rinaldi (eds.) Proceedings 13$^{\text{th}}$ Conference on Integer Programming and Combinatorial Optimization, pp. 463–475. Springer-Verlag (2007)
16. Dey, S.S., Wolsey, L.A.: Two row mixed integer cuts via lifting. Tech. Rep. 30, CORE DP, Université catholique de Louvain, Louvain-la-Neuve, Belgium (2008)
17. Fischetti, M., Saturni, C.: Mixed integer cuts from cyclic groups. Math. Prog. **109**, 27–53 (2007)
18. Gomory, R.E.: An algorithm for the mixed-integer problem. Tech. Rep. RM-2597, Rand Report (1960)
19. Gomory, R.E.: Some polyhedra related to combinatorial problems. Linear Algebra Appl. **2**, 451–558 (1969)
20. Gomory, R.E., Johnson, E.L.: Some continuous functions related to corner polyhedra, part I. Math. Prog. **3**, 23–85 (1972)
21. Gomory, R.E., Johnson, E.L.: Some continuous functions related to corner polyhedra, part II. Math. Prog. **3**, 359–389 (1972)
22. Gomory, R.E., Johnson, E.L.: T-space and cutting planes. Math. Prog. **96**, 341–375 (2003)
23. Gomory, R.E., Johnson, E.L., Evans, L.: Corner polyhedra and their connection with cutting planes. Math. Prog. **96**, 321–339 (2003)
24. Johnson, E.L.: On the group problem for mixed integer programming. Math. Prog. Study **2**, 137–179 (1974)
25. Johnson, E.L.: Characterization of facets for multiple right-hand choice linear programs. Math. Prog. Study **14**, 112–142 (1981)

26. Lovász, L.: Geometry of numbers and integer programming. In: M. Iri, K. Tanabe (eds.) Mathematical Programming: Recent Developments and Applications. Kluwer, Dordrecht (1989)
27. Miller, L.A., Li, Y., Richard, J.-P.P.: New facets for finite and infinite group problems from approximate lifting. Naval Res. Logis. **55**, 172–191 (2008)
28. Nemhauser, G.L., Wolsey, L.A.: Integer and Combinatorial Optimization. Wiley-Interscience, New York, NY (1988)
29. Nemhauser, G.L., Wolsey, L.A.: A recursive procedure to generate all cuts for 0-1 mixed integer programs. Math. Prog. **46**, 379–390 (1990)
30. Richard, J.-P.P., Li, Y., Miller, L.A.: Valid inequalities for MIPs and group polyhedra from approximate liftings. To appear in Math. Prog. (2008)
31. Rockafeller, G.T.: Convex Analysis. Princeton University Press, New Jersey, NJ (1970)

Appendix 1

In this section we show that if a function $\tilde{\pi} : \mathbb{Q}^2 \to \mathbb{R}_+$ as defined in (4) is minimal (extreme resp.) for (1) and $f \in \text{interior}(P(\pi))$, then π as defined in (5) is minimal (extreme resp.) for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$.

The proof of the following proposition is exactly the same as the proof of Theorem 1 from Borozan and Cornuéjols [6] and related to the proof of Theorem 8 from Johnson [24].

Proposition 17 *If $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ is a minimal inequality for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$, then*

1. *π is positively homogenous.*
2. *π is subadditive.*

Since the function $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ is positively homogenous and subadditive, it is convex, see Rockafeller [31]. Moreover, if $\pi(w)$ is finite for every $w \in \mathbb{R}^2$, then it is continuous.

Proposition 18 ([31]) *If $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ is a finite, subadditive, and positively homogenous function, then π is a continuous function.*

Proposition 19 *If $P(\pi)$ is a maximal lattice-free set with $f \in$ interior$(P(\pi))$ and $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ is defined as (5), then π is minimal.*

Proof: Note that π is a continuous function by construction. Assume by contradiction that π is not minimal. Then there exists a minimal valid function $\pi' : \mathbb{R}^2 \to \mathbb{R}_+$ such that $\pi > \pi'$. Since π is a finite function, this implies that π' is finite. Using Proposition 17, π' is positively homogenous and subadditive. Thus using Proposition 18, π' is continuous. However, by Theorem 1, $\pi(u) = \pi'(u)\ \forall u \in \mathbb{Q}^2$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, continuity of π and π' implies that $\pi = \pi'$. □

Proposition 20 *Let $\tilde{\pi} : \mathbb{Q}^2 \to \mathbb{R}_+$ be an inequality for (1) corresponding to a maximal lattice-free convex set $P(\pi)$ with $f \in$ interior$(P(\pi))$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ be as defined in (5). If $\tilde{\pi}$ is extreme for (1), then π is extreme for $MI(\{\bar{0}\}, \mathbb{R}^2, r)$.*

Proof: Observe that $\pi(u) = \tilde{\pi}(u)\ \forall u \in \mathbb{Q}^2$. Assume by contradiction that there exist two valid functions $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}_+$ such that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ and $\pi_1 \neq \pi_2$. By Proposition 19, π is a minimal inequality. This implies that π_1

and π_2 are minimal. By Proposition 17, π_1 and π_2 are positively homogenous and subadditive. Moreover since π is finite by construction, π_1 and π_2 are finite. Thus by Proposition 18, π_1 and π_2 are continuous.

Since $\tilde{\pi}$ is extreme, $\tilde{\pi}(u) = \pi_i(u)\ \forall u \in \mathbb{Q}^2$. Since $f \in \text{interior}(P(\pi))$, π is a continuous function. However since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, this implies that $\pi_1 = \pi_2$, a contradiction. □