Cutting Planes in Mixed Integer Programming

Santanu S. Dey

School of Industrial and Systems Engineering,
Georgia Institute of Technology.

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(Many Thanks to Q. Louveaux for sharing some Images)

Outline

Mixed Integer Programming (MIP)

MIP are useful!

MIP are difficult to solve

How we go about solving MIPs

Cutting Planes

Part 1

Mixed Integer Programming

Mixed Integer Programming: 'Linear Problem' with Discrete Variables

Mixed Integer Program (MIP)

$$\begin{array}{llllllll} \max & c_1 x_1 & + & \cdots & + & c_n x_n & & \\ \text{s.t.} & a_{11} x_1 & + & \cdots & + & a_{1n} x_n & \leq & b_1 \\ & \vdots & & \ddots & & \vdots & & \vdots \\ & a_{m1} x_1 & + & \cdots & + & a_{mn} x_n & \leq & b_m \\ & \multicolumn{7}{l}{x_1, x_2, \cdots, x_k \in \mathbb{Z}_+ \rightarrow \text{nonnegative integers}} \end{array}$$

I will typically use "x", "y" to represent decision variables and all the other letters to represent data.

If $k = n$, then Integer Program.

If $k = 0$, then Linear Program.

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- I want to illustrate its expressive power with an elementary example

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x_1, x_2, x_3, x_4, x_5 : The amount I choose to produce in each month.
y_1, y_2, y_3, y_4, y_5 : Whether I produce or not each month (0 = Do not Produce/ 1 = I decide to Produce).

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$$x_1 \geq 5$$
$$x_1 + x_2 \geq 5 + 13$$

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$x_1 \leq 20$

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$$\min 6x_1 + 7x_2 + 2x_3 + 3x_4 + 6x_5 + 8y_1 + 8y_2 + 8y_3 + 8y_4 + 8y_5$$

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$$x_3 \leq 20y_3$$
$$x_4 \leq 20y_4$$
$$x_5 \leq 20y_5$$
$$y_1 \leq 1$$
$$y_2 \leq 1$$
$$\ldots$$
$$x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}_+$$
$$y_1, y_2, y_3, y_4, y_5 \in \mathbb{Z}_+$$

MIP is a Flexible Tool

- Logistics: Traveling Salesman problem, Vehicle Routing.
- Inventory (and Production) Planning.
- Facilities Location.
- Capacity planning: Matching, Assignment Problem.
- Data Mining: Classification, Regression.

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- Data Mining: Classification, Regression.
- Airline Industry: Schedule Planning, Fleet Assignment, Aircraft Rotation, Crew-pairing.
- Mining and Forestry Industry: Covering Models, partitioning models.
- National Security Planning.
- VLSI Chip Design.
- Computational Biology: Sequence Alignment, Genome Rearrangement.
- Health care: IMRT, Scheduling.
- Sports Scheduling, Timetabling.

Some "Slightly Dated" Applications of Integer Programming

Papers from Interfaces.

Company	Year	Type of Model	Revenue
Air New Zealand	2001	Crew Scheduling	NZ $15.6 million
AT&T	2000	Network Restoration	'Hundreds of millions of dollars'
NBC	2002	Product Mix/ Commercials/Schedule	$200 million
Procter & Gamble	2006	Expressive Bidding	$298.4 million
Schindler Elevator	2003	Routing planning	$1 million
UPS	2004	Network Design	$87 million
Ford	2001	Set-covering/ Product optimization	$250 million
Hewlett-Packard	2004	Inventory Optimization	$130 million
Merrill-Lynch	2002	Integrated Choice Strategy	$80 million
Motorola	2005	Bidding/ Supplier Negotiation	$200 million

Part 2

MIPs are Difficult to Solve

A Natural Way to Solve MIPs

1. Enumerate all possible integer solution vectors.

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1. Enumerate all possible integer solution vectors.
2. For each potential solution vectors, check if it is feasible.
3. If feasible, then compute objective function value.
4. Pick the best solution.

Some Calculations

Suppose we can evaluate 10^6 potential solution vectors in a second.

Number of binary variables	Number of vectors 2^n	Time
10	1024	0.001 seconds

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However, we routinely solve problems with thousands of integer variables.

Part 3

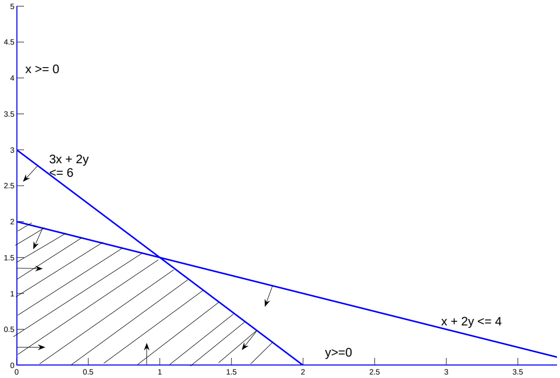
How we go about solving MIPs

Main Ideas

- Branch and Bound: Smart Enumerate
- Cutting Planes

A little bit of Linear Programming (LP) Geometry

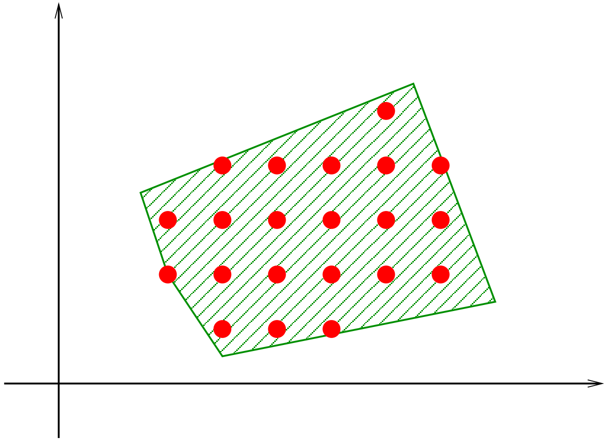
Polyhedron



$$3x + 2y \leq 6$$
$$x + 2y \leq 4$$
$$x \geq, y \geq 0.$$

Using Linear Programming Techniques To Solve MIPs

Convex Hull



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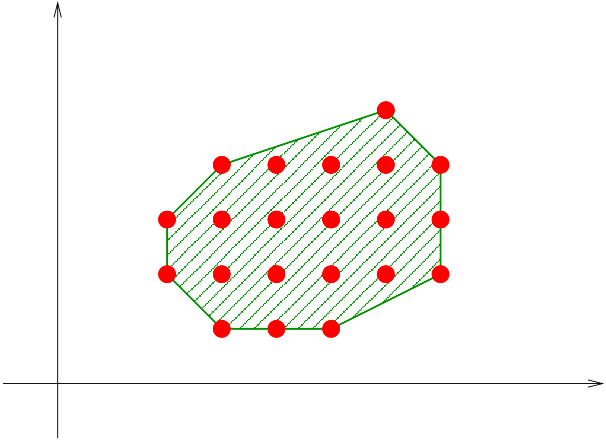
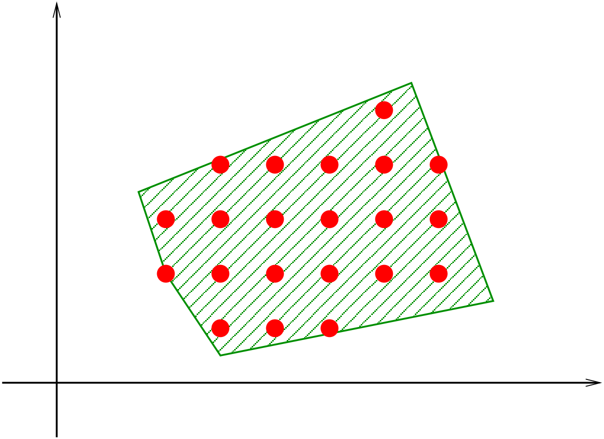


Cutting Planes (Cuts)

A Simple Algorithm To Solve Integer Programs



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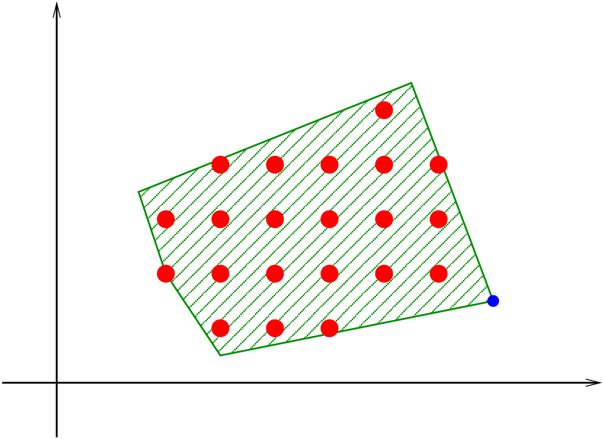

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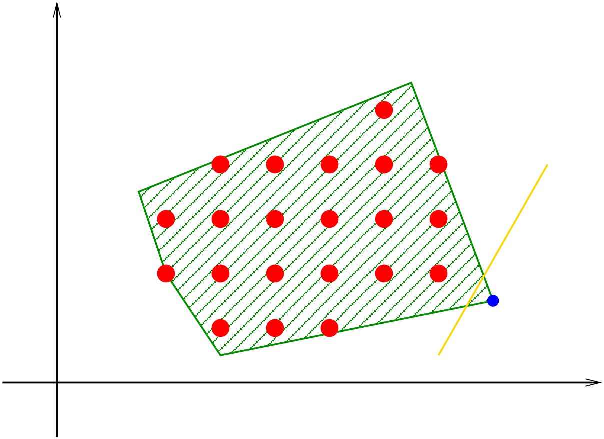

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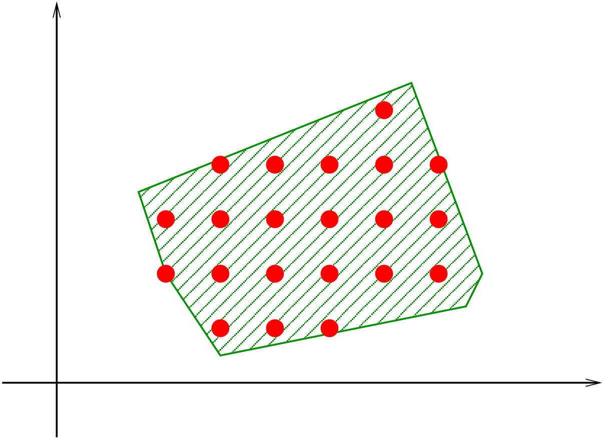

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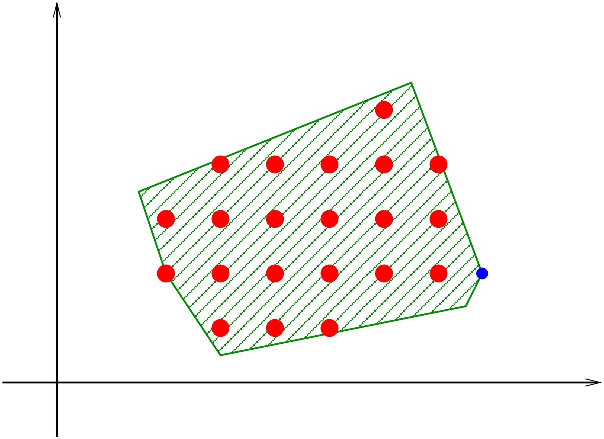

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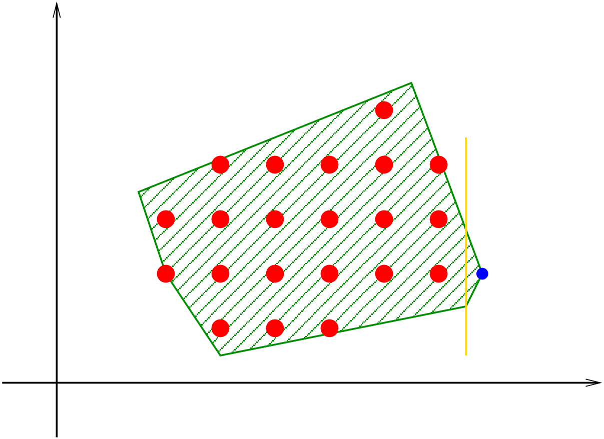

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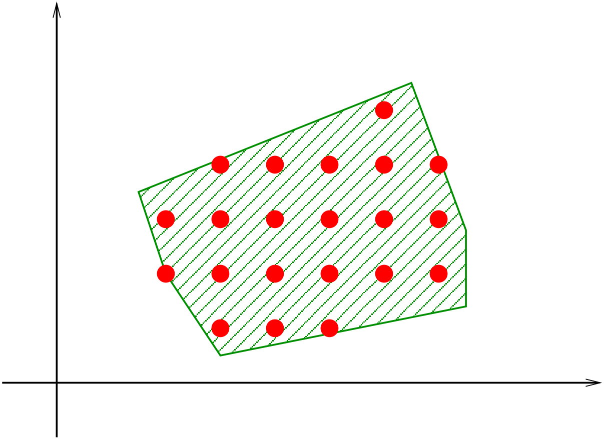

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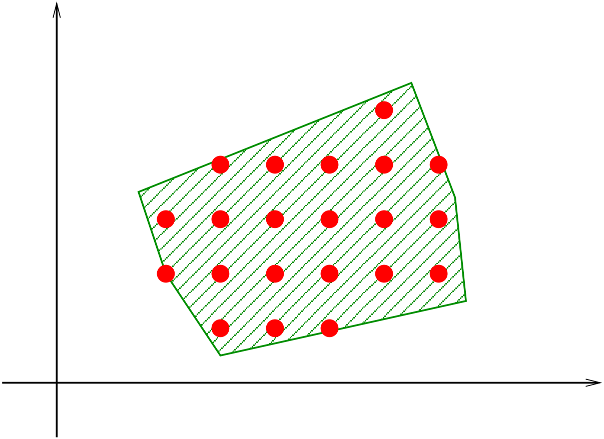

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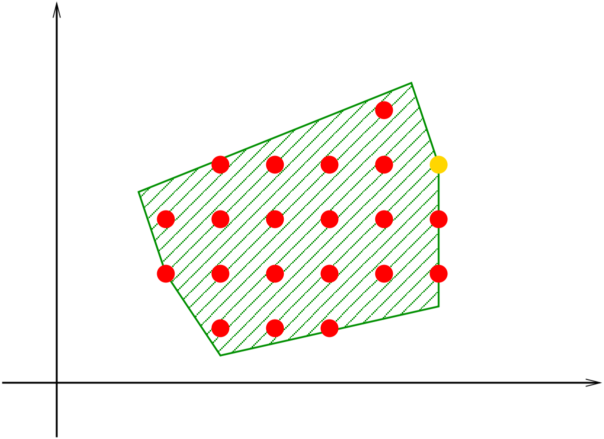

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2. Generic Cutting Planes: These are useful for solving any Mixed Integer Programs.

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Two paradigms to generate cutting planes

1. Problem Specific Cutting Planes: These are useful for solving specific problems.
2. Focus of the rest of the talk $\longrightarrow$ Generic Cutting Planes: These are useful for solving any Mixed Integer Programs.

Most Successful Cutting Planes for Generic Problems

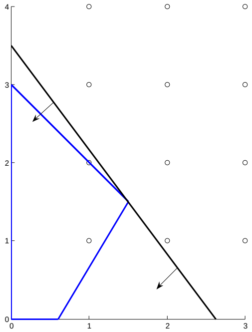
[Bixby, Rothberg (2007)]

Disabled Cut	Year	Mean Performance Degradation
Gomory Mixed Integer (GMIC)	1960	2.52X
Mixed Integer Rounding	2001	1.83X
Knapsack Cover	1983	1.40X
Flow Cover	1985	1.22X
Implied Bound	1991	1.19X
Flow path	1985	1.04X
Clique	1983	1.02X
GUB Cover	1998	1.02X
Disjunctive	1979	0.53X

Chvátal-Gomory Cutting Planes

$$x_1 \geq, x_2 \geq 0, x_1 + x_2 \leq 3, 5x - 3y \leq 3$$
$$x_1, x_2 \in \mathbb{Z}$$

Valid inequality for Continuous Relaxation: $\underbrace{4x_1 + 3x_2}_{\in \mathbb{Z}} \leq 10.5.$



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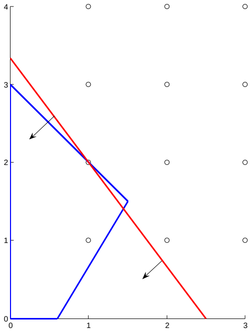
Valid inequality for Continuous Relaxation: $\underbrace{4x_1 + 3x_2}_{\in \mathbb{Z}} \leq 10.5$.

This gives the following nontrivial valid inequality: $\boxed{4x_1 + 3x_2 \leq \lfloor 10.5 \rfloor.}$



Cover Cuts

$$
\begin{aligned}
3x_1 + 5x_2 + 4x_3 + 2x_4 + 7x_5 &\leq 8, \\
0 \leq x_1, x_2, x_3, x_4, x_5 &\leq 1, \\
x_1, x_2, x_3, x_4, x_5 &\in \mathbb{Z}_+
\end{aligned}
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\end{aligned}
$$

$$5 + 4 > 8$$

x_2 and x_3 cannot simultaneously be equal to 1

Cover Cuts

$$\begin{aligned} 3x_1 + 5x_2 + 4x_3 + 2x_4 + 7x_5 &\le 8, \\ 0 \le x_1, x_2, x_3, x_4, x_5 &\le 1, \\ x_1, x_2, x_3, x_4, x_5 &\in \mathbb{Z}_+ \end{aligned}$$

$$\boxed{x_2 + x_3 \le 1}$$

Cover Cuts

$$3x_1 + 5x_2 + 4x_3 + 2x_4 + 7x_5 \leq 8,$$
$$0 \leq x_1, x_2, x_3, x_4, x_5 \leq 1,$$
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$$\boxed{x_2 + x_3 \leq 1}$$
$$\boxed{x_4 + x_5 \leq 1}$$
$$\boxed{x_1 + x_2 + x_3 \leq 2}$$
$$\boxed{x_1 + x_2 + x_4 \leq 2}$$
$$\boxed{x_1 + x_2 + x_5 \leq 2}$$

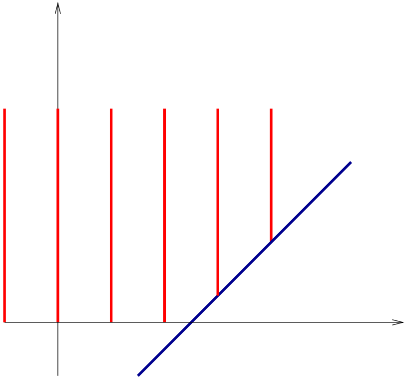
$\vdots$

Basic Mixed-Integer Rounding

Consider the basic set

$$x - y \leq 2.5$$

$$x \in \mathbb{Z}, y \geq 0.$$



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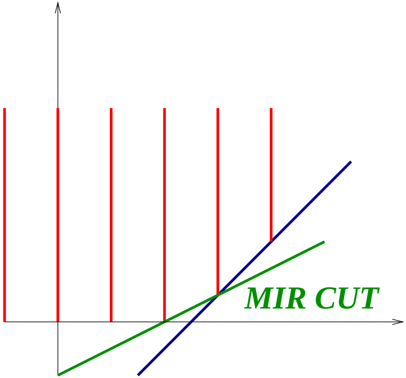


It is called the Mixed-Integer-Rounding Inequality (MIR)

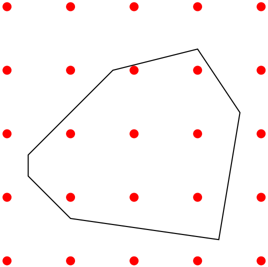
$$x \leq \lfloor 2.5 \rfloor + \frac{y}{1-f},$$

where $f = 2.5 - \lfloor 2.5 \rfloor$ is the fractional part of 2.5.

The Disjunctive Cuts

Split Disjunctive cuts

General family of cutting planes that include the MIR cuts.
Based on a disjunction $\pi^T x \leq \pi_0$ or $\pi^T x \geq \pi_0 + 1$.



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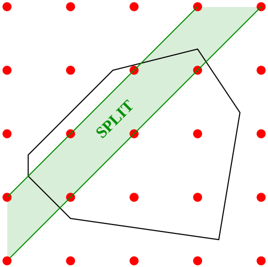

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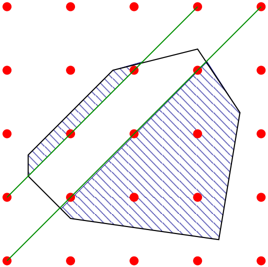

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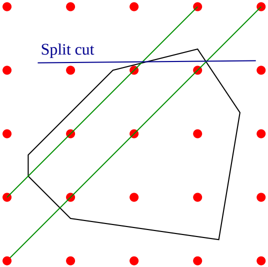
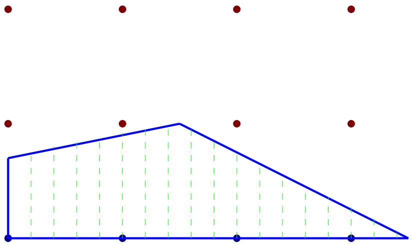


Some recent trends...

Single Constraint Relaxation

A simple example



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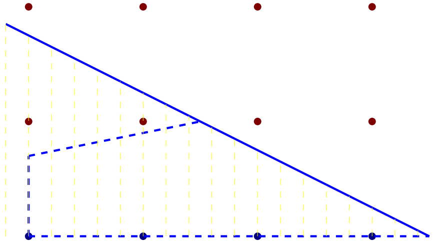

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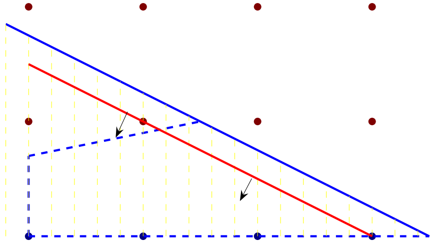
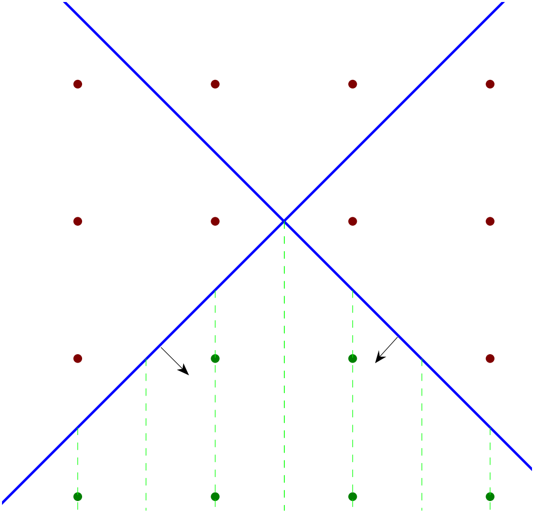


Question: Can we expect any improvement by looking at *multiple constraints simultaneously?*

Single Constraint Vs Multiple Constraints

'Extra' information by considering multiple constraints simultaneously!



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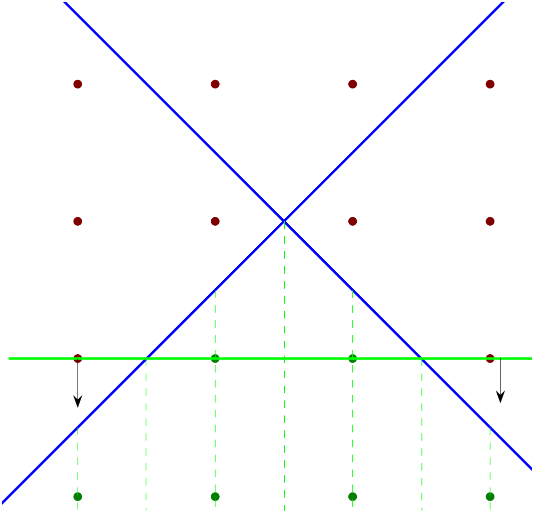

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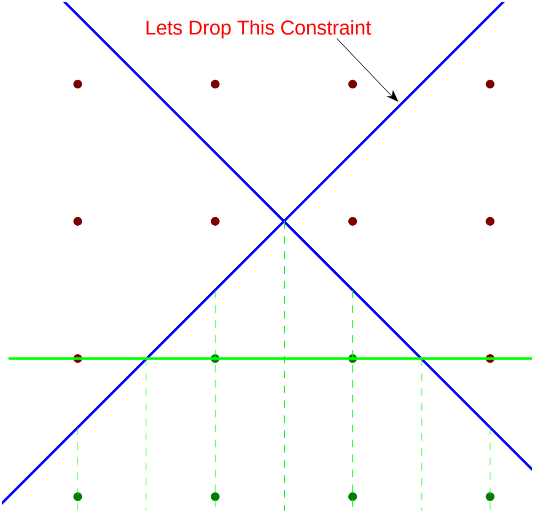

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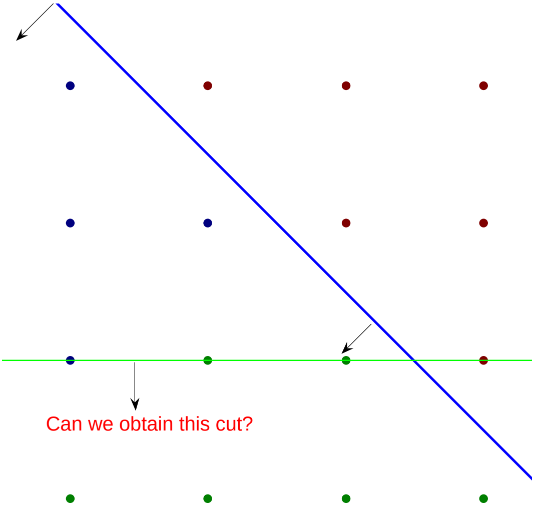

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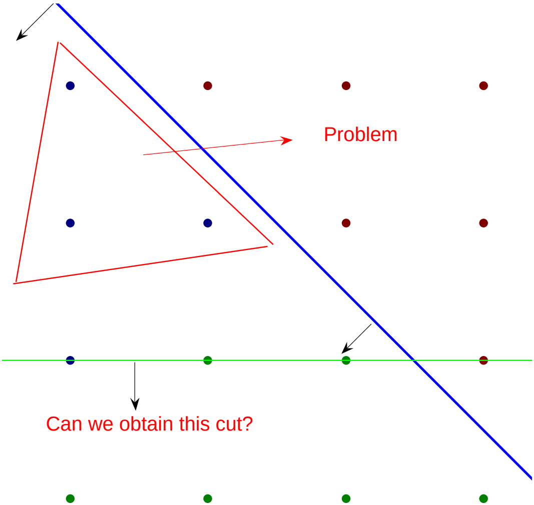

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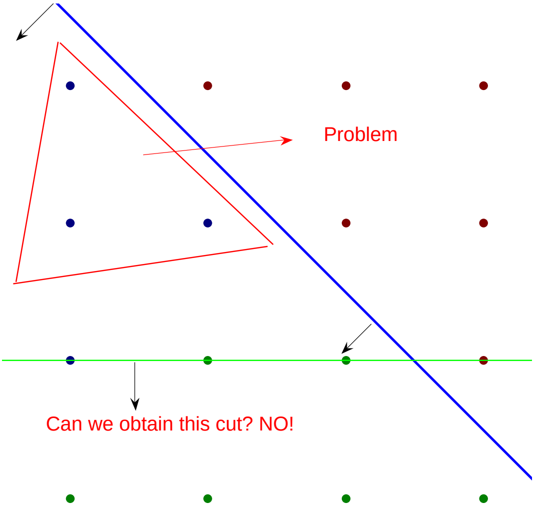

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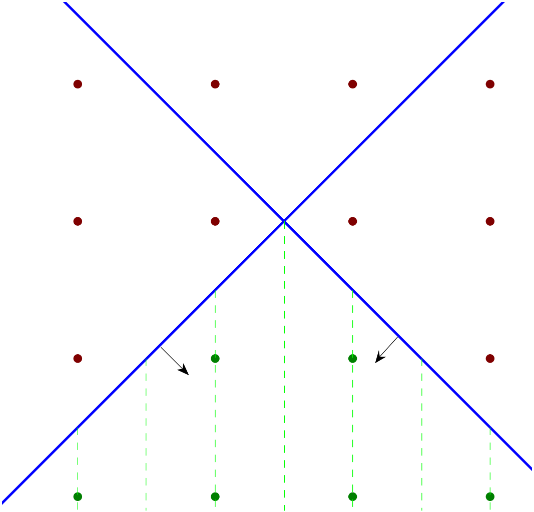

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