Integer Quadratic Programming is in NP

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Outline

1

Introduction and Main Result

Integer Quadratic Program: Definition

Definition (IQP)

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\min \quad & x^\top Q x + c^\top x \\
\text{s.t.} \quad & Ax \leq b \\
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Decision Version of IQP

Does there exist x satisfying:

$$\left.\begin{array}{rcl} x^\top Qx + c^\top x + d & \leq & 0 \\ Ax & \leq & b \\ x & \in & \mathbb{Z}^n, \end{array}\right\} \quad \mathcal{F}(Q, c, d, A, b)$$

where we assume all the data is rational.

Main Result

Theorem

Let $n, m \in \mathbb{Z}_{++}$. Let $Q \in \mathbb{Q}^{n\times n}$, $c \in \mathbb{Q}^n$, $d \in \mathbb{Q}$, $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$.

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If $\mathcal{F}(Q, c, d, A, b)$ is non-empty, then there exists $x^0 \in \mathcal{F}(Q, c, d, A, b)$ such that the binary encoding size of x^0 is bounded from above by a polynomial function of the size of binary encoding of Q, c, d, A, b.

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Consequences

1. Integer Quadratic Programming is in NP . In particular, the decision version of IQP is NP-complete.
2. Broadly speaking, this implies that these exists an algorithm to solve IQP, i.e. not undecidable.

Comparison 1: More quadratic inequalities?

Undecidable!

Determing the feasibility of a system with

1. Number of quadratic inequalities: $2\left(\binom{58}{2} + 58 + 1\right) = 3424.$
2. Number of linear inequalities: 58
3. Number of integer variables: $\left(\binom{58}{2} + 2 * 58\right) = 1769.$

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Reduction from undecidability of determining the feasibility of a quartic equation in 58 non-negative integer variables.

[Jones (1982)], See discussion and additional references in [Köppe (2012)].

Comparison 2: Two quadratic inequalities?

Exponential size solution!

Consider the system for $d = 5^{2n+1}$:

$$\begin{aligned} x^2 - dy^2 + 1 &\leq 0, \\ -x^2 + dy^2 - 1 &\leq 0 \\ x, y &\in \mathbb{Z}. \end{aligned}$$

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[Lagarias (1980)], See discussion and additional references in [Köppe (2012)]

Comparison 3: More convex quadratic inequalities?

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Consider the system:

$$\begin{aligned} x_1 &\geq 2 \\ x_j &\geq x_{j-1}^2 \ \forall j \in \{2, \dots, n\} \\ x_j &\in \mathbb{Z} \ \forall j \in \{1, \dots, n\}. \end{aligned}$$

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2. With even two inequality, the binary encoding of the smallest solution may be exponential in size.
3. With “many" inequalities, (a) the problem become undecidables with general quadratics, or (b) binary encoding of all solutions may be exponential in size in the convex quadratics case.

2

Proof Outline

Overview of the proof

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Definition: Simplicial cone

A simplicial cone is a cone generated by a simplex.

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   - $\rightarrow$ Standard techniques to show Integer linear programming is in NP.
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2. Step 2: Verify the result for the case where $\mathcal{P}$ is a full-dimensional simplicial cone.

2.1
Step 2

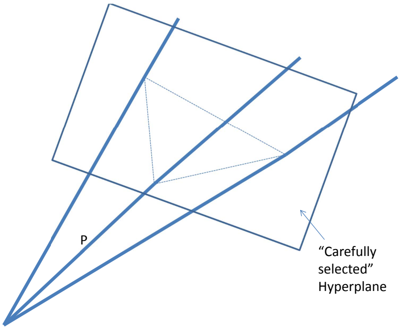
Getting Started

$$\begin{aligned} x^\top Q x + c^\top x + d &\leq 0 \\ Ax &\leq 0 \quad \ldots\ldots\ldots\ldots\ (\mathcal{P}) \\ x &\in \mathbb{Z}^n \end{aligned}$$

1. $\{x \mid Ax \leq 0\}$ is a simplicial cone.
2. We may assume $d > 0$.

$\mathcal{P}$ is a full-dimensional simplicial cone.

- "Slice " the cone $\mathcal{P}$ with a "carefully selected" hyperplane $\mathcal{H}$



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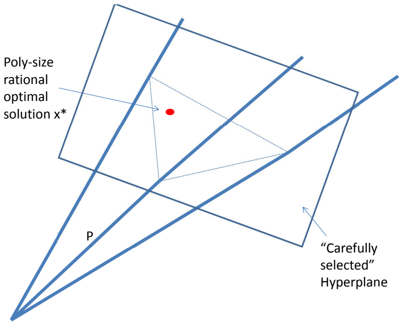
- Let x^* be a poly-size rational optimal solution to the problem

$$x^{*\top} Q x^* := \min \quad x^\top Q x \quad \text{s.t.} \quad x \in \mathcal{P} \cap \mathcal{H}$$



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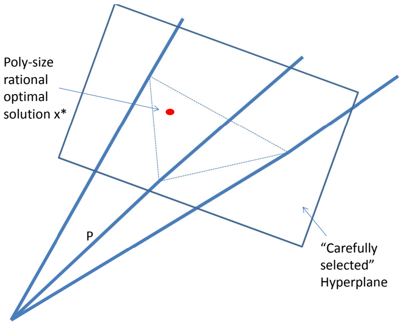
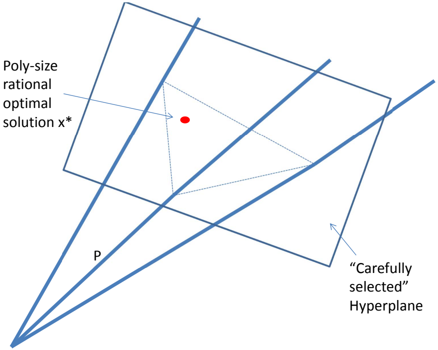
- The quadratic problem $\min\{x^\top V x \mid x \in \text{rational polytope}\}$ (where V is a rational matrix) has a rational globally optimal solution of poly-size with respect to the size of the instance. [Vavasis 1990]



Case analysis based on sign of ${x^*}^\top Q x^*$

$$\begin{aligned} {x^*}^\top Q x^* := \quad & \min \quad x^\top Q x \\ & \text{s.t.} \quad x \in \mathcal{P} \cap \mathcal{H} \end{aligned}$$



Case 1: ${x^*}^\top Q x^* < 0$

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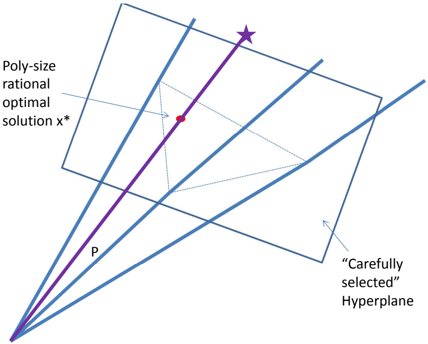
Scale and find a solution

1. First scale x^* to $\bar{x}$ so that $\bar{x} \in \mathcal{P} \cap \mathbb{Z}^n$.
2. $\bar{\lambda} = \left\lceil \left| \frac{c^\top \bar{x}}{(\bar{x})^\top Q \bar{x}} \right| + \sqrt{-\frac{d}{(\bar{x})^\top Q \bar{x}}} \right\rceil$
3. Then $\lambda \bar{x} \in \mathcal{P} \cap \mathbb{Z}^n$ and

$$(\bar{\lambda}\bar{x})^\top Q(\bar{\lambda}\bar{x}) + c^\top(\bar{\lambda}\bar{x}) + d \leq 0.$$



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2. Let $x \in \mathcal{P}$ and $x = \lambda \tilde{x}$ where $\tilde{x}$ belongs to slice of $\mathcal{P}$.

$$x^\top Q x + c^\top x + d = \lambda^2 \tilde{x}^\top Q \tilde{x} + \lambda c^\top \tilde{x} + d$$

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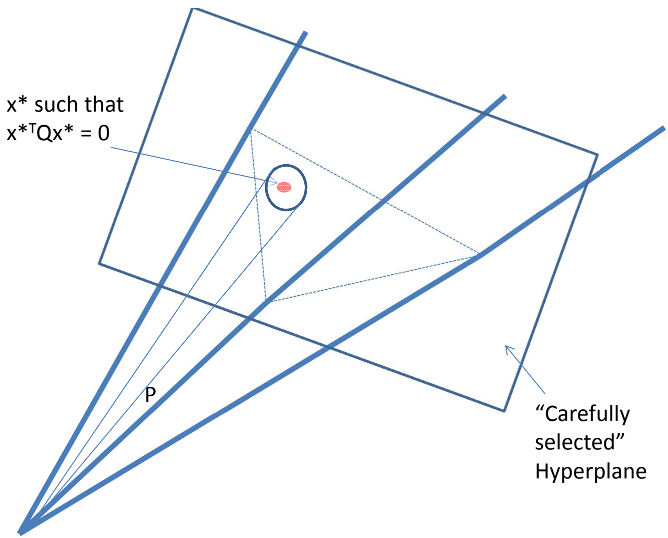
Bound size of all potential solution

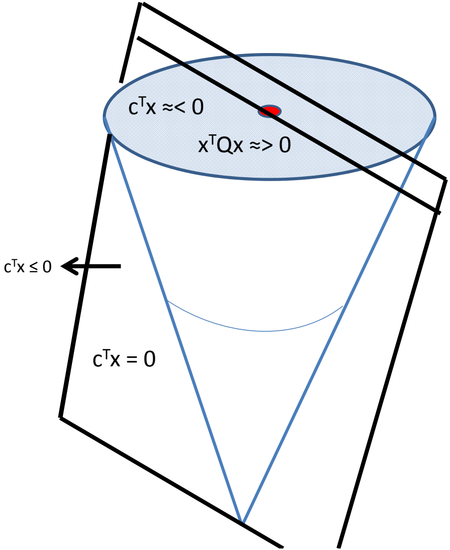
If $\|x\| > \|x^*\| \|c\| (R)^3$ and $x \in \mathcal{P}$, then $x^\top Q x + c^\top x + d > 0$ (R is the size of the largest extreme ray of $\mathcal{P}$.).

Case 3: $x^{*\top} Q x^* = 0$



Looking around x^* ...

$$(x^\top Qx + c^\top x + d \leq 0)$$



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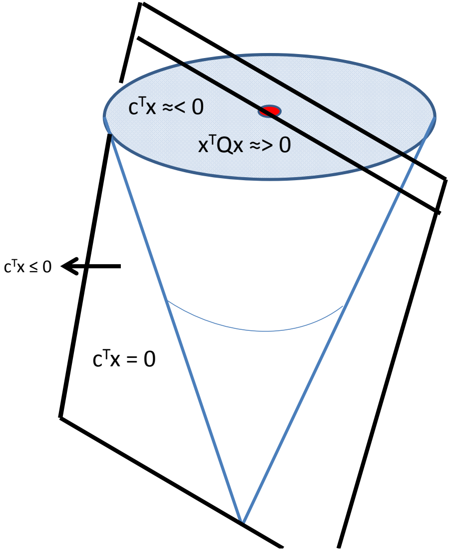

1. Not easy to bound size of feasible solution.
2. Not easy to find feasible solution of small size.

So far we have

If

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Lets see one possible approach.

Decomposing $\mathcal{P}$ further

Lemma

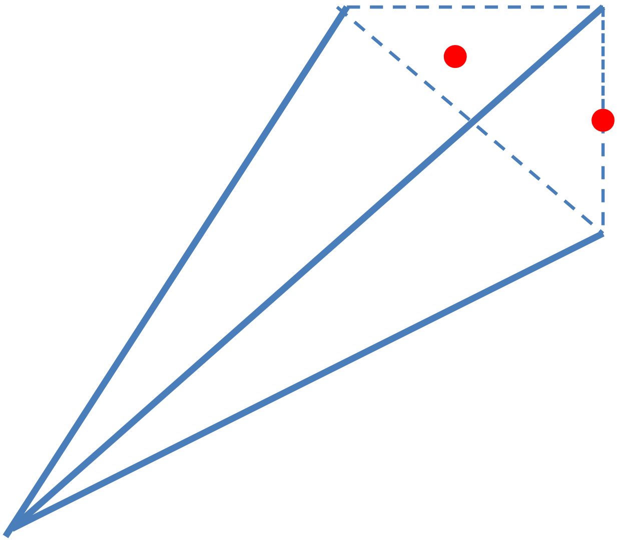
Let $\mathcal{P}$ be a full-dimensional simplicial cone such that $x^\top Qx \geq 0$ for every $x \in C$. Let $\mathcal{H}$ be a hyperplane such that $\mathcal{P} \cap \mathcal{H}$ is a simplex. Then there exist a finite family of full-dimensional simple cones C^i, $i \in I$ such that

(a) $\bigcup_{i \in I} C^i = \mathcal{P}$,

(b) for every $i \in I$, if a face F of C^i satisfies $\min\{x^\top Hx : x \in F \cap \mathcal{H}\} = 0$, then there exists an extreme ray v of F with $v^\top Hv = 0$,

(c) for every $i \in I$, the size of C^i is polynomial in the size of $\mathcal{P}$.

Illustration of Lemma



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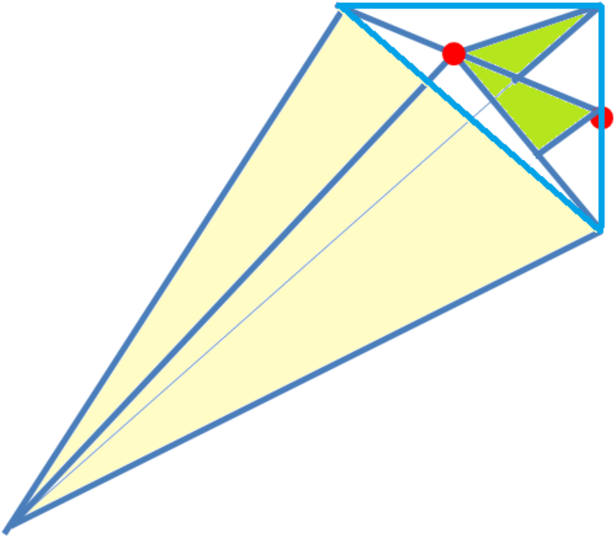

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5. If we are not in case (4.1) above for all $j \in J$ then: If $x \in \mathcal{C} \cap \mathbb{Z}$ and $x^\top Q x + c^\top x + d \leq 0$, then there exists $\tilde{x} \in \mathcal{C} \cap \mathbb{Z}$ such that

$$x = x_0 + \sum_{j=1}^{n} r^j y_j, y_j \in \mathbb{Z}_+ \; \forall j \in \{1, \dots, n\} \setminus J.$$

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6. Finally, using the structure of the cone, (5) implies all the solutions are bounded, where the bound is polynomial size.

Open Problem

Is Integer Quadratic Programming in P for fixed dimension?

Thank You!