Split Rank of Triangle and Quadrilateral Inequalities

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Outline

Lets look at one relaxation of MIPs...

Relaxation of MIP

Simplex Tableau

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$r_{1,1}x_1$	$\cdots +$	$r_{1,k}x_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
x_{B_m}	$=$	f_m	$+$	$r_{m,1}x_1$	$\cdots +$	$r_{m,k}x_k$	$+$	$r_{m,k+1}s_{k+1}$	$\cdots +$	$r_{m,n}s_n$
$s_{B_{m+1}}$	$=$	f_{m+1}	$+$	$r_{m+1,1}x_1$	$\cdots +$	$r_{m+1,k}x_k$	$+$	$r_{m+1,k+1}s_{k+1}$	$\cdots +$	$r_{m+1,n}s_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
s_{B_p}	$=$	f_p	$+$	$r_{p,1}x_1$	$\cdots +$	$r_{p,k}x_k$	$+$	$r_{p,k+1}s_{k+1}$	$\cdots +$	$r_{pn}s_n$

1. $x_{B_1}, \ldots, x_{B_m} \in \mathbb{Z}_+$
2. $s_{B_{m+1}}, \ldots, s_{B_p} \in \mathbb{R}_+$
3. $x_1, \ldots, x_k \in \mathbb{Z}_+$
4. $s_{k+1}, \ldots, s_n \in \mathbb{R}_+$

Solution is ‘fractional’, i.e. $f_1, \ldots, f_m$ are not all integer.

Relaxation of MIP

Relaxation Step 1: Drop Some Constraints

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$r_{1,1}x_1$	$\cdots +$	$r_{1,k}x_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
x_{B_m}	$=$	f_m	$+$	$r_{m,1}x_1$	$\cdots +$	$r_{m,k}x_k$	$+$	$r_{m,k+1}s_{k+1}$	$\cdots +$	$r_{m,n}s_n$
$s_{B_{m+1}}$	$=$	f_{m+1}	$+$	$r_{m+1,1}x_1$	$\cdots +$	$r_{m+1,k}x_k$	$+$	$r_{m+1,k+1}s_{k+1}$	$\cdots +$	$r_{m+1,n}s_n$
$\vdots$		$\vdots$		$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
s_{B_p}	$=$	f_p	$+$	$r_{p,1}x_1$	$\cdots +$	$r_{p,k}x_k$	$+$	$r_{p,k+1}s_{k+1}$	$\cdots +$	$r_{pn}s_n$

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Relaxation of Simplex Tableau

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x_{B_2}	$=$	f_2	$+$	$r_{2,1}x_1$	$\cdots +$	$r_{2,k}x_k$	$+$	$r_{2,k+1}s_{k+1}$	$\cdots +$	$r_{2,n}s_n$

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$(f_1, f_2) \notin \mathbb{Z}^2$.

Relaxation of MIP

Relaxation Step 2: Drop Integrality Requirement

Basic Variable		rhs		Columns Corresponding to Integer Non-Basic Variable				Columns Corresponding to Continuous Non-Basic Variable		
x_{B_1}	$=$	f_1	$+$	$r_{1,1}x_1$	$\cdots +$	$r_{1,k}x_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
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2. $x_1, \ldots, x_k \in \mathbb{Z}_+ \xrightarrow{\text{Relaxation}} x_1, \ldots, x_k \in \mathbb{R}_+$
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$(f_1, f_2) \notin \mathbb{Z}^2$.

Relaxation of MIP

Relaxation Step 2: Drop Integrality Requirement

Basic Variable		rhs				Columns Corresponding to Continous Variables				
x_{B_1}	$=$	f_1	$+$	$r_{1,1}s_1$	$\cdots+$	$r_{1,k}s_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots+$	$r_{1,n}s_n$
x_{B_2}	$=$	f_2	$+$	$r_{2,1}s_1$	$\cdots+$	$r_{2,k}s_k$	$+$	$r_{2,k+1}s_{k+1}$	$\cdots+$	$r_{2,n}s_n$

1. $x_{B_1}, x_{B_2} \in \mathbb{Z}_+$
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$(f_1, f_2) \notin \mathbb{Z}^2$.

Continuous Group Relaxation

Basic Variable		rhs		Columns With Continous Variables						
x_{B_1}	$=$	f_1	$+$	$r_{1,1}s_1$	$\cdots +$	$r_{1,k}s_k$	$+$	$r_{1,k+1}s_{k+1}$	$\cdots +$	$r_{1,n}s_n$
x_{B_2}	$=$	f_2	$+$	$r_{2,1}s_1$	$\cdots +$	$r_{2,k}s_k$	$+$	$r_{2,k+1}s_{k+1}$	$\cdots +$	$r_{2,n}s_n$

1. $x_{B_1}, x_{B_2} \in \mathbb{Z}_+ \xrightarrow{\text{Relaxation}} x_{B_1}, x_{B_2} \in \mathbb{Z}$
2. $s_1, \ldots, s_k,\ s_{k+1}, \ldots, s_n \in \mathbb{R}_+$

$(f_1, f_2) \notin \mathbb{Z}^2$.

The valid inequalities for the above are valid for the original simplex tableau!

Model studied in Andersen, Louveaux, Weismantel, Wolsey, IPCO2007 (for the finite case),
Cornuéjols and Margot, 2009.

Related to Group Relaxation of Gomory and Johnson (1972), Johnson (1974).

The 2 row-model

The model

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \sum_{j=1}^{n} \begin{pmatrix} r_1^j \\ r_2^j \end{pmatrix} s_j, \qquad x_1, x_2 \in \mathbb{Z}, s_j \in \mathbb{R}_+$$

As a short hand we call this set $P(R, f)$, where $R = [r^1, ..., r^n] \in \mathbb{Q}^{2\times n}$

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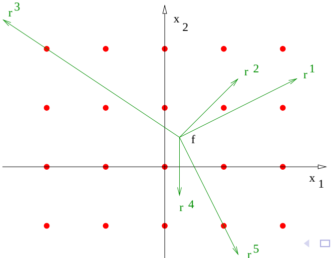
Projection on x-space

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} -3 \\ 2 \end{pmatrix} s_3 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_4 + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The green rays correspond to points of the form $\{w \in \mathbb{R}^2 \mid w = f + \lambda r^i, \lambda \geq 0\}$.



Valid inequalities for $P(R, f)$...

Valid inequality and lattice-free convex set

A set $B \subseteq \mathbb{R}^2$ is called lattice-free if $\mathrm{interior}(B) \cap \mathbb{Z}^2 = \emptyset$.

Proposition (Lattice-free convex set $\mapsto$ Valid inequality)

Let $R \in \mathbb{Q}^{2\times k}$ and $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$. Let B be a closed lattice-free convex set containing f in its interior. Let ∂B represent the boundary of B. Define the vector $\phi(B) \in \mathbb{R}^k_+$ as

$$\phi(B)_i = \begin{cases} 0 & \text{if } r^i \in \text{reccesion cone of } B \\ \lambda & \text{if } \lambda > 0 \text{ and } f + \frac{r^i}{\lambda} \in \partial B. \end{cases} \tag{1}$$

Then the inequality

$$\sum_{i=1}^{k} \phi(B)_i s_i \geq 1, \tag{2}$$

is a valid inequality for $conv(P(R, f))$. □

Proposition (Valid inequality $\mapsto$ Lattice-free convex set)

All non-trivial valid inequalities for $conv(P(R, f))$ can be written in the form $\sum_{i=1}^{k} \alpha_i s_i \geq 1$ where $\alpha_i \geq 0 \ \forall 1 \leq i \leq k$. Then the set

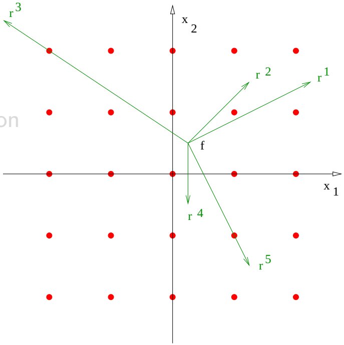
$$L_\alpha = conv\left(\cup_{\alpha_i > 0}\left\{f + \frac{r^i}{\alpha_i}\right\} \cup f\right) + cone\left(\cup_{\alpha_i = 0}\{r^i\}\right) \tag{3}$$

is lattice-free and convex. □

Lattice-free set from nontrivial valid inequality

$$2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7}s_5 \geq 1$$

- Mark the point $f + \frac{1}{\text{Coefficients of } s^i} r^i$
  α_1 α_3
- There is no integer point in the interior of L_α
- The facet is represented by a polygon L_α: The induced lattice-free set.



Lattice-free sets and Valid Inequalities

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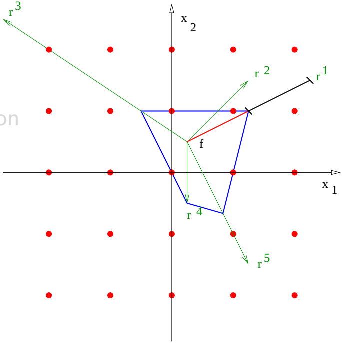

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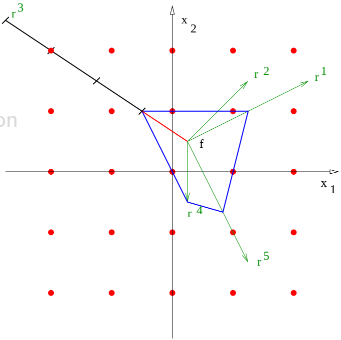

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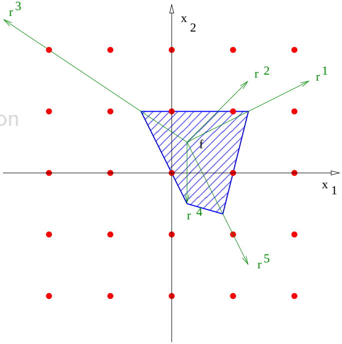

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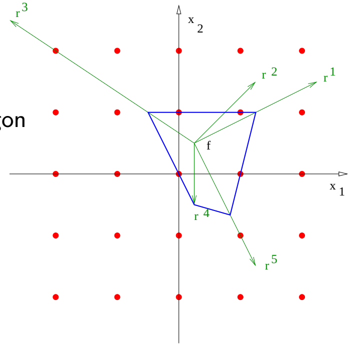
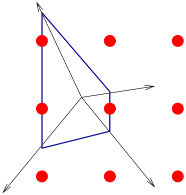


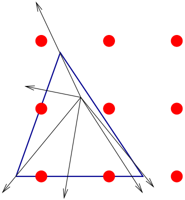
Classification of all possible facet-defining inequalities

Theorem (Andersen et al. 2007)

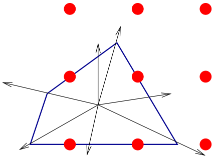
If $cone_{r \in R}(R) = \mathbb{R}^2$, and $\sum_{i=1}^{n} \alpha_i s_i \geq 1$ is facet-defining for $conv(P(R,f))$, then L_α is either (1) subset of split set, or (2) a triangle or (3) a quadrilateral.



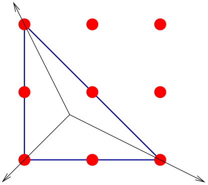
Split Cut



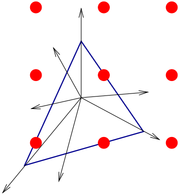
Triangle Cut



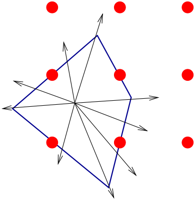
Quadrilateral Cut



Cook−Kannan−Schrijver



Disection Triangle

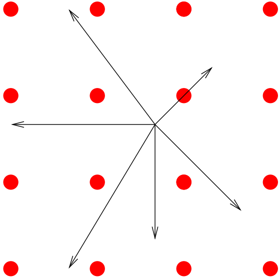


Disection Quadrilateral

The split rank question

- Split cut: Let π be an integer vector. Applying a disjunction $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$ to a polyhedron P

$$
\begin{aligned}
x &= f + RS \\
s_1 &\geq 0 \\
&\vdots \\
s_n &\geq 0
\end{aligned}
$$



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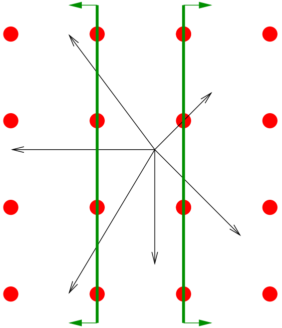
$$\begin{array}{rl} x & = f + RS \\ s_1 & \geq 0 \\ \vdots & \\ s_n & \geq 0 \\ \pi^T x & \leq \pi_0 \end{array} \quad \text{OR} \quad \begin{array}{rl} x & = f + RS \\ s_1 & \geq 0 \\ \vdots & \\ s_n & \geq 0 \\ \pi^T x & \geq \pi_0 + 1 \end{array}$$



- The first split closure P^1 of P is what you obtain after having applied all

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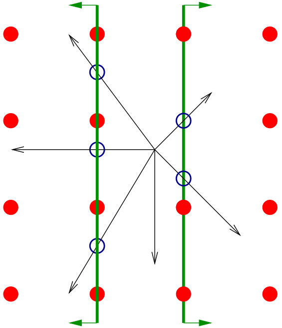

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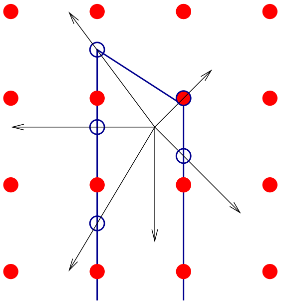

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The split rank question

- Split cut: Let π be an integer vector. Applying a disjunction $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$ to a polyhedron P
- The first split closure P^1 of P is what you obtain after having applied all possible split disjunctions π: This set is a polyhedron.
- Recursively we define P^i for $i = 2, 3, \ldots$.
- The split rank of a valid inequality is the minimum i such that the inequality is valid for P^i
- Why do we care?
  - Most inequalities (for general integer variables) used in commercial softwares are split cuts
  - The Cook-Kannan-Schrijver triangle has infinite rank, i.e. it is not possible to obtain the Cook-Kannan-Schrijver inequality by a finite sequence of split inequalities.

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Question and ∽Answer

Question: What is the split rank of the 2 row-inequalities?
(In how many rounds of split cuts only can we generate the inequalities?)

Theorem
Let $R = [r^1, ..., r^k] \in \mathbb{Q}^{2\times k}$ and $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$. Let $\sum_{i=1}^k \alpha_i s_i \geq 1$ be a non-trivial facet-defining inequality for $conv(P(R, f))$. The split rank of $\sum_{i=1}^k \alpha_i s_i \geq 1$ is finite if and only if its induced lattice-free set L_α is not a Cook-Kannan-Schrijver triangle.

The proof is constructive and gives an upper bound on the split rank of these inequalities.

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2

Outline Of Proof

Proof Outline

1. Restricting the proof to the case where $\text{cone}_{r^i \in R}\{r^i\} = \mathbb{R}^2$

Proposition

Let $\sum_{i=1}^{k} \alpha_i s_i \geq 1$ be a facet-defining inequality for $conv(P([r^1, \dots, r^k], f))$ that is not dominated by any split inequality. If $dim(cone\{r^1, \dots, r^k\}) = 2$ and $cone\{r^1, \dots, r^k\} \subsetneq \mathbb{R}^2$, then there exists a column $r^{k+1} \in \mathbb{R}^2$ and $\alpha_{k+1} > 0$ such that

- $cone\{r^1, \dots, r^k, r^{k+1}\} = \mathbb{R}^2$,
- $\sum_{i=1}^{k+1} \alpha_i s_i \geq 1$ *is a facet-defining inequality for* $conv(P([r^1, \dots, r^{k+1}], f))$,
- *The induced lattice-free set of the inequality $\sum_{i=1}^{k+1} \alpha_i s_i \geq 1$ is not a Cook-Kannan-Schrijver triangle.*

Proposition

Let $\sum_{i=1}^{k+1} \alpha_i s_i \geq 1$ be an inequality for $conv(P([r^1, \dots, r^{k+1}], f))$ with split rank η. Then the projected inequality $\sum_{i=1}^{k} \alpha_i s_i \geq 1$ for $conv(P([r^1, \dots, r^k], f))$ has a split rank of at most η for the projected problem.

Thus it is sufficient to verify that the split rank of facet-defining inequalities for $\text{conv}(P(R, f))$ such that $\text{cone}_{r^i \in R}\{r^i\} = \mathbb{R}^2$ is finite.

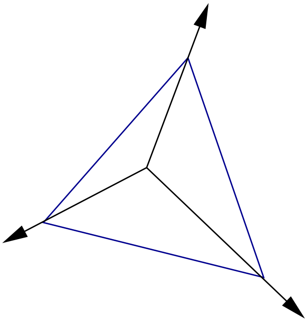
2. Restricting the proof to sets with at most four continuous variables.
3. *Restricting the proof to 'standard' triangles and quadrilaterals.*
4. *It is sufficient to prove that the split rank is finite for inequalities for problem upto 4 continuous variables, where the induced lattice-free set of the inequality is a 'Standard' Triangle or Quadrilateral...*

Proof Outline

1. Restricting the proof to the case where $\mathrm{cone}_{r^i \in R}\{r^i\} = \mathbb{R}^2$
2. Restricting the proof to sets with at most four continuous variables.

Proposition

Consider a triangle (resp. quadrilateral) inequality for a 3-variable problem (resp. 4-variable problem). If we consider an n-variable problem with the same shape of the L_α, the split rank does not increase.



L_α has at most 4 vertices.

3. *Restricting the proof to 'standard' triangles and quadrilaterals.*
4. *It is sufficient to prove that the split rank is finite for inequalities for problem upto 4 continuous variables, where the induced lattice-free set of the inequality is a 'Standard' Triangle or Quadrilateral...*

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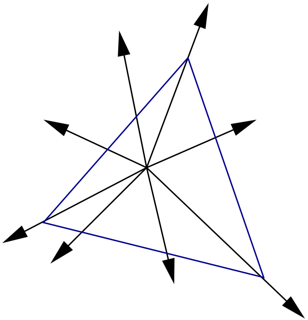


L_α has at most 4 vertices.
It is sufficient to show that the split rank of facet-defining inequalities for $\text{conv}(P(R, f))$ such that $\text{cone}_{r^i \in R}\{r^i\} = \mathbb{R}^2$ and $R \in \mathbb{Q}^{2\times k}$ where $k \leq 4$ is finite.

3. *Restricting the proof to 'standard' triangles and quadrilaterals.*
4. *It is sufficient to prove that the split rank is finite for*

Proof Outline

1. Restricting the proof to the case where $\mathrm{cone}_{r^i \in R}\{r^i\} = \mathbb{R}^2$
2. Restricting the proof to sets with at most four continuous variables.
3. *Restricting the proof to 'standard' triangles and quadrilaterals.*

Proposition

Let $w \in \mathbb{Z}^2$ and $M \in \mathbb{Z}^{2\times 2}$ be a unimodular matrix. Then

3.1 *A valid inequality $\sum_{i=1}^k \alpha_i s_i \geq \gamma$ for $P(R, f)$ is facet-defining for $P(R, f)$ if and only if $\sum_{i=1}^k \alpha_i s_i \geq \gamma$ is valid and facet-defining for $P(MR, M(f + w))$.*

3.2 *The split rank of $\sum_{i=1}^k \alpha_i s_i \geq \gamma$ wrt $P(R, f)^0$ is η if and only if the split rank of $\sum_{i=1}^k \alpha_i s_i \geq \gamma$ wrt $P(MR, M(f + w))^0$ is η.* □

We need to work with only 'standard' shapes of triangles and quadrilaterals.

4. *It is sufficient to prove that the split rank is finite for inequalities for problem upto 4 continuous variables, where the induced lattice-free set of the inequality is a 'Standard' Triangle or Quadrilateral...*

Proof Outline

1. Restricting the proof to the case where $\text{cone}_{r^i \in R}\{r^i\} = \mathbb{R}^2$
2. Restricting the proof to sets with at most four continuous variables.
3. *Restricting the proof to 'standard' triangles and quadrilaterals.*
4. *It is sufficient to prove that the split rank is finite for inequalities for problem upto 4 continuous variables, where the induced lattice-free set of the inequality is a 'Standard' Triangle or Quadrilateral...*

A useful lemma..

Lemma (Shape)

Let $\sum_{i=1}^{k_1} \alpha_i s_i \geq 1$ *be a valid inequality for* $\text{conv}(P(R^a, f))$ *with* $R^a \in \mathbb{Q}^{2\times k_1}$ *and let* $\sum_{i=1}^{k_2} \beta_i s_i \geq 1$ *be a valid inequality for* $\text{conv}(P(R^b, f))$ *with* $R^b \in \mathbb{Q}^{2\times k_2}$*. We denote by* η_a *and* η_b *the split rank of* $\sum_{i=1}^{k_1} \alpha_i s_i \geq 1$ *and* $\sum_{i=1}^{k_2} \beta_i s_i \geq 1$ *respectively. If* $cone(R^b) = \mathbb{R}^2$ *and* $L_\alpha \subseteq L_\beta$*, then* $\eta_a \leq \eta_b$*.*

3

Finiteness proofs for the triangles for three-continuous-variable problems

The triangle case

Several cases to consider, after suitable unimodular transformation.

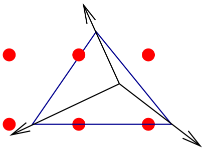
$T^{2C} - LHS$ $T^{2C} - RHS$ T^{2B}

T^{3} T^{2A} T^{1}

Lets start with this case.

The triangle case

Several cases to consider, after suitable unimodular transformation.



$T^{2C} - RHS$

T^{2B}

$T^{2C} - LHS$

T^3

T^{2A}

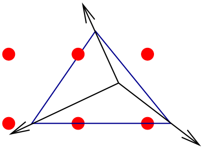
T^1

Lets start with this case.

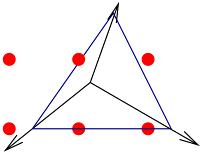
The triangle case

Several cases to consider, after suitable unimodular transformation.

T^{2B}



$T^{2C} - LHS$



$T^{2C} - RHS$

T^3 T^{2A} T^1

Lets start with this case.

The triangle case

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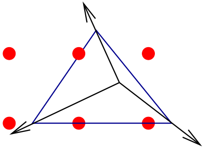
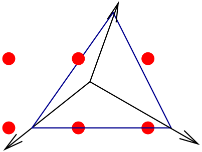
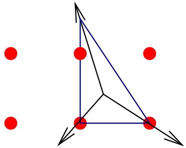


$T^{2C} - LHS$



$T^{2C} - RHS$



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The triangle case

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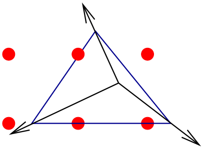
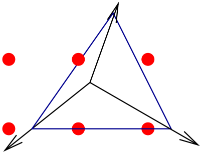
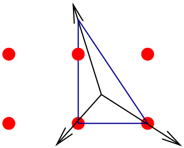


$T^{2C} - LHS$



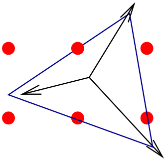
$T^{2C} - RHS$



T^{2B}

T^{2A}

T^{1}

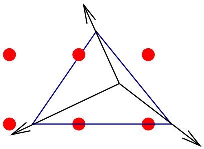


T^{3}

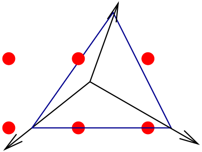
Lets start with this case.

The triangle case

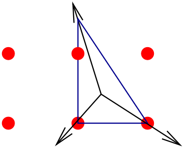
Several cases to consider, after suitable unimodular transformation.



$T^{2C} - LHS$

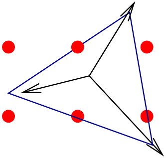


$T^{2C} - RHS$

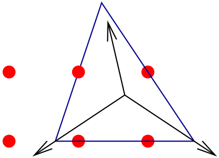


T^{2B}

T^1



T^3

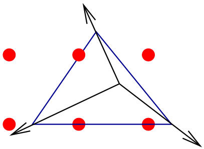


T^{2A}

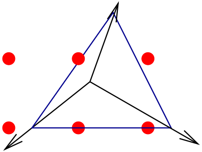
Lets start with this case.

The triangle case

Several cases to consider, after suitable unimodular transformation.



$T^{2C} - LHS$



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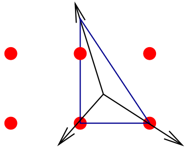
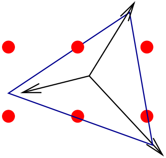
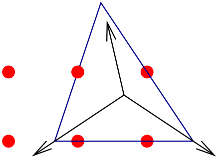


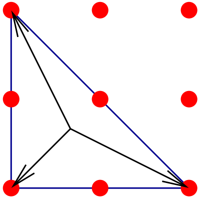
T^{2B}



T^{3}



T^{2A}

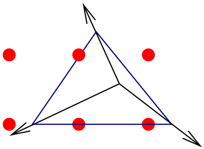


T^{1}

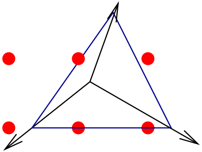
Lets start with this case.

The triangle case

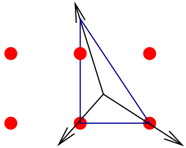
Several cases to consider, after suitable unimodular transformation.



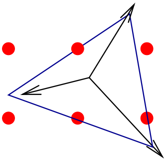
$T^{2C} - LHS$



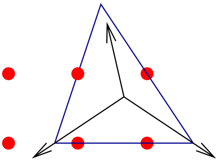
$T^{2C} - RHS$



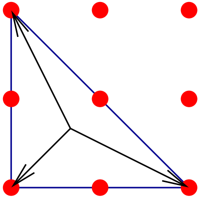
T^{2B}



T^{3}



T^{2A}

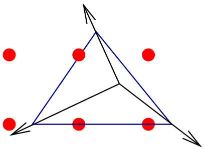


T^{1}infinite split rank

Lets start with this case.

The triangle case

Several cases to consider, after suitable unimodular transformation.



$T^{2C} - RHS$

T^{2B}

$T^{2C} - LHS$

T^3

T^{2A}

T^1

Lets start with this case.

Idea of the proof of upper bounds

- Procedure: We apply a sequence of two split disjunctions. Successively: $x_1 \leq 0 \vee x_1 \geq 1$ and $x_2 \leq 0 \vee x_2 \geq 1$
- At step i, we keep one inequality of rank at most i and proceed to the next disjunction.
- We prove that this procedure converges in finite time to the desired inequality.

Idea of the proof of upper bounds

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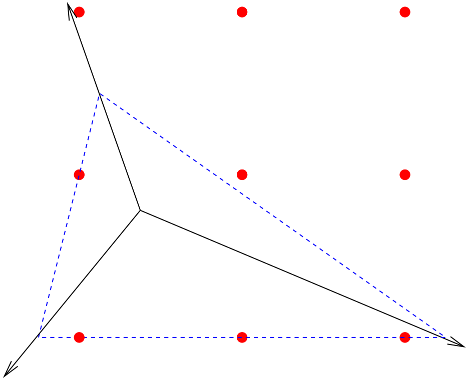
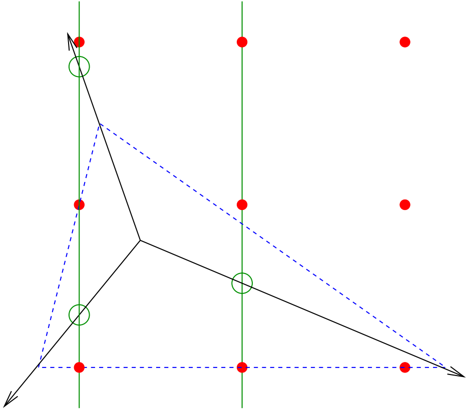
Proof for $T^{2C} - LHS$ triangle



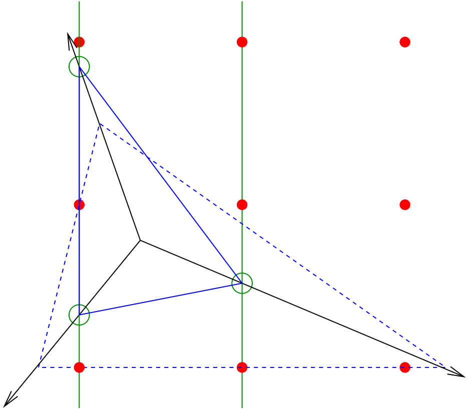
Rank 0

Proof for $T^{2C} - LHS$ triangle



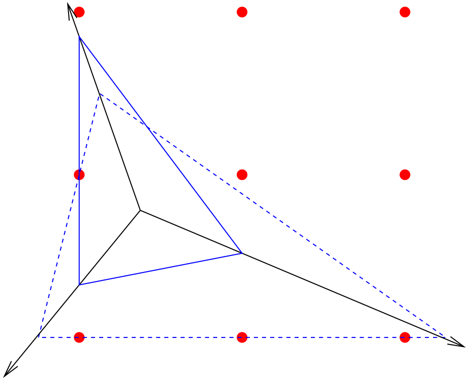
Rank 0

Proof for $T^{2C} - LHS$ triangle



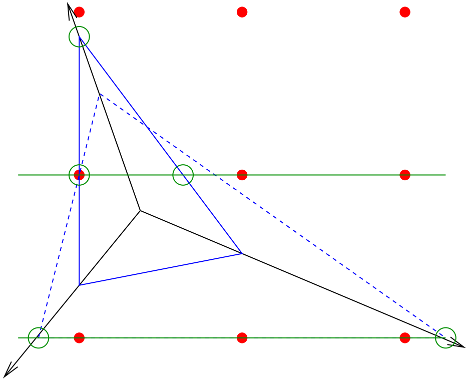
Rank 1

Proof for $T^{2C} - LHS$ triangle



Rank 1

Proof for $T^{2C} - LHS$ triangle



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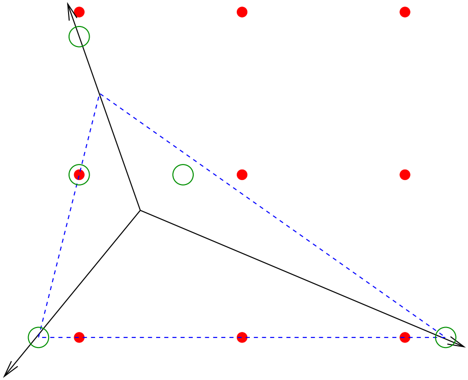
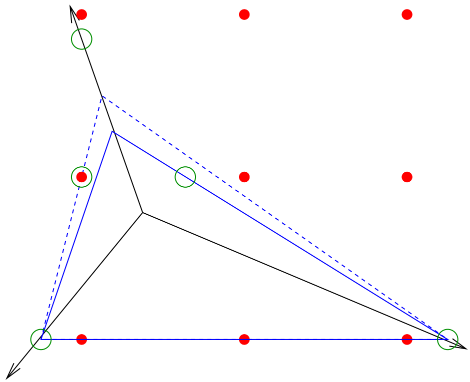
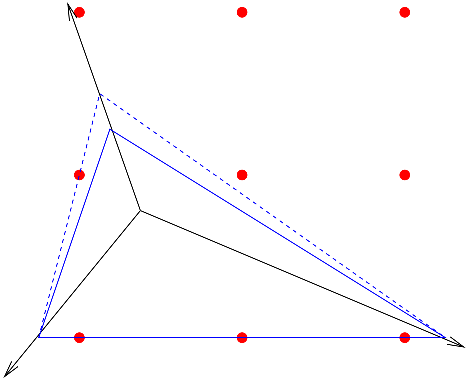


Proof for $T^{2C} - LHS$ triangle



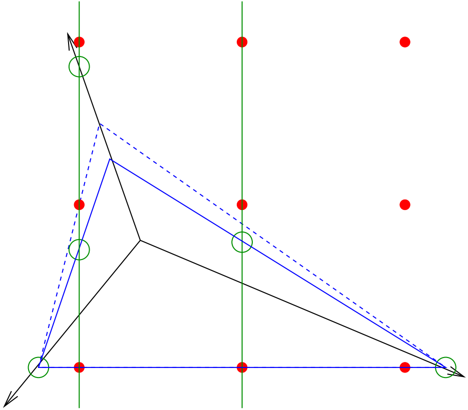
Rank 2

Proof for $T^{2C} - LHS$ triangle



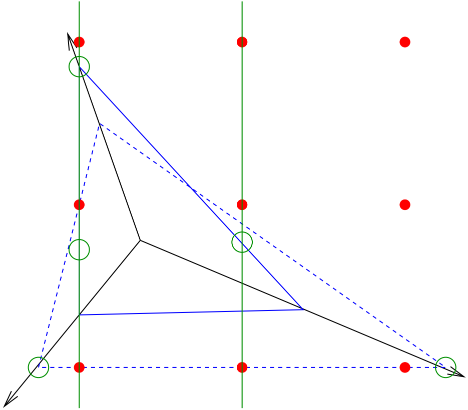
Rank 2

Proof for $T^{2C} - LHS$ triangle



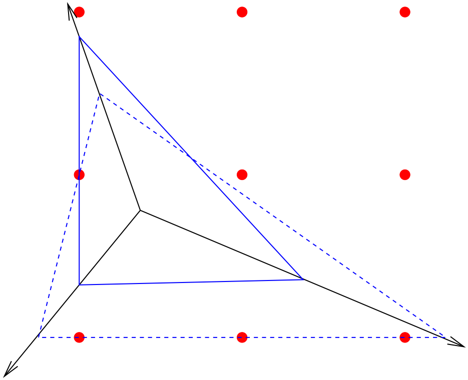
Rank 2

Proof for $T^{2C} - LHS$ triangle



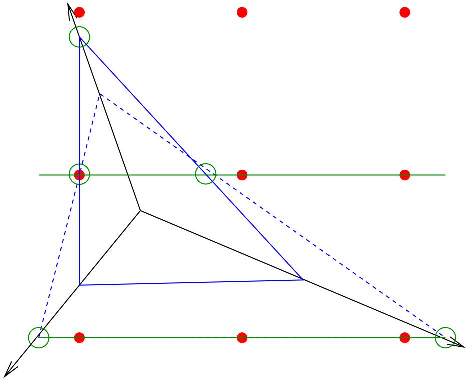
Rank 3

Proof for $T^{2C} - LHS$ triangle



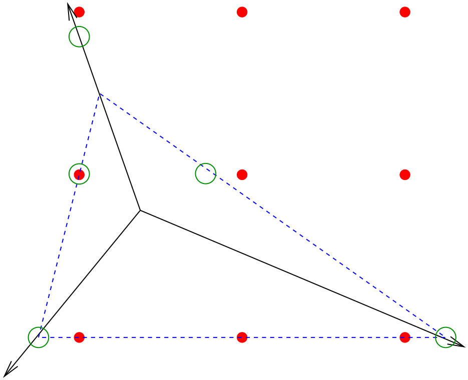
Rank 3

Proof for $T^{2C} - LHS$ triangle

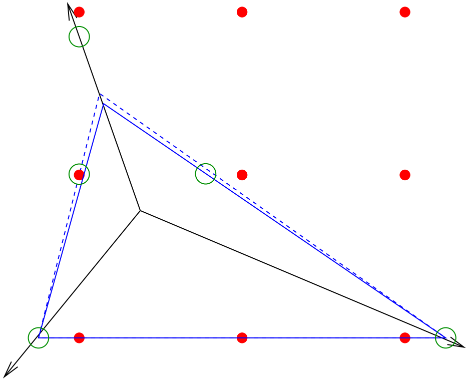


Rank 3

Proof for $T^{2C} - LHS$ triangle

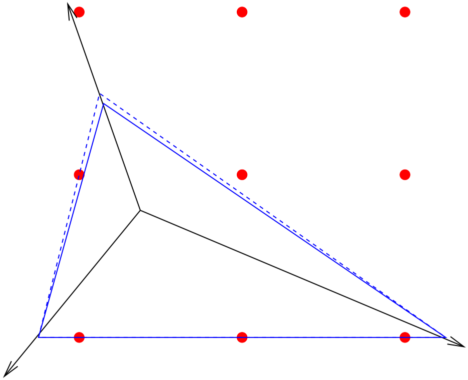


Proof for $T^{2C} - LHS$ triangle



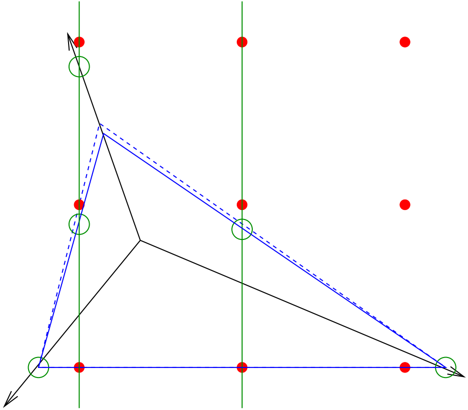
Rank 4

Proof for $T^{2C} - LHS$ triangle



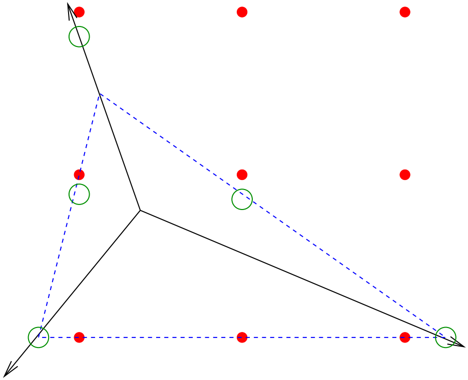
Rank 4

Proof for $T^{2C} - LHS$ triangle

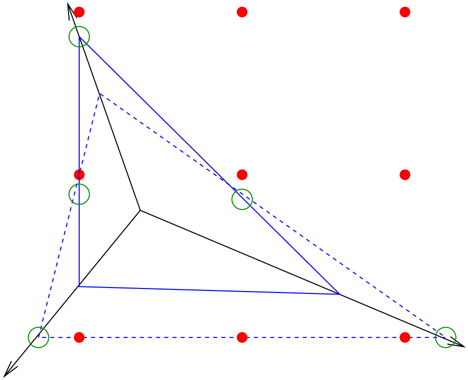


Rank 4

Proof for $T^{2C} - LHS$ triangle

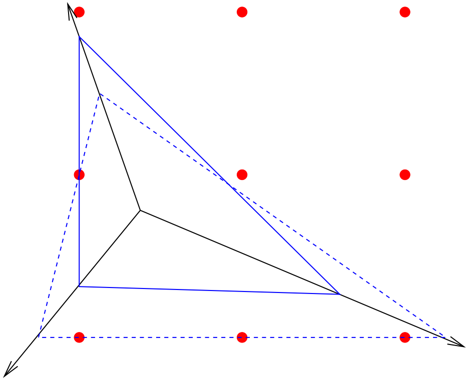


Proof for $T^{2C} - LHS$ triangle



Rank 5

Proof for $T^{2C} - LHS$ triangle



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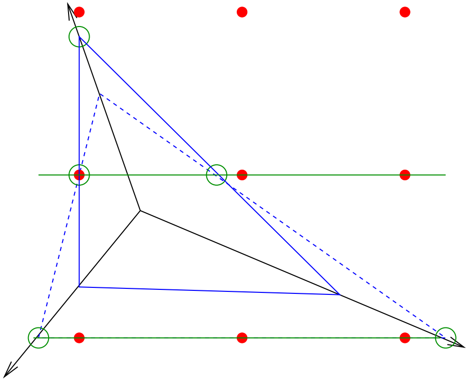

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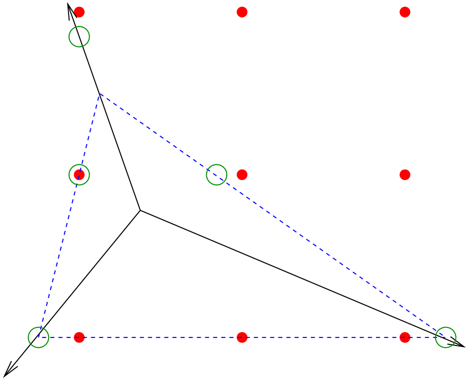

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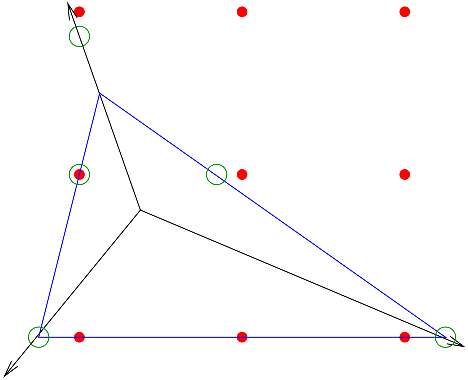
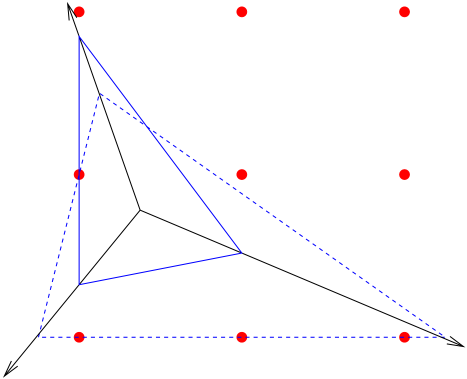
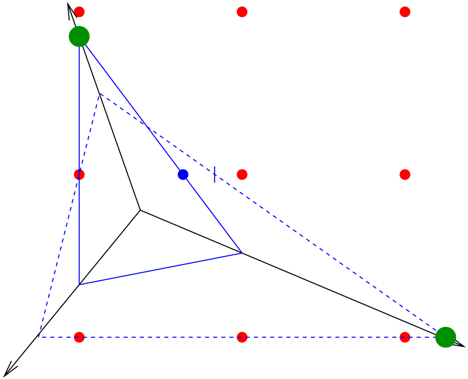


The goal inequality has a split rank of at most 6

The geometry behind the convergence



The geometry behind the convergence



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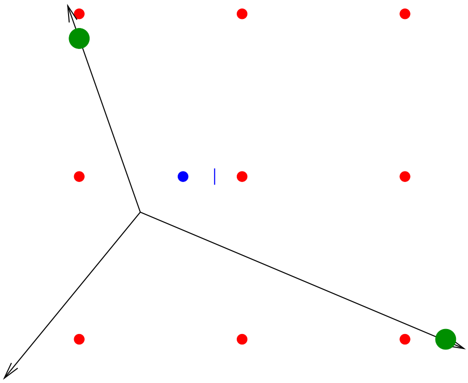

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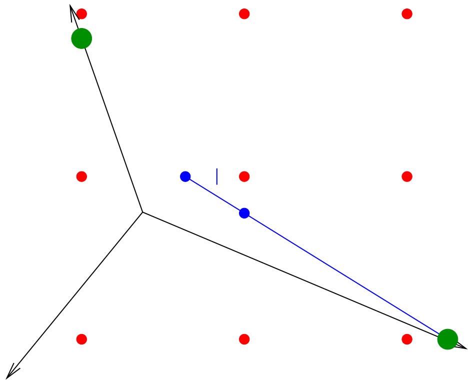

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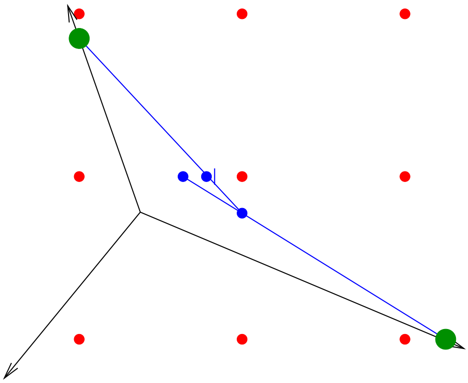
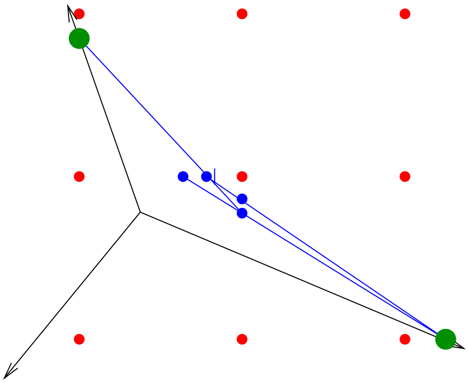
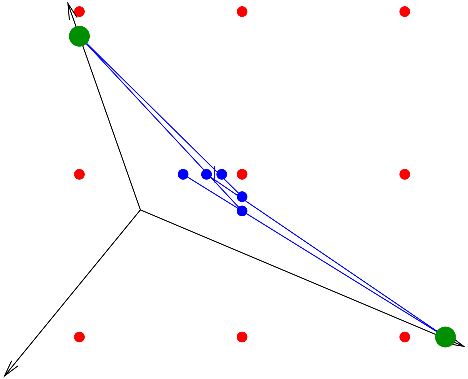


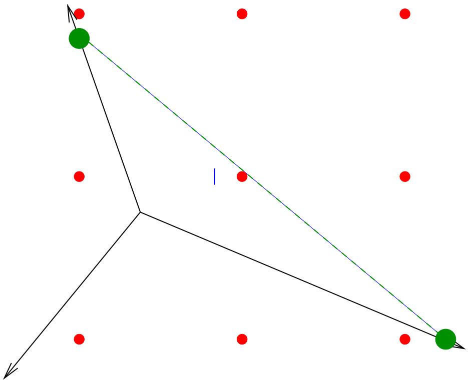
The geometry behind the convergence



The geometry behind the convergence



The geometry behind the convergence

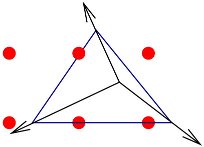


Where do we stand.

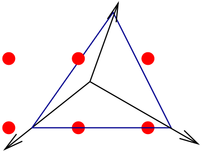
- We have proven that a inequality corresponding to $T^{2C} - LHS$ triangle has a finite rank.
- We can prove that the constructed bound is logarithmic in the number of bits of the input.

The triangle case

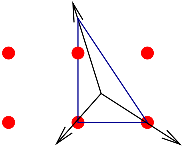
Several cases to consider, after suitable unimodular transformation.



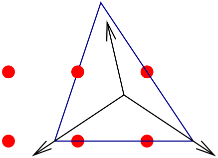
$T^{2C} - LHS$: Done



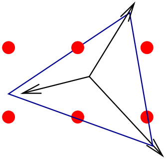
$T^{2C} - RHS$:
The proof is more complex
But similar



T^{2B} : By Shape Lemma

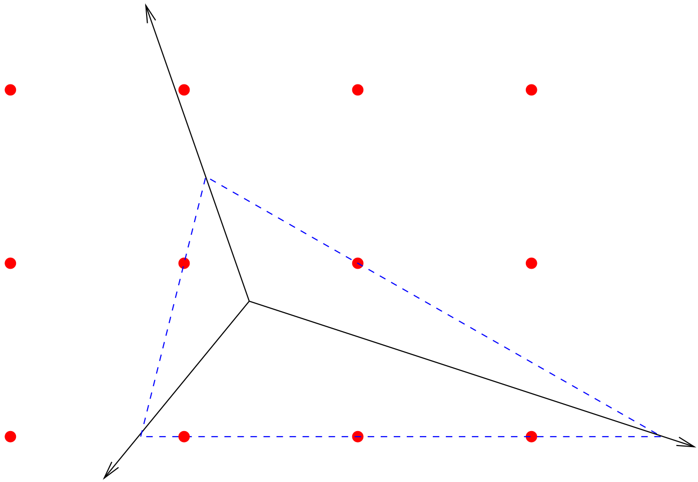


T^{2A}

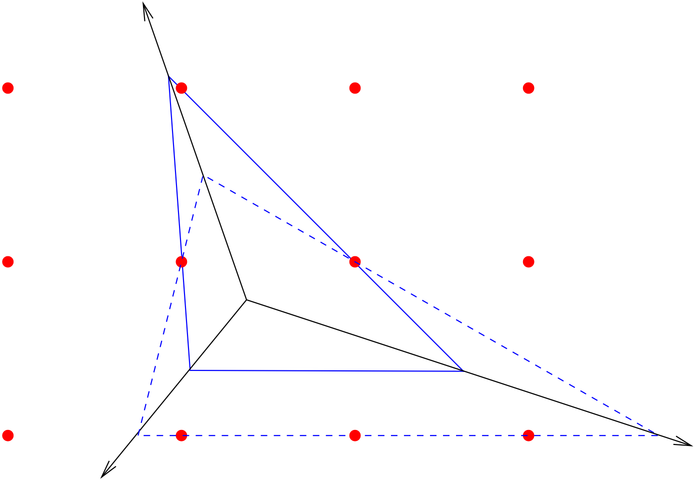


T^3

T^{2A}

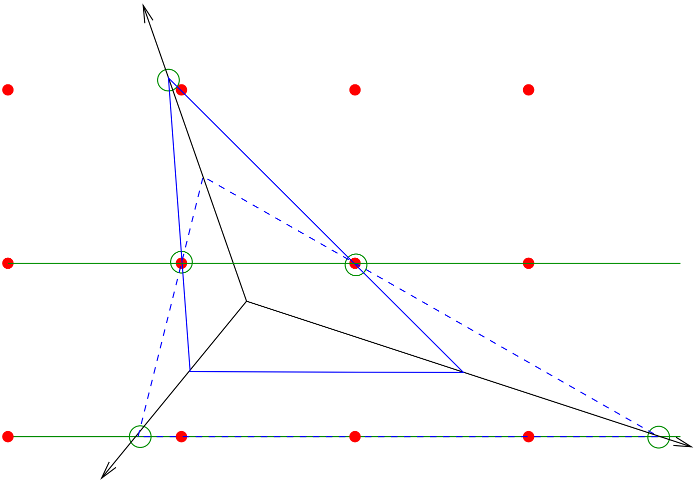


T^{2A}

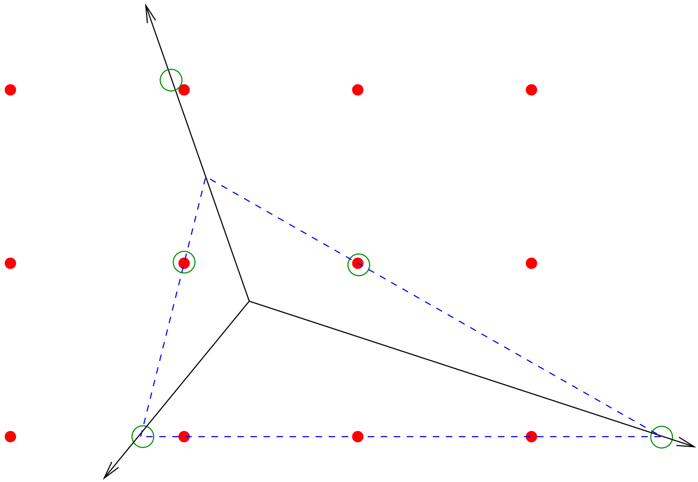


This inequality has finite rank It is either T^{2C} or T^{2B}!

T^{2A}

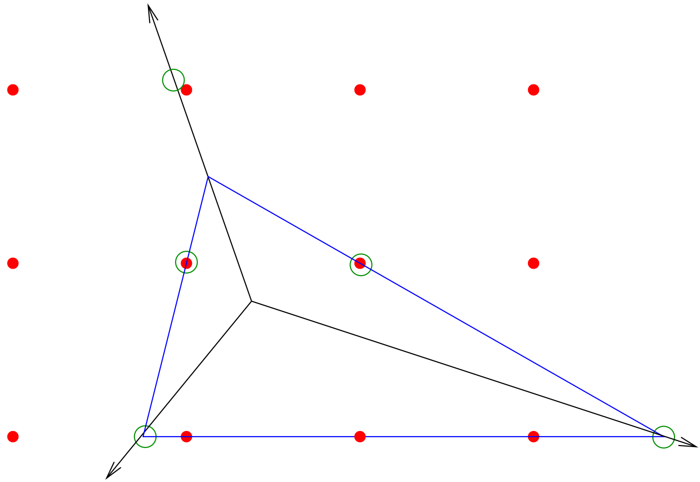


T^{2A}



The goal inequality is “valid for the disjunction”.

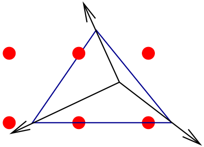
T^{2A}



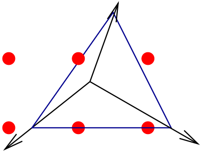
The goal inequality has a finite rank

The triangle case

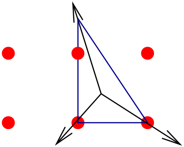
Several cases to consider, after suitable unimodular transformation.



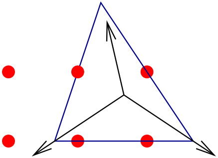
$T^{2C} - LHS$: Done



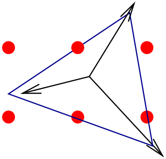
$T^{2C} - RHS$:
The proof is more complex
~~But similar~~



T^{2B} : By Shape Lemma



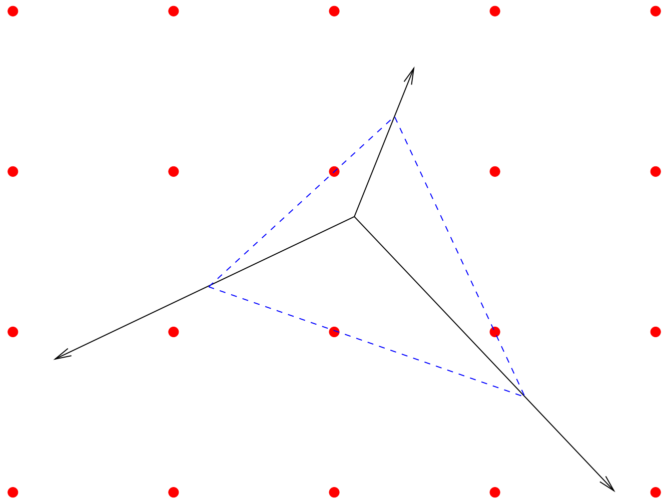
T^{2A} : Done



T^3

The T^3 triangle

$T^3 \equiv$ each side is tight at exactly one integer point



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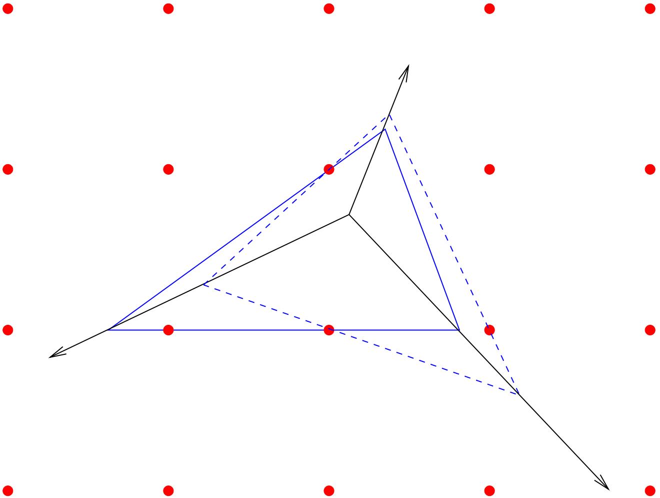

This inequality has finite rank!

The T^3 triangle

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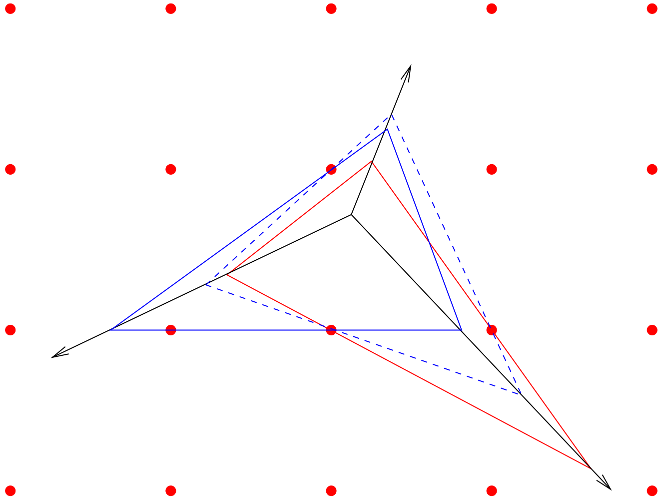

Similarly this inequality has a finite rank!

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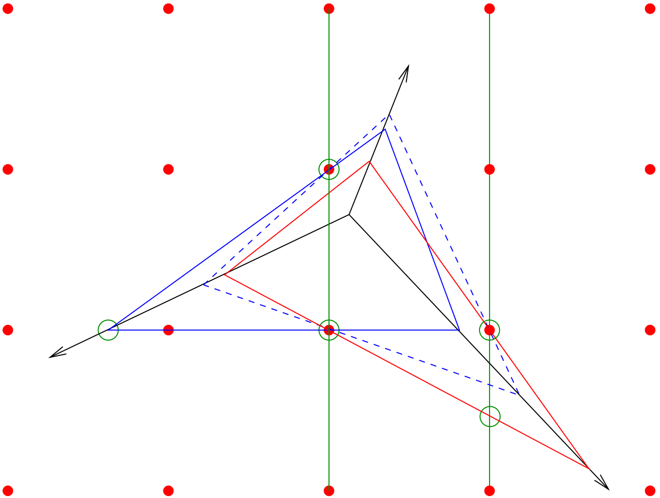

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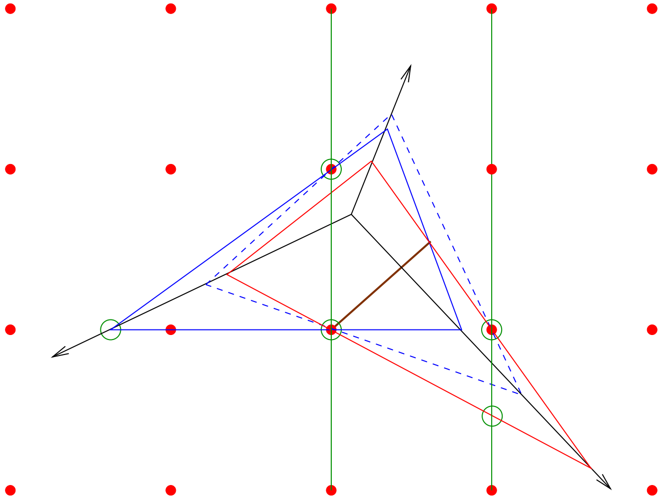

Brown line : set of points with a representation that satisfy both inequalities with equality

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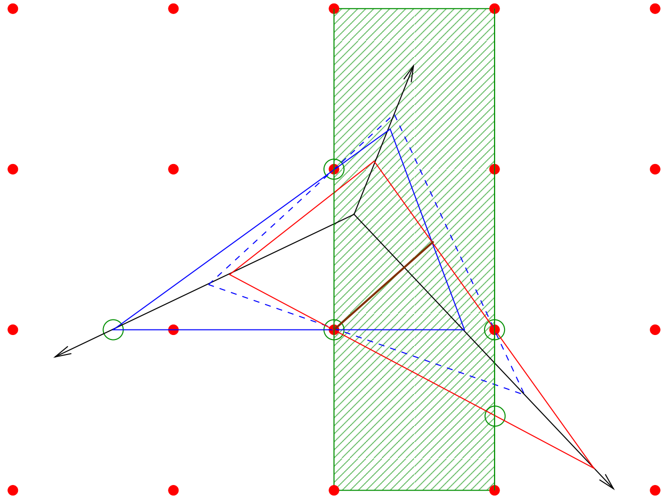

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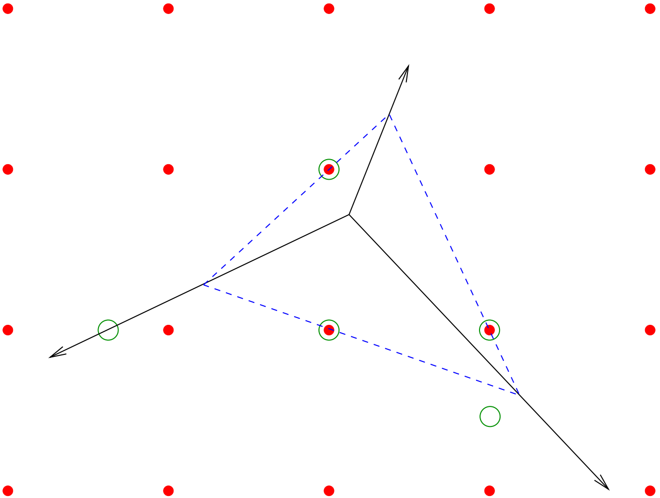

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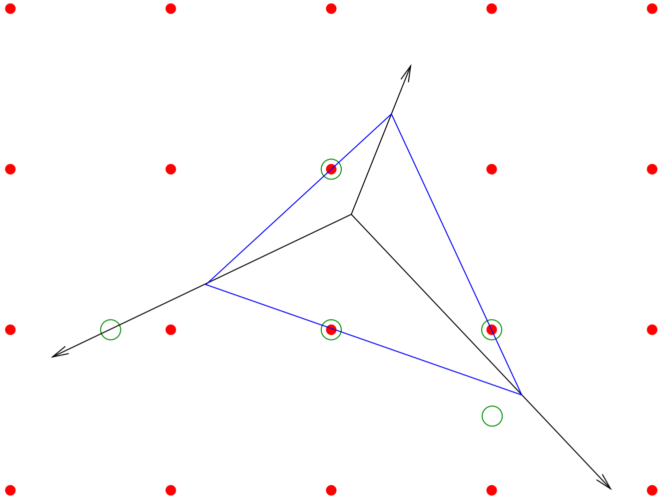

The disection cut has a finite rank

4

Finiteness proofs for the quadrilaterals for four-continuous-variable problems

The quadrilateral cuts

- Two cases: non-maximal quadrilateral and disection quadrilateral.
- By the Shape Lemma, we can deal with most non-maximal quadrilaterals
- One exception: if the lifted triangle has infinite rank.

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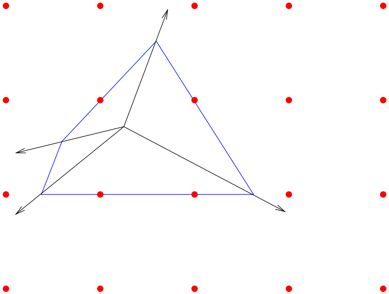

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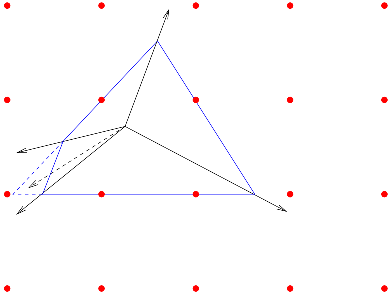

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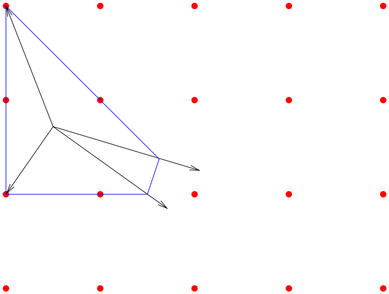

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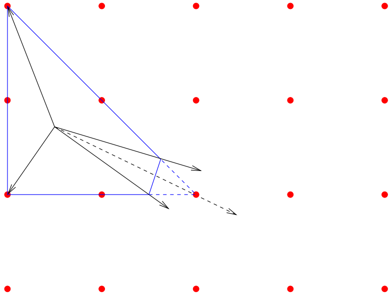

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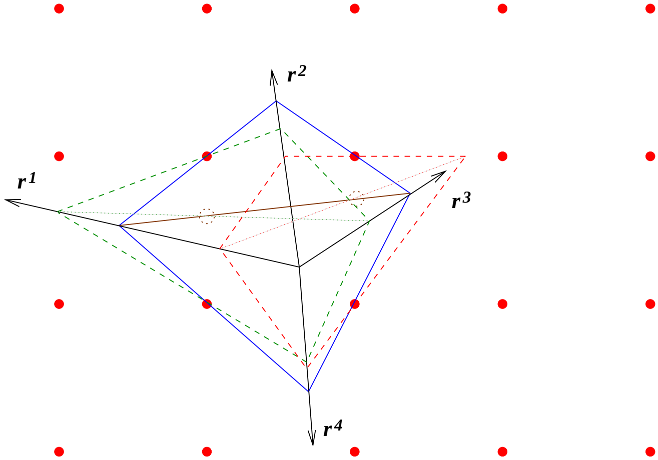
The proof is more technical, using two finite rank triangles and one disjunction.

The disection quadrilateral



Conclusion

- All triangles except those whose induced lattice-free set is the Cook-Kannan-Schrijver triangle have a finite rank.
- We provide a constructive split proof of that fact.
- Split cuts can essentially achieve "all quadrilateral" and "most triangles" .

Some Questions:

- Lower bounds on the split rank.
- Is there a pattern? Extensions to higher dimension?

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