The Chvátal-Gomory Closure of Strictly Convex Body

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Outline

1
Chvátal-Gomory Procedure

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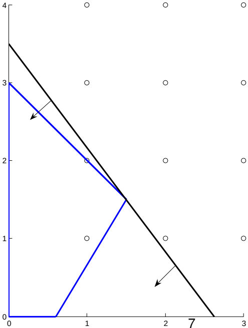
2. Since $x \in \mathbb{Z}^n$, $\langle a, x \rangle \in \mathbb{Z}$. Therefore the inequality

$$\langle a, x \rangle \leq \lfloor r \rfloor$$

is a valid inequality for $C \cap \mathbb{Z}^n$.

Chvátal-Gomory Cuts for Polyhedron

$\{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 \geq, x_2 \geq 0, x_1 + x_2 \leq 3, 5x - 3y \leq 3\}$
Valid inequality for Continuous Relaxation: $\underbrace{4x_1 + 3x_2}_{\in \mathbb{Z}} \leq 10.5.$



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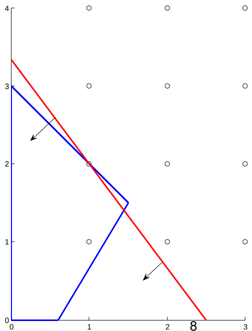

Strictly Convex Sets

Definition

1. We say a set C is strictly convex if for all $u, v \in C$, $u \neq v$ we have that $\lambda u + (1 - \lambda)v \in \text{rel.int}(C)$ for all $0 < \lambda < 1$.
2. We say C is a strictly convex body if C is a full dimensional, strictly convex and a compact set.

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Almost always I will be concerned with strictly convex body.

Chvátal-Gomory Cuts for a Strictly Convex Body

$$\{(x_1, x_2) \in \mathbb{Z}^2 \mid \| \begin{bmatrix} 0.375 & 0.125 \\ 0.125 & 0.375 \end{bmatrix} (x - \begin{bmatrix} 1 \\ 3 \end{bmatrix}) \| \leq 1\}$$

Valid inequality for Continuous Relaxation: $\underbrace{x_1 + x_2}_{\in \mathbb{Z}} \leq \sqrt{8} + 4$.

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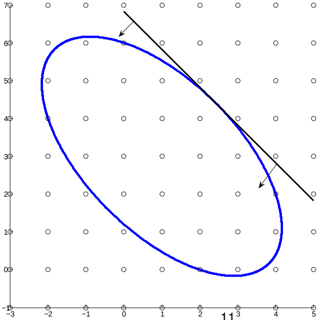

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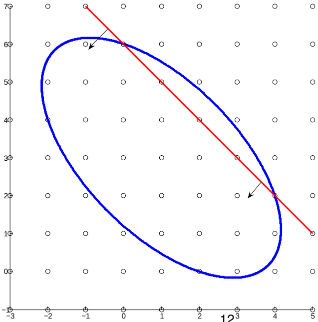

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3. For any subset $D \subseteq \mathbb{Z}^n$ we define,

$$\mathrm{CGC}(D, C) := \{x \in \mathbb{R}^n \mid \bigcap_{a \in D} \langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor\}$$

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$$\boxed{\text{Is } \texttt{CGC}(\mathbb{Z}^n, C) \text{ a polyhedron?}}$$

What We Know?

Theorem (Chvátal (1973), Schrijver (1980))

Let C be a rational polyhedron. Then $\mathrm{CGC}(\mathbb{Z}^n, C)$ *is a polyhedron.*

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Let $\mathrm{C}^2 := \mathrm{CGC}(\mathbb{Z}^n, \mathrm{C}^1)$.
Let $\mathrm{C}^3 := \mathrm{CGC}(\mathbb{Z}^n, \mathrm{C}^2)$.

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Let C be a rational polyhedron or any bounded convex set, then there exists an integer k such that C^k is the convex hull of the integer feasible points.

Unfortunately, Theorem 2 does not imply Theorem 1. For example:

$\boxed{\text{If } C \text{ is irrational polytope, then we do not know if } \mathrm{CGC}(\mathbb{Z}^n, C) \text{ a polytope.}}$

Main Results

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Let C be a strictly convex body. Then $\mathit{CGC}(C)$ is a rational polytope.

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Theorem

Let C be a strictly convex body and P a rational polyhedron. Then $\mathit{CGC}(C \cap P)$ is a rational polytope.

2
Proof outline

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Figure: A procedure to generate the CG closure for a strictly convex body C

Step 1 Show that it is possible to construct a polytope Q defined by a finite number of CG cuts such that:

- $Q \subset C$.
- $Q \cap \text{bnd}(C) \subset \mathbb{Z}^n$.

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Outline of Step 1: Separate All Points on the Boundary 'Smartly'

Lemma (Fractional Points on the Boundary)

If $u \in \mathrm{bnd}(C) \setminus \mathbb{Z}^n$, then there exists a CG cut that separates point u.

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Lemma (Fractional Points on the Boundary)

If $u \in \operatorname{bnd}(C) \setminus \mathbb{Z}^n$, then there exists a CG cut that separates point u.

In case of rational polyhedron, there is a similar result that essentially says that if P is a rational polyhedron and F is an face, then

$$\mathrm{CGC}(\mathbb{Z}^n, P) \cap F = \mathrm{CGC}(\mathbb{Z}^n, F).$$

However, this proof does not carry though here.

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Lemma (Fractional Points on the Boundary)

If $u \in \text{bnd}(C) \setminus \mathbb{Z}^n$, then there exists a CG cut that separates point u.

Lemma (Integral Points on the Boundary)

Let $C \subseteq \mathbb{R}^n$ be a strictly convex body and $u \in \text{bnd}(C) \cap \mathbb{Z}^n$. Then

1. *There exists a polyhedral cone $T'(u) = \{x \in \mathbb{R}^n : \langle c_i, x \rangle \leq 0, 1 \leq i \leq k\}$ such that*

$$c_i \in \mathbb{Z}^n \quad \text{and} \quad \lfloor \sigma_C(c_i) \rfloor = \langle c_i, u \rangle \quad \forall\, i, 1 \leq i \leq k \tag{1}$$

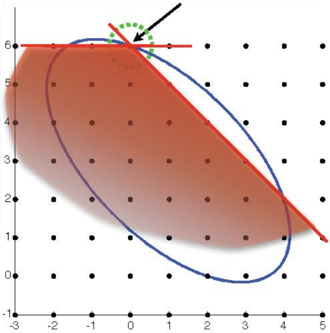
and

$$T'(u) \subseteq \text{int}(T_C(u)) \cup \{0\} \tag{2}$$

2. *There exists an open neighborhood $\mathcal{N}$ of u such that*

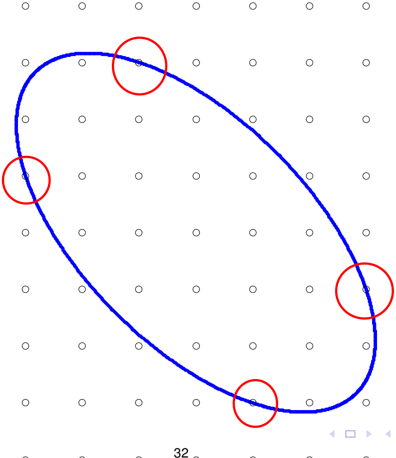
$$\mathcal{N} \cap \text{bnd}(C) \cap (u + T'(u)) = \{u\}$$

Illustration of Second Lemma



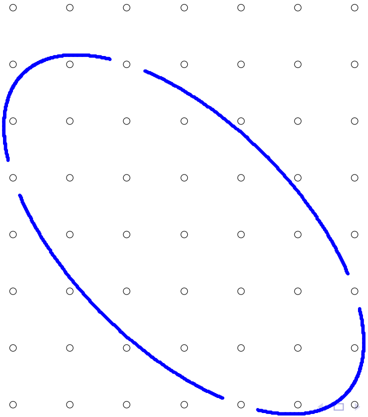
Constructing Q Using a Compactness Argument

The Boundary is compact



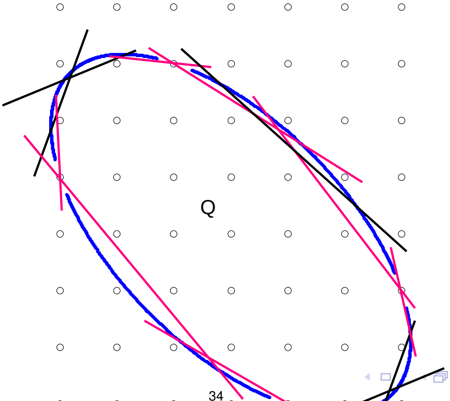
Constructing Q Using a Compactness Argument

The new set obtained by removing some open sets from the boundary is compact



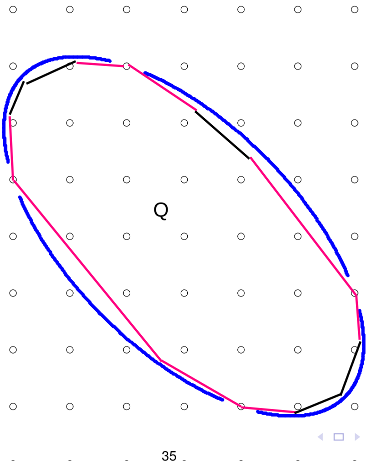
Constructing Q Using a Compactness Argument

Since every point is separated, there exists a finite sub collection that separates every point + finite number of cuts separating neighborhood of integer points



Constructing Q Using a Compactness Argument

$Q \subset C$, $Q \cap \mathrm{Bnd}(C) \subseteq \mathbb{Z}^n$



2.1
Proof of the First Separation Lemma

Separation: Easy Case

1. Select a fractional point u on the boundary and let $s(u)$ be a vector in the normal cone, i.e.

   $\langle s(u), x\rangle \leq \sigma_C(s(u)) = \langle s(u), u\rangle$ be the valid inequality which is tight at u

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2. If there exists $\lambda > 0$ such that
   - 2.1 $\lambda s(u) \in \mathbb{Z}^n$ and
   - 2.2 $\sigma_C(\lambda s(u)) \notin \mathbb{Z}$
3. Then generate the CG cut

$$\langle \lambda s(u), u\rangle \leq \lfloor \sigma_C(\lambda s(u))\rfloor$$

   that separates u.

Separation: Difficult Cases

Given $u \in \text{bnd}(C)$, for all $\lambda > 0$ such that

1. $\lambda s(u) \in \mathbb{Z}^n$
2. but $\sigma_C(\lambda s(u)) \in \mathbb{Z}$.

Example

Let $C := \{x \in \mathbb{R}^2 \mid \sqrt{x_1^2 + x_2^2} \leq 5\}$ and $u = (25/13, 60/13)^T \in \text{bnd}(C)$. We have that the supporting inequality for u can be is $5x_1 + 12x_2 \leq 65$. Because 5 and 12 are coprime and $\sigma_C(\cdot)$ is positively homogeneous, $(5, 12)$ cannot be scaled so that $\lambda(5, 12) \in \mathbb{Z}^2$ and $\sigma_C(\lambda(5, 12)) \notin \mathbb{Z}$.

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Given $u \in \text{bnd}(C)$, there does not exist $\lambda > 0$ such that $\lambda s(u) \in \mathbb{Z}^n$.

Example

Let $C := \{x \in \mathbb{R}^2 \mid \sqrt{x_1^2 + x_2^2} \leq 1\}$ and $u = (1/2, \sqrt{3}/2)^T \in \text{bnd}(C)$. We have that the supporting inequality for u is $\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 \leq 1$. Since $(1/2, \sqrt{3}/2)$ is irrational in only one component, observe that it cannot be scaled to be integral.

Separation in Difficult Cases

Given $u \in \text{bnd}(C)$ such that the supporting hyperplane does not provide a CG cut.
Idea: Construct a sequence $\{s^i\}_{i=1}^{\infty}$ of vectors in $\mathbb{Z}^n$ such that there exists some N such that

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$$\begin{aligned} & \langle s^N, u\rangle > \lfloor \sigma_C(s^N) \rfloor \\ \Leftrightarrow\ & \langle s^N, u\rangle > \sigma_C(s^N) - \mathcal{F}(\sigma_C(s^N)) \\ \Leftrightarrow\ & \langle s^N, u\rangle - \sigma_C(s^N) + \mathcal{F}(\sigma_C(s^N)) > 0. \end{aligned} \tag{3}$$

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To obtain (3), we construct sequence that satisfy the following property:

1. $\lim_{i\to\infty} \langle s^i, u\rangle - \sigma_C(s^i) = 0$
2. $\lim_{i\to\infty} \mathcal{F}(\sigma_C(s^i)) = \delta > 0$

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- We can select $s = (5,12)^T \in N_C(u)$ and approximate s with sequence $\{s^i\}_{i \in \mathbb{N}}$ given by $s^i = (65i^2, 26i + 156i^2)^T$. Then

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$$\lim_{i\to\infty} \frac{s^i}{||s^i||} = \frac{s}{||s||},$$

but $\lim_{i\to\infty}\langle s^i, u\rangle - \sigma_C(s^i) = -\infty$.

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but $\lim_{i \to \infty} \langle s^i, u \rangle - \sigma_C(s^i) = -\infty$.
- Consider $s^i = (65i, 26 + 156i)^T$. Then $\lim_{i \to \infty} \langle s^i, u \rangle - \sigma_C(s^i) = 0$. Unfortunately, $\langle s^i, u \rangle - \sigma_C(s^i) \leq -F(\sigma(s^i))$ and hence $\langle s^i, u \rangle \leq \lfloor \sigma(s^i) \rfloor$ for all i.

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- Perturb slightly $s^i = (65i, 25 + 156i)^T$, which works: $\langle s^3, u\rangle > \lfloor \sigma(s^3) \rfloor$.

Separation: Difficult case

Theorem (Dirichlet)

Let $s \in \mathbb{R}^n$. There exists $\{p^i, q_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}^n \times \mathbb{Z}$ such that $1 \leq q_i \leq i^n$ for all $i \in \mathbb{N}$, $\max_{1\leq j\leq n} \left|p^i_j - q_i s_j\right| \leq \frac{1}{i}$ and $\lim_{i\to\infty} q_i = +\infty$.

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Construction

- Since $u \notin \mathbb{Z}^n$, let $u_l \notin \mathbb{Z}$.
- Scale $s(u)$, that is let $s = \lambda s(u)$ such that $\langle s, u \rangle \in \mathbb{Z}$.

Separation: Difficult case

Theorem (Dirichlet)

Let $s \in \mathbb{R}^n$. There exists $\{p^i, q_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}^n \times \mathbb{Z}$ such that $1 \leq q_i \leq i^n$ for all $i \in \mathbb{N}$, $\max_{1\leq j\leq n} \left|p^i_j - q_i s_j\right| \leq \frac{1}{i}$ and $\lim_{i\to\infty} q_i = +\infty$.

Construction

- Since $u \notin \mathbb{Z}^n$, let $u_l \notin \mathbb{Z}$.
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$$\begin{aligned}&\mathcal{F}(\sigma_C(s^i))\\ &\approx \mathcal{F}(\langle s^i, u\rangle) = \mathcal{F}(\langle p^i + e^l, u\rangle) = \mathcal{F}(\langle p^i, u\rangle + \langle e^l, u\rangle)\\ &\approx \mathcal{F}(q_i\langle s, u\rangle + \langle e^l, u\rangle)) = \mathcal{F}(\underbrace{q_i\langle s, u\rangle}_{\in\mathbb{Z}}) + \mathcal{F}(u_l).\end{aligned}$$

2.2
Proof of the Second Separation Lemma

The Second Separation Lemma

Proposition

Let $C \subseteq \mathbb{R}^n$ be a strictly convex body. Take $u \in \mathrm{bnd}(C) \cap \mathbb{Z}^n$. Then for every $v \in \mathbb{R}^n \setminus \mathrm{int}(T_C(u))$, $v \neq 0$, there exists $a \in \mathbb{Z}^n$ such that

$$\langle a, u \rangle = \lfloor \sigma_C(a) \rfloor \quad \textit{and} \quad \langle a, v \rangle > 0$$

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- The construction of a uses Dirichlet approximation.
- We use another compactness argument to obtain the second separation lemma.

Back to Proof Outline

Figure: A procedure to generate the CG closure for a strictly convex body C

Step 1 Show that it is possible to construct a polytope Q defined by a finite number of CG cuts such that:

- $Q \subset C$.
- $Q \cap \text{bnd}(C) \subset \mathbb{Z}^n$.

Step 2 Update Q with CG cuts that separate points of $Q \setminus \text{CGC}(\mathbb{Z}^n, C)$: Show that there can exist only a finite number of such cuts are possible.

Proof of Step 2

1. Let V be the set of vertices of Q. We have that

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$$B(v, \varepsilon) \subseteq C \quad \forall v \in V \setminus \text{bnd}(C). \tag{4}$$
Now, if a CG cut $\langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor$ has $||a|| > \frac{1}{\varepsilon}$ then it cannot separate v.

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Now, if a CG cut $\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$ has $||a|| > \frac{1}{\varepsilon}$ then it cannot separate v.
5. The set of integer vectors such that $||a|| \leq \frac{1}{\varepsilon}$ is finite.

Open Questions

1. General bounded convex sets.
2. Unbounded sets, where convex hull of integer feasible points is a polyhedron.

Thank You.