# New SOCP relaxation and branching rule for bipartite bilinear programs

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# 1
# Introduction

# Bilinear Bipartite Problems (BBP)

$$\begin{array}{ll} \min_{x,y} & x^\top Q_0 y + d_1^\top x + d_2^\top y \\ \text{s.t.} & x^\top Q_k y + a_k^\top x + b_k^\top y + c_k = 0,\ k \in \{1,\dots,m\} \\ & l \le (x,y) \le u \\ & (x,y) \in \mathbb{R}^{n_1+n_2}, \end{array} \quad (1)$$

where

1. $n_1, n_2 \in \mathbb{Z}_+$,
2. $Q_0, Q_k \in \mathbb{R}^{n_1 \times n_2}$,
3. $d_1, a_k \in \mathbb{R}^{n_1}$,
4. $d_2, b_k \in \mathbb{R}^{n_2}$,
5. $c_k \in \mathbb{R}$,

for $k \in \{1, \dots, m\}$.

Without loss of generality, we assume that $l_i = 0,\ u_i = 1,\ i \in \{1, \dots, n_1 + n_2\}$.

# Graph corresponding to BBP

For a given instance of BBP, we associate a graph $G(V, E)$:

1. The set of vertices corresponds to the variables in the instance, and
2. There is an edge between two vertices if there is a degree two term involving the corresponding variables in the instance formulation.

# Outline

2

## A Second order cone programming relaxation of BBP

## 2.1
## Convex hull of one-constraint relaxation

# Convex hull of one-constraint relaxation

A very successful approach in integer linear programing is to generate cutting-planes implied by single constraint relaxations.

# Convex hull of one-constraint relaxation

## Theorem

*Let $n_1, n_2 \in \mathbb{Z}_+$, $V_1 \in \{1, \ldots, n_1\}$, $V_2 \in \{1, \ldots, n_2\}$, and $E \subseteq V_1 \times V_2$. Consider the one-constraint BBP set*

$$S := \left\{ (x, y, w) \;\middle|\; \begin{array}{l} \sum_{(i,j)\in E} q_{ij} w_{ij} + \sum_{i \in V_1} a_i x_i + \sum_{j \in V_2} b_j y_j + c = 0, \\ w_{ij} = x_i y_j, \; (i, j) \in E \\ (x, y, w) \in [0, 1]^{n_1 + n_2 + |E|} \end{array} \right\}.$$

*Then:*

$$\text{conv}(S) \text{ is SOCP-representable.}$$

We give a constructive proof.

# Convex hull of one-constraint relaxation

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*Then:*

conv($S$) is SOCP-representable.

We give a constructive proof.
Related results:

- M. Tawarmalani, N.V. Sahinidis [2001]
- M. Tawarmalani, J.-P. P. Richard, K. Chung [2010]
- K. M. Anstreicher, S. Burer [2010]
- T. T. Nguyen, J.-P. P. Richard, M. Tawarmalani [2012]
- A. Gupte Thesis [2012]
- B. Kocuk, SSD, and X. A. Sun [2018]
- And many others...(especially on convex envelopes of bilinear functions)

2.2

## Proof of convex hull result

# High-level Sketch: Disjunctive argument

1. If $S$ is compact, then $\text{conv}(S) = \text{conv}(\text{extr}(S))$.

# High-level Sketch: Disjunctive argument

1. If $S$ is compact, then $\text{conv}(S) = \text{conv}(\text{extr}(S))$.
2. $\Rightarrow$ If $S \supseteq \bigcup_{i=1}^{k} T_i \supseteq \text{extr}(S)$, then

$$\text{conv}(S) = \text{conv}\left(\bigcup_{i=1}^{k} \underbrace{\text{conv}(T_i)}_{\text{SOCP Representable}}\right)$$

$$\Downarrow$$

$$\underbrace{\text{conv}(S)}_{\text{SOCP representable}} = \text{conv}\left(\bigcup_{i=1}^{k} \underbrace{\text{conv}(T_i)}_{\text{SOCP Representable}}\right)$$

## 2.2.1
## Step 1: "Support" of extreme points

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## Proposition

*Let $(\bar{x}, \bar{y}, \bar{w})$ be an extreme point of the set $S$. Then, there exists $U \subseteq V_1 \cup V_2$, where*

1. *$U = \{i_0, j_0\}$ where $(i_0, j_0) \in E$, or,*
2. *$U = \{i_0\}$ where $i_0 \in V_1$ is an isolated node, or,*
3. *$U = \{j_0\}$ where $j_0 \in V_2$ is an isolated node,*

*such that $\bar{x}_i \in \{0,1\}$, $\forall i \in V_1 \setminus U$, and $\bar{y}_j \in \{0,1\}$, $\forall j \in V_2 \setminus U$.*

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*such that $\bar{x}_i \in \{0,1\}$, $\forall i \in V_1 \setminus U$, and $\bar{y}_j \in \{0,1\}$, $\forall j \in V_2 \setminus U$.*

$\approx$ No two $x$ (or $y$s) can be strictly between bounds in an extreme point.

# Step 1: "Support" of extreme points

Consider a feasible point $(\bar{x}, \bar{y}, \bar{w})$ such that $0 < \bar{x}_1, \bar{x}_2 < 1$:

$$q_{11}x_1y_1 + q_{12}x_1y_2 + q_{21}x_2y_1 + q_{22}x_2y_2 \quad + \cdots + \sum_{i \in V_1} a_i x_i + \sum_{j \in V_2} b_j y_j = -c$$

$$w_{ij} = x_i y_j$$

$$x, y, w \in [0, 1].$$

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$$\bar{w}_{1j} = \bar{x}_1\bar{y}_j \ \forall j$$

$$\bar{w}_{2j} = \bar{x}_2\bar{y}_j \ \forall j$$

$$\bar{w}_{ij} = \bar{x}_i\bar{y}_j \qquad i \neq 2, \forall j$$

$$\bar{x}, \bar{y}, \bar{w} \in [0, 1].$$

# Step 1: "Support" of extreme points

Consider a feasible point $(\bar{x}, \bar{y}, \bar{w})$ such that $0 < \bar{x}_1, \bar{x}_2 < 1$:

$$\tilde{q}_1 \bar{x}_1 + \tilde{q}_2 \bar{x}_2 = \text{constant}$$

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# Step 1: "Support" of extreme points

Consider a feasible point $(\bar{x}, \bar{y}, \bar{w})$ such that $0 < \bar{x}_1, \bar{x}_2 < 1$:

$$\underset{\substack{\uparrow \\ +\epsilon}}{\tilde{q}_1 \bar{x}_1} + \tilde{q}_2 \underset{\substack{\downarrow \\ -\delta}}{\bar{x}_2} = \text{constant}$$

$$\overset{\uparrow}{\bar{w}_{1j}} = \overset{\substack{+\epsilon \\ \uparrow}}{\bar{x}_1} \bar{y}_j \quad \forall j \qquad \underset{\downarrow}{\bar{w}_{2j}} = \underset{\substack{\downarrow \\ -\delta}}{\bar{x}_2} \bar{y}_j \quad \forall j$$

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## 2.2.2
## Step 2: Analyzing each fixing

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At most one $x_i$ and at most one $y_j$ unfixed in an extreme point:

$$
\begin{aligned}
\tilde{q}x_1y_1 + \tilde{a}x_1 + \tilde{b}y_1 &= \tilde{c} \\
w_{11} &= x_1y_1 \\
w_{1j} &= x_1\bar{y}_j \qquad \forall j \neq 1 \\
w_{i1} &= \bar{x}_iy_1 \qquad \forall i \neq 1 \\
x_1, y_1, w &\in [0,1]^2 \times [0,1]^{n_1+n_2+1}.
\end{aligned}
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\end{aligned}$$

Case 1: $\tilde{q} \neq 0$

$$\begin{aligned}
\tilde{q}x_1y_1 + \tilde{a}x_1 + \tilde{b}y_1 &= \tilde{c} \\
w_{11} &= \frac{\tilde{c}-(\tilde{a}x_1+\tilde{b}y_1)}{\tilde{q}} \\
w_{1j} &= x_1\bar{y}_j && \forall j \\
w_{i1} &= \bar{x}_iy_1 && \forall i \\
0 \leq x_1, y_1 &\leq 1
\end{aligned}$$

Case 2: $\tilde{q} = 0,\ \tilde{a} \neq 0$

$$\begin{aligned}
w_{11} &= \tilde{c}/\tilde{a}y_1 - \tilde{b}/\tilde{a}y_1^2 \\
x_1 &= \tilde{c}/\tilde{a} - \tilde{b}/\tilde{a}y_1 \\
w_{1j} &= x_1\bar{y}_j && \forall j \neq 1 \\
w_{i1} &= \bar{x}_iy_1 && \forall i \neq 1 \\
0 \leq w_1 \leq 1, \alpha \leq y_1 &\leq \beta
\end{aligned}$$

Other cases “easy".

# Step 2: Analyzing each fixing

At most one $x_i$ and at most one $\bar{y}_j$ in an extreme point:

$$\begin{aligned} \tilde{q}x_1y_1 + \tilde{a}x_1 + \tilde{b}y_1 &= \tilde{c} \\ w_{11} &= x_1y_1 \\ w_{1j} &= x_1\bar{y}_j && \forall j \neq 1 \\ w_{i1} &= \bar{x}_iy_1 && \forall i \neq 1 \\ x_1, y_1, w &\in [0,1]^2 \times [0,1]^{n_1+n_2+1}. \end{aligned}$$

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Case 2: $\underline{\tilde{q} = 0,\ \tilde{a} \neq 0}$

$$\begin{aligned} w_{11} &= \tilde{c}/\tilde{a}y_1 - \tilde{b}/\tilde{a}y_1^2 \quad \leftarrow \boxed{\text{Quad}} \\ x_1 &= \tilde{c}/\tilde{a} - \tilde{b}/\tilde{a}y_1 \\ w_{1j} &= x_1\bar{y}_j \quad \forall j \neq 1 \\ w_{i1} &= \bar{x}_iy_1 \quad \forall i \neq 1 \\ 0 \leq w_1 \leq 1, &\ \alpha \leq y_1 \leq \beta \end{aligned}$$

Other cases “easy".

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- Quadratic equation in two variables $x \in \mathbb{R}^2$ with box constraints,
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## Lemma

*Let* $B = \{(x, w) \in [0,1]^n \times \mathbb{R} \,|\, x \in B_0,\ w = l^\top x + l_0\}$*, where* $B_0 \subseteq \mathbb{R}^n$*, and* $l^\top x + l_0$ *is an affine function of* $x$*. Then,*

$$\text{conv}(B) = \{(x, w) \in [0,1]^n \times \mathbb{R} \,|\, x \in \text{conv}(B_0),\ w = l^\top x + l_0\}.$$

# Step 2: Analyzing each fixing

We are left with:

- Quadratic equation in two variables $x \in \mathbb{R}^2$ with box constraints.

## Lemma (Richard, Tawarmalani (2013))

*Let $f : [0,1]^n \to \mathbb{R}$ be a continuous function. Then*

$$\text{conv}\left(\{x \in [0,1]^n \mid f(x) = 0\}\right) = \text{conv}\left(\{x \in [0,1]^n \mid f(x) \leq 0\}\right) \bigcap \text{conv}(\{x \in [0,1]^n \mid f(x) \geq 0\}).$$

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In all case the convex hull of resulting set can be shown to SOCP representable.

## 2.2.3
## Wrap-up of proof

# Wrap-up

$$S := \left\{ (x,y,w) \;\middle|\; \begin{array}{l} \sum_{(i,j)\in E} q_{ij} w_{ij} + \sum_{i\in V_1} a_i x_i + \sum_{j\in V_2} b_j y_j + c = 0, \\ w_{ij} = x_i y_j,\ (i,j) \in E \\ (x,y,w) \in [0,1]^{n_1+n_2+|E|} \end{array} \right\}.$$

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1. In order to obtain extreme points fix variables, so that no more than one $x$ and one $y$ are left free.
2. For each fixing, obtain a SOCP relaxation of the extreme points.

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Remarks:

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2. Key strength constructing convex hull in the extended $w$ space.

2.3
## SOCP Relaxation of BBP

# SOCP relaxation and its strength

## Definition (SOCP relaxation of BBP)

$$S^{SOCP} = \bigcap_{k=1}^{m} \text{conv}(S_k),$$

where $S_k = \{(x, y, w) \in [0, 1]^{n_1 \times n_2 \times |E|} \mid x^\top Q_k y + a_k^\top x + b_k^\top y + c_k = 0, w_{ij} = x_i y_j \ \forall (i, j) \in E\}$ and $E$ is the edge set of the graph corresponding to the BBP instance (and not just of one row).

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## Theorem

*For any BBP, we have that*

$$\text{proj}_{x,y,w}\left(S^{SDP}\right) \bigcap S^{QBP} \supseteq S^{SOCP}.$$

$$\begin{aligned} S^{QBP} &:= \{(x,y,w) \mid \sum_{(ij) \in E} (Q_k)_{ij} w_{ij} + a_k^\top x + b_k^\top y + c_k = 0 \ k \in [m]\} \\ & \bigcap \text{conv}\left(\{(x,y,w) \in [0,1]^{n_1+n_2+|E|} \mid w_{ij} = x_i y_j \ \forall (i,j) \in E\}\right). \\ S^{SDP} &:= \text{standard SDP relaxation.} \end{aligned}$$

# 3
# Branching rules

# Branching rule: key ideas

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- Collect all such ideal rages corresponding to all the two-dimensional sets involving $x_1$.

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- We develop a simple heuristic to find *an “ideal" range*.
- Collect all such ideal rages corresponding to all the two-dimensional sets involving $x_1$.
- We develop a heuristic to select one of these points (based on corresponding volume reduction) to finally partition the domain of $x_1$.

# Illustration of Ideal range: hyperbola, two branches

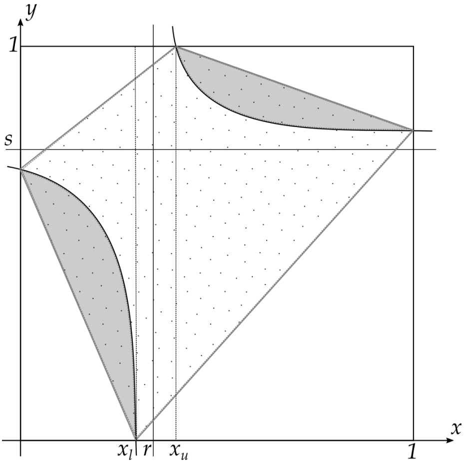


Figure: Convex hull of the set defined by the intersection of two branches of a hyperbola with the $[0, 1]^2$ box.

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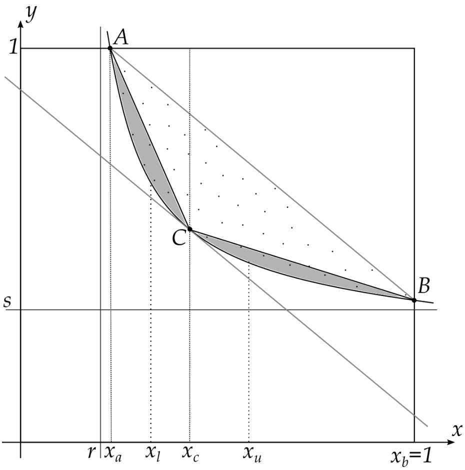


Figure: Convex hull of the set defined by the intersection of a single branch of a hyperbola with the $[0, 1]^2$ box. $A$ and $B$ are the intersection points of the curve with the $[0, 1]^2$ box and $C$ is the point of the curve at which the tangent line is parallel to $AB$.

4

# Computational results

## 4.1
## Instances

# FEM Model Updating problem

Find $x$ and $y^l$ so that:

$$\left[K_0 + \sum_{i=1}^{n_1} x_i K_i - \lambda_l M\right] \begin{bmatrix} \bar{y}^l \\ y^l \end{bmatrix} = 0, \; l \in \{1, \ldots, n_3\},$$

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Minimizing $l_1$ error:

$$\begin{aligned} \min \quad & \sum_{k=1}^{m} z_k \\ \text{s.t.} \quad & |x^\top Q_k y + a_k^\top x + b_k^\top y + c_k| = z_k, \; k \in \{1, \ldots, m\} \\ & x \in [0,1]^{n_1}, y \in [0,1]^{n_2}, \end{aligned} \tag{2}$$

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$$\begin{aligned} \min \quad & \sum_{k=1}^{m} z_k \\ \text{s.t.} \quad & |x^\top Q_k y + a_k^\top x + b_k^\top y + c_k| = z_k, \ k \in \{1, \dots, m\} \\ & x \in [0,1]^{n_1}, y \in [0,1]^{n_2}, \end{aligned} \tag{2}$$

BBP:

$$\begin{aligned} \min \quad & \sum_{k=1}^{m} z'_k + z''_k \\ \text{s.t.} \quad & x^\top Q_k y + a_k^\top x + b_k^\top y + c_k = z'_k - z''_k, \ k \in \{1, \dots, m\} \\ & x \in [0,1]^{n_1}, y \in [0,1]^{n_2}. \\ & 0 \le z'_k, z''_k \le u, \ k \in \{1, \dots, m\}. \end{aligned} \tag{3}$$

# Size of instances

Table: Instances description

| Inst | # var.s | # eqn.s | # bilinear terms | # max var.s per row |
|---|---|---|---|---|
| inst1 | 186 | 312 | 990 | 9 |
| inst2 | 186 | 312 | 954 | 9 |
| inst3 | 174 | 312 | 966 | 9 |
| inst4 | 174 | 312 | 972 | 9 |
| inst5 | 162 | 312 | 900 | 9 |
| inst6 | 150 | 312 | 780 | 10 |

## 4.2
## A more tractable relaxation

# A more tractable relaxation

## SOCP-light

1. The number of disjunction needed to model the convex hull of a single bilinear equation can be computationally prohibitive.
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## Polyhedral outer-approximation

1. Basically piece-wise linear out approximation of the convex quadratic inequalities.
2. We use only the tangents where the curve intersects the $[0, 1]^2$ box.

# Illustration of outer-approximation

Figure: Hyperbola outer-approx

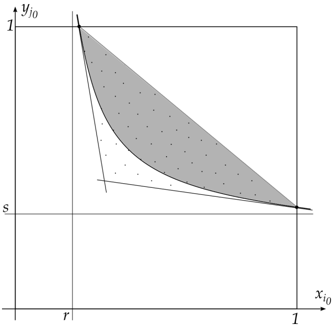
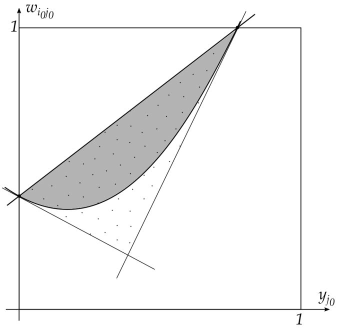



Figure: Parabola outer-approx



4.3

## Computational results

# Software and Hardware

Machine and OS: Windows 10 machine with 64-bit operating system, x64 based processor with 2.19GHz, and 32GB RAM.

1. LP and IP: Gurobi 7.5.1.
2. SDP: MOSEK ver 9.
3. Global solver: BARON 15.6.5.
4. Nonlinear solver: IPOPT.

Table: Root relaxations

| | Mc | | SDP+Mc | | SOCP | |
|---|---|---|---|---|---|---|
| Inst | Bound | Time | Bound | Time | Bound | Time |
| 1 | 0.1777 | 0.07 | 0.1777 | 35.89 | 0.1779 | 17.59 |
| 2 | 0.0000 | 0.05 | 0.0000 | 38.98 | 0.0000 | 20.93 |
| 3 | 0.2754 | 0.07 | 0.2754 | 44.02 | 0.2820 | 16.22 |
| 4 | 0.1009 | 0.08 | 0.1009 | 36.13 | 0.1010 | 20.71 |
| 5 | 0.3477 | 0.05 | 0.3477 | 31.58 | 0.3493 | 13.17 |
| 6 | 0.9776 | 0.05 | 0.9776 | 28.47 | 1.0027 | 11.64 |

Table: Branch-and-bound methods: dual bounds

| Inst | BB-SOCP-1 | BB-SOCP-2 | BB-SOCP-3 | BB-Mc | Mc-Descr ($2^{16}$) |
|---|---|---|---|---|---|
| 1 | 2.5074 | 0.1847 | 0.1823 | 0.1834 | 1.8539 |
| 2 | 2.8644 | 0.00000 | 0.0000 | 0.0000 | 2.1471 |
| 3 | 3.1308 | 0.2911 | 0.2898 | 0.2888 | 2.1427 |
| 4 | 3.1115 | 0.1053 | 0.1025 | 0.1041 | 2.4485 |
| 5 | 3.7896 | 0.3526 | 0.3539 | 0.3540 | 3.4027 |
| 6 | 4.6399 | 1.1110 | 1.0954 | 1.1519 | 4.1650 |

Running-time: 10 hours.

- *SOCP-1*: Uses the polyhedral relaxation with new branching rule.
- *SOCP-2*: Uses the same relaxation of BB-SOCP-1 above. The branching variable is selected according to the *gap-error-rule*. Then uses the *incumbent-rule* for branching point selection, whenever possible, otherwise uses the *maximum-deviation-rule*.
- *SOCP-3*: Same as BB-SOCP-2 except that uses *bisection for branching point selection*.
- *BB-Mc*: Uses McCormick relaxation with gap-error-rule as branching variable selection rule and *bisection for branching point selection*.

Table: Comparison with BARON

| | BB-SOCP-1 | | | BARON | | |
|---|---|---|---|---|---|---|
| Inst | Dual | Primal | % Gap | Dual | Primal | % Gap |
| 1 | 2.5074 | 3.4785 | 27.91 | 0.3312 | 3.4789 | 90.47 |
| 2 | 2.8644 | 3.4998 | 18.15 | 0.5245 | 3.4993 | 85.01 |
| 3 | 3.1308 | 3.6810 | 14.94 | 0.4760 | 3.6831 | 87.07 |
| 4 | 3.1115 | 3.7522 | 17.07 | 0.7863 | 3.7530 | 79.04 |
| 5 | 3.7896 | 4.1328 | 8.30 | 0.3840 | 4.1354 | 90.71 |
| 6 | 4.6399 | 5.6609 | 18.03 | 2.2657 | 5.6605 | 59.97 |

Running-time: 10 hours.

$$\%\text{gap} = \frac{\text{Primal} - \text{Dual}}{\textit{Primal}}$$

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   - Strength of SOCP-light?

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    - Strength of SOCP-light?
    - Better methods to obtain SOCP-light: theoretical and computational?
    - Taking row combinations?
    - Using more structure: e.g. pooling problem?
2. Multi-linear: As long as we can finally reduce (by fixing) to sets whose convex hull can be described?
3. Multi-row extensions?

Thank You!