Aggregation-based cutting-planes for packing and covering integer programs

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Discrete Optimization Workshop, ETH Zürich

Outline

Introduction

Main results

- Single row-aggregation cuts for packing and covering IPs
- Split cuts
- Aggregation cuts for non-packing/non-covering instances
- Multi-row aggregation cuts for aggregations

Proof sketch

- Packing
- Covering

1
Introduction

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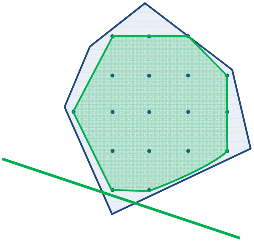

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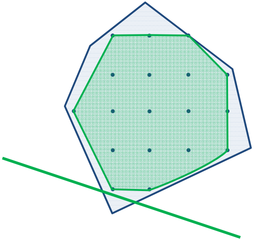

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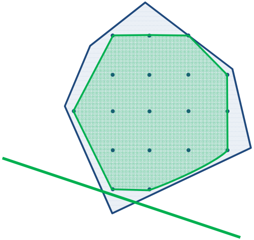
3. Significantly lesser research on evaluating the strength of cuts and (theory of) cutting-plane selection.



Cutting-plane via “easy" relaxation

Integer program (IP),

$$\left\{ x \in \mathbb{Z}^n \;\middle|\; \begin{array}{c} Ax \leq b \\ x \in [0, u]^n \end{array} \right\}$$

Relaxation →

R $\supseteq$ **IP**, where

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Generate valid inequality for R

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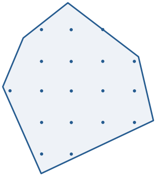
A common relaxation:

- Aggregate constraints into a single constraint.

Example of aggregation: Chvátal-Gomory (CG) cuts

Integer Program
$Ax \leq b$



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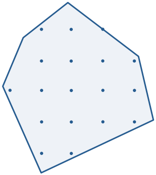
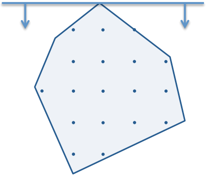


Relaxation
$\alpha^T x \leq \beta, \alpha \in \mathbb{Z}^n$



By Farkas' Lemma,
$\exists \lambda \in \mathbb{R}^m_+$ such that

1. $\alpha^T = \lambda^T A$
2. $\beta = \lambda^T b$

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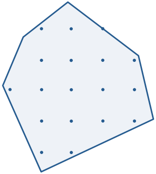

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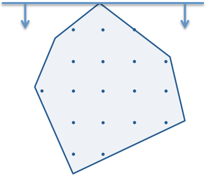

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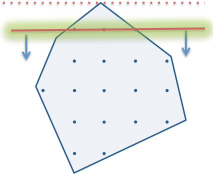
Cutting-plane

$\alpha^T x \leq \lfloor \beta \rfloor$



conv $\left(\{x \,|\, \alpha^T x \leq \beta\} \cap \mathbb{Z}^n\right)$
$= \{x \,|\, \alpha^T x \leq \lfloor \beta \rfloor\}$

Aggregation cuts are ubiquitous in integer programming

A more common relaxation: single constraint with bounds

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$$\begin{array}{rcl} \text{(non-trivial) } Ax & \leq & b \\ x & \in & [0,\ u]^n \\ x & \in & \mathbb{Z}^n \end{array} \rightarrowtail \text{Relaxation} \rightarrowtail \begin{array}{rcl} \lambda^T Ax & \leq & \lambda^T b \\ x & \in & [0,\ u]^n \\ x & \in & \mathbb{Z}^n \end{array}$$

Cuts from single constraint with bounds

1. Lifted cover inequalities.
2. Weight inequalities.
3. c-MIR inequalities.
4. Any facet of knapsack polytope.

A natural question

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Analyze strength of cuts from the original constraints vs (same family of) cuts generated from aggregation

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Literature survey:

1. R. Fukasawa and M. Goycoolea, *On the exact separation of mixed integer knapsack cuts*, Math. Prog., 2011.
2. H. Marchand and L. A. Wolsey, *Aggregation and Mixed Integer Rounding to Solve MIPs*, Oper. Research, 2001.
3. K. Andersen and R. Weismantel, *Zero-coefficient cuts*, IPCO, 2010.

2.1
Single row-aggregation cuts for packing and covering IPs

Some definitions: Packing integer programs

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Packing: CG and aggregation closures can be *2-approximated* by generating the respective cuts for original formulation constraints

Theorem (Packing)

Consider a packing polyhedron (with bounds) Q. Let $\mathcal{M} \in \{\mathcal{A}(\text{aggregation}), \mathcal{C}(\text{CG})\}$. *Then*

$$z^{1\mathcal{M}} \leq 2z^{\mathcal{M}}.$$

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$$z^{1\mathcal{M}} \leq 2z^{\mathcal{M}}.$$

Moreover, this bound is tight, namely for every $\varepsilon > 0$ *there is a packing polyhedron Q such that* $z^{1\mathcal{M}} \geq (2-\epsilon)z^{\mathcal{M}}$.

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + Mx_2 \leq M \\ & Mx_1 + x_2 \leq M \\ & x \in \mathbb{Z}^2_+, \end{aligned}$$

where M is an integer with $M \geq 1$.

- The constraints are integral $\Rightarrow z^{1\mathcal{C}} = z^{1\mathcal{A}} = z^{LP} = \frac{2M}{M+1}$
- $x_1 + x_2 \leq \frac{2M}{M+1}$ is valid $\Rightarrow x_1 + x_2 \leq 1$ is a CG cut. $\mathcal{C}(P) \subseteq \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 + x_2 \leq 1\} \Rightarrow z^{\mathcal{C}} = z^{\mathcal{A}} = 1$.

$\therefore \lim_{M\to\infty} \frac{z^{1\mathcal{C}}}{z^{\mathcal{C}}} \to 2$.

Covering (with bounds): CG and aggregation closures can be *2-approximated* by generating the respective cuts for original formulation constraints

Theorem (Covering)

Consider a covering polyhedron (with bounds) Q. Let $\mathcal{M} \in \{\mathcal{A}(\text{aggregation}), \mathcal{C}(\text{CG})\}$. *Then*

$$2z^{1\mathcal{M}} \geq z^{\mathcal{M}}.$$

Moreover, this bound is tight, namely for every $\varepsilon > 0$ *there is a covering polyhedron such that* $(2-\epsilon)z^{1\mathcal{M}} \leq z^{\mathcal{M}}$.

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} x_j \\ \text{s.t.} \quad & x_i + \sum_{j \in [n] \setminus \{i\}} 2x_j \geq 2, \ \forall i \in [n], \qquad x \in \{0,1\}^n. \end{aligned}$$

- The constraints are integral $\Rightarrow z^{1\mathcal{A}} = z^{1\mathcal{C}} = z^{LP} = \frac{2n}{2n-1}$
- $\sum_{j \in [n]} x_j \geq \frac{2n}{2n-1}$ is valid $\Rightarrow \sum_{j \in [n]} x_j \geq 2$ is a CG cut $\Rightarrow z^{\mathcal{C}}, z^{\mathcal{A}} \geq 2$

$\therefore \lim_{n \to \infty} \frac{z^{1\mathcal{C}}}{z^{\mathcal{C}}} \to \frac{1}{2}$.

Packing and covering do not behave exactly the same

Proposition (Packing vs Covering)

- *For a packing polyhedron Q, $1\mathcal{C}(Q)$ is a 2-approximation of $\mathcal{A}(Q)$.*
- *For every $\alpha > 0$, there exists a covering polyhedron Q such that $\mathcal{C}(Q) \not\subseteq \frac{1}{\alpha}1\mathcal{A}(Q)$.*

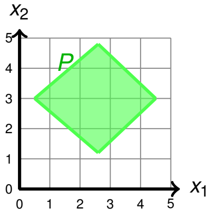
Let $t \in \mathbb{Z}_+$ and $t \geq \max\{3M, 1\}$. Consider the instance:

$$\begin{aligned} \min \quad & tx_1 + x_2 && (1)\\ \text{s.t.} \quad & t^2 x_1 + x_2 \geq t && (2)\\ & x_1, x_2 \in \mathbb{Z}_+. && (3) \end{aligned}$$

$z^{1A} = t$ and $z^{1C} < 3$.

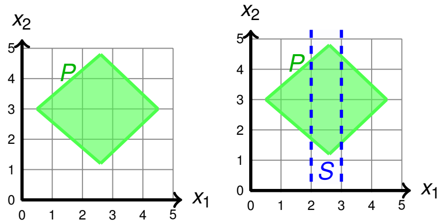
2.2
Split cuts

Split disjunction



$$P := \{x \in \mathbb{R}^n \,|\, Ax \leq b\}$$

Split disjunction



$P := \{x \in \mathbb{R}^n \,|\, Ax \leq b\}$ Integer points satisfy:

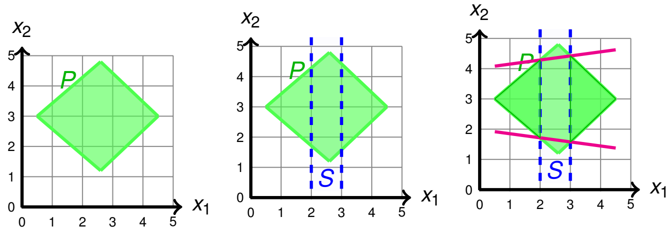
$$\pi^T x \leq \pi_0$$

or

$$\pi^T x \geq \pi_0 + 1,$$

where $\pi \in \mathbb{Z}^n$, $\pi_0 \in \mathbb{Z}$.

Split disjunction



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New inequalities valid for:

$P \cap \{\pi^T x \leq \pi_0\}$

and

$P \cap \{\pi^T x \geq \pi_0 + 1\}$

Some definitions

Packing

1. $Q := \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$ where $A \in \mathbb{Z}^{m \times n}_+$ and $b \in \mathbb{Z}^m_+$.
2. $\mathcal{S}(Q)$:= *Split closure* of Q. ($z^{\mathcal{S}}$)

Some definitions

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2. $\mathcal{S}(Q)$:= *Split closure* of Q. ($z^{\mathcal{S}}$)
3. The *1-row split closure* is defined as ($z^{1\mathcal{S}}$)

$$1\mathcal{S}(Q) = \bigcap_{i \in [m]} \mathcal{S}(\{x \in \mathbb{R}^n_+ \,|\, A^i x \leq b_i\}),$$

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Covering

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where A^i denotes the i^{th} row of A.

2-approx result for split disjunction

Theorem (Packing)

Consider a packing polyhedron Q. Then $1\mathcal{S}(Q)$ is a 2-approximation of $\mathcal{S}(Q)$. Moreover, this bound is tight.

Theorem (Covering)

Consider a covering polyhedron Q. Then $1\mathcal{S}(Q)$ is a 2-approximation of $\mathcal{S}(Q)$. Moreover, this bound is tight.

2.3
Aggregation cuts for non-packing/non-covering instances

Experiments with random instances

We generate instances of the following form:

$$\max \left\{ \sum_{j \in [n]} x_j \mid Ax = b,\ 0 \leq x \leq u \right\},$$

where

1. We consider instances with $n \in \{10, 12, 14, 16\}$ variables and $m = \lfloor n/2 \rfloor$ equality constraints.
2. We choose $M = 50$ and set $u_j = M/2$ for all $j \in [n]$.
3. For any $i \in [m], j \in [n]$, we let $A_{ij} = 0$ with probability 0.5. Otherwise, we set A_{ij} to an integer in $\{-M, \ldots, M\}$ with equal probability.
4. We construct b by first generating a binary solution $\hat{x}$ uniformly at random, and then letting $b = A\hat{x}$.

Experiments with random instances - II

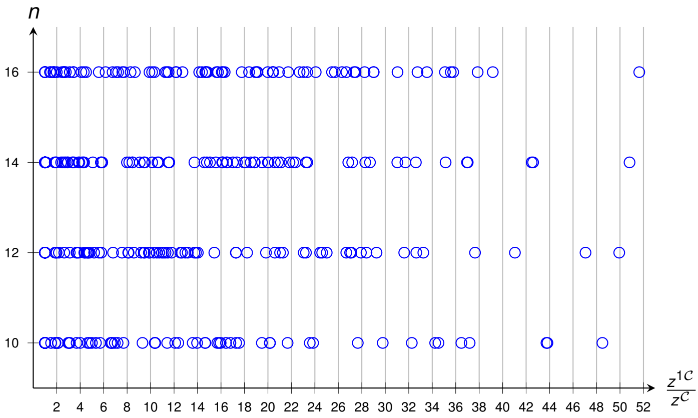


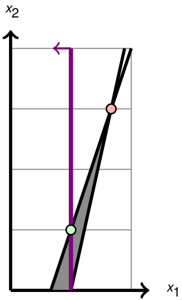
Figure: Multiplicative gap between 1-row CG closure and CG closure of randomly generated instances

Non-packing/covering instances

Proposition

Let $\mathcal{M}$ be any of the closures $\mathcal{A}$ (aggregation) or $\mathcal{C}$ (CG). Then there is a family of (non-packing/non-covering) polyhedra for which $1\mathcal{M}$ is an arbitrarily bad approximation to $\mathcal{M}$, namely for each $\alpha \geq 0$ there is a polyhedron P such that $1\mathcal{M}(P)$ is not an α-approximation of $\mathcal{M}(P)$.

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & k^2 x_1 - (k-1)x_2 \leq k^2 \\ & -kx_1 + x_2 \leq -k + 1, \\ & x \in \mathbb{Z}^2_+ \end{aligned}$$



2.4
Multi-row aggregation cuts for packing/covering problems

Some definitions: Packing

1. A *packing set* is one of the form $\{x \in \mathbb{R}^n_+ \mid A^i x \leq b_i \ \forall i \in I\}$ where each $(A^i, b_i) \in (\mathbb{R}^{1 \times n}_+, \mathbb{R}_+)$ and I is an arbitrary set.

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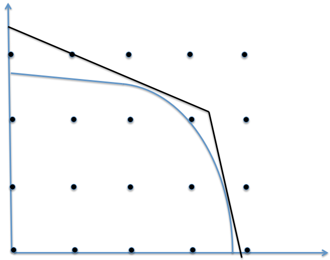

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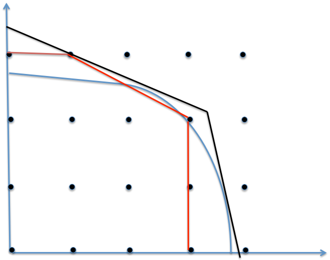

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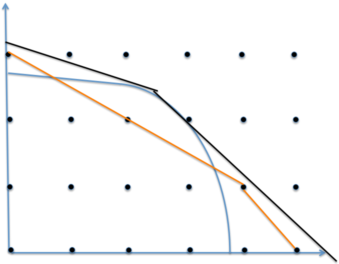

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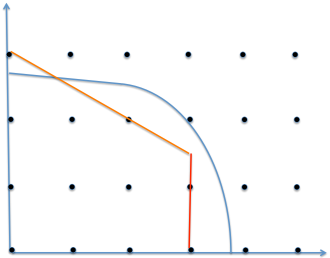

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Theorem (Packing multi-row rank)

Let Q be a well-behaved, packing set. Then,

$$\text{rank}_{\mathcal{A}_k}(Q) \geq \left\lceil \frac{\log_2\left(\frac{z^{LP}}{z^I}\right)}{\log_2(k+1)} \right\rceil$$

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3

Proof sketch

Proof sketch - packing: CG and aggregation

Proposition

If $Q := \{x \in R^n_+ \mid Ax \leq b\}$ *is a well-behaved packing polytope, then* $\mathcal{A}(Q) \subseteq Q \subseteq 2\mathcal{A}(Q)$.

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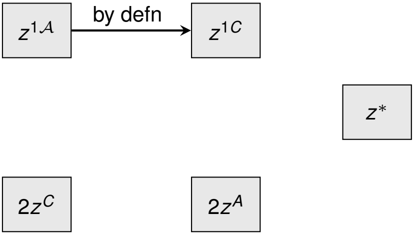
- 

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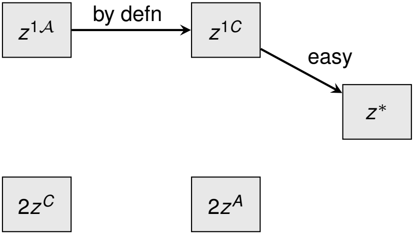
- 

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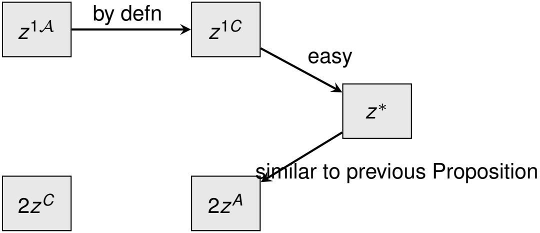
- 

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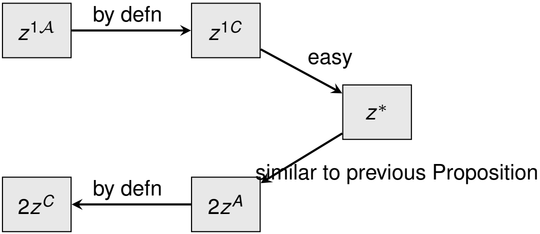
- 

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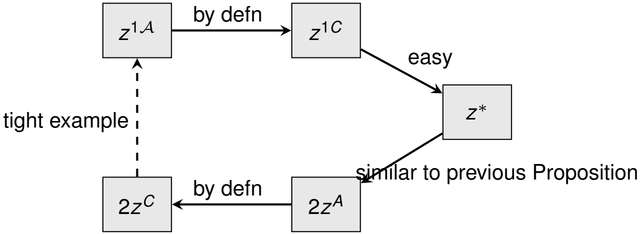
- 

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- $$\begin{array}{llllll} (\lambda^1) & Ax \leq b & (\lambda^2) & Ax \leq b & \\ (\mu_1) & \pi^\top x \leq \pi_0 & (\mu_2) & -\pi^\top x \leq -(\pi_0 + 1) & \\ (\sigma_j^1) & -x_j \leq 0 & (\sigma_j^2) & -x_j \leq 0 & j \in [n] \end{array}$$

  such that:

  $$\alpha_j = \sum_{i=1}^m \lambda_i^1 A_j^i + \mu_1 \pi_j - \sigma_j^1 = \sum_{i=1}^m \lambda_i^2 A_j^i - \mu_2 \pi_j - \sigma_j^2.$$

  $$\beta \geq (\lambda^1)^\top b + \mu_1 \pi_0, \beta \geq (\lambda^2)^\top b - \mu_2(\pi_0 + 1).$$
- $\alpha^\top x \leq \beta$ is a valid inequlity for $\tilde{Q}^I$ where

  $$\tilde{Q} := \{x \in \mathbb{R}^n_+ \,|\, (\lambda^1)^\top Ax \leq (\lambda^1)^\top b, (\lambda^2)^\top Ax \leq (\lambda^2)^\top b\} \supseteq Q$$
- $\tilde{Q} \subseteq 3(\tilde{Q}^I)$
- $\Rightarrow \alpha^\top x \leq 3\beta$ is a v.i. for $\tilde{Q} \Rightarrow \alpha^\top x \leq 3\beta$ is a v.i. for Q.

Proof sketch - packing: Multi-row aggregation cut rank

If P is *well-behaved knapsack polytope*, then $P \subseteq 2P^I$

$$\Downarrow$$

Proposition

If $Q := \{x \in R^n_+ \mid Ax \leq b\}$ is a well-behaved packing set, then $Q \subseteq 2\mathcal{A}(Q)$.

Proof sketch - packing: Multi-row aggregation cut rank

Similarly we obtain:
If P is *well-behaved packing polyhedron with k-constraints*, then $P \subseteq (k+1)P^I$

$$\Downarrow$$

Proposition

If Q is a well-behaved packing set, then $Q \subseteq (k+1)\mathcal{A}_k(Q)$.

Proof sketch - packing: Multi-row aggregation cut rank

Proposition

If Q is a well-behaved packing set, then $Q \subseteq (k+1)\mathcal{A}_k(Q)$.

Proposition

Let Q be a well-behaved, packing set. Then,

$$\text{rank}_{\mathcal{A}_k}(Q) \geq \left\lceil \frac{\log_2\left(\frac{z^{LP}}{z^I}\right)}{\log_2(k+1)} \right\rceil.$$

Proof:

- First argue $A_k(Q)$ is also a well behaved packing set (using Generalized Farkas Lemma).
- $$\underbrace{\left(\frac{z^{LP}}{z^1}\right)}_{\leq k+1} \cdot \underbrace{\left(\frac{z^1}{z^2}\right)}_{\leq k+1} \cdots \underbrace{\left(\frac{z^{\text{rank -1}}}{z^{\text{rank}}}\right)}_{\leq k+1} = \frac{z^{LP}}{z^I}$$
- $\text{rank}_{\mathcal{A}_k}(Q) \geq \left\lceil \frac{\log_2\left(\frac{z^{LP}}{z^I}\right)}{\log_2(k+1)} \right\rceil$

Proof sketch - covering: CG and aggregation

Proposition

Consider a covering polyhedron (with bounds) Q. Let $\mathcal{M}$ be any of the closures $\mathcal{A}$ (aggregation) or $\mathcal{C}$ (CG). Then $1\mathcal{M}(Q)$ is a 2-approximation of $\mathcal{M}(Q)$.

Proof sketch - covering: CG and aggregation

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$\frac{1}{2}z^A$	z^{KC} (Knapsack cover)	z^{1A}

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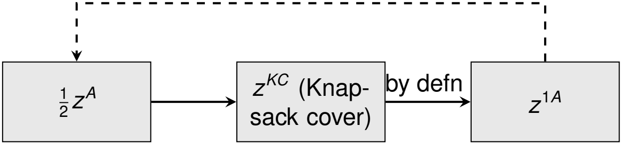

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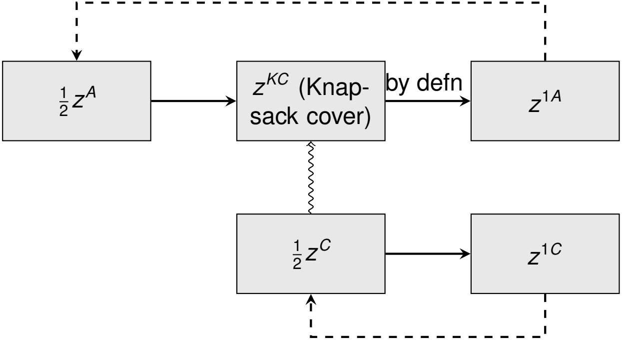

Proof sketch - covering: Multi-row aggregation cut rank

In case of packing:

Proposition

If Q is a well-behaved packing set with k constraints, then $Q \subseteq (k+1)\mathcal{A}_k(Q)$.

$$\Downarrow$$

$$\text{rank}_{\mathcal{A}_k}(Q) \geq \left\lceil \frac{\log_2\left(\frac{z^{LP}}{z^I}\right)}{\log_2(k+1)} \right\rceil$$

Proof sketch - covering: Multi-row aggregation cut rank

Similarly, in case of covering:

Proposition

If Q is a well-behaved covering set with k constraints, then $Q \subseteq \frac{1}{\eta}\mathcal{A}_k(Q)$ where $\eta = O(\log(k))$.

$$\Downarrow$$

$$\text{rank}_{\mathcal{A}_k}(Q) = \Omega\left(\frac{\log_2\left(\frac{z^{LP}}{z^I}\right)}{\log_2 \log_2(k)}\right)$$