New Formulation and Strong MISOCP Relaxations for AC Optimal Transmission Switching Problem

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Abstract

As the modern transmission control and relay technologies evolve, transmission line switching has become an important option in power system operators' toolkits to reduce operational cost and improve system reliability. Most recent research has relied on the DC approximation of the power flow model in the optimal transmission switching problem. However, it is known that DC approximation may lead to inaccurate flow solutions and also overlook stability issues. In this paper, we focus on the optimal transmission switching problem with the full AC power flow model, abbreviated as AC OTS. We propose a new exact formulation for AC OTS and its mixed-integer second-order conic programming (MISOCP) relaxation. We improve this relaxation via several types of strong valid inequalities inspired by the recent development for the closely related AC Optimal Power Flow (AC OPF) problem [21]. We also propose a practical algorithm to obtain high quality feasible solutions for the AC OTS problem. Extensive computational experiments show that the proposed formulation and algorithms efficiently solve IEEE standard and congested instances and lead to significant cost benefits with provably tight bounds.

1 Introduction

Transmission switching, as an emerging operational scheme, has gained considerable attention in both industry and academia in the recent years [28, 11, 13, 19, 14]. Switching on and off transmission lines, therefore, changing the network topology in the real-time operation, may bring several benefits that the traditional economic dispatch cannot offer, such as

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reducing the total operational cost, mitigating transmission congestion, and clearing contingencies.

Previous literature on OTS mainly relies on the DC approximation of the power flow model to avoid the mathematical complexity induced by the non-convexity of AC power flow equations (see e.g. [28, 11, 29, 30]). This DC version of the OTS problem can be modeled as a mixed-integer linear program (MILP), which is a computationally challenging problem and several heuristic method are proposed [4, 12, 33]. In a recent work [22], the authors propose a new formulation and a class of valid inequalities to exactly solve the MILP problem. However, even if this problem can be solved quickly, it has been recognized that the optimal topology obtained by solving DC transmission switching is not guaranteed to be AC feasible, also it may over-estimate cost improvements and overlook stability issues [15].

The AC optimal transmission switching problem (AC OTS) is much less explored. In [15], a convex relaxation of AC OTS is proposed based on trigonometric outer-approximation. The problem is formulated as a mixed integer nonlinear program (MINLP) and solved using the solver BONMIN to obtain upper bounds. In [31], a new ranking heuristic is proposed based on the economic dispatch solutions and the corresponding dual variables. In [5], DC OTS is solved first and then a heuristic correction mechanism is utilized to restore AC feasibility of the solutions. In this paper, we aim to push the control scheme for transmission switching closer to the real-world power system operation by proposing a new exact formulation and an efficient algorithm for the AC OTS problem.

Our study starts from the recent advances in a related fundamental problem in power system analysis, namely the AC Optimal Power Flow (AC OPF) problem, which minimizes the generation cost to satisfy load and various physical contraints represented in the AC power flow constraints, while the power network topology is kept unchanged. It is demonstrated by several authors that convex relaxations, especially semidefinite programming (SDP) relaxations, of the AC OPF problem provide tight lower bounds on standard IEEE test instances [3, 23, 25, 26]. However, the computational burden of solving large-scale SDP relaxations is still unwieldy. To solve for large-scale systems, one may need to turn to computationally less demanding alternatives such as quadratic convex [9, 15, 8] or linear programming relaxations [6].

In a recent work [21], we proposed several strong second-order cone programming (SOCP) relaxations for AC OPF, which produce extremely high quality feasible AC solutions (not dominated by the SDP relaxations) in a time that is an order of magnitude faster than solving the SDP relaxations. In this paper, we extend these new techniques to the more challenging AC OTS problem. In particular, we first formulate the AC OTS problem as an MINLP problem. Then, we propose an MISOCP relaxation, which relaxes the nonconvex AC power flow constraints to a set of convex quadratic constraints, represented in the form of SOCP constraints. The paper then provides several techniques to strengthen this MISOCP relaxation by adding several types of valid inequalities. Some of these valid inequalities have demonstrated to have excellent performance for the AC OPF in [21], and some others are specifically developed for the AC OTS problem. Finally, we also propose practical algorithms that utilize the solutions from the MISOCP relaxation to obtain high quality feasible solutions for the AC OTS problem.

The rest of the paper is organized as follows: In Section 2 we formally define AC OPF and present two exact formulations. In Section 3, we present AC OTS as an MINLP problem and discuss its MISOCP relaxation. Then, we propose several valid inequalities in Section 4 and develop a practical algorithm to solve AC OTS in Section 5. We present the results of our extensive computational experiments in Section 6. Finally, some concluding remarks are given in Section 7.

2 AC Optimal Power Flow

Consider a power network $\mathcal{N} = (\mathcal{B}, \mathcal{L})$, where $\mathcal{B}$ and $\mathcal{L}$ respectively denote the set of buses and transmission lines. Generation units are connected to a subset of buses, denoted as $\mathcal{G} \subseteq \mathcal{B}$. The aim of the AC optimal power flow (OPF) problem is to satisfy demand at all buses with the minimum total production costs of generators such that the solution obeys the physical laws (e.g., Ohm's and Kirchoff's Law) and other operational restrictions (e.g., transmission line flow limit constraints).

Let $Y \in \mathbb{C}^{|\mathcal{B}|\times|\mathcal{B}|}$ denote the nodal admittance matrix, which has components $Y_{ij} = G_{ij} + \mathrm{i}B_{ij}$ for each line $(i,j) \in \mathcal{L}$, and $G_{ii} = g_{ii} - \sum_{j\neq i} G_{ij}$, $B_{ii} = b_{ii} - \sum_{j\neq i} B_{ij}$, where g_{ii} (resp. b_{ii}) is the shunt conductance (resp. susceptance) at bus $i \in \mathcal{B}$ and $\mathrm{i} = \sqrt{-1}$. Let p_i^g, q_i^g (resp. p_i^d, q_i^d) be the real and reactive power output of the generator (resp. load) at bus i. The complex voltage V_i at bus i can be expressed either in the rectangular form as $V_i = e_i + \mathrm{i}f_i$ or in the polar form as $V_i = |V_i|(\cos\theta_i + \mathrm{i}\sin\theta_i)$, where $|V_i| = \sqrt{e_i^2 + f_i^2}$ is the voltage magnitude and θ_i is the phase angle. Real and reactive power on line (i,j) are denoted by p_{ij} and q_{ij}, respectively and computed as follows:

$$
\begin{aligned}
p_{ij} &= -G_{ij}(e_i^2 + f_i^2) + G_{ij}(e_ie_j + f_if_j) - B_{ij}(e_if_j - e_jf_i) \\
q_{ij} &= \quad B_{ij}(e_i^2 + f_i^2) - B_{ij}(e_ie_j + f_if_j) - G_{ij}(e_if_j - e_jf_i).
\end{aligned} \tag{1}
$$

With the above notation, the AC OPF problem is given in the so-called rectangular formulation as follows:

$$\min \sum_{i\in\mathcal{G}} C_i(p_i^g) \tag{2a}$$

$$\text{s.t. } p_i^g - p_i^d = g_{ii}(e_i^2 + f_i^2) + \sum_{j\in\delta(i)} p_{ij} \qquad i \in \mathcal{B} \tag{2b}$$

$$q_i^g - q_i^d = -b_{ii}(e_i^2 + f_i^2) + \sum_{j\in\delta(i)} q_{ij} \qquad i \in \mathcal{B} \tag{2c}$$

$$\underline{V}_i^2 \le e_i^2 + f_i^2 \le \overline{V}_i^2 \qquad i \in \mathcal{B} \tag{2d}$$

$$p_{ij}^2 + q_{ij}^2 \le (S_{ij}^{\max})^2 \qquad (i,j) \in \mathcal{L} \tag{2e}$$

$$p_i^{\min} \le p_i^g \le p_i^{\max} \qquad i \in \mathcal{G} \tag{2f}$$

$$q_i^{\min} \le q_i^g \le q_i^{\max} \qquad i \in \mathcal{G}, \tag{2g}$$

(1).

The objective function $C_i(p_i^g)$ is typically linear or convex quadratic in the real power output p_i^g of generator i. Constraints (2b) and (2c) correspond to the conservation of active and reactive power flows at each bus, respectively. Here, $\delta(i)$ denotes the set of neighbor buses of bus i. Constraint (2d) restricts voltage magnitude at each bus. Constraint (2e) puts an upper bound on the apparent power on each line. Finally, constraints (2f) and (2g), respectively, limit the active and reactive power output of each generator to respect its physical capability.

Note that the rectangular formulation (2) is a non-convex quadratic optimization problem. However, we note that all the nonlinearity and non-convexity comes from one of the following three forms: (1) $e_i^2 + f_i^2 = |V_i|^2$, (2) $e_ie_j + f_if_j = |V_i||V_j|\cos(\theta_i - \theta_j)$, (3) $e_if_j - f_ie_j = -|V_i||V_j|\sin(\theta_i - \theta_j)$. We define new variables c_{ii}, c_{ij} and s_{ij} for each bus i and each transmission line (i,j) to capture the non-convexity. In particular, we define for each $i \in \mathcal{B}$ and $(i,j) \in \mathcal{L}$,

$$c_{ii} := e_i^2 + f_i^2, \quad c_{ij} := e_ie_j + f_if_j, \quad s_{ij} := e_if_j - e_jf_i. \tag{3}$$

Now, we introduce an alternative formulation of the OPF problem as follows:

$$\min \sum_{i\in\mathcal{G}} C_i(p_i^g) \tag{4a}$$

$$\text{s.t. } p_i^g - p_i^d = g_{ii}c_{ii} + \sum_{j\in\delta(i)} p_{ij} \qquad i \in \mathcal{B} \tag{4b}$$

$$q_i^g - q_i^d = -b_{ii}c_{ii} + \sum_{j\in\delta(i)} q_{ij} \qquad i \in \mathcal{B} \tag{4c}$$

$$p_{ij} = -G_{ij}c_{ii} + G_{ij}c_{ij} - B_{ij}s_{ij} \qquad (i,j) \in \mathcal{L} \tag{4d}$$
$$q_{ij} = \quad B_{ij}c_{ii} - B_{ij}c_{ij} - G_{ij}s_{ij} \qquad (i,j) \in \mathcal{L} \tag{4e}$$
$$\underline{V}_i^2 \leq c_{ii} \leq \overline{V}_i^2 \qquad i \in \mathcal{B} \tag{4f}$$
$$c_{ij} = c_{ji}, \quad s_{ij} = -s_{ji} \qquad (i,j) \in \mathcal{L} \tag{4g}$$
$$c_{ij}^2 + s_{ij}^2 = c_{ii}c_{jj} \qquad (i,j) \in \mathcal{L} \tag{4h}$$
$$\theta_j - \theta_i = \text{atan2}(s_{ij}, c_{ij}) \qquad (i,j) \in \mathcal{L}, \tag{4i}$$
$$\text{(2e)-(2g)}.$$

A variant of this formulation without (4i) was previously proposed in [10] and [16] for radial networks (also see [20]) while it was later adapted to general networks in [17, 18].

3 AC Optimal Transmission Switching

AC Optimal Transmission Switching (AC OTS) is a variant of the AC OPF problem in which transmission lines are allowed to be switched on and off to reduce the total cost of dispatch. AC OTS can be formulated as an optimization problem, which aims to find a topology with the least cost while achieving feasible AC power flow solutions. In this section, we first formulate AC OTS as an MINLP and then, propose an MISOCP relaxation to obtain lower bounds. We will use OTS (resp. OPF) to denote AC OTS (resp. AC OPF) for brevity, unless stated otherwise.

3.1 MINLP Formulation

Mathematical programming formulation of OTS can be posed with the same variables as used in OPF with the addition of a set of binary variables, denoted by x_{ij}, for each line. The variable x_{ij} takes the value one if the corresponding line (i,j) is switched on, and zero otherwise. Then, OTS is formulated as the following MINLP problem:

$$\min \sum_{i \in \mathcal{G}} C_i(p_i^g) \tag{5a}$$
$$\text{s.t. } p_{ij} = (-G_{ij}c_{ii} + G_{ij}c_{ij} - B_{ij}s_{ij})x_{ij} \qquad (i,j) \in \mathcal{L} \tag{5b}$$
$$q_{ij} = \quad (B_{ij}c_{ii} - B_{ij}c_{ij} - G_{ij}s_{ij})x_{ij} \qquad (i,j) \in \mathcal{L} \tag{5c}$$
$$(c_{ij}^2 + s_{ij}^2 - c_{ii}c_{jj})x_{ij} = 0 \qquad (i,j) \in \mathcal{L} \tag{5d}$$
$$(\theta_j - \theta_i - \text{atan2}(s_{ij}, c_{ij}))x_{ij} = 0 \qquad (i,j) \in \mathcal{L} \tag{5e}$$
$$x_{ij} \in \{0, 1\} \qquad (i,j) \in \mathcal{L}, \tag{5f}$$

$$(2e)\text{-}(2g), (4b)\text{-}(4c), (4f)\text{-}(4g).$$

Here, constraints (5b) and (5c) guarantee that real and reactive flow on every line takes the associated values if the line is switched on and zero otherwise. Similarly, cone constraint (5d) and angle constraint (5e) are active only when the corresponding binary variable takes the value one.

3.2 MISOCP Relaxation

Now, we propose an MISOCP relaxation of OTS (5). For notational convenience, let $\underline{c}_{ii} = \underline{V}_i^2$ and $\overline{c}_{ii} = \overline{V}_i^2$. Here, we extend the definition of variables c_{ij} and s_{ij}, which now take the values as before when the corresponding line is switched on and zero otherwise. We also denote lower and upper bounds of c_{ij} (resp. s_{ij}) as $\underline{c}_{ij}$ (resp. $\underline{s}_{ij}$) and $\overline{c}_{ij}$ (resp. $\overline{s}_{ij}$), respectively, when the line is switched on. Next, we define new variables $c_{ii}^j := c_{ii}x_{ij}$. Using this notation, we present an MISOCP relaxation as follows:

$$
\begin{aligned}
\min \ & \sum_{i \in \mathcal{G}} C_i(p_i^g) && && \text{(6a)}\\
\text{s.t.} \ & p_{ij} = -G_{ij}c_{ii}^j + G_{ij}c_{ij} - B_{ij}s_{ij} && (i,j) \in \mathcal{L} && \text{(6b)}\\
& q_{ij} = \quad B_{ij}c_{ii}^j - B_{ij}c_{ij} - G_{ij}s_{ij} && (i,j) \in \mathcal{L} && \text{(6c)}\\
& \underline{c}_{ij}x_{ij} \le c_{ij} \le \overline{c}_{ij}x_{ij} && (i,j) \in \mathcal{L} && \text{(6d)}\\
& \underline{s}_{ij}x_{ij} \le s_{ij} \le \overline{s}_{ij}x_{ij} && (i,j) \in \mathcal{L} && \text{(6e)}\\
& \underline{c}_{ii}x_{ij} \le c_{ii}^j \le \overline{c}_{ii}x_{ij} && (i,j) \in \mathcal{L} && \text{(6f)}\\
& c_{ii} - \overline{c}_{ii}(1 - x_{ij}) \le c_{ii}^j && (i,j) \in \mathcal{L} && \text{(6g)}\\
& c_{ii}^j \le c_{ii} - \underline{c}_{ii}(1 - x_{ij}) && (i,j) \in \mathcal{L} && \text{(6h)}\\
& c_{ij}^2 + s_{ij}^2 \le c_{ii}^j c_{jj}^i && (i,j) \in \mathcal{L}, && \text{(6i)}\\
& (2e)\text{-}(2g), (4b)\text{-}(4c), (4f)\text{-}(4g), (5f).
\end{aligned}
$$

Here, constraints (6b) and (6c) again guarantee that flow variables takes the correct value when the line is switched on and zero otherwise, due to constraints (6d)-(6f). On the other hand, (6g)-(6h) restrict that c_{ii}^j takes value c_{ii} when line in switched on. We note that constraints (6f)-(6h) are precisely the McCormicks envelopes [24] applied to $c_{ii}^j = c_{ii}x_{ij}$. Finally, (6i) is the SOCP relaxation of (5d).

We note that the non-convex constraint (5e) is dropped altogether to obtain the MISOCP relaxation (6). In the next section, we propose three ways to incorporate the constraint (5e) back into the MISOCP relaxation.

4 Valid Inequalities

In this section, we propose three methods to strengthen the MISOCP relaxation (6). They are based on the strengthening methods we recently proposed for the SOCP relaxation of the AC OPF problem in [21], which are combined with integer programming techniques. In Section 4.1, we construct a polyhedral envelope for the arctangent constraint (5e) in 3-dimension. In Section 4.2, we propose a disjunctive cut generation scheme that separates a given SOCP solution from the SDP cones. In Section 4.3, we propose another disjunctive cut generation scheme that separates a given SOCP solution from the McCormick relaxation of a newly-proposed cycle based formulation of the OPF problem. Finally, in Section 4.4, we propose variable bounding techniques that provide tight variable bounds, which is essential for the success of the proposed approach.

4.1 Arctangent Envelopes

First, we propose a convex outer-approximation of the angle condition (5e) to the MISOCP relaxation. Our construction uses four linear inequalities to approximate the convex envelope for the following set defined by the arctangent constraint (5e) for each line $(i, j) \in \mathcal{L}$,

$$\mathcal{AT} := \left\{(c, s, \theta) \in \mathbb{R}^3 : \theta = \arctan\left(\frac{s}{c}\right), (c, s) \in B\right\}, \tag{7}$$

where we denote $\theta = \theta_j - \theta_i$ and drop (i, j) indices for brevity and define the box $B := [\underline{c}, \overline{c}] \times [\underline{s}, \overline{s}]$. We also assume $\underline{c} > 0$. The four corners of the box correspond to four points in the (c, s, θ) space:

$$\begin{aligned} z^1 &= (\underline{c}, \overline{s}, \arctan(\overline{s}/\underline{c})), \quad & z^2 &= (\overline{c}, \overline{s}, \arctan(\overline{s}/\overline{c})), \\ z^3 &= (\overline{c}, \underline{s}, \arctan(\underline{s}/\overline{c})), \quad & z^4 &= (\underline{c}, \underline{s}, \arctan(\underline{s}/\underline{c})). \end{aligned} \tag{8}$$

Let us first focus on the upper envelopes. Proposition 4.1 is adapted from [21] to the case of OTS:

Proposition 4.1. *Let $\theta = \gamma_1 + \alpha_1 c + \beta_1 s$ and $\theta = \gamma_2 + \alpha_2 c + \beta_2 s$ be the planes passing through points $\{z^1, z^2, z^3\}$, and $\{z^1, z^3, z^4\}$, respectively. Then, for $k = 1, 2$, we have*

$$\gamma_k' + \alpha_k c + \beta_k s + (2\pi - \gamma_k')(1 - x) \geq \arctan\left(\frac{s}{c}\right) \tag{9}$$

for all $(c, s) \in B$ with $\gamma_k' = \gamma_k + \Delta\gamma_k$ where

$$\Delta\gamma_k = \max_{(c,s)\in B} \left\{\arctan\left(\frac{s}{c}\right) - (\gamma_k + \alpha_k c + \beta_k s)\right\}. \tag{10}$$

The nonconvex optimization problem (10) can be solved by enumerating all possible Karush-Kuhn-Tucker (KKT) points. A similar argument can be used to construct lower envelopes as well. See [21] for details.

4.2 SDP Disjunction

In the second method to strengthen the MISOCP relaxation (6), we propose a cutting plane approach to separate a given SOCP relaxation solution from the feasible region of the SDP relaxation of cycles. To start with, let us consider a cycle with the set of lines C and the set of buses $\mathcal{B}_C$. Let $v \in \mathbb{R}^{2|C|}$ be a vector of bus voltages defined as $v = [e; f]$ such that $v_i = e_i$ for $i \in \mathcal{B}$ and $v_{i'} = f_i$ for $i' = i + |C|$. Observe that if we have a set of c, s variables satisfying the definitions in (3) and a matrix variable $W = vv^T$, then the following linear relationship holds between c, s, x and W,

$$
\begin{aligned}
& c_{ij} = (W_{ij} + W_{i'j'})x_{ij} && (i,j) \in C && \text{(11a)}\\
& s_{ij} = (W_{ij'} - W_{ji'})x_{ij} && (i,j) \in C && \text{(11b)}\\
& c_{ii} = W_{ii} + W_{i'i'} && i \in \mathcal{B}_C && \text{(11c)}\\
& \underline{c}_{ij}x_{ij} \le c_{ij} \le \overline{c}_{ij}x_{ij} && (i,j) \in C && \text{(11d)}\\
& \underline{s}_{ij}x_{ij} \le s_{ij} \le \overline{s}_{ij}x_{ij} && (i,j) \in C && \text{(11e)}\\
& \underline{c}_{ii} \le c_{ii} \le \overline{c}_{ii} && i \in \mathcal{B}_C && \text{(11f)}\\
& c_{ii}^{j} = c_{ii}x_{ij} && (i,j) \in C && \text{(11g)}\\
& x_{ij} \in \{0,1\} && (i,j) \in C && \text{(11h)}\\
& W \succeq 0. && && \text{(11i)}
\end{aligned}
$$

Let us define $\mathcal{S} := \{(c, s, x) : \exists W : (11)\}$. Clearly, any feasible solution to the OTS formulation (5) must also satisfy (11). Therefore, any valid inequality for $\mathcal{S}$ is also valid for the formulation (5).

Note that $\mathcal{S}$ is a mixed-integer set. Ideally, one would be interested in finding conv($\mathcal{S}$) to generate strong valid inequalities. However, this is a quite computationally challenging task, no easier than solving the original MINLP. Instead, we outer-approximate conv($\mathcal{S}$) and obtain cutting planes by utilizing a simple disjunction for a cycle C: Either every line is active, that is $\sum_{(i,j)\in C} x_{ij} = |C|$, or at least one line is disconnected, that is $\sum_{(i,j)\in C} x_{ij} \le |C| - 1$. Below, we approximate these two disjunctions.

Disjunction 1: In the first disjunction, we have $x_{ij} = 1$ for all $(i,j) \in C$. Let us consider the following constraints

$$
\begin{aligned}
& c_{ij} = W_{ij} + W_{i'j'} && (i,j) \in C && \text{(12a)}\\
& s_{ij} = W_{ij'} - W_{ji'} && (i,j) \in C && \text{(12b)}\\
& c_{ii} = c_{ii}^{j} && (i,j) \in C && \text{(12c)}\\
& x_{ij} = 1 && (i,j) \in C, && \text{(12d)}
\end{aligned}
$$

and define $\mathcal{S}_1 := \{(c,s,x) : \exists W : (12), (11c) - (11f), (11i)\}$.

Disjunction 0: In the second disjunction, $x_{ij} = 0$ for some $(i,j) \in C$. Let us consider the following constraints

$$
\begin{aligned}
& c_{ij}^2 + s_{ij}^2 \le c_{ii}^{j} c_{jj}^{i} && (i,j) \in C && \text{(13a)}\\
& \underline{c}_{ii} x_{ij} \le c_{ii}^{j} \le \overline{c}_{ii} x_{ij} && (i,j) \in C && \text{(13b)}\\
& c_{ii} - \overline{c}_{ii}(1 - x_{ij}) \le c_{ii}^{j} && (i,j) \in C && \text{(13c)}\\
& c_{ii}^{j} \le c_{ii} - \underline{c}_{ii}(1 - x_{ij}) && (i,j) \in C && \text{(13d)}\\
& 0 \le x_{ij} \le 1 && (i,j) \in C && \text{(13e)}\\
& \sum_{(i,j) \in C} x_{ij} \le |C| - 1, && && \text{(13f)}
\end{aligned}
$$

and define $\mathcal{S}_0 := \{(c,s,x) : (13), (11d)\text{-}(11f)\}$.

We note that both $\mathcal{S}_1$ and $\mathcal{S}_0$ are conic representable. In particular, these bounded sets are respectively semidefinite and second-order cone representable. Therefore, $\text{conv}(\mathcal{S}_1 \cup \mathcal{S}_0)$ is also conic representable (see Appendix A on how to obtain a representation as an extended formulation), and by construction, contains $\mathcal{S}$.

Now, suppose a point (c^*, s^*, x^*) is given. We want to decide whether this point belongs to $\text{conv}(\mathcal{S}_1 \cup \mathcal{S}_0)$ or otherwise, find a separating hyperplane. Given that we have an extended semidefinite representation for $\text{conv}(\mathcal{S}_1 \cup \mathcal{S}_0)$, we can solve an SDP separation problem to achieve this. See Appendix B.

4.3 McCormick Disjunction

The last method to strengthen the MISOCP relaxation (6) is based on a new cycle-based OPF formulation we propose in [21]. The key observation is as follows: instead of satisfying the angle condition (5e) for each $(i,j) \in \mathcal{L}$, it is equivalent to guarantee that angle differences sum up to 0 modulo 2π over every cycle C in the power network if all the lines of the cycle

C are switched on, i.e.

$$\Big(\sum_{(i,j)\in C} \theta_{ij} - 2\pi k\Big) \prod_{(i,j)\in C} x_{ij} = 0, \qquad \text{for some } k \in \mathbb{Z}, \tag{14}$$

where $\theta_{ij} := \theta_j - \theta_i$.

Next, we consider

$$\begin{aligned}
&\Big[\cos\Big(\sum_{(i,j)\in C} \theta_{ij}\Big) - 1\Big] \prod_{(i,j)\in C} x_{ij} = 0 && && \text{(15a)}\\
&c_{ij} = \sqrt{c_{ii}c_{jj}} \cos\theta_{ij} x_{ij} && (i,j) \in C && \text{(15b)}\\
&s_{ij} = \sqrt{c_{ii}c_{jj}} \sin\theta_{ij} x_{ij} && (i,j) \in C, && \text{(15c)}\\
&\text{(11d)} - \text{(11h)}.
\end{aligned}$$

Here, (15a) is equivalent to (14) and (15b)-(15c) follow from the definition of c, s variables. Let us define $\mathcal{M} := \{(c, s, x) : \exists\theta : (15), (11d) - (11h)\}$. Again, observe that any feasible solution to the OTS formulation (5) must also satisfy (15). Therefore, any valid inequality for $\mathcal{M}$ is also valid for the formulation (5).

We again follow a similar procedure to the previous section and consider two disjunctions for a cycle C.

Disjunction 1: In the first disjunction, we have $x_{ij} = 1$ for all $(i, j) \in C$. Note that (15a) reduces to

$$\cos\Big(\sum_{(i,j)\in C} \theta_{ij}\Big) = 1.$$

Now, we can expand the cosine appropriately and replace $\cos(\theta_{ij})'s$ and $\sin(\theta_{ij})$'s in terms of c, s variables following (15b)-(15c). This transformation yields a homogeneous polynomial, denoted by p_C, in terms of only c, s variables, and an equivalent constraint $p_C = 0$. However, p_C can have up to $2^{|C|-1} + 1$ monomials and each monomial of degree $|C|$. In [21], we propose a method, which is used to "bilinearize" this high degree polynomial by decomposing larger cycles into smaller ones by the addition of artificial lines and corresponding variables. We refer the reader to [21] for details.

Using the proposed decomposition scheme, we obtain a set of bilinear polynomials $q_k(c, s, \tilde{c}, \tilde{s}) = 0,\ k \in \mathcal{K}_C$, for a given cycle C. Here, $\tilde{c}, \tilde{s}$ denote the extra variables used in the construction.

Finally, we use McCormick envelopes for each bilinear constraint to linearize the system of polynomials. For a given cycle C, consider the McCormick relaxation of the bilinear cycle constraints, which can be written compactly as follows:

$$Az + \tilde{A}\tilde{z} + By \le c \tag{16a}$$

$$Ey = 0. \tag{16b}$$

Here, z is a vector composed of the c, s variables, $\tilde{z}$ is a vector composed of the additional $\tilde{c}, \tilde{s}$ variables introduced in the cycle decomposition, and y is a vector of new variables defined to linearize the bilinear terms in the cycle constraints. Constraint (16a) contains the McCormick envelopes of the bilinear terms and bounds on the c, s variables, while (16b) includes the linearized cycle equality constraints. Finally, we define the set $\mathcal{M}_1 := \{(c, s, x) : \exists(\tilde{c}, \tilde{s}) : (16), (11d)\text{-}(11f), (12c)\text{-}(12d)\}$.

Disjunction 0: In the second disjunction, $x_{ij} = 0$ for some $(i, j) \in C$. We take $\mathcal{M}_0 := \mathcal{S}_0$.

We note that both $\mathcal{M}_1$ and $\mathcal{M}_0$ are conic representable. In particular, these bounded sets are respectively polyhedral and second-order cone representable. Therefore, $\text{conv}(\mathcal{M}_1 \cup \mathcal{M}_0)$ is also conic representable, and by construction, contains $\mathcal{M}$.

Now, suppose a point (c^*, s^*, x^*) is given. We want to decide whether this point belongs to $\text{conv}(\mathcal{M}_1 \cup \mathcal{M}_0)$ or otherwise, find a separating hyperplane. Given that we have an extended second-order cone representation for $\text{conv}(\mathcal{M}_1 \cup \mathcal{M}_0)$, we can solve an SOCP separation problem.

In our computations, we observed that stronger cuts are obtained by combining SDP and McCormick Disjunction. In particular, we separate cutting planes from $\text{conv}((\mathcal{S}_1 \cap \mathcal{M}_1) \cup \mathcal{S}_0)$ by solving SDP separation problems.

4.4 Obtaining Variable Bounds

Note that the arctangent envelopes and the McCormick relaxations are more effective when tight variable upper/lower bounds are available for the c and s variables. Now, we explain how we obtain good bounds for these variables, which is the key ingredient in the success of our proposed methods.

Observe that c_{ij} and s_{ij} do not have explicit variable bounds except the implied bounds due to (4f) and (4h) as

$$-\overline{V}_i\overline{V}_j \leq c_{ij}, s_{ij} \leq \overline{V}_i\overline{V}_j \quad (i, j) \in \mathcal{L}.$$

However, these bounds may be quite loose, especially when the phase angle differences are small, implying $c_{ij} \approx 1$ and $s_{ij} \approx 0$ when the corresponding line is switched on. Therefore, one should try to improve these bounds.

We adapt the procedure proposed in [21] (which dealt only with OPF) to the case of OTS in order to obtain variable bounds, that is, we solve a reduced version of the full MISOCP relaxation to efficiently compute bounds. In particular, to find variable bounds for c_{kl} and

s_{kl} for some $(k,l) \in \mathcal{L}$, consider the buses which can be reached from either k or l in at most r steps. Denote this set of buses as $\mathcal{B}_{kl}(r)$. For instance, $\mathcal{B}_{kl}(0) = \{k,l\}$, $\mathcal{B}_{kl}(1) = \delta(k) \cup \delta(l)$, etc. We also define $\mathcal{G}_{kl}(r) = \mathcal{B}_{kl}(r) \cap \mathcal{G}$ and $\mathcal{L}_{kl}(r) = \{(i,j) \in \mathcal{L} : i \in \mathcal{B}_{kl}(r) \text{ or } j \in \mathcal{B}_{kl}(r)\}$. Then, we consider the following SOCP relaxation:

$$p_i^g - p_i^d = g_{ii}c_{ii} + \sum_{j \in \delta(i)} p_{ij} \qquad i \in \mathcal{B}_{kl}(r) \tag{17a}$$

$$q_i^g - q_i^d = -b_{ii}c_{ii} + \sum_{j \in \delta(i)} q_{ij} \qquad i \in \mathcal{B}_{kl}(r) \tag{17b}$$

$$p_{ij} = -G_{ij}c_{ii}^j + G_{ij}c_{ij} - B_{ij}s_{ij} \qquad (i,j) \in \mathcal{L}_{kl}(r) \tag{17c}$$

$$q_{ij} = \quad B_{ij}c_{ii}^j - B_{ij}c_{ij} - G_{ij}s_{ij} \qquad (i,j) \in \mathcal{L}_{kl}(r) \tag{17d}$$

$$p_{ij}^2 + q_{ij}^2 \leq (S_{ij}^{\max})^2 \qquad (i,j) \in \mathcal{L}_{kl}(r) \tag{17e}$$

$$\underline{V}_i^2 \leq c_{ii} \leq \overline{V}_i^2 \qquad i \in \mathcal{B}_{kl}(r+1) \tag{17f}$$

$$p_i^{\min} \leq p_i^g \leq p_i^{\max} \qquad i \in \mathcal{G}_{kl}(r) \tag{17g}$$

$$q_i^{\min} \leq q_i^g \leq q_i^{\max} \qquad i \in \mathcal{G}_{kl}(r) \tag{17h}$$

$$\underline{c}_{ij}x_{ij} \leq c_{ij} \leq \overline{c}_{ij}x_{ij} \qquad (i,j) \in \mathcal{L}_{kl}(r) \tag{17i}$$

$$\underline{s}_{ij}x_{ij} \leq s_{ij} \leq \overline{s}_{ij}x_{ij} \qquad (i,j) \in \mathcal{L}_{kl}(r) \tag{17j}$$

$$\underline{c}_{ii}x_{ij} \leq c_{ii}^j \leq \overline{c}_{ii}x_{ij} \qquad (i,j) \in \mathcal{L}_{kl}(r) \tag{17k}$$

$$c_{ii} - \overline{c}_{ii}(1 - x_{ij}) \leq c_{ii}^j \qquad (i,j) \in \mathcal{L}_{kl}(r) \tag{17l}$$

$$c_{ii}^j \leq c_{ii} - \underline{c}_{ii}(1 - x_{ij}) \qquad (i,j) \in \mathcal{L}_{kl}(r) \tag{17m}$$

$$c_{ij} = c_{ji}, \quad s_{ij} = -s_{ji} \qquad (i,j) \in \mathcal{L}_{kl}(r) \tag{17n}$$

$$c_{ij}^2 + s_{ij}^2 \leq c_{ii}^j c_{jj}^i \qquad (i,j) \in \mathcal{L}_{kl}(r) \tag{17o}$$

$$0 \leq x_{ij} \leq 1 \qquad (i,j) \in \mathcal{L}_{kl}(r) \tag{17p}$$

$$x_{kl} = 1. \tag{17q}$$

Essentially, (17) is the continuous relaxation of MISOCP relaxation applied to the part of the power network within r steps of the buses k and l. c_{kl} and s_{kl} can be minimized and maximized subject to (17) for each edge (k,l) to obtain lower and upper bounds, respectively. These SOCPs can be solved in parallel, since they are independent of each other. It is observed that a good tradeoff between accuracy and speed is to select $r = 2$[21]. Constraint (17q) may seem to restrict the feasible region, however, the way we defined c_{kl} and s_{kl} variables, they are the values for cosine and sine components when $x_{kl} = 1$ (otherwise, they are 0). Therefore, it is enough for the bounds to be valid for $x_{kl} = 1$ only.

Bounds on an artificial edge (i,j) used in the construction of McCormick envelopes are chosen as follows:

$$\overline{c}_{ij} = -\underline{c}_{ij} = \overline{s}_{ij} = -\underline{s}_{ij} = \overline{V}_i \overline{V}_j. \tag{18}$$

A similar idea can be used to fix some of the binary variables as well. In particular, we can minimize x_{kl} over (17) after omitting (17q). If the optimal value turns out to be strictly positive, then x_{kl} can be fixed to one.

5 Algorithm

In this section, we propose an algorithm to solve OTS. The algorithm has two phases. The first phase involves solving a sequence of SOCPs obtained by relaxing integrality restriction of the binary variables in MISOCP (6), and incorporates cycle inequalities generated from the extended SDP and McCormick relaxations in Section 4.2 and 4.3. In this phase, the aim is to strengthen the lower bound on the MISOCP relaxation. The second phase involves solving a sequence of MISOCP relaxations strengthened by cycle inequalities. The aim in this phase is to obtain high quality feasible solutions for OTS. In particular, this is achieved by solving OPF subproblems with fixed topologies obtained from the integral solutions found during the branch-and-cut process of solving the MISOCP (6). This procedure is repeated by "forbidding" the topologies already considered in order to obtain different network configurations in the subsequent iterations.

Now we formally define the ingredients of the algorithm. First, let $SOCP(\mathcal{V})$ be the continuous relaxation of MISOCP (6) with a set of valid inequalities $\mathcal{V}$ obtained from cycle inequalities using extended SDP and McCormick relaxations. The set $\mathcal{V}$ is dynamically updated T_1 times. Similarly, we define $MISOCP(\mathcal{V}, \mathcal{F})$ as the MISOCP relaxation of OTS with a set of valid cycle inequalities $\mathcal{V}$ and forbidden topologies $\mathcal{F}$. Here, we forbid a topology $x^* \in \mathcal{F}$ by adding the following "no-good" cut (see [2] for generalizations) to the formulation:

$$\sum_{(i,j):x^*_{ij}=1} (1 - x_{ij}) + \sum_{(i,j):x^*_{ij}=0} x_{ij} \geq 1. \tag{19}$$

We denote by LB_t as the optimal value of $MISOCP(\mathcal{V}, \mathcal{F})$ and $\mathcal{P}_t$ as the set of all integral solutions found by the MIP solver at the t-th iteration. For instance, CPLEX offers this option called *solution pool*. In a practical implementation, this part is repeated T_2 times.

Let $OPF(x)$ denote the value of a feasible solution to OPF problem (4) for the fixed topology induced by the integral vector x. Finally, UB is the best upper bound on OTS. Now, we present Algorithm 1.

Algorithm 1 OTS algorithm.

Input: T_1, T_2, ϵ.
Phase I: Set $\mathcal{V} \leftarrow \emptyset$, $\mathcal{F} \leftarrow \emptyset$, $UB \leftarrow \infty$.
for $\tau = 1, \ldots, T_1$ **do**
  Solve $SOCP(\mathcal{V})$.
  Separate cycle inequalities for each cycle in a cycle basis to obtain a set of valid inequalities $\mathcal{V}_\tau$.
  Update $\mathcal{V} \leftarrow \mathcal{V} \cup \mathcal{V}_\tau$.
end for
Phase II: Set $t \leftarrow 0$.
repeat
  $t \leftarrow t + 1$
  Solve $MISOCP(\mathcal{V}, \mathcal{F})$ to obtain a pool of integral solutions $\mathcal{P}_t$ and record the optimal cost as LB_t.
  for all $x \in \mathcal{P}_t$ **do**
    if $OPF(x) < UB$ **then**
      $UB \leftarrow OPF(x)$
    end if
  end for
  Update $\mathcal{F} \leftarrow \mathcal{F} \cup \mathcal{P}_t$.
until $LB_t \geq (1 - \epsilon)UB$ or $t \geq T_2$

Observation 5.1. *If $OPF(x)$ returns globally optimal solution for every topology x, $\epsilon = 0$ and $T_2 = \infty$, then Algorithm 1 converges to the optimal solution of OTS in finitely many iterations.*

Observation 5.1 follows from the fact that there are finitely many topologies and by the hypothesis that $OPF(x)$ can be solved globally. Although Observation 5.1 states that Algorithm 1 can be used to solve OTS to global optimality in finitely many iterations, the requirement of solving $OPF(x)$ to global optimality may not be satisfied. In practice, we can only solve OPF subproblems using local solver, in which case we have Observation 5.2.

Observation 5.2. *If $OPF(x)$ is solved by a local solution method, then we have $LB_1 \leq z^* \leq UB$ upon termination of Algorithm 1, where z^* is the optimal value of OTS.*

6 Computational Experiments

In this section, we present the results of our extensive computational experiments on standard IEEE instances available from MATPOWER [34] and instances from NESTA 0.3.0 archive with congested operating conditions [7]. The code is written in the C# language with Visual Studio 2010 as the compiler. For all experiments, we used a 64-bit computer with Intel Core i5 CPU 2.50GHz processor and 16 GB RAM. Time is measured in seconds. We use three different solvers:

- MIP solver CPLEX 12.6 [1] to solve MISOCPs.
- Conic interior point solver MOSEK 7.1 [27] to solve SDP separation problems.
- Nonlinear interior point solver IPOPT [32] to find local optimal solutions to $OPF(x)$.

We use a Gaussian elimination based approach to construct a cycle basis proposed in [22] and use this set of cycles in the separation phase.

6.1 Methods

We report the results of three algorithmic settings:

- SOCP: MISOCP formulation (6) in Phase II without Phase I (i.e. $T_1 = 0$).
- SOCPA: SOCP strengthened by the arctangent envelopes introduced in Section 4.1.
- SOCPA_Disj: SOCPA strengthened further by dynamically generating linear valid inequalities obtained from separating an SOCP feasible solution from the SDP and McCormick relaxation over cycles using a disjunctive argument T_1 times. In particular, a separation oracle is used to separate a given point from $\text{conv}((\mathcal{S}_1 \cap \mathcal{M}_1) \cup \mathcal{S}_0)$.

The following four performance measures are used to assess the accuracy and the efficiency of the proposed methods:

- "%OG" is the percentage optimality gap proven by our algorithm calculated as $100 \times (1 - \widetilde{LB}_1/UB)$. Here, $\widetilde{LB}_1$ is the lower bound proven, which may be strictly smaller than LB_1 due to optimality gap tolerance and time limit.
- "%CB" is the percentage cost benefit obtained by line switching calculated as $100 \times (1 - UB/OPF(e))$, where e is the vector of ones so that $OPF(e)$ corresponds to the OPF solution with the initial topology.
- "#off" is the number of lines switched off in the topology which gives UB.
- "TT" is the total time in seconds, including preprocessing (bound tightening), solution of $T_1 = 5$ rounds of SOCPs to improve lower bound and separation problems to generate cutting planes (in the case of SOCPA_Disj), solution of T_2 rounds of MISOCPs and several calls to local solver IPOPT with given topologies. MISOCPs are solved under 720 seconds time limit so that 5 iterations take about 1 hour (optimality gap for integer programs is 0.01%). Preprocessing and separation subproblems are parallelized.

We choose parameter $T_2 = 5$ and pre-terminate Algorithm 1 if 0.1% optimality gap is proven.

6.2 Results

The results of our computational experiments are presented in Tables 1 and 2 for standard IEEE and NESTA instances, respectively. We considered instances up to 300-bus since Phase II of the Algorithm 1 does not scale up well for larger instances. Let us start with the former: IEEE instances are a relatively easy set since transmission line limits are generally not binding. Therefore, cost benefits obtained by switching are also limited. The largest cost reduction is obtained for case30Q with 2.24%. Among the three methods, the most successful one is SOCPA_Disj, on average proving 0.05% optimality gap and providing 0.31% cost savings. In terms of computational time, SOCP is the fastest, however, its performance is not as good as the other two. Quite interestingly, SOCPA_Disj is faster than SOCPA, on average, for this set of instances. In terms of comparison with other methods, unfortunately, there is limited literature for this purpose. In [15], nine of these instances (except for cases 9Q and 30Q) are considered and a quadratic convex (QC) relaxation based approach is used. On average, their approach proves 0.14% optimality gap, which is worse than any of our methods over the same nine instances. The only instance QC approach is better is 118ieee with 0.11% optimality gap, while it is worse than our methods for case300 with a 0.47% optimality gap.

Table 1: Results summary for standard IEEE Instances.

	SOCP				SOCPA				SOCPA_Disj			
case	%OG	%CB	#off	TT(s)	%OG	%CB	#off	TT(s)	%OG	%CB	#off	TT(s)
6ww	0.16	0.48	2	1.29	0.02	0.48	2	0.67	0.01	0.48	2	1.28
9	0.00	0.00	0	0.26	0.00	0.00	0	0.22	0.00	0.00	0	0.55
9Q	0.04	0.00	0	0.42	0.04	0.00	0	0.33	0.04	0.00	0	0.97
14	0.08	0.00	0	0.66	0.09	0.00	0	0.70	0.01	0.00	1	1.81
ieee30	0.05	0.00	1	1.95	0.05	0.00	0	1.67	0.02	0.00	1	3.84
30	0.07	0.52	1	4.60	0.06	0.52	2	5.01	0.03	0.51	2	9.39
30Q	0.44	2.05	2	24.43	0.43	2.03	5	25.80	0.13	2.24	5	44.16
39	0.03	0.00	0	2.53	0.01	0.02	1	3.17	0.01	0.02	1	4.48
57	0.07	0.02	4	6.18	0.07	0.02	4	8.72	0.08	0.01	1	13.59
118	0.19	0.08	4	3065.64	0.15	0.12	10	2553.59	0.17	0.08	16	3174.01
300	0.16	0.02	9	2318.89	0.15	0.03	12	3624.12	0.10	0.05	15	2803.31
Avg.	**0.12**	**0.29**	**2.1**	**493.35**	**0.10**	**0.29**	**3.3**	**565.82**	**0.05**	**0.31**	**4.0**	**550.67**

Now let us consider NESTA instances with congested operating conditions. This set is particularly suited for line switching as more stringent transmission line limits are imposed. In fact, large cost improvements are observed for some test cases. For instance, about 45% and 39% cost reductions are possible for cases 30fsr and 118ieee, respectively. Other instances with sizable cost reductions include cases 6ww and 30as. SOCPA_Disj is again the most successful method if we look at averages of optimality gap (1.16%) and cost savings (6.21%). It certifies that the best topology is within 1.17% of the optimal for all the cases except for 118ieee and 189edin. In terms of computational time, SOCP is again the fastest, however, its performance is significantly worse than the other two. We also note that SOCPA improves quite a bit over SOCP in terms of optimality and cost benefits with 70% increase in computational time. SOCPA_Disj takes about only 10% more time than SOCPA. As we go from SOCP to SOCPA_Disj, problems get more complicated and sometimes, MISOCPs are not solved to optimality within time limit. Consequently, for cases 189edin and 300ieee, the optimality gaps proven and cost benefits obtained by SOCPA_Disj can be slightly worse.

Table 2: Results summary for NESTA Instances from Congested Operating Conditions.

	SOCP				SOCPA				SOCPA_Disj			
case	%OG	%CB	#off	TT(s)	%OG	%CB	#off	TT(s)	%OG	%CB	#off	TT(s)
3lmbd	3.30	0.00	0	0.14	2.00	0.00	0	0.14	1.17	0.00	0	0.30
4gs	0.65	0.00	0	0.11	0.16	0.00	0	0.13	0.00	0.00	0	0.27
5pjm	0.18	0.27	1	0.61	0.01	0.27	1	0.41	0.02	0.27	1	0.89
6ww	6.06	7.74	1	1.23	1.34	7.74	1	1.64	1.05	7.74	1	1.97
9wscc	0.00	0.00	0	0.19	0.00	0.00	0	0.20	0.00	0.00	0	0.30
14ieee	1.02	0.33	1	2.86	0.89	0.45	2	3.48	0.41	0.45	2	4.49
29edin	0.43	0.00	2	12.79	0.24	0.18	13	299.82	0.33	0.08	21	181.74
30as	1.81	3.13	2	14.82	0.35	3.30	5	19.52	0.34	3.30	5	24.93
30fsr	3.24	44.20	2	9.72	0.05	44.98	2	4.76	0.03	44.98	3	6.97
30ieee	0.54	0.46	1	12.28	0.40	0.48	2	10.61	0.15	0.48	2	13.37
39epri	1.92	1.10	1	11.56	0.80	1.41	2	13.20	0.70	1.52	2	12.65
57ieee	0.12	0.10	3	41.48	0.12	0.10	2	58.97	0.09	0.10	3	29.86
118ieee	41.67	4.33	3	225.57	21.51	27.98	30	3838.62	7.50	39.09	21	3856.76
162ieee	0.57	1.05	9	3675.75	0.63	1.00	15	3861.29	0.60	1.00	15	3855.50
189edin	5.31	1.10	3	540.02	4.81	0.13	2	2194.80	5.58	0.00	0	3634.02
300ieee	1.00	0.10	12	3655.10	0.65	0.37	21	3640.14	0.61	0.35	21	3651.95
Avg.	**4.24**	**3.99**	**2.6**	**512.76**	**2.12**	**5.52**	**6.1**	**871.73**	**1.16**	**6.21**	**6.1**	**954.75**

Finally, we note that that optimality gaps can be explained by two non-convexities: 1) integrality, 2) power flow equations. For instance, in case 3lmbd, the optimality gap can only be explained by the non-convexity of power flow equation since all the relevant topologies are considered. Similarly, at least some portion of the relatively large optimality gaps for cases 118ieee and 189edin may be attributed to non-convexity of power flow equations. Consequently, any future improvements on strengthening the convex relaxations of OPF problem can be useful in closing more gaps in OTS as well.

7 Conclusions

In this paper, we proposed a systematic approach to solve the AC OTS problem. In particular, we presented an alternative formulation for OTS and constructed a MISOCP relaxation. We improved the strength of this relaxation by the addition of arctangent envelopes and cutting planes obtained using disjunctive techniques. The use of these disjunctive cuts help in closing gap significantly. Our experiments on standard and congested instances suggest that the proposed methods are effective in obtaining strong lower bounds and producing provably good feasible solutions.

We remind the reader that AC OTS is a challenging problem since it embodies two types of non-convexities due to AC power flow constraints and integrality of variables. We hope that the methodology developed in this paper can eventually be further improved to solve AC OTS problem in real life operations.

A Convex Hull of Union of Two Conic Representable Sets

Let S_1 and S_2 be two bounded, conic representable sets

$$S_i = \{x : \exists u^i : A_i x + B_i u^i \succeq_{K_i} b_i\} \quad i = 1, 2.$$

Here, K_i's are regular (closed, convex, pointed with non-empty interior) cones. Then, a conic representation for $\text{conv}(S_1 \cup S_2)$ is given as follows:

$$\begin{aligned} & x = x^1 + x^2,\ \lambda_1 + \lambda_2 = 1,\ \lambda_1, \lambda_2 \geq 0 \\ & A_i x^i + B_i u^i \succeq_{K_i} b_i \lambda_i \quad i = 1, 2. \end{aligned}$$

B Separation from an Extended Conic Representable Set

Let S be a conic representable set $S = \{x : \exists u : Ax + Bu \succeq_K b\}$. Here, K is a regular cone. Suppose we want to decide if a given point x^* belongs to S and find a separating hyperplane $\alpha^\top x \geq \beta$ if $x^* \notin S$. This problem can be formulated as $\max_{\alpha,\beta} \left\{\beta - \alpha^\top x^* : \alpha^\top x \geq \beta \ \forall x \in S\right\}$, where the constraint can be further dualized as

$$Z^* := \max_{\alpha,\beta,\mu}\{\beta - \alpha^\top x^* \ : b^\top \mu \geq \beta, A^\top \mu = \alpha, B^\top \mu = 0,$$
$$\mu \in K^*, -e \leq \alpha \leq e, -1 \leq \beta \leq 1\},$$

where K^* is the dual cone of K. If $Z^* \leq 0$, then $x^* \in S$, otherwise, the optimal α, β from the above program gives the desired separating hyperplane. For details, please see [21].

References

[1] *User's Manual for CPLEX Version 12.6*. IBM, 2014.

[2] Gustavo Angulo, Shabbir Ahmed, Santanu S Dey, and Volker Kaibel. Forbidden vertices. *Mathematics of Operations Research*, 40(2):350–360, 2014.

[3] X. Bai, H. Wei, K. Fujisawa, and Y. Wang. Semidefinite programming for optimal powerflow problems. *International Journal on Electric Power Energy Systems*, 30(6-7):383–392, 2008.

[4] C. Barrows, S. Blumsack, and R. Bent. Computationally efficient optimal transmission switching: Solution space reduction. In *Power and Energy Society General Meeting, 2012 IEEE*, pages 1–8, July 2012.

[5] C. Barrows, S. Blumsack, and P. Hines. Correcting optimal transmission switching for AC power flows. In *47th Hawaii International Conference on System Sciences (HICSS)*, page 2374–2379, 2014.

[6] D. Bienstock and G. Munoz. On linear relaxations of OPF problems. *arXiv preprint arXiv:1411.1120*, 2014.

[7] C. Coffrin, D. Gordon, and P. Scott. NESTA, The NICTA energy system test case archive. *arXiv preprint arXiv:1411.0359*, 2014.

[8] C. Coffrin, H. L. Hijazi, and P. Van Hentenryck. The QC relaxation: Theoretical and computational results on optimal power flow. *arXiv preprint arXiv:1502.07847*, 2015.

[9] C. Coffrin and P. Van Hentenryck. A linear-programming approximation of AC power flows. *INFORMS Journal on Computing*, 26(4):718–734, 2014.

[10] A. G. Expósito and E. R. Ramos. Reliable load flow technique for radial distribution networks. *IEEE Trans. on Power Syst.*, 14(3):1063 – 1069, 1999.

[11] E.B. Fisher, R.P. O'Neill, and M.C. Ferris. Optimal transmission switching. *IEEE Trans. on Power Syst.*, pages 1346–1355, 2008.

[12] J.D. Fuller, R. Ramasra, and A. Cha. Fast heuristics for transmission-line switching. *IEEE Trans. on Power Syst.*, 27(3):1377–1386, Aug 2012.

[13] K.W. Hedman, R.P. O'Neill, E.B. Fisher, and S.S. Oren. Optimal transmission switching with contingency analysis. *IEEE Trans. on Power Syst.*, 24(3):1577–1586, Aug 2009.

[14] K.W. Hedman, S.S. Oren, and R.P. O'Neill. A review of transmission switching and network topology optimization. In *Power and Energy Society General Meeting, 2011 IEEE*, pages 1–7, July 2011.

[15] HL Hijazi, C Coffrin, and P Van Hentenryck. Convex quadratic relaxations of mixed-integer nonlinear programs in power systems. Technical report, NICTA, Canberra, ACT Australia, 2013.

[16] R. A. Jabr. Radial distribution load flow using conic programming. *IEEE Trans. on Power Syst.*, 21(3):1458–1459, 2006.

[17] R. A. Jabr. A conic quadratic format for the load flow equations of meshed networks. *IEEE Trans. on Power Syst.*, 22(4):2285–2286, 2007.

[18] R. A. Jabr. Optimal power flow using an extended conic quadratic formulation. *IEEE Trans. on Power Syst.*, 23(3):1000–1008, 2008.

[19] A. Khodaei, M. Shahidehpour, and S. Kamalinia. Transmission switching in expansion planning. *IEEE Trans. on Power Syst.*, 25(3):1722–1733, Aug 2010.

[20] B. Kocuk, S. S. Dey, and X. A. Sun. Inexactness of SDP relaxation and valid inequalities for optimal power flow. *To appear in IEEE Trans. on Power Syst.*, 2015.

[21] B. Kocuk, S. S. Dey, and X. A. Sun. Strong SOCP relaxations for optimal power flow. *arXiv preprint arXiv:1504.06770*, 2015.

[22] B. Kocuk, H. Jeon, S. S. Dey, J. Linderoth, J. Luedtke, and X. A. Sun. A cycle-based formulation and valid inequalities for DC power transmission problems with switching. *arXiv preprint arXiv:1412.6245*, 2014.

[23] J. Lavaei and S. H. Low. Zero duality gap in optimal power flow problem. *IEEE Trans. on Power Syst.*, 27(1):92–107, 2012.

[24] G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I – convex underestimating problems. *Math. Prog.*, 10(1):147–175, 1976.

[25] D. K. Molzahn, J. T. Holzer, B. C. Lesieutre, and C. L DeMarco. Implementation of a large-scale optimal power flow solver based on semidefinite programming. *IEEE Trans. on Power Syst.*, 28(4):3987–3998, 2013.

[26] D.K. Molzahn and I.A. Hiskens. Sparsity-exploiting moment-based relaxations of the optimal power flow problem. *To appear in IEEE Trans. on Power Syst.*, 2015.

[27] MOSEK. *MOSEK Modeling Manual.* MOSEK ApS, 2013.

[28] R. O' Neill, R. Baldick, U. Helman, M. Rothkopf, and J. Stewart. Dispatchable transmission in RTO markets. *IEEE Trans. on Power Syst.*, 20(1):171–179, 2005.

[29] P. Ruiz, J. M. Forster, A. Rudkevich, and M. C. Caramanis. On fast transmission topology control heuristics. *IEEE Power and Energy Society General Meeting*, pages 1–8, 2011.

[30] P. Ruiz, J. M. Forster, A. Rudkevich, and M. C. Caramanis. Tractable transmission topology control using sensitivity analysis. *IEEE Trans. on Power Syst.*, 27(3):1550–1559, 2012.

[31] M. Soroush and J.D. Fuller. Accuracies of optimal transmission switching heuristics based on DCOPF and ACOPF. *IEEE Trans. on Power Syst.*, 29(2):924–932, March 2014.

[32] A. Wächter and L. T. Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Math. Prog.*, 106(1):25–57, 2006.

[33] J. Wu and K.W. Cheung. On selection of transmission line candidates for optimal transmission switching in large power networks. In *Power and Energy Society General Meeting (PES), 2013 IEEE*, pages 1–5, July 2013.

[34] R.D. Zimmerman, C.E. Murillo-Sanchez, and R.J. Thomas. MATPOWER: Steady-state operations, planning, and analysis tools for power systems research and education. *IEEE Trans. Power Syst.*, 26(1):12–19, Feb 2011.