Coherent Risk Measures in Inventory Problems

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Abstract

We analyze an extension of the classical multi-period, single-item, linear cost inventory problem where the objective function is a coherent risk measure. Properties of coherent risk measures allow us to offer a unifying treatment of risk averse and min-max type formulations. For the single period newsvendor problem, we show that the structure of the optimal solution of the risk averse model is similar to that of the classical expected value problem. For a finite horizon dynamic inventory model, we show that, again, the optimal policy has a similar structure as that of the expected value problem. This result carries over even to the case when there is a fixed ordering cost. We also analyze monotonicity properties of the optimal order quantity with respect to the degree of risk aversion for certain risk measures.

Key words: inventory models, newsvendor problem, coherent risk measures, mean-absolute deviation, conditional-value-at-risk, dynamic programming

1 Introduction

Common inventory models are based upon two important assumptions: (i) a risk-neutral setting of optimizing expected cost or profit; and (ii) complete knowledge of the distribution of the underlying random parameters. For the single period newsvendor problem, risk aversion has been addressed through the use of an expected utility objective [6, 12, 13] or a mean-variance criterion [3, 5]. Expected utility objectives have also been considered in some classes of dynamic inventory models [2, 4]. To deal with the second issue regarding imprecision in the underlying distribution, min-max inventory models with the objective of optimizing worst-case expected costs or profits over a given family of distributions has been addressed by many authors [8, 9, 10, 11, 14, 17, 18]. In this paper, we offer a unifying treatment of risk-averse and min-max inventory models using the recent theory of coherent risk measures (Artzner et al. [1]).

The notion of coherent risk measures arose from an axiomatic approach for quantifying the *risk* of a financial position. Consider a random outcome Z viewed as an element of a linear space $\mathcal{Z}$ of measurable functions, defined on an appropriate sample space. According to [1], a function $\rho : \mathcal{Z} \to \mathbb{R}$ is said to be coherent risk measure for Z if it satisfies the following axioms:

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A1. *Convexity*: $\rho(\alpha Z_1 + (1-\alpha)Z_2) \leq \alpha\rho(Z_1) + (1-\alpha)\rho(Z_2)$ for all $Z_1, Z_2 \in \mathcal{Z}$ and all $\alpha \in [0,1]$.

A2. *Monotonicity*: If $Z_1, Z_2 \in \mathcal{Z}$ and $Z_2 \succeq Z_1$, then $\rho(Z_2) \geq \rho(Z_1)$.

A3. *Translation Equivariance*: If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z + a) = \rho(Z) + a$.

A4. *Positive Homogeneity*: If $\alpha > 0$ and $Z \in \mathcal{Z}$, then $\rho(\alpha Z) = \alpha\rho(Z)$.

(The notation $Z_2 \succeq Z_1$ means that $Z_2(\omega) \geq Z_1(\omega)$ for all elements ω of the corresponding sample space.) Two particular examples of coherent risk measures, which we discuss in more details later, are the mean-absolute deviation

$$\rho_\lambda[Z] := \mathbb{E}_F[Z] + \lambda\mathbb{E}_F\big|Z - \mathbb{E}_F[Z]\big|, \tag{1.1}$$

and the conditional-value-at-risk

$$CVaR_\alpha[Z] := \inf_{t\in\mathbb{R}} \left\{ t + (1-\alpha)^{-1}\mathbb{E}_F\big[Z - t\big]_+ \right\}. \tag{1.2}$$

In the above examples F is a reference probability distribution, $\lambda \in [0, 1/2]$ and $\alpha \in (0,1)$ are the corresponding parameters and Z has a finite mean $\mathbb{E}_F|Z|$.

The relevance and significance of the axioms A1-A4 are by now well-established in the risk-management literature. An important consequence of these axioms is that with every coherent risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is associated a (convex) set $\mathcal{A}$ of probability measures, depending on the dual space to $\mathcal{Z}$, such that the following dual representation holds

$$\rho[Z] = \sup_{F\in\mathcal{A}} \mathbb{E}_F[Z]. \tag{1.3}$$

Conversely, for every convex set $\mathcal{A}$ of probability measures such that the right-hand-side of (1.3) is real-valued, the corresponding function ρ is a coherent risk measure. We refer to [1, 7, 16] for a thorough discussion of mathematical properties of coherent risk measures.

In this paper we consider risk-averse inventory models where cost variability is controlled using coherent risk measures. Then the representation (1.3) immediately establishes a one-to-one correspondence between the risk averse formulation and min-max type formulations. For the single period newsvendor problem, we show that the structure of the optimal solution of the risk averse model is similar to that of the classical expected value problem. We also analyze monotonicity properties of the optimal order quantity with respect to the degree of risk aversion for certain risk measures. Next, we extend our analysis to a finite horizon dynamic inventory model. We show that, again, the optimal policy has a similar structure as that of the expected value problem. This result carries over even to the case when there is a fixed ordering cost.

2 The single period newsvendor problem

2.1 The models

Consider the classical newsvendor problem in a cost minimization setting. The newsvendor has to decide an order quantity x so as to satisfy uncertain demand d. The cost of ordering is $c_0 \geq 0$ per unit. Once demand d is realized, if the demand exceeds order, i.e., $d \geq x$, a back order penalty of

$b \geq 0$ per unit is incurred. On the other hand, if $d \leq x$ then a holding cost of $h \geq 0$ per unit is incurred. The remaining inventory $x - d$ incurs a (discounted) cost of $-\gamma c_1(x - d)$, where $c_1 \geq 0$ and $\gamma \in (0, 1]$ are salvage value and discount parameter respectively. The total cost is then

$$c_0 x - \gamma c_1(x - d) + b[d - x]_+ + h[x - d]_+ = cx + \Psi(x, d),$$

where $[a]_+ := \max\{a, 0\}$, $c := c_0 - \gamma c_1$, and

$$\Psi(x, d) := \gamma c_1 d + b[d - x]_+ + h[x - d]_+. \tag{2.1}$$

Note that the function $\Psi(x, d)$, and hence the cost function, are convex in x for any d. In the subsequent analysis we view the uncertain demand as a random variable, denoted D, to distinguish it from its particular realization d.

In the risk neutral setting the corresponding optimization problem is formulated as minimization of the expected value of the total cost with respect to the probability distribution of the demand D, say given by cumulative distribution function $F(\cdot)$. That is (cf., [19, section 9.4.1]),

$$\operatorname*{Min}_{x \in \mathbb{R}} \mathbb{E}_F \left[cx + \Psi(x, D) \right], \tag{2.2}$$

Let us emphasize that in the above formulation (2.2) the optimization is performed on *average* and it is assumed that the distribution F of the demand is *known*. Let us consider the following risk averse formulation of the newsvendor problem:

$$\operatorname*{Min}_{x \in \mathbb{R}} \rho \left[cx + \Psi(x, D) \right]. \tag{2.3}$$

Here $\rho[Z]$ is a coherent risk measure corresponding to a random outcome Z. By using the dual representation (1.3) of ρ we can write problem (2.3) in the following min-max form:

$$\operatorname*{Min}_{x \in \mathbb{R}} \sup_{F \in \mathcal{A}} \mathbb{E}_F \left[cx + \Psi(x, D) \right]. \tag{2.4}$$

Thus with ρ and $\mathcal{A}$ appropriately chosen, there is a one-to-one correspondence between risk averse (2.3) and min-max (2.4) formulations of the newsvendor problem.

2.2 Optimal solution structure

In the following we show that the risk-averse problem (2.3), and equivalently the min-max problem (2.4), has an optimal solution structurally similar to that of the classical newsvendor problem (2.2). We assume that the reference cdf F^* is such that $F^*(t) = 0$ for any $t < 0$. It follows then that any $F \in \mathcal{A}$ is also like that, i.e., $F(t) = 0$ for any $t < 0$. We also assume $b \geq c$ and $b + h > 0$ to avoid trivial solutions.

Theorem 1 *With any coherent risk measure ρ is associated cdf $\bar{F}$, depending on ρ and $\beta := (b+\gamma c_1)/(b+h)$, such that $\bar{F}(t) = 0$ for any $t < 0$, and the objective function $\psi(x) := \rho[cx+\Psi(x, D)]$ of the newsvendor problem can be written in the form*

$$\psi(x) = (b + \gamma c_1)\rho[D] + (c - b)x + (b + h) \int_{-\infty}^{x} \bar{F}(t)dt. \tag{2.5}$$

Proof. Recall that by the dual representation (1.3) we have that

$$\psi(x) = \sup_{F \in \mathcal{A}} \mathbb{E}_F \left[cx + \Psi(x, D) \right]. \tag{2.6}$$

Using integration by parts to evaluate $\mathbb{E}_F[\Psi(x,D)]$, we can write $\psi(x) = (c-b)x + (b+h)g(x)$, where

$$g(x) := \sup_{F \in \mathcal{A}} \left\{ \beta\, \mathbb{E}_F[D] + \int_{-\infty}^{x} F(t)dt \right\} \tag{2.7}$$

(recall that $\beta := \frac{b+\gamma c_1}{b+h}$). Since $F(\cdot)$ is a monotonically nondecreasing function, we have that $x \mapsto \int_{-\infty}^{x} F(t)dt$ is a convex function. It follows that the function $g(x)$ is given by the maximum of convex functions and hence is convex. Moreover, $g(x) \geq 0$ and

$$g(x) \leq \beta \sup_{F \in \mathcal{A}} \mathbb{E}_F[D] + [x]_+ = \beta\rho[D] + [x]_+, \tag{2.8}$$

and hence $g(x)$ is finite valued for any $x \in \mathbb{R}$. Also for any $F \in \mathcal{A}$ and $t < 0$ we have that $F(t) = 0$, and hence $g(x) = \beta \sup_{F \in \mathcal{A}} \mathbb{E}_F[D] = \beta\rho[D]$ for any $x < 0$.

Consider the right hand side derivative of $g(x)$:

$$g^+(x) := \lim_{t \downarrow 0} \frac{g(x+t) - g(x)}{t},$$

and define $\bar{F}(\cdot) := g^+(\cdot)$. Since $g(x)$ is convex, its right hand side derivative $g^+(x)$ exists, is finite and for any $x \geq 0$ and $a < 0$,

$$g(x) = g(a) + \int_a^x g^+(t)dt = \beta\rho[D] + \int_{-\infty}^{x} \bar{F}(t)dt. \tag{2.9}$$

Note that definition of the function $g(\cdot)$, and hence $\bar{F}(\cdot)$, involves the constant β and set $\mathcal{A}$ only. Let us also observe that the right hand side derivative $g^+(x)$, of a real valued convex function, is monotonically nondecreasing and right side continuous. Moreover, $g^+(x) = 0$ for $x < 0$ since $g(x)$ is constant for $x < 0$. We also have that $g^+(x)$ tends to one as $x \to +\infty$. Indeed, since $g^+(x)$ is monotonically nondecreasing it tends to a limit, denoted r, as $x \to +\infty$. We have then that $g(x)/x \to r$ as $x \to +\infty$. It follows from (2.8) that $r \leq 1$, and by (2.7) that for any $F \in \mathcal{A}$,

$$\liminf_{x \to +\infty} \frac{g(x)}{x} \geq \liminf_{x \to +\infty} \frac{1}{x} \int_{-\infty}^{x} F(t)dt \geq 1,$$

and hence $r \geq 1$. It follows that $r = 1$.

We obtain that $\bar{F}(\cdot) = g^+(\cdot)$ is a cumulative distribution function of some probability distribution and the representation (2.5) holds. ■

Consider the number

$$\kappa := \frac{b-c}{b+h}. \tag{2.10}$$

Recall that it was assumed that $b + h > 0$. Therefore, $\kappa \geq 0$ iff $b \geq c\,(= c_0 - \gamma c_1)$. Note that by (2.5) we have that for $x < 0$ the objective function $\psi(x)$, of the newsvendor problem, is equal to a constant (independent of x) plus the linear term $(c-b)x$. Therefore, if $b < c$, then the objective function $\psi(x)$ can be made arbitrary small by letting $x \to -\infty$. If $b = c$, i.e., $\kappa = 0$, then $\psi(x)$ is constant for $x < 0$. Now if $\kappa > 1$, i.e., $b - c > b + h$, then the objective function $\psi(x)$ can be made arbitrary small by letting $x \to +\infty$. If $\kappa = 1$ and $\bar{F}(t) < 1$ for all t, then $\psi(x)$ is monotonically decreasing as $x \to +\infty$. Therefore, situations where $b \le c$ or $h + c \le 0$ are somewhat degenerate, and we assume that $\kappa \in (0, 1)$. Consider the corresponding quantile (also called value-at-risk) of the cdf $\bar{F}$:

$$VaR_\kappa(\bar{F}) = \bar{F}^{-1}(\kappa) := \inf\big\{t \in \mathbb{R} : \bar{F}(t) \ge \kappa\big\}. \tag{2.11}$$

Note that for $\kappa \in (0, 1)$ this quantile is well defined and finite valued. It follows from the representation (2.5) that if $\kappa \in (0, 1)$, then $\bar{x} := VaR_\kappa(\bar{F})$ is always an optimal solution the newsvendor problem (2.3). More precisely, we have the following result.

Corollary 1.1 *Suppose that $\kappa \in (0, 1)$. Then the set of optimal solutions of the newsvendor problem* (2.3) *is a nonempty closed bounded interval $[a, b]$, where $a := VaR_\kappa(\bar{F})$ and $b := \sup\{t \in \mathbb{R} : \bar{F}(t) \le \kappa\}$. For $\kappa = 0$, any $x < 0$ is an optimal solution of the newsvendor problem* (2.3).

In some cases it is possible to calculate the corresponding cdf $\bar{F}$ in a closed form. Consider the conditional-value-at-risk measure $\rho[Z] := CVaR_\alpha[Z]$ defined with respect to a reference cdf $F^*(\cdot)$. The corresponding set $\mathcal{A}$ of probability measures is given by cumulative distribution functions $F(\cdot)$ such that $\mathbb{P}_F(S) \le (1 - \alpha)^{-1}\mathbb{P}_{F^*}(S)$ for any Borel set $S \subset \mathbb{R}$ (here $\mathbb{P}_F$ denotes the probability measure corresponding to cdf F). It follows that the cdf

$$\hat{F}_\alpha(t) := \min\big\{(1 - \alpha)^{-1}F^*(t), 1\big\}$$

belongs to the set $\mathcal{A}$ and dominates any other cdf in $\mathcal{A}$. Suppose now that the parameters $b = 0$ and $c_1 = 0$. Then the function $g(x)$, defined in (2.7), can be written as follows

$$g(x) = \sup_{F \in \mathcal{A}} \int_{-\infty}^{x} F(t)dt = \int_{-\infty}^{x} \hat{F}_\alpha(t)dt.$$

That is, in this case $\bar{F} = \hat{F}_\alpha$. Of course, as it was discussed above, the case of $b = 0$ and $c_1 = 0$ is not very interesting since then $\kappa < 0$.

2.3 Monotonicity with respect to risk aversion

Consider now risk measures of the form

$$\rho_{\lambda,\mathbb{D}}[Z] = \mathbb{E}[Z] + \lambda\mathbb{D}[Z], \tag{2.12}$$

where $\mathbb{E}$ is the usual expectation operator, taken with respect to a reference distribution F, and $\mathbb{D}$ is a measure of variability, and λ is a nonnegative weight to trade off expectation with variability. Higher values of λ reflects a higher degree of risk aversion. A risk measure of the form (2.12) is called a mean-risk function. Not all variability measures $\mathbb{D}$ and/or values of λ result in the risk measure $\rho_{\lambda,\mathbb{D}}$ to be coherent. Consider the p-th semideviation as the variability measure

$$\delta_p[Z] := \big(\mathbb{E}[Z - \mathbb{E}Z]_+^p\big)^{\frac{1}{p}}. \tag{2.13}$$

Then ρ_{λ,δ_p} is a coherent risk measure for any $p \ge 1$ and $\lambda \in [0, 1]$. For $p = 1$ and λ changed to $\lambda/2$, the corresponding mean-absolute semideviation risk function coincides with the mean-absolute deviation risk function defined in (1.1). On the other hand if we use variance (or standard deviation) as the dispersion measure, then the corresponding mean-risk function typically does not satisfy the monotonicity condition, and hence is not a coherent risk measure, for any $\lambda > 0$.

In the following we investigate the behavior of optimal solutions to the risk-averse model (2.3), involving coherent mean-risk objectives, with respect to the risk aversion parameter λ.

Lemma 1 *Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be two convex functions and $S_i := \arg\min_{x\in\mathbb{R}} f_i(x)$, $i = 1, 2$, be their sets of minimizers. Suppose that S_1 and S_2 are nonempty, and hence are closed intervals $S_1 = [a_1, b_1]$ and $S_2 = [a_2, b_2]$. Consider function $f_\lambda(x) := (1-\lambda)f_1(x) + \lambda f_2(x)$ and let $S_\lambda := \arg\min_{x\in\mathbb{R}} f_\lambda(x)$. Then for any $\lambda \in [0, 1]$ the following holds:* (i) *If $b_1 < a_2$ then the set S_λ is nonempty and monotonically nondecreasing in $\lambda \in [0, 1]$,* (ii) *If $b_2 < a_1$, then the set S_λ is nonempty and monotonically nonincreasing in $\lambda \in [0, 1]$,* (iii) *If the sets S_1 and S_2 have a nonempty intersection, then $S_\lambda = S_1 \cap S_2$ for any $\lambda \in (0, 1)$.*

Proof. We only prove the assertion (i), the other assertion (ii) is analogous. We have that for every $\lambda \in [0, 1]$ the function $f_\lambda(x)$ is convex. Note that since $f_1(x)$ and $f_2(x)$ are real valued and convex, and hence continuous, S_1 and S_2 are closed intervals. Since $f_1(x)$ is convex, it has finite left and right side derivatives, denoted $f_1^-(x)$ and $f_1^+(x)$, respectively. Since b_1 is a minimizer of $f_1(\cdot)$, we have that $f_1^-(b_1) \geq 0$. Also since a_2 is the smallest minimizer of $f_2(\cdot)$ and $a_2 > b_1$ we have that $f_2^-(b_1) > 0$. Consequently, for any $\lambda \in (0, 1)$ we have that $f_\lambda^-(b_1) > 0$, and hence $f_\lambda(x) > f_\lambda(b_1)$ for all $x < b_1$. By similar arguments we have that $f_\lambda(x) > f_\lambda(a_2)$ for all $x > a_2$. By convexity arguments this implies that the set S_λ is nonempty and $S_\lambda \subset [b_1, a_2]$ for all $\lambda \in (0, 1)$. Note that the set S_λ is a closed interval, say $S_\lambda = [a_\lambda, b_\lambda]$. Also by similar arguments it is not difficult to show that for any $0 \leq \lambda < \lambda' \leq 1$, it holds that $b_\lambda \leq a_{\lambda'}$.

In order to prove (iii) note that for any $\lambda \in (0, 1)$, $y \in \mathbb{R}$ and $x \in S_1 \cap S_2$ we have

$$(1-\lambda)f_1(y) + \lambda f_2(y) \geq (1-\lambda)f_1(x) + \lambda f_2(x),$$

and that the above inequality is strict if $y \notin S_1 \cap S_2$. ■

Now let us make the following observations. If $\rho_i : \mathcal{Z} \to \mathbb{R}$, $i = 1, 2$, are two coherent risk measures, then their convex combination $\rho_\lambda[Z] := (1-\lambda)\rho_1[Z] + \lambda\rho_2[Z]$ is also a coherent risk measure for any $\lambda \in [0, 1]$. Since, for any d, the function $x \mapsto cx + \Psi(x, d)$ is convex, the functions $f_i(x) := \rho_i[cx + \Psi(x, D)]$, $i = 1, 2$, are convex real valued (convexity of the composite functions f_i follows by convexity and monotonicity of ρ_i). Therefore, by the above lemma, we have that if functions $f_1(\cdot)$ and $f_2(\cdot)$ have disjoint sets of minimizers S_1 and S_2, respectively (recall that by Corollary 1.1 these sets are nonempty), then the set S_λ, of minimizers of $\rho_\lambda[cx + \Psi(x, D)]$ is monotonically nondecreasing or nonincreasing in $\lambda \in [0, 1]$, depending on whether $S_2 > S_1$ or $S_1 < S_2$.

Theorem 2 *Consider the newsvendor problem with a mean-risk objective $\rho_{\lambda,\mathbb{D}}$ of the form* (2.12). *Suppose that $\rho_{\lambda,\mathbb{D}}$ is a coherent risk measure for all $\lambda \in [0, 1]$ and let S_λ be the set of optimal solutions of the corresponding problem. Suppose that the sets S_0 and S_1 are nonempty. Then the following holds.*

(i) *If $S_0 \cap S_1 = \emptyset$, then S_λ is monotonically nonincreasing or monotonically nondecreasing in $\lambda \in [0, 1]$ depending upon whether $S_0 > S_1$ or $S_0 < S_1$. If $S_0 \cap S_1 \neq \emptyset$, then $S_\lambda = S_0 \cap S_1$ for any $\lambda \in (0, 1)$.*

(ii) *Consider some* $x_0 \in S_0$ *such that* $f_0(x) := cx + \mathbb{E}[\Psi(x, D)]$ *is twice continuous differentiable near* x_0 *with* $f_0''(x_0) \neq 0$ *and* $v(x) := \mathbb{D}[\Psi(x, D)]$ *is continuously differentiable near* x_0*. If* $v'(x_0) > 0$ *then* S_λ *is monotonically nonincreasing; if* $v'(x_0) < 0$ *then* S_λ *is monotonically nondecreasing; if* $v'(x_0) = 0$ *then* $S_\lambda = \{x_0\}$ *for all* $\lambda \in [0, 1]$.

Proof.

(i) Note that for any $\lambda \in [0, 1]$ the objective function of the newsvendor problem with a mean-risk objective $\rho_{\lambda,\mathbb{D}}$ is

$$f_\lambda(x) = (1 - \lambda)f_1(x) + \lambda f_2(x),$$

where $f_1(x) := cx + \mathbb{E}[\Psi(x, D)]$ and $f_2(x) := cx + \mathbb{E}[\Psi(x, D)] + \mathbb{D}[\Psi(x, D)]$. The result then follows from Lemma 1.

(ii) Note that since f_0 is convex and $f_0''(x_0) \neq 0$, it follows that $f_0''(x_0) > 0$ and hence x_0 is the unique minimizer of $f_0(x)$. Since $v(x)$ is continuously differentiable at x_0, we have then that for all $\lambda > 0$ small enough, a minimizer $x_\lambda \in S_\lambda$ is a solution of the equation $f_0'(x) + \lambda v'(x) = 0$. It follows by the Implicit Function Theorem that for $\lambda > 0$ small enough the minimizer x_λ is unique and

$$\frac{dx_\lambda}{d\lambda} = -\frac{v'(x_0)}{f_0''(x_0)}.$$

Combining the above with (i) the result follows. ■

Let us check the sign of $v'(x_0)$ corresponding to the p-th semideviation risk measure (2.13) taken with respect to cdf F. It is sufficient to check the sign of

$$\frac{d}{dx}\left(\mathbb{E}\big[\Psi(x, D) - \mathbb{E}[\Psi(x, D)]\big]_+^p\right)\Big|_{x=x_0}. \tag{2.14}$$

Suppose that the reference cdf $F(\cdot)$ is continuous. Then the above derivative exists and the derivative can be taken inside the expectation. Letting $\Delta(x, d) := p[\Psi(x, d) - \mathbb{E}\Psi(x, D)]_+^{p-1}$, the derivative (2.14) is equal to:

$$\begin{aligned}
&= \int \Delta(x_0, t)\frac{d}{dx}\big[\Psi(x_0, t) - \mathbb{E}[\Psi(x_0, D)]\big]_+ dF(t) \\
&= \int_{\Psi(x_0,t)\geq\mathbb{E}[\Psi(x_0,D)]} \Delta(x_0, t)\frac{d}{dx}\big[\Psi(x_0, t) - \mathbb{E}[\Psi(x_0, D)]\big] dF(t) \\
&= \int_{\Psi(x_0,t)\geq\mathbb{E}[\Psi(x_0,D)]} \Delta(x_0, t)\left[\Psi_x'(x_0, t) + c\right] dF(t), \qquad (2.15)
\end{aligned}$$

where the last line follows from the optimality conditions for x_0.

Consider the case $\gamma c_1 - h > 0$(which will be the case if salvage value is higher than holding cost and discount factor is close to 1). Then $\Psi(x, t)$ is monotonically non-decreasing in t. Note that $\Psi_x'(x_0, t) = -b$ if $t > x_0$ and $\Psi_x'(x_0, t) = h$ if $t < x_0$. Now let

$$t_0 := \inf\big\{t : \Psi(x_0, t) \geq \mathbb{E}[\Psi(x_0, D)]\big\}.$$

If $t_0 \geq x_0$ then (2.15) reduces to:

$$= \int_{t \geq t_0} \Delta(x_0, t)[c - b] dF(t) \leq 0, \tag{2.16}$$

where the inequality follow since $\Delta(x_0, t) \geq 0$ and $b > c$ for the problem to be nontrivial. On the other hand if $t_0 < x_0$ we can use the following inequality

$$-\int_{t \geq t_0} [c - b] dF(t) \geq \int_{x_0 \geq t \geq t_0} [c + h] dF(t) \tag{2.17}$$

which follows from the optimality conditions and the fact that $\Delta(x_0, t) \geq 0$ is non-decreasing in t to conclude that $v'(x_0) \leq 0$. Therefore the minimizer x_λ of the newsvendor problem with a mean p-th semideviation objective is monotonically nondecreasing with λ (note that by an arbitrary small change of the cdf F we can ensure that the corresponding second order derivative $f_0''(x_0)$ exists and is nonzero). We have thus established the following result.

Corollary 2.1 *If $\gamma c_1 - h > 0$ the minimizer x_λ of the newsvendor problem with a mean p-th semideviation objective $\rho_\lambda[Z] := \mathbb{E}[Z] + \lambda \left(\mathbb{E}[Z - \mathbb{E}Z]_+^p\right)^{\frac{1}{p}}$ (with $p \geq 1$) is monotonically nondecreasing with λ.*

Let us check the sign of $v'(x_0)$ corresponding to $\mathbb{D}[Z] := CV@R_\alpha[Z]$. We assume again that the reference cdf $F(\cdot)$ is continuous. Let t_0 be the minimizer (assumed to be unique) of the right hand side of (1.2) corresponding to $Z = \Psi(x_0, D)$. Then

$$\frac{d}{dx} CVaR_\alpha[\Psi(x_0, D)] = \frac{d}{dx} \left(t_0 + (1 - \alpha)^{-1} \mathbb{E}_F[\Psi(x_0, D) - t_0]_+\right). \tag{2.18}$$

Note that the variable t in the above formula can be fixed to the constant value t_0 by the so-called Danskin Theorem since the minimizer t_0 is assumed to be unique. It follows that we need to check the sign of

$$\frac{d}{dx} \mathbb{E}\big[[\Psi(x_0, D) - t_0]_+\big] = \int_{\Psi(x_0, \tau) \geq t_0} \Psi_x'(x_0, \tau) dF(\tau). \tag{2.19}$$

Assuming that $\gamma c_1 - h > 0$, then $\Psi(x, \tau)$ is monotonically nondecreasing in τ. It is possible to use an argument similar to the one used to obtain (2.17) and conclude that $v'(x_0) \leq 0$. Then we have the following result.

Corollary 2.2 *If $\gamma c_1 - h > 0$ the minimizer x_λ of the newsvendor problem with a mean-CVaR objective $\rho_\lambda[Z] := \mathbb{E}[Z] + \lambda\, CV@R_\alpha[Z]$ is monotonically nondecreasing with λ.*

If $\gamma c_1 - h < 0$ then $\Psi(x, t)$ is no longer guaranteed to monotonic in t, and the sign of $v'(x_0)$ may be positive or negative. Intuitively, Corollaries 2.1 and 2.2 show that if the discounted salvage value is higher than the holding cost then increased risk aversion implies higher order quantity. Similar monotonicity results for the profit maximizing newsvendor model has been discussed in [6].

2.4 Numerical illustration

We now present some numerical results for the newsvendor problem with the mean-absolute deviation objective (1.1). The problem parameters are as follows: ordering cost $c = 100$, holding cost $h = 20$, backordering cost $b = 60$ and discount factor $\gamma = 0.9$. The demand D is distributed according to a lognormal distribution with mean $\mu = 50$ and standard deviation $\sigma = 90$. Table 1 presents the optimal order quantity x^* for five different values of the mean risk trade-off λ. Note that here $\gamma c_1 > h$, hence, as per Corollary 2.1, the optimal order quantity is increasing with λ.

Table 1: Optimal order quantity

λ	0	0.1	0.2	0.3	0.4
x^*	52.591	53.225	53.694	54.130	54.549

Next we compare the cost distribution of the risk neutral solution $x^* = 52.591$ (for $\lambda = 0$) and that of the risk averse solution $x^* = 54.549$ (for $\lambda = 0.4$) over a sample of 5000 demand scenarios generated for lognormal with $\mu = 50$ and $\sigma = 90$. To test the effect imprecision in the underlying distribution, we also considered demand scenarios by changing μ and σ. Figure 1 presents the mean and range of the cost distribution for the two solutions corresponding to $\mu = 45, 50, 55$ and $\sigma = 90$. An immediate observation is the proportionality between the cost and the mean value. Clearly a bigger mean implies bigger costs. One can see that the difference between the average values for risk neutral and risk averse solutions is larger when the actual mean is less than the predicted mean. On the other hand if the actual mean is underestimated than risk averse solution yields a lower average cost. In all cases the risk averse solution gives costs with smaller dispersion. As a result the risk averse solution dominate the risk neutral solution when the actual mean is underestimated. Figure 2 presents the mean and range of the cost distribution for the two solutions corresponding to $\mu = 50$ and $\sigma = 60, 90, 120$. Figure 2 shows the maximum, minimum and average cost for 5000 different scenarios generated using three different standard deviation values ($\sigma = 60, 90, 120$) and a fixed mean ($\mu = 50$). Note that the average value is not significantly affected by the change in standard deviation however the maximum value increases and the minimum value decreases as we increase the standard deviation. In all cases we end up with a smaller dispersion if we use the risk averse solution. These results demonstrate that the risk averse solution will yield costs with smaller dispersion even if the forecasted mean and standard deviation of the distribution is not accurate.

3 The multi-period problem

3.1 The models

Let us now extend the single period model of Section 2 to multiple periods. In each period $t \in \{0, \ldots, T\}$, the decision maker first observes the inventory level y_t and then places an order to replenish the inventory level to x_t ($\geq y_t$), i.e., the order quantity is $x_t - y_t$. The ordering cost is $c_t \geq 0$ per unit. After the inventory is replenished, demand d_t is realized and, accordingly, either (if $d_t < x_t$) an inventory holding cost of h_t per unit or (if $d_t \geq x_t$) a backorder penalty cost of b_t per unit is incurred. The inventory holding and backorder penalty cost will be denoted by the function

$$\Psi_t(x_t, d_t) = b_t[d_t - x_t]_+ + h_t[x_t - d_t]_+. \tag{3.1}$$

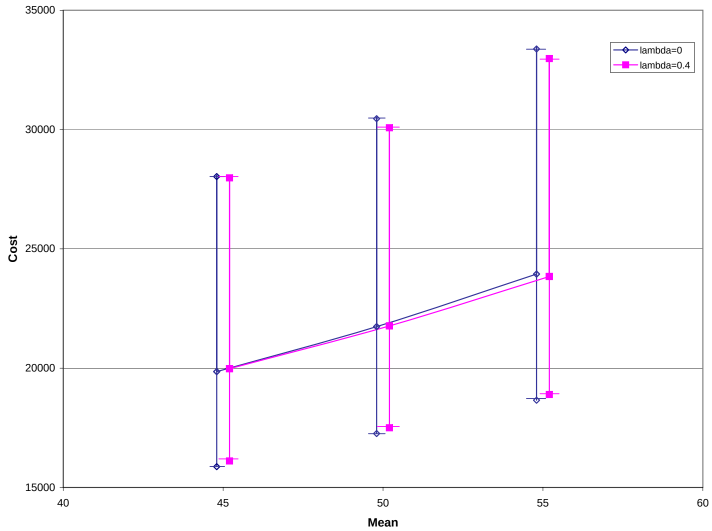

Figure 1: Maximum, minimum and average cost for different mean values

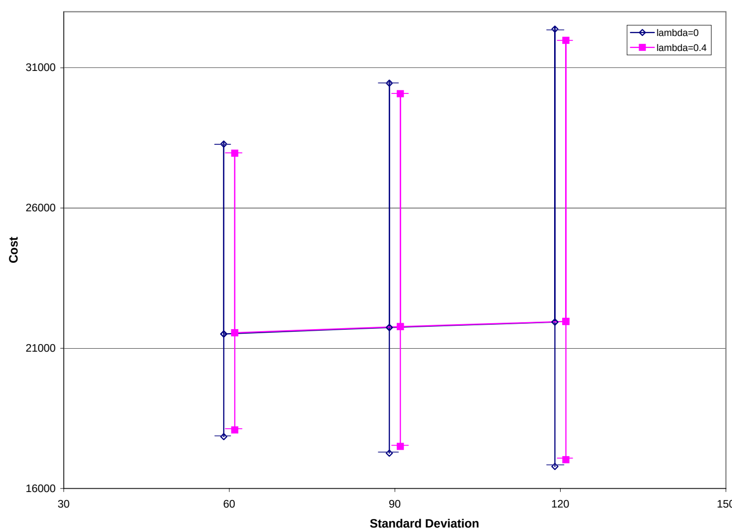

Figure 2: Maximum, minimum and average cost for different standard deviation values

Thus the total cost incurred in period t is $c_t(x_t - y_t) + \Psi_t(x_t, d_t)$. After demand is satisfied, the inventory level at the end of period t, i.e., at the beginning of period $t+1$ is $y_{t+1} = x_t - d_t$. It will be assumed that $b_t + h_t \geq 0$, $t = 0, ..., T$, and hence functions $\Psi_t(x_t, d_t)$ are convex in x_t for any d_t.

We view the demand, considered as a function of time (period) t, as a random process D_t (as in the previous section we denote by d_t a particular realization of D_t). Unless stated otherwise we assume that the random process D_t is *across periods independent*, i.e., D_t is (stochastically) independent of $(D_0, ..., D_{t-1})$ for $t = 1, ..., T$. This assumption of across periods independence considerably simplifies the analysis. The cost of period t is discounted by a factor of γ^t where $\gamma \in (0, 1]$ is a given parameter. The remaining inventory y_{T+1} at the end of the planning horizon incurs a (discounted) cost of $-\gamma^{T+1} c_{T+1} y_{T+1}$.

In the classical risk neutral setting, the goal is to find an ordering policy to minimize expected total discounted cost. We consider a generalization of this classical model, where the expectation operation is replaced by a coherent risk measure ρ. Let us start our discussion with one period model. In the risk neutral setting the corresponding optimization problem can be formulated in the following form (compare with problem (2.2)):

$$\operatorname*{Min}_{x_0\in\mathbb{R}} c_0(x_0 - y_0) + \mathbb{E}\left[\Psi_0(x_0, D_0) - \gamma c_1(x_0 - D_0)\right] \text{ subject to } x_0 \geq y_0, \tag{3.2}$$

where $y_0 \geq 0$ is a given initial value and the expectation is taken with respect to the probability distribution of D_0. Note that by linearity of the expectation functional, the second term of the objective function of (3.2) can be written in the following equivalent form

$$\mathbb{E}\left[\Psi_0(x_0, D_0) - \gamma c_1(x_0 - D_0)\right] = \mathbb{E}[\Psi_0(x_0, D_0)] - \gamma c_1(x_0 - \mathbb{E}[D_0]).$$

Now for a specified coherent risk measure $\rho_0(\cdot)$ we can formulate the following risk-averse analogue of problem (3.2):

$$\operatorname*{Min}_{x_0\in\mathbb{R}} c_0(x_0 - y_0) + \rho_0\left[\Psi_0(x_0, D_0) - \gamma c_1(x_0 - D_0)\right] \text{ subject to } x_0 \geq y_0. \tag{3.3}$$

Note that since $\rho_0(Z + a) = \rho_0(Z) + a$ for any constant $a \in \mathbb{R}$, we can write

$$\rho_0\left[\Psi_0(x_0, D_0) - \gamma c_1(x_0 - D_0)\right] = \rho_0\left[\Psi_0(x_0, D_0) + \gamma c_1 D_0\right] - \gamma c_1 x_0.$$

We discuss now an extension of the static problem (3.3) to a dynamic (multistage) setting. Let ρ_t, $t = 0, ..., T$, be a sequence of coherent risk measures. We assume that risk measures ρ_t are *distribution invariant* in the sense that $\rho_t[Z]$ depends on the distribution of the random variable Z only. For example, we can use mean-absolute deviation risk measures

$$\rho_t(Z) := \mathbb{E}[Z] + \lambda_t \mathbb{E}\big|Z_t - \mathbb{E}[Z_t]\big|, \tag{3.4}$$

where $\lambda_t \in [0, 1/2]$, $t = 0, ..., T$, is a chosen sequence of numbers. Consider the following (nested) formulation of the corresponding multistage risk-averse problem:

$$\begin{array}{ll}\text{Min} & c_0(x_0 - y_0) + \rho_0\Big[\Psi_0(x_0, D_0) + \gamma\rho_1\big[c_1(x_1 - y_1) + \Psi_1(x_1, D_1) + ... \\ & \qquad \gamma\rho_{T-1}[c_{T-1}(x_{T-1} - y_{T-1}) + \Psi_{T-1}(x_{T-1}, D_{T-1}) + \\ & \qquad \gamma\rho_T[c_T(x_T - y_T) + \Psi_T(x_T, D_T) - \gamma c_{T+1} y_{T+1}]]\big]\Big] \\ \text{s.t.} & x_t \geq y_t,\ y_{t+1} = x_t - D_t,\ t = 0, ..., T.\end{array} \tag{3.5}$$

Using the min-max representation (1.3) of ρ_t, $t = 0, ..., T$, with $\mathcal{A}_t$ being the corresponding set of cdf's, we can write (3.5) as

$$\begin{array}{ll}\text{Min} & c_0(x_0 - y_0) + \sup\limits_{F\in\mathcal{A}_0} \mathbb{E}_F\Big[\Psi_0(x_0, D_0) + \gamma \sup\limits_{F\in\mathcal{A}_1} \mathbb{E}_F\big[c_1(x_1 - y_1) + \Psi_1(x_1, D_1) + ... \\ & \qquad \gamma \sup\limits_{F\in\mathcal{A}_{T-1}} \mathbb{E}_F[c_{T-1}(x_{T-1} - y_{T-1}) + \Psi_{T-1}(x_{T-1}, D_{T-1}) + \\ & \qquad \gamma \sup\limits_{F\in\mathcal{A}_T} \mathbb{E}_F[c_T(x_T - y_T) + \Psi_T(x_T, D_T) - \gamma c_{T+1} y_{T+1}]]\big]\Big] \\ \text{s.t.} & x_t \geq y_t,\ y_{t+1} = x_t - D_t,\ t = 0, ..., T.\end{array} \tag{3.6}$$

We can write the corresponding dynamic programming equations for (3.5) as follows. At the last stage we need to solve the problem:

$$\operatorname*{Min}_{x_T \in \mathbb{R}} \rho_T\big[c_T(x_T - y_T) + \Psi_T(x_T, D_T) - \gamma c_{T+1}(x_T - D_T)\big] \text{ s.t. } x_T \geq y_T. \tag{3.7}$$

Its optimal value, denoted $V_T(y_T)$, is a function of y_T. At stage $T-1$ we solve the problem:

$$\begin{array}{ll} \operatorname*{Min}_{x_{T-1}\in\mathbb{R}} & \rho_{T-1}\left[c_{T-1}(x_{T-1} - y_{T-1}) + \Psi_{T-1}(x_{T-1}, D_{T-1}) + \gamma V_T(x_{T-1} - D_{T-1})\right] \\ \text{s.t.} & x_{T-1} \geq y_{T-1}. \end{array} \tag{3.8}$$

Its optimal value is denoted by $V_{T-1}(y_{T-1})$. And so on at stage $t = T-1, ..., 0$, we can write dynamic programming equations:

$$V_t(y_t) = \min_{x_t \geq y_t} \big\{c_t(x_t - y_t) + \rho_t\left[\Psi_t(x_t, D_t) + \gamma V_{t+1}(x_t - D_t)\right]\big\}. \tag{3.9}$$

Note that if each ρ_t is taken to be usual expectation operator (e.g., if we use ρ_t of the form (3.4) with all $\lambda_t = 0$), then the above becomes the standard formulation of a multistage inventory model (cf., [19]).

3.2 Optimal policy structure

A policy $x_t = x_t(d_0, ..., d_{t-1})$, $t = 0, ..., T$, is a function of a realization of the demand process up to time $t-1$ (with $d_{-1} := 0$). Recalling that $y_t = x_{t-1} - d_{t-1}$, we can view a policy $x_t = x_t(y_t)$ as a function of y_t, $t = 0, ..., T$. A policy is feasible if it satisfies the corresponding constraints $x_t \geq y_t$, $t = 0, ..., T$. Because of the across periods independence of the demand process, we have that, for a chosen policy $\hat{x}_t$, $\hat{y}_t = \hat{x}_{t-1} - D_{t-1}$, the total cost is given here by

$$\begin{array}{l} c_0(\hat{x}_0 - y_0) + \rho_0\big[\Psi_0(\hat{x}_0, D_0) + \gamma c_1(\hat{x}_1 - \hat{y}_1)\big] + ... \\ \rho_{T-1}\big[\Psi_{T-1}(\hat{x}_{T-1}, D_{T-1}) + \gamma c_T(\hat{x}_T - \hat{y}_T)\big] + \rho_T\big[\Psi_T(x_T, D_T) - \gamma c_{T+1}\hat{y}_{T+1}\big]. \end{array} \tag{3.10}$$

That is, the nested problem (3.5) can be formulated as minimization of the cost function (3.10) over all feasible policies.

By the dynamic programming equations (3.9) we have that a policy $\bar{x}_t = \bar{x}_t(y_t)$ is optimal iff

$$\bar{x}_t \in \arg\min_{x_t \geq y_t} \Lambda_t(x_t), \tag{3.11}$$

where

$$\Lambda_t(x_t) := \left\{ \begin{array}{ll} c_t x_t + \rho_t\left[\Psi_t(x_t, D_t) + \gamma V_{t+1}(x_t - D_t)\right], & t = 0, ..., T-1, \\ (c_T - \gamma c_{T+1})x_T + \rho_T\left[\Psi_T(x_T, D_T) + \gamma c_{T+1} D_T\right], & t = T. \end{array} \right. \tag{3.12}$$

Theorem 3 *For $t = 0, \ldots, T$, let $x_t^* \in \arg\min_{x_t \in \mathbb{R}} \Lambda_t(x_t)$ be an unconstrained minimizer of $\Lambda_t(\cdot)$. Then the basestock policy $\bar{x}_t := \max\{y_t, x_t^*\}$ solves the dynamic programming equations* (3.10)*, and hence is optimal.*

Proof. Since functions $\Psi_t(x_t, d_t)$ are convex in x_t, for any d_t, and ρ_t are convex and nondecreasing, it is straightforward to show by the induction that the value functions $V_t(\cdot)$ are convex, and hence functions $\Lambda_t(\cdot)$ are convex as well for all $t = 0, ..., T$. By convexity of $\Lambda_t(\cdot)$ we have that if an unconstrained minimizer of $\Lambda_t(\cdot)$ is bigger than y_t, then it solves the right hand side of (3.10), otherwise $\bar{x}_t = y_t$ solves (3.10). ■

The result of the above theorem is based on convexity properties and does not require the assumption of across periods independence. It is also possible to write dynamic programming type equations for a general (not necessarily across periods independent) process (cf., [15]). In such a case the corresponding value functions $V_t(y_t, d_0, ..., d_{t-1})$ will involve a history of the demand process. Again, optimality of the corresponding basestock policy will follow by convexity arguments.

Theorem 4 *Suppose that the costs $c_t = c$, $t = 0, \dots, T+1$, and parameters $b_t = b$, $h_t = h$, $t = 0, \dots, T$, are constant, and hence $\Psi_t(\cdot,\cdot) = \Psi(\cdot,\cdot)$ does not depend on t, that risk measures $\rho_t = \rho$, $t = 0, \dots, T$, are the same and that the demand process $D_0, ..., D_T$ is iid (independent identically distributed). Let*

$$x^* \in \arg\min_{x\in\mathbb{R}} \big\{(1-\gamma)cx + \rho[\Psi(x, D) + \gamma cD]\big\}. \tag{3.13}$$

Then the myopic basestock policy $\bar{x}_t := \max\{y_t, x^*\}$ solves the dynamic programming equations (3.11)*, and hence is optimal.*

Proof. We have that x^* is an unconstrained minimizer of $\Lambda_T(\cdot)$, and hence by Theorem 3 the claim is true for $t = T$. We use now backward induction by t. Suppose the claim is true for some period t. Then by Theorem 3, $\Lambda_t(x^*) \le \Lambda_t(x_t)$ for any x_t. Now, consider the period $t-1$. By Theorem 3, the optimal policy is $\bar{x}_{t-1} = \max\{x^*_{t-1}, y_{t-1}\}$ where x^*_{t-1} is an unconstrained minimizer of

$$\begin{aligned}\Lambda_{t-1}(x_{t-1}) &= cx_{t-1} + \rho[\Psi(x_{t-1}, D) + \gamma V_t(x_{t-1} - D)] \\ &= (1-\gamma c)x_{t-1} + \rho[\Psi(x_{t-1}, D) + \gamma cD + \gamma\Lambda_t(\max\{x^*, x_{t-1} - D\})],\end{aligned} \tag{3.14}$$

where the second line follows from the induction hypothesis and the translation equivariance property of ρ. Since the demand D is nonnegative, we clearly have that $\max\{x^*, x^* - D\} = x^*$ for a.e. D. Then by (3.14) we have

$$\begin{aligned}\Lambda_{t-1}(x^*) &= (1-\gamma c)x^* + \rho[\Psi(x^*, D) + \gamma cD + \gamma\Lambda_t(\max\{x^*, x^* - D\})] \\ &= (1-\gamma c)x^* + \rho[\Psi(x^*, D) + \gamma cD] + \gamma\Lambda_t(x^*) \\ &\le (1-\gamma c)x_{t-1} + \rho[\Psi(x_{t-1}, D) + \gamma cD] + \gamma\Lambda_t(x^*) \\ &\le (1-\gamma c)x_{t-1} + \rho[\Psi(x_{t-1}, D) + \gamma cD] + \gamma\Lambda_t(\max\{x^*, x_{t-1} - D\}) \ \text{ for a.e. } D,\end{aligned} \tag{3.15}$$

where the second line follows from the translation equivariance property of ρ; the third line from the definition of x^*; and the fourth line follows from the induction hypothesis. Let us observe now that if Z is a random variable such that $Z \ge \alpha$ w.p.1, then by the monotonicity property of ρ we have that $\rho[Z_1 + \alpha] \le \rho[Z_1 + Z]$. Applying this with $Z := \Lambda_t(\max\{x^*, x_{t-1} - D\})$ and $Z_1 := \Psi(x_{t-1}, D) + \gamma cD$, we obtain by the last line of (3.15) that

$$\Lambda_{t-1}(x^*) \le (1-\gamma c)x_{t-1} + \rho[\Psi(x_{t-1}, D) + \gamma cd + \gamma\Lambda_t(\max\{x^*, x_{t-1} - D\})] = \Lambda_{t-1}(x_{t-1}), \tag{3.16}$$

for any x_{t-1}, where the last equality in (3.16) holds by (3.14). Thus $x^*_{t-1} = x^*$ minimizes $\Lambda_{t-1}(x_{t-1})$, and hence the result follows by Theorem 3. ■

We obtain that under the assumptions of the above theorem one can apply monotonicity results of the previous section to the optimal (myopic) policy in a straightforward way.

3.3 The multiperiod problem with setup cost

We now consider the case when the ordering cost includes a fixed cost of k in each period. Thus the total cost incurred in period t is $k\delta(x_t - y_t) + c_t(x_t - y_t) + \Psi_t(x_t, d_t)$, where $\delta(x) = 1$ if $x > 0$ and 0 otherwise. In this case the dynamic programming recursion takes the following form.

$$V_t(y_t) = -c_t y_t + \min \big\{ k\delta(x_t - y_t) + \Lambda_t(x_t) : \ x_t \geq y_t \big\} \ \text{ for } t = 0, \dots, T, \tag{3.17}$$

where $\Lambda_t(x_t)$ are defined in the same way as in (3.12), and a policy $\bar{x}_t = \bar{x}_t(y_t)$ is optimal iff

$$\bar{x}_t \in \arg\min_{x_t \geq y_t} \big\{ k\delta(x_t - y_t) + \Lambda_t(x_t) \big\}, \ \ t = 0, \dots, T. \tag{3.18}$$

Theorem 5 *Let for all* $t = 0, \dots, T$,

$$x^*_t \in \arg\min_{x \in \mathbb{R}} \Lambda_t(x) \ \ \text{and} \ \ r^*_t := \max\{x : \ \Lambda_t(x) = k + \Lambda_t(x^*_t)\}. \tag{3.19}$$

Then the following policy is optimal for the dynamic program (3.17):

$$\bar{x}_t(y_t) := \begin{cases} x^*_t, & \text{if } y_t \leq r^*_t, \\ y_t, & \text{otherwise.} \end{cases} \tag{3.20}$$

If $k = 0$ then the above policy is a base-stock policy with base-stock level x^*_t, and if $k > 0$ the above policy is a (s, S) policy with reorder point $s = r^*_t$ and replenishment level $S = x^*_t$.

Theorem 5 follows from classical results if we can verify that, for all t, the functions $V_t(y_t)$ and $\Lambda_t(x_t)$ are k-convex in y_t and x_t respectively (cf., [19, section 9.5.2]). We shall need the following result.

Lemma 2 *If $f(x, d)$ is k-convex in x for all d (with $k \geq 0$) and ρ is a coherent risk measure, then $g(x) = \rho[f(x, D)]$ is k-convex.*

Proof. By the definition of k-convexity we have that $f(x, d)$ is k-convex in x iff for all $u, v > 0$ the following inequality holds:

$$\left(1 + \frac{u}{v}\right) f(x, d) - \frac{u}{v} f(x - v, d) \leq f(x + u, d) + k.$$

By the monotonicity and translation equivariance property of ρ it follows

$$\rho\left[\left(1 + \frac{u}{v}\right) f(x, D) - \frac{u}{v} f(x - v, D)\right] \leq \rho[f(x + u, D)] + k. \tag{3.21}$$

By convexity and positive homogeneity of ρ we have

$$\rho\left[\left(1 + \frac{u}{v}\right) f(x, D) - \frac{u}{v} f(x - v, D)\right] \geq \left(1 + \frac{u}{v}\right) \rho\left[f(x, D)\right] - \frac{u}{v} \rho\left[f(x - v, D)\right].$$

Together with (3.21) this implies

$$\left(1 + \frac{u}{v}\right)\rho[f(x,d)] - \frac{u}{v}\rho[f(x-v,d)] \le \rho[f(x+u,d)] + k.$$

Thus $g(x) = \rho[f(x,D)]$ is k-convex. ■

Proof of Theorem 5. We only need to verify that, for all t, the functions $V_t(y_t)$ and $\Lambda_t(x_t)$ are k-convex in y_t and x_t respectively. By the nondecreasing convexity property of ρ, $\Lambda_T(x_T)$ is convex in x_T, and hence $V_T(y_T)$ is k-convex in y_T. Now suppose $V_t(y_t)$ and $\Lambda_t(x_t)$ are k-convex in y_t and x_t, respectively. Then $V_t(x_t - d_t)$ is k-convex in x_t since k-convexity is not affected by parallel shifts. Invoking Lemma 2 we have that $\Lambda_{t-1}(x_{t-1})$ is k-convex in x_{t-1}. Consequently $V_{t-1}(y_{t-1})$ is k-convex in y_{t-1}. ■

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