ISyE 6669: Deterministic Optimization

The Optimization Process: An example of portfolio optimization

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1 Introduction

Optimization can be roughly defined as a quantitative approach for decision making, where we seek to determine a "best" decision from a "set" of possible decisions. We need to specify what we mean by decisions, the set of possible decisions, and the criteria according to which one decision is better than another. Let us illustrate these notions through a concrete decision problem.

Example: Portfolio Optimization
Suppose we are considering investing \$1000.00 in three non-dividend paying stocks, IBM (IBM), Walmart (WMT), and Southern Electric (SEHI), for a one-month period. This means we will use the \$1000.00 to buy shares of the three stocks at the current market prices, hold these for one month, and sell the shares off at the prevailing market prices at the end of the month. As a rational investor, we hope to make some profit out of this endeavor, i.e., the return on our investment should be positive. Suppose we bought a stock at \$ p/share in the beginning of the month, and sold it off at \$$s$/share at the end of the month. Then the one-month return on a share of the stock is $\frac{s-p}{p}$ \$/\$. Since the stock prices are quite uncertain, so is the end-of-month return on our investment. Our goal is to invest in such a way that the *expected* end-of-month return is at least \$50.00 or 5%. Furthermore, we want to make sure that the "risk" of not achieving our desired return is minimum.

In the above example, our decision constitutes determining how much of the \$1000.00 to invest in each of the three stocks; the set of possible decisions are the investment amounts that will provide an expected return of \$50.00; and the criteria for comparing two decisions is the risk of not achieving the desired return.

An optimization approach to the above decision problem consists of the following steps:

- Build a mathematical model of the decision problem.
- Analyze available quantitative data to use in the mathematical model.
- Use a numerical method to solve the mathematical model.
- Infer the actual decision from the solution to the mathematical model.

In this class you will learn important concepts and tools for each of the above steps in the context of optimization of various decisions. In the following, we briefly describe these steps in the context of our portfolio optimization example.

Note that fortunately the actual decision problem in our example is quite well formulated. This is already great progress. Most often than not, in practical circumstances, the right question to ask is far from being well-posed.

2 Modeling

This step involves approximating the underlying decision problem using mathematical expressions suitable for quantitative analysis. A key trade-off in this step is deciding how much of the detail of the actual problem to consider while maintaining numerical tractability of the mathematical model. For example, should we consider the fact that investing in more stocks requires us to pay more transaction costs, or are these costs negligible? Should we explicitly consider the fact that shares can only be traded in lots of certain sizes, or should we neglect this restriction? The modeling step often requires us to make simplifying assumptions regarding the actual problem, either because some of the problem characteristics are not well-defined mathematically, or because we wish to develop a model that can actually be solved. One needs to exercise great caution in these assumptions and not loose sight of the true underlying problem.

The three key components of an optimization model are:

(a) **The decision variables** representing the actual decisions we are seeking. In our portfolio optimization example, these represent the investment levels in each of the three stocks.

(b) **The constraints** that specify the restrictions and interactions between the decision variables, thus defining the set of possible decisions. In our example, one constraint corresponds to the restriction that our investment should provide an expected return of $50.

(c) **The objective function** quantifies the criteria for choosing the best decision. The values of the decision variables that maximize or minimize the objective function is the "best" among the set of decision values defined by the constraints in the optimization model. In our example, the objective function is the risk level of the investment.

The standard statement of an optimization model has the form:

$$\begin{array}{rl} \text{min or max} & \text{objective function} \\ \text{s.t.} & \text{constraints} \end{array}$$

where “s.t.” stands for “subject to.” Let us now develop an optimization model for our example problem.

Example (contd.):
It will be convenient to adopt an indexing scheme on the set of stocks. Let us use the index i to denote a particular stock, where $i = 1, 2$, and 3 correspond to the stocks IBM, WMT, and SEHI, respectively.

Our decision variables are x_i, $i = 1, 2, 3$, denoting the dollars invested in stock i. Since we have a total of \$ 1000.00 to invest, thus x_i's must satisfy:

$$\sum_{i=1}^{3} x_i \leq 1000.00.$$

Let us make the following assumptions:

(i) We can trade any continuum of shares.

(ii) No short-selling is allowed.

These assumptions restrict the variables x_i to take on non-negative real values, i.e.

$$x_i \geq 0 \quad i = 1, 2, 3.$$

Note that the end-of-month return of a stock i is uncertain, and therefore, so is the return on our investment. Let us denote by $\tilde{r}_i$ the random variable corresponding to the monthly return (increase in the stock price) per dollar for stock i. Then the return (or profit) on x_i dollars invested in stock i is $\tilde{r}_i x_i$, and the total (random) return on our investment is $\sum_{i=1}^{3} \tilde{r}_i x_i$. Note that we have assumed that:

(iii) There are no transaction costs.

The expected return on our investment is then $\mathbb{E}[\sum_{i=1}^{3} \tilde{r}_i x_i] = \sum_{i=1}^{3} \bar{r}_i x_i$, where $\bar{r}_i$ is the expected value of the $\tilde{r}_i$. Since we want to have an expected return of at least \$ 50.00, thus x_i's have to be such that:

$$\sum_{i=1}^{3} \bar{r}_i x_i \geq 50.00.$$

Now we need to quantify the notion of “risk” in our investment. Markowitz, in his Nobel prize winning work, showed that a rational investor's notion of minimizing risk can be closely approximated by minimizing the variance of the return of the investment portfolio. This variance is given by:

$$\begin{aligned}
\mathrm{Var}[\sum_{i=1}^{3} \tilde{r}_i x_i] &= \mathbb{E}\left[\left(\sum_{i=1}^{3} \tilde{r}_i x_i - \sum_{i=1}^{3} \bar{r}_i x_i\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{3} (\tilde{r}_i - \bar{r}_i) x_i\right)\left(\sum_{j=1}^{3} (\tilde{r}_j - \bar{r}_j) x_j\right)\right] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \mathbb{E}[(\tilde{r}_i - \bar{r}_i)(\tilde{r}_j - \bar{r}_j)] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \sigma_{ij},
\end{aligned}$$

where σ_{ij} is the covariance of the return of stock i with stock j.

We can now state the following optimization model for our example problem:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \sigma_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{3} x_i \leq 1000.00, \\
& \sum_{i=1}^{3} \bar{r}_i x_i \geq 50.00, \\
& x_i \geq 0 \quad i = 1, 2, 3.
\end{aligned}$$

For a complete description of the model, we need to specify the values of $\bar{r}_i$ and σ_{ij}. Although we do not know the exact distribution of the random return, we can obtain some statistical inference regarding it through analysis of historical data. This will be discussed in Section 3.

Using matrices and vectors, we can compactly represent the above optimization model as follows:

$$\begin{aligned}
\min \quad & x^T Q x \\
\text{s.t.} \quad & e^T x \leq 1000.00 \\
& \bar{r}^T x \geq 50.00 \\
& x \geq 0,
\end{aligned}$$

where x is the decision vector of size n (n is the number of stocks, $n = 3$ in our example), e is an n-vector of ones, $\bar{r}$ is the n-vector of expected returns of the stocks, and Q is the $n \times n$ covariance matrix (whose i-jth element $Q_{ij} = \sigma_{ij}$).

Note that all equations and inequalities are defined component-wise.

Exercise 1: *Verify that the above model with vector notation is equivalent to the original portfolio optimization model.*

3 Data Collection

Before attempting to solve an optimization model numerically, we need to quantify all components of the model except for the decisions variables (whose values are to be determined through solving the model). These are known as the model **parameters**. For our example problem, the model parameters are the investment budget of \$1000.00, the desired return \$50.00, the expected returns for the stocks $\bar{r}_i$, and the covariances σ_{ij}. In this case, the parameters $\bar{r}_i$ and σ_{ij} need to be quantified.

Reliable quantification of model parameters can only be done through careful observation and analysis of the actual data processes underlying the decision problem. This is one of the most crucial steps in the optimization approach. One of the key reasons why optimization models can fail to provide a useful decision (or even provide a wrong decision) in practice is improper data specification. This is the *garbage in garbage out* (GIGO) principle.

Specification of model parameters again involves approximations and assumptions. *Descriptive* models are used to describe the actual data processes, and provide numerical estimates[1] For example, the approximation of risk via variance (in Markowitz's model) is a descriptive model. Another common descriptive model is the probability distribution used to describe uncertain data processes. Once a descriptive model of the data process is decided upon, we need to collect numerical data to calibrate this model and obtain the desired information. Let us illustrate this process using our example.

Example (contd.):
As mentioned before, we need to determine the expected values and the covariances for the end-of-month returns of the three stocks. We will assume that

(iv) The end-of-month stock prices have a stationary probability distribution.

Under this model of the stock price process, the end-of-month returns on the stocks, defined as the ratio on the change in stock price to the last month's stock price, has also a stationary distribution. We can then estimate the expected values and covariances of this distribution through statistical analysis of historical data. Table 1 presents the closing prices of the three stocks in the last trading day of the month from November 2000 to November 2001. Using this data, the one-month return on the stocks are calculated in Table 2. This table also presents the average one-month return for each of the three stocks. The

[1]Note, that an optimization model is *prescriptive* and involves a variety of descriptive models in its description.

covariances among the returns of the three stocks are presented in Table 3. We can now use these values for $\bar{r}_i$ and σ_{ij} in our model.

Note that the analysis used above is quite naive. We have made a very simplistic assumption of the stock price distribution, no temporal effects are considered here. Furthermore, the expectation and covariances are estimated based on a very small sample. In practice, the data collection and analysis process would involve very sophisticated statistical and time series models to obtain reliable estimates of the means and covariances.

Month	IBM	WMT	SEHI
November-00	93.043	51.826	1.063
December-00	84.585	52.823	0.938
January-01	111.453	56.477	1.000
February-01	99.525	49.805	0.938
March-01	95.819	50.287	1.438
April-01	114.708	51.521	1.700
May-01	111.515	51.531	2.540
June-01	113.211	48.664	2.390
July-01	104.942	55.744	3.120
August-01	99.827	47.916	2.980
September-01	91.607	49.438	1.900
October-01	107.937	51.336	1.750
November-01	115.590	55.081	1.800

Table 1: Stock price data ($)

4 Solving the Model

Once the model has been established and all parameters quantified, we are now ready to solve the model to obtain the values of the decisions variables that are consistent with the model constraints and optimize the specified objective function.

Rarely do optimization models of practical decision problems lend themselves to analytical solutions. These models have to be solved using iterative schemes (or algorithms) with the help of a computer. Depending on the *structure* (eg. linear of non-linear) of the objective function, constraints, and the domain of the decision variables, a wide variety of optimization algorithms have been developed. The most widely used optimization algorithms have been implemented in robust software packages (like GAMS and CPLEX) and can be used to solve very large models. Unfortunately, software implementations are not available for all possible model structures. In that case, one can either approximate the complicated model using the more common structures, or implement special

Month	IBM	WMT	SEHI
December-00	-0.091	0.019	-0.118
January-01	0.318	0.069	0.067
February-01	-0.107	-0.118	-0.063
March-01	-0.037	0.010	0.533
April-01	0.197	0.025	0.183
May-01	-0.028	0.000	0.494
June-01	0.015	-0.056	-0.059
July-01	-0.073	0.145	0.305
August-01	-0.049	-0.140	-0.045
September-01	-0.082	0.032	-0.362
October-01	0.178	0.038	-0.079
November-01	0.071	0.073	0.029
Average	0.026	0.008	0.074

Table 2: Stock Returns ($ / $)

	IBM	WMT	SEHI
IBM	0.017087987	0.003298885	0.001224849
WMT	0.003298885	0.005900944	0.004488271
SEHI	0.001224849	0.004488271	0.063000818

Table 3: The Covariance Matrix

purpose algorithms. Thus, one needs to knowledge of the various model structures and the algorithms and software available for these, as well as skills to modify and develop new optimization algorithms.

In this class, we shall learn about: the classification of optimization models based upon structure, a wide variety of optimization algorithms and software, practical computational issues in using existing software for solving large problems, as well as modifying existing optimization algorithms to solve problems with special structures.

Example (contd.):
Our portfolio optimization model involves minimizing a quadratic objective function subject to linear constraints. Fortunately algorithms and software for this class of optimization problems are widely available. We used the GAMS optimization system to solve this model, and for the data of Section 3, obtained the following solution:

$$x_1 = 497.669, \quad x_2 = 0.00, \quad x_3 = 502.331.$$

The return on the investment is \$50.00 and the variance of the return is 20742.077[2].

[2] The standard deviation is 144.02. Thus if the end-of-month wealth is assumed to be

5 Solution Analysis

This step involves inferring the actual decisions from the solution to the optimization model of the decision problem. Note that the optimization process involves various approximations and assumptions at modeling, data collection, and solution stage. Therefore, it is crucial to check the solution for robustness and sensitivity to the underlying approximations. This step can be quite subjective and requires a clear understanding of the real problem and its surrounding circumstances.

Example (contd.):
From the optimization problem solution, our optimal decision is to invest \$497.669 in IBM stocks, and \$502.331 in SEHI stocks.

Suppose we made our investment on the last day of November 2001. The closing prices on the last day of November were:

$$\text{IBM} := \$115.59, \text{ WMT} := \$55.081, \text{ and SEHI} := \$1.8.$$

Thus, the optimal decision is to buy (497.669/115.59 =) 4.305 shares of IBM, and 279.073 shares of SEHI. Unfortunately, shares can only be traded in whole numbers, so we can consider buying either 4 or 5 IBM shares, and 279 or 280 SEHI shares.

Exercise 2: *Consider each of the four possible combinations of the number of shares of IBM and SEHI stocks, i.e. (4, 279), (4, 280), (5, 279), and (5, 280).*

1. *For each of the four investment possibilities, compute the expected return and variance based on the data given in Section 3. Comment on these values in relation to the expected return and variance values obtained from the optimization model. Choose one investment strategy from the four, and outline your reasoning.*

2. *Obtain the closing prices for the three stocks on December 31, 2001 (from* `http://finance.yahoo.com` *for example). What is the true return of the investment strategy chosen in part 1?*

6 Concluding Remarks

In this handout, we have tried to illustrate the steps in an optimization approach to solve a decision problem. It is important to realize that the steps outlined here is not a straight-cut recipe. Each of the steps of modeling, data analysis, solution, and solution analysis is very involved and may need to be revisited and revised as more is learnt about the problem.

Normal distributed about the mean \$1050.00, the 3σ range of the end-of-period wealth is between \$617.94 and \$1482.06.