2.(a)

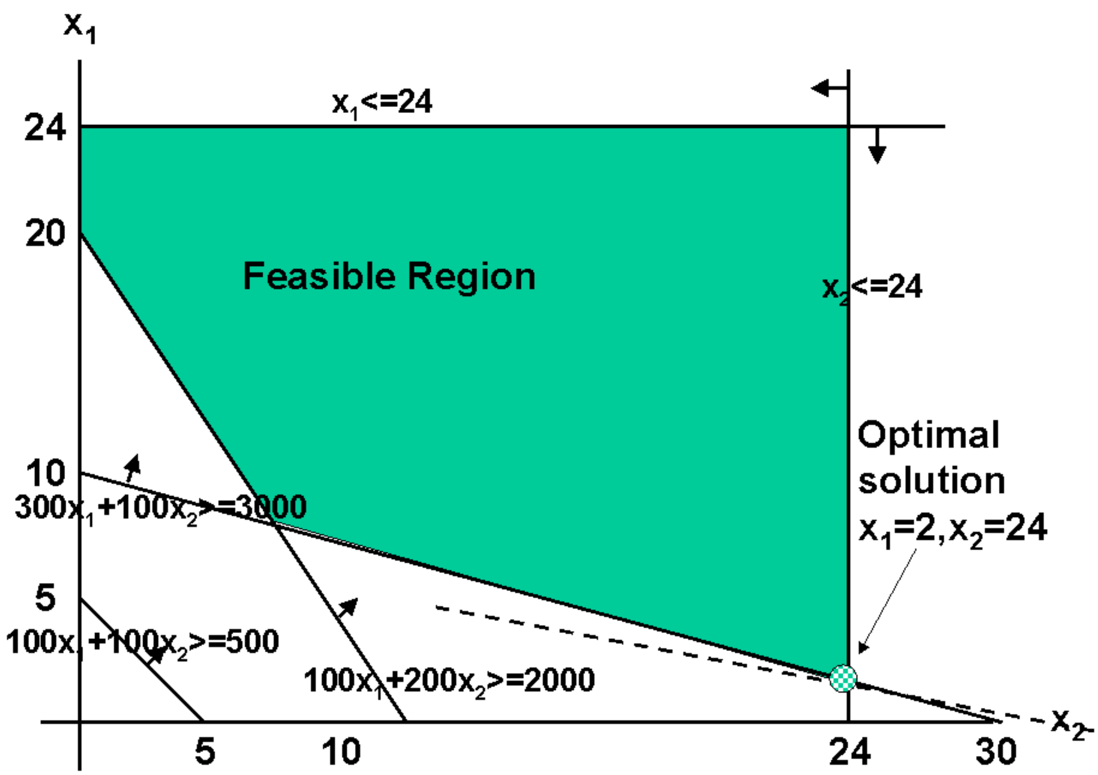
Variables: x_j = hours per day that machine j runs

Minimize $4000\ x_1 + 1000\ x_2$

Subject to

$300\ x_1 + 100\ x_2 \geq 3000$	(chemical A minimum)
$100\ x_1 + 100\ x_2 \geq 500$	(chemical B minimum)
$100\ x_1 + 200\ x_2 \geq 2000$	(chemical C minimum)
$x_1 \leq 24$	(at most 24 hours in a day)
$x_2 \leq 24$	(at most 24 hours in a day)
$x_1, x_2 \geq 0$	(nonnegativity)

(b)



(c)

Before solving the problem, we have to put it into standard form. That means we need to add one slack variable for each ≤ constraint, and one excess (surplus) variable for each ≥ constraint.

Minimize $4000\ x_1 + 1000\ x_2$

Subject to
$$300\,x_1 + 100\,x_2 - e_1 = 3000$$
$$100\,x_1 + 100\,x_2 - e_2 = 500$$
$$100\,x_1 + 200\,x_2 - e_3 = 2000$$
$$x_1 + s_4 = 24$$
$$x_2 + s_5 = 24$$
$$x_1,\ x_2,\ e_1,\ e_2,\ e_3,\ s_4,\ s_5 \geq 0$$

b will be the same at every iteration:

$$b = \begin{bmatrix} 3000 \\ 500 \\ 2000 \\ 24 \\ 24 \end{bmatrix}$$

Since we're starting with the basis $\{e_1, e_2, e_3, x_1, x_3\}$,

$$x_B = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ x_1 \\ x_2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 & 300 & 100 \\ 0 & -1 & 0 & 100 & 100 \\ 0 & 0 & -1 & 100 & 200 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad c_B = [\ 0\ \ 0\ \ 0\ \ 4000\ \ 1000\ ]$$

ITERATION 1

First, we need to find B^{-1}:

$$\left[\begin{array}{ccccc|ccccc} -1 & 0 & 0 & 300 & 100 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 100 & 100 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 100 & 200 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & -300 & -100 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 100 & 100 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 100 & 200 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & -300 & -100 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -100 & -100 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 100 & 200 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & -300 & -100 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -100 & -100 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -100 & -200 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & -100 & -1 & 0 & 0 & 300 & 0 \\ 0 & 1 & 0 & 0 & -100 & 0 & -1 & 0 & 100 & 0 \\ 0 & 0 & 1 & 0 & -200 & 0 & 0 & -1 & 100 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 300 & 100 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 100 & 100 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 100 & 200 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

So,

$$B^{-1} = \begin{bmatrix} -1 & 0 & 0 & 300 & 100 \\ 0 & -1 & 0 & 100 & 100 \\ 0 & 0 & -1 & 100 & 200 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Reduced costs for nonbasic variables:

It will be easier to calculate $c_B B^{-1}$ once, instead of repeating the calculation for every nonbasic variable.

$$c_B B^{-1} = [\,0\ \ 0\ \ 0\ \ 4000\ \ 1000\,] \begin{bmatrix} -1 & 0 & 0 & 300 & 100 \\ 0 & -1 & 0 & 100 & 100 \\ 0 & 0 & -1 & 100 & 200 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = [\,0\ \ 0\ \ 0\ \ 4000\ \ 1000\,]$$

s_4 : $c_{s4} - c_B B^{-1} a_{s4} = 0 - 4000 = -4000$
s_5 : $c_{s5} - c_B B^{-1} a_{s5} = 0 - 1000 = -1000$

Since we're minimizing, variables with negative reduced cost will help the objective. So, we can choose either one to be our entering variable. Let's choose s_4.

To find the variable that leaves the basis, we need to find the current basic variable that reaches zero first.

$$x_B = B^{-1}b = \begin{bmatrix} -1 & 0 & 0 & 300 & 100 \\ 0 & -1 & 0 & 100 & 100 \\ 0 & 0 & -1 & 100 & 200 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3000 \\ 500 \\ 2000 \\ 24 \\ 24 \end{bmatrix} = \begin{bmatrix} 6600 \\ 4300 \\ 5200 \\ 24 \\ 24 \end{bmatrix}$$

We also need to find the rate of change of each basic variable if s_4 enters.

$$\text{rate} = B^{-1}a_{s4} = -\begin{bmatrix} -1 & 0 & 0 & 300 & 100 \\ 0 & -1 & 0 & 100 & 100 \\ 0 & 0 & -1 & 100 & 200 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -300 \\ -100 \\ -100 \\ -1 \\ 0 \end{bmatrix}$$

So, how far can s_4 enter?

Limit from $e_1 = 6600/300 = 22$
Limit from $e_2 = 4300/100 = 43$

Limit from e_3 = 5200/100 = 52
Limit from x_1 = 24/1 = 24
Limit from x_2 = 24/0 = infinite

So the smallest limit is from e_1. Therefore, when s_4 enters the basis, e_1 leaves.

ITERATION 2

Now, our new set of basic variables is $\{s_4, e_2, e_3, x_1, x_2\}$.

The new B^{-1} is just E times the old B^{-1}, where E is the identity matrix, with the first column (because the first variable left the basis) replaced by a special column based on variable s_4 (because s_4 entered the basis).

From last iteration, we know that

$$B^{-1}a_{s4} = \begin{bmatrix} 300 \\ 100 \\ 100 \\ 1 \\ 0 \end{bmatrix}$$

So to get the special column, we divide each entry but the first by -300 (because the leaving variable was the first one) and the first entry is just 1/(300).

$$E = \begin{bmatrix} 1/300 & 0 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 & 0 \\ -1/3 & 0 & 1 & 0 & 0 \\ -1/300 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The new B^{-1} is just E times the old B^{-1}:

$$B^{-1} = \begin{bmatrix} 1/300 & 0 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 & 0 \\ -1/3 & 0 & 1 & 0 & 0 \\ -1/300 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 300 & 100 \\ 0 & -1 & 0 & 100 & 100 \\ 0 & 0 & -1 & 100 & 200 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/300 & 0 & 0 & 1 & 1/3 \\ 1/3 & -1 & 0 & 0 & 66\,2/3 \\ 1/3 & 0 & -1 & 0 & 166\,2/3 \\ 1/300 & 0 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(Whew! Wasn't that much easier than inverting B again from scratch?)

Reduced costs for nonbasic variables:

It will be easier to calculate $c_B B^{-1}$ once, instead of repeating the calculation for every nonbasic variable.

$$c_B B^{-1} = [\,0\;\;0\;\;0\;\;4000\;\;1000\,] \begin{bmatrix} {}^{-1}/_{300} & 0 & 0 & 1 & {}^{1}/_{3} \\ {}^{1}/_{3} & -1 & 0 & 0 & 66^{2}/_{3} \\ {}^{1}/_{3} & 0 & -1 & 0 & 166^{2}/_{3} \\ {}^{1}/_{300} & 0 & 0 & 0 & {}^{-1}/_{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = [\,13^{1}/_{3}\;\;0\;\;0\;\;0\;\;-333^{1}/_{3}\,]$$

e_1 : $c_{e1} - c_B B^{-1} a_{e1} = 0 - -13^{1}/_{3} = 13^{1}/_{3}$

s_5 : $c_{s5} - c_B B^{-1} a_{s5} = 0 - -333^{1}/_{3} = 333^{1}/_{3}$

There are no variables with negative reduced costs. Therefore, this is the optimal basis. The optimal solution $x_B = B^{-1}b$.

$$x_B = B^{-1}b = \begin{bmatrix} {}^{-1}/_{300} & 0 & 0 & 1 & {}^{1}/_{3} \\ {}^{1}/_{3} & -1 & 0 & 0 & 66^{2}/_{3} \\ {}^{1}/_{3} & 0 & -1 & 0 & 166^{2}/_{3} \\ {}^{1}/_{300} & 0 & 0 & 0 & {}^{-1}/_{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3000 \\ 500 \\ 2000 \\ 24 \\ 24 \end{bmatrix} = \begin{bmatrix} 22 \\ 2100 \\ 3000 \\ 2 \\ 24 \end{bmatrix}$$

So, $s_4 = 22$, $e_2 = 2100$, $e_3 = 3000$, $x_1 = 2$, and $x_2 = 24$. Our two nonbasic variables, e_1 and s_5, are both zero.

The objective value is $c_B x_B$ = \$8000 + \$24000 = \$32000.