

1. Every year, the United States Congress must approve a budget for the country. In order to be approved, the budget must get a majority of the votes in the Senate, a majority of votes in the House, and the signature of the President.

Most of the budget is easy to agree on – everyone supports things like basic operating expenses for the government, diplomats in foreign countries, and the armed forces; emergency relief funds in case of earthquakes, hurricanes, etc.; maintenance of the interstate highway system; and so on.

Other aspects of the budget, however, are less easy to agree on. For example, should there be a tax cut? Should the government spend money on defense research? Should they use some of the budget surplus to solidify social security? To pay off our debt to the United Nations? To give aid to farmers unable to make a living due to low food prices?

Because of monetary restrictions, the budget can have money allocated for at most 4 of the 5 issues.

One Senator has asked his staff to categorize all of Congress (Senators and members of the House), as well as the President, by their views on the five controversial issues. The Senator’s staff has compiled the following table.

	Issues					Number of Votes
	Tax Cut	Defense Research	Social Security	United Nations	Farm Aid	
<b>Senate (Total = 100)</b>						
Senate Group 1	Y	Y	N	N	Y	32
Senate Group 2	Y	Y	D	N	D	16
Senate Group 3	N	N	Y	Y	Y	30
Senate Group 4	D	Y	N	D	Y	22
<b>House of Representatives (Total = 435)</b>						
House Group 1	Y	Y	N	N	Y	140
House Group 2	Y	Y	D	N	D	75
House Group 3	N	N	Y	Y	Y	155
House Group 4	D	Y	N	D	Y	65
<b>President (Total = 1)</b>						
President	N	D	Y	Y	Y	1

**Table. Stance on Budget Issues: “Y” = yes, “N” = no, “D” = don’t care**

In order for a particular group to vote for a budget proposal, the proposal must allocate money in agreement with the group's views on *more than half* of the issues that the group cares about. For example, the Senate Group 2 only cares about 3 of the issues. Therefore, any budget proposal that agrees with their views on at least two out of the three issues they care about (allocating money for a Tax Cut and Defense Research, but not for the United Nations) will get their 16 votes in the Senate.

In order to be approved, a budget must get a majority of the votes (at least 51 out of 100) in the Senate, a majority of the votes (at least 218 out of 435) in the House of Representatives, and the approval of the President.

Formulate an integer program that the Senator could use to devise a budget proposal that would be approved by all three (Senate, House, and President) while funding as many budgetary issues as possible.

**Variables:**  $x_i = 1$  if issue  $i$  is allocated money, and 0 if not  
 $s_j = 1$  if Senate group  $j$  supports the budget, and 0 if not  
 $h_k = 1$  if House group  $k$  supports the budget, and 0 if not  
 $p = 1$  if the President supports the budget, and 0 if not

**Maximize**  $x_T + x_D + x_S + x_U + x_F$

**Subject to**  $x_T + x_D + (1 - x_S) + (1 - x_U) + x_F \geq 3 s_1$   
 $x_T + x_D + (1 - x_U) \geq 2 s_2$   
 $(1 - x_T) + (1 - x_D) + x_S + x_U + x_F \geq 3 s_3$   
 $x_D + (1 - x_S) + x_F \geq 2 s_4$

$x_T + x_D + (1 - x_S) + (1 - x_U) + x_F \geq 3 h_1$   
 $x_T + x_D + (1 - x_U) \geq 2 h_2$   
 $(1 - x_T) + (1 - x_D) + x_S + x_U + x_F \geq 3 h_3$   
 $x_D + (1 - x_S) + x_F \geq 2 h_4$

$(1 - x_T) + x_S + x_U + x_F \geq 3 p$

$32 s_1 + 16 s_2 + 30 s_3 + 22 s_4 \geq 51$

$140 h_1 + 75 h_2 + 155 h_3 + 65 h_4 \geq 218$

$p \geq 1$

**all  $x_i, s_j, h_k, p$  are binary**

2. A soccer team is looking to sign free agents during the off-season. They have just hired a new coach, who is trying to decide whether the team should become an offense-oriented team or a defense-oriented team. If he decides to make the team offense-oriented, then the team should sign at least three times as many strikers as fullbacks. If he decides to make the team defense-oriented, then the team should

sign at least two times as many fullbacks as strikers. The team can sign a total of up to 6 free agents.

Let  $x_s$  be the number of strikers the team signs, and let  $x_f$  be the number of fullbacks the team signs. Both  $x_s$  and  $x_f$  are integer variables. Let  $y$  be a binary variable that is 1 if the team is offense-oriented and 0 if it is defense-oriented.

Pick the *strongest* set of constraints that accurately describe all of the team's restrictions.

$$\begin{aligned} \text{(i)} \quad x_s &\geq 3x_f - 24(1 - y) \\ x_f &\geq 2x_s - 16y \\ x_s + x_f &\leq 6 \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad x_s &\geq 3x_f - 24(1 - y) \\ x_f &\geq 2x_s - 16y \\ x_s + x_f &\leq 6 \\ x_f &\leq 1 + 5(1 - y) \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad x_s &\geq 3x_f - 18(1 - y) \\ x_f &\geq 2x_s - 12y \\ x_s + x_f &\leq 6 \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad x_s &\geq 3x_f - 18(1 - y) \\ x_f &\geq 2x_s - 12y \\ x_s + x_f &\leq 6 \\ x_f &\leq 1 + 5(1 - y) \end{aligned}$$

**All four sets of constraints are “correct” in that they accurately represent the restrictions. However, the constraints in choice (i) are not as strong as the constraints in choice (iii); using 18 and 12 instead of 24 and 16 gives a tighter feasible set without eliminating any integer solutions. In fact, 18 and 12 are the smallest numbers that we could use in these constraints:**

$$x_s \geq 3x_f - M(1 - y)$$

**If  $y = 0$ , then we could possibly have 6 fullbacks and 0 strikers, so  $x_s \geq 3x_f - M$  becomes  $0 \geq 18 - M$ , so the best  $M$  would be 18.**

$$x_f \geq 2x_s - My$$

**If  $y = 1$ , then we could possibly have 0 fullbacks and 6 strikers, so  $x_f \geq 2x_s - M$  becomes  $0 \geq 12 - M$ , so the best  $M$  would be 12.**

**However, consider the restriction “if the team is offense-oriented, they should sign at least three times as many strikers as fullbacks”. Here’s a table representing that constraint:**

Number of Fullbacks $x_f$	$3 x_f$	Minimum number of strikers $x_s$
0	0	0
1	3	3
2	6	6

So, we can't sign 2 fullbacks on an offense-oriented team, because then we would need 6 strikers for a total of 8, and we're only allowed to sign 6.

However, if we were to solve the linear programming relaxation, we could have something like  $x_f = 1.5$ ,  $x_s = 4.5$ ; we should sign 1.5 fullbacks and 4.5 strikers on an offense-oriented team.

In order to eliminate this fractional possibility, we can add the constraint

$$x_f \leq 1 + 5(1 - y)$$

If the team is not offense-oriented ( $y = 0$ ), the constraint says that we cannot sign more than 6 fullbacks, which must be true anyway since we can only sign up to 6 players of any sort.

On the other hand, if the team is offense-oriented ( $y = 1$ ), then the constraint says that we can only hire a maximum of 1 fullback (as we saw from the table), and eliminates the fractional solution mentioned above.

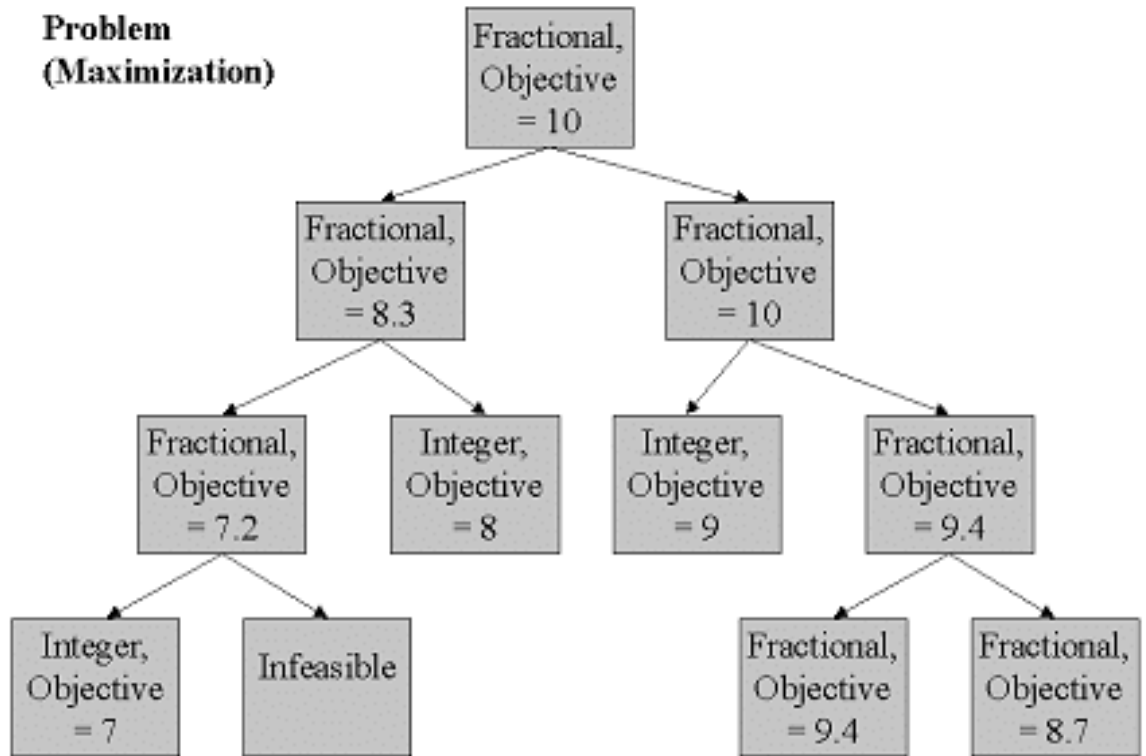
Therefore, adding this constraint gives us a tighter formulation, and so choice (iv)

$$\begin{aligned} x_s &\geq 3x_f - 18(1 - y) \\ x_f &\geq 2x_s - 12y \\ x_s + x_f &\leq 6 \\ x_f &\leq 1 + 5(1 - y) \end{aligned}$$

is the best set of constraints.

3. In the following branch-and-bound tree, answer the following questions:
  - (a) Which nodes do you still need to branch from? Why?
  - (b) Which nodes do you not need to branch from? Why?
  - (c) What is the gap between the best solution and the best bound found so far?

- (d) In what order were the three integer solutions found in the branch-and-bound process? Explain why your answer must be correct.



- (a), (b) There are 4 nodes from which we have already branched. Since we have already branched from these nodes, we need not branch from them anymore.

Of the remaining 5 nodes, three of them are integer and one is infeasible. We need not branch from any of these four nodes either.

The final remaining nodes are fractional, with objectives 9.4 and 8.7. Since the first objective (9.4) is better than any of the integer nodes' objectives, we do need to branch from this node, since there may be an integer solution with an objective as high as 9.4, and the current best one only has an objective of 9. On the other hand, since our integer solution with objective 9 is better than the fractional solution with objective 8.7, we do not need to branch from the second fractional solution.

- (c) The current best solution is the integer node with an objective of 9. The other two integer solutions have worse objectives (7 and 8). The only remaining node to branch from has objective 9.4, so we know that no solution we find from branching on it can be better than 9.4.

So, the best solution is 9 and there can not be any solution better than 9.4. Therefore, the gap is  $9.4 - 9 = 0.4$ .

- (d) This one takes a little bit of optimization detective work, but you've learned enough to be able to figure out the answer.

First of all, suppose we had found the integer node with objective 9 before branching from the fractional node with objective 8.3. Since the integer node with objective 9 has a better objective value than the fractional node with objective 8.3, we wouldn't have needed to branch on the fractional node with objective 8.3. Since we did indeed branch on it, we must've done so before we found the integer objective 9 node. Therefore, we must have found the integer objective 8 node before the integer objective 9 node.

Now, suppose we had found the integer objective 8 node before branching from the fractional objective 7.2 node. If this was the case, then we would not have needed to ever branch from the fractional objective 7.2 node, since the integer objective 8 node is better than anything we might find branching from the fractional objective 7.2 node. Therefore, we must have branched from the fractional objective 7.2 node before discovering the integer objective 8 node. Therefore, we must have found the integer objective 7 node before the integer objective 8 node.

Putting the two pieces together indicates that we must have found the integer objective 7 node first, then the integer objective 8 node, and finally the integer objective 9 node.

If you got this one, you're on your way to becoming a true optimization Sherlock Holmes!