FREE MATERIAL DESIGN VIA SEMIDEFINITE PROGRAMMING. THE MULTI-LOAD CASE WITH CONTACT CONDITIONS*

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Abstract. Free material design deals with the question of finding the stiffest structure with respect to one or more given loads which can be made when both the distribution of material as the material itself can be freely varied. The case of one single load has been discussed in several recent papers and an efficient numerical approach was presented in [7]. We attack here the multi-load situation (understood in the worst-case sense) which is of much more interest for applications but also significantly more challenging, both from the theoretical and the numerical point of view. After a series of transformation steps we reach a problem formulation for which we can prove existence of a solution; a suitable discretization leads to a semidefinite programming problem for which modern polynomial time algorithms of interior-point type are available. A number of numerical examples demonstrates the efficiency of our approach.

1. Introduction. One of the basic problems of structural engineering is to design the stiffest structure of a given volume, occupying some fixed domain $\Omega \subset \mathbb{R}^{dim}$ ($dim = 2, 3$) with boundary Γ, which is capable of carrying a given set of external loads. The desired optimal structure is considered to be a continuum elastic body and the design variables are the *material properties* which may vary from point to point. Thus the aim is to optimize not only the distribution of material but also the material properties themselves and we are looking for the ultimately best structure among all possible elastic continua, in a framework of what is now usually referred to as "free material design".

Optimization of structures is traditionally performed through the variation of sizing variables (e.g., thicknesses of bars in a truss) and shape variables (e.g., splines defining the boundary of a body). With the appearance of composites and other advanced man-made materials it has been natural to extend this variation to the material choice itself. The basic problem setting of "free material design" we will deal with goes back to the work of Bendsøe et al. [3] and Ringertz [9], where it was suggested to represent material properties as elements of the unrestricted set of positive semi-definite constitutive tensors with the trace of the stiffness tensor as a measure of resource ("weight"). In mathematical language this leads to an optimization problem with an objective function (stiffness) which is the result of an inner optimization. More precisely, one minimizes (with respect to material properties) the compliance (a certain global measure of the stiffness of the structure), where the compliance itself is the outcome of a lower optimization level (minimization of potential energy). The resulting minimax-problem looks rather complicated: in two (three) space dimensions, the design variables are the six (twenty-one) defining elements of the symmetric elasticity tensor and these variables are allowed to vary pointwise throughout the structure. The case of single-load design (SLD in short) was treated in [3], see also [10]. It is shown that one can analytically reduce the problem to one with only a single design variable at each point (in addition to the displacement vector), namely, the *trace* of the elasticity tensor. The elements of the optimal tensor itself are then fully recoverable from

*Supported by GIF-contract I0455-214.06/95 (ABT, AN, and JZ) and BMBF-project 03ZO7BAY (MK and JZ).

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the optimal trace and the related displacements. A finite element discretization of the above reduced problem leads to a mathematical programming formulation, which in form is identical to maximal stiffness optimization problems for trusses, and the very efficient interior-point based software developed for truss problems (see e.g. [1, 6, 7]) can be almost immediately used in this framework of material optimization. In [10] this computational approach to SLD is discussed in detail and a number of examples demonstrate its efficiency.

For most applications, however, the assumption of a single acting load is too restrictive and may lead to a structure which is highly unstable with respect to small load perturbations; hence one is interested in a structure which is stable with respect to a whole scenario of independent loads and which is the stiffest one in the worst-case sense. This multi-load feature complicates the situation substantially since it leads to a blow-up in the dimension and further, the above mentioned reduction process leads now to an integral over an eigenvalue problem which is hard to eliminate when discretizing for a numerical approach. All this excludes a direct transfer of the tools, which are successful in the SLD-case, to the multi-load situation. *Multi-load design* (in short: MLD) requires essentially new tools. Only some first steps in the direction of a theoretical treatment of the MLD can be found in literature [2]; reports on numerical approaches are not known to us. Our paper tries to fill this gap.

2. Problem formulation and existence theorem. We study the optimization of the design of a *continuum* structure that is loaded by multiple independent forces. In order to deal with the problem in a very general form, we consider the *distribution of the material in space* as well as the *material properties at each point* as design variables. The idea to treat the material itself as a function of the space variable goes back to the works [8,24] and has also been studied in various other context in [6,7,9]. This present text develops in this framework a theory for the MLD-case with additional contact conditions. We start from the infinite-dimensional problem setting, prove existence of a solution after a reformulation of the problem and, after discretization, reach a finite-dimensional formulation expressed as a *semidefinite program*, and as such accessible to modern numerical interior point methods.

For an easier understanding of the physical background we begin with a sketch of the single-load model. Let $\Omega \subset \mathbb{R}^{dim}$, $dim = 2,3$, be a bounded domain (the elastic body) with a Lipschitz boundary Γ. We use the standard notation $[H^1(\Omega)]^{dim}$ and $[H_0^1(\Omega)]^{dim}$ for Sobolev spaces of functions $v : \Omega \to \mathbb{R}^{dim}$. By $u(x) = (u_1(x), \ldots, u_{dim}(x))$ with $u \in [H^1(\Omega)]^{dim}$ (in short $u \in H^1(\Omega)$) we denote the *displacement vector* at point x of the body under load. Also

$$e_{ij}(u(x)) = \frac{1}{2}\left(\frac{\partial u_i(x)}{\partial x_j} + \frac{\partial u_j(x)}{\partial x_i}\right) \qquad \text{for } i,j = 1,\ldots,dim$$

denotes the *(small-)strain tensor*, and $\sigma_{ij}(x), i,j = 1,\ldots,dim$, the *stress tensor.* We assume that our system is governed by linear Hooke's law, i.e., the stress is a linear function of the strain

$$\sigma_{ij}(x) = E_{ijkl}(x)e_{kl}(u(x)) \quad \text{(in tensor notation)}, \tag{2.1}$$

where $E(x)$ is the so-called (plain-stress) *elasticity tensor* of order 4; this tensor characterizes the behaviour of material at point x. To unburden the notation we will often skip the variable x in u, e, E, etc. The strain and stress tensors are symmetric (e.g., $e_{ij} = e_{ji}$) and also E is symmetric in the following sense:

$$E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij} \qquad \text{for } i,j,k,l = 1,\ldots,dim.$$

These symmetries allow us to avoid the tensor notation which is not so common in the optimization community and interpret the 2-tensors e and σ as vectors

$$e = (e_{11}, e_{22}, \sqrt{2}e_{12})^T \in \mathbb{R}^3, \qquad \sigma = (\sigma_{11}, \sigma_{22}, \sqrt{2}\sigma_{12})^T \in \mathbb{R}^3$$

for $dim = 2$ and analogously as vectors in $\mathbb{R}^6$ for $dim = 3$. Correspondingly, the 4-tensor E can be written as a symmetric 3×3 matrix

$$E = \begin{pmatrix} E_{1111} & E_{1122} & \sqrt{2}E_{1112} \\ & E_{2222} & \sqrt{2}E_{2212} \\ \text{sym.} & & 2E_{1212} \end{pmatrix} \tag{2.2}$$

for $dim = 2$ and as a symmetric 6×6 matrix for $dim = 3$. In this notation, equation (2.1) reads as

$$\sigma(x) = E(x)e(u(x)).$$

Since E will be understood as a matrix in our paper, we will use double indices for the elements of E; the correspondence between E_{ij} and the tensor components E_{ijkl} is clear from (2.2). To allow switches from material to no-material, it is natural to work with ($d = 3$ or 6)

$$E \in [L^\infty(\Omega)]^{d\times d} \qquad \text{(in short: } E \in L^\infty(\Omega)\text{)}.$$

For a consistent notation, we will always use $d = 3$ in connection with $dim = 2$ and $d = 6$ when $dim = 3$.

Further we put

$$\mathcal{H} = \{u \in [H^1(\Omega)]^{dim} \mid e^T(s)u(s) = 0, s \in \Gamma\},$$

$e(s)$ being a measurable matrix-valued function defining the *boundary conditions*, so that $[H_0^1(\Omega)]^{dim} \subset \mathcal{H} \subset [H^1(\Omega)]^{dim}$; we assume that the admissible displacement fields belong to $\mathcal{H}$.

Part of the body is under an external load

$$f \in [L^2(\Gamma')]^{dim} \qquad \text{(in short: } f \in L^2(\Gamma')\text{)},$$

$\Gamma' \subseteq \Gamma$ being open in Γ, which leads to the displacement $u \in \mathcal{H}$ of the body. To allow for more general situations, we require that u stays within a closed convex set $U \subset \mathcal{H}$. This U can be given, e.g., by *unilateral contact condition* (for details cf. [18,21]). Our body under load deformates and capacitates a certain potential energy (the so-called *compliance*), which is a measure for the stiffness of the structure, i.e., its ability to withstand the load; the less is compliance the more rigid will be the structure with respect to the considered load. For given elasticity matrix E and acting load f, the potential energy as a function of the displacement $u \in U$ is given by

$$-\frac{1}{2}\int_\Omega \langle Ee(u), e(u)\rangle\, dx + F(u), \tag{2.3}$$

where we have put

$$F(u) := \int_{\Gamma'} f \cdot u\, dx. \tag{2.4}$$

We recall once more that E, u and f in (2.3), (2.4) are functions of x which is omitted only to economize the notation. The system is in equilibrium (outer and inner forces balance each other) for u which maximizes (2.3), i.e., u which solves

$$\sup_{u \in U} \left\{ -\frac{1}{2} \int_\Omega \langle Ee(u), e(u) \rangle \, dx + F(u) \right\}. \tag{2.5}$$

Nature always tries to reach the equilibrium (2.5). It is now the interest of the designer to choose under physical and economical constraints the material function E such that the "sup" in (2.5) becomes as small as possible, that is, the body responds with minimal displacements and strains to the load. Physics tells us that $E(x)$ has to to be a symmetric and positive semidefinite matrix for each $x \in \Omega$, what we write as

$$E(x) = E(x)^T \succeq 0 \qquad \text{for } x \in \Omega \qquad (\text{in short: } E = E^T \succeq 0). \tag{2.6}$$

To introduce a resource (cost) constraint for E, we use the *trace* of E (with $d = 3$ or 6 according to $dim = 2$ or 3)

$$tr(E(x)) := \sum_{i=1}^{d} E_{ii}(x) \tag{2.7}$$

and require with some given positive α

$$\int_\Omega tr(E(x)) \, dx \le \alpha. \tag{2.8}$$

Further, to exclude singularities at isolated points (e.g., at boundary points of Γ') we demand that, with some fixed $r^+, r^- \in L^\infty(\Omega)$, $0 \le r^- < r^+$,

$$r^-(x) \le tr(E(x)) \le r^+(x) \qquad \text{for } x \in \Omega. \tag{2.9}$$

It is convenient to summarize the feasible design functions in a set

$$\mathcal{E} := \left\{ E \in L^\infty(\Omega) \mid \begin{array}{l} E \text{ is of form (2.2) and} \\ \text{satisfies (2.6), (2.8) and (2.9)} \end{array} \right\}. \tag{2.10}$$

With this definition, the SLD-problem becomes

$$\inf_{E \in \mathcal{E}} \sup_{u \in U} \left\{ -\frac{1}{2} \int_\Omega \langle Ee(u), e(u) \rangle \, dx + F(u) \right\}. \tag{2.11}$$

Obviously, a minimizing E in (2.11) will only be optimal for the *one* considered load f and might be extremely unstable (may even collapse) under other loads than f (even of small magnitude). Hence a more realistic approach requires to look for a structure which can withstand a whole collection of independent loads $f^1, \dots, f^L$ from $L^2(\Gamma')$, acting at different times; further, the design should be the "best possible" one. In an engineering context, the worst-case aspect makes most sense. This leads to the following MLD-problem, in which we seek the design function E which yields the smallest possible worst-case compliance

$$\inf_{E \in \mathcal{E}} \sup_{\ell = 1, \dots, L} \sup_{u^\ell \in U^\ell} \left\{ -\frac{1}{2} \int_\Omega \langle Ee(u^\ell), e(u^\ell) \rangle \, dx + F^\ell(u^\ell) \right\}; \tag{2.12}$$

here we have put in accordance with (2.4)

$$F^\ell(u) := \int_{\Gamma'} f^\ell \cdot u\,dx \qquad \text{for } \ell = 1,\dots,L. \tag{2.13}$$

Further, the sets U^ℓ in (2.12) allow individual contact conditions for the loads f^ℓ; hence we can work in (2.12) with different rigid obstacles and we solve indeed a coupled *multiple-load* and *multiple-obstacle* problem.

To be more precise for the numerical part later on we assume that the sets U^ℓ in (2.12) can be written in the form

$$U^\ell := \left\{u \in H^1(\Omega) \mid g^\ell(u) \le \delta^\ell\right\} \tag{2.14}$$

with linear functions g^ℓ, $\ell = 1,\dots,L$.

All our forthcoming efforts aim at finding an efficient analytical and computational way to solve the MLD (2.12). We start with two steps which convert (2.12) to an "equivalent" but easier accessible problem. First let us eliminate the discrete inner "$\sup\limits_{\ell=1,\dots,L}$" in (2.12). With a *weight vector* λ for the loads, which runs over the unit simplex

$$\Lambda := \left\{\lambda \in \mathbb{R}^L \mid \sum_{\ell=1}^L \lambda_\ell = 1,\ \lambda_\ell \ge 0 \text{ for } \ell = 1,\dots,L\right\},$$

we get from a standard convexity argument the following equivalent representation of (2.12):

$$\inf_{E\in\mathcal{E}} \sup_{\substack{\lambda\in\Lambda\\ (u^1,\dots,u^L)\in U^1\times\cdots\times U^\ell}} \sum_{\ell=1}^L \left\{-\frac{1}{2}\int_\Omega \lambda_\ell \langle Ee(u^\ell), e(u^\ell)\rangle\,dx + \lambda_\ell F^\ell(u^\ell)\right\}. \tag{2.15}$$

The objective function in (2.15) is linear (thus convex) in the inf-variable E; it is, however, not concave in the sup-argument $(u^1,\dots,u^\ell;\lambda)$. This is in sharp contrast to the SLD-case, where λ reduces to 1 and (2.15) specializes to (2.11) which is convex-concave in (E,u). However, we will show here that a simple change of variable can recover a convex-concave formulation of the MLD as well.

We begin by noting that the inf-sup value in (2.15) remains the same when restricting λ to the half-open set

$$\Lambda^0 := \{\lambda \in \Lambda \mid \lambda_\ell > 0 \text{ for } \ell = 1,\dots,L\}$$

and passing from the variable $(u^1,\dots,u^L;\lambda)$ to

$$(v^1 := \lambda_1 u^1,\dots,v^L := \lambda_L u^L;\lambda).$$

This step converts (2.12) – (2.15) to

$$\inf_{E\in\mathcal{E}} \sup_{(\boldsymbol{v},\lambda)\in\mathcal{V}} \sum_{\ell=1}^L \left\{-\frac{1}{2}\int_\Omega \lambda_\ell^{-1}\langle Ee(v^\ell), e(v^\ell)\rangle\,dx + F^\ell(v^\ell)\right\}, \tag{2.16}$$

where we have put $\boldsymbol{v} := (v^1,\dots,v^L)$ and

$$\mathcal{V} := \left\{(\boldsymbol{v};\lambda) \mid \lambda \in \Lambda^0, g^\ell(v^\ell) - \lambda_\ell\delta^\ell \le 0 \text{ for } \ell = 1,\dots,L\right\}$$

with g^ℓ and δ^ℓ from (2.13). $\mathcal{V}$ is again a convex set. Further, and this is the crucial observation, the objective function in (2.16)

$$\mathcal{F}(E;(\boldsymbol{v};\lambda)) := \sum_{\ell=1}^{L}\left\{-\frac{1}{2}\int_\Omega \lambda_\ell^{-1}\langle Ee(v^\ell), e(v^\ell)\rangle\,dx + F^\ell(v^\ell)\right\} \tag{2.17}$$

is now concave in $(\boldsymbol{v},\lambda) = (v^1,\ldots,v^L;\lambda) \in \mathcal{V}$. This follows easily from the concavity of $-x^2/y$ in $(x,y) \in \mathbb{R}\times\mathbb{R}_+\setminus\{0\}$ and linearity of $\mathcal{F}$ in E. We are thus in a position to apply tools of Convex Analysis. Using the Minimax Theorem (cf., e.g., [, Thm. 2.7.1]) one gets the following existence result.

THEOREM 2.1 (Existence of an optimal design tensor for MLD). *There exists $E^* \in \mathcal{E}$ such that*

$$\sup_{(\boldsymbol{v};\lambda)\in\mathcal{V}} \mathcal{F}(E^*;(\boldsymbol{v};\lambda)) = v^* = \beta^*$$

where

$$v^* = \inf_{E\in\mathcal{E}} \sup_{(\boldsymbol{v};\lambda)\in\mathcal{V}} \mathcal{F}(E;(\boldsymbol{v};\lambda))$$
$$\beta^* = \sup_{(\boldsymbol{v};\lambda)\in\mathcal{V}} \inf_{E\in\mathcal{E}} \mathcal{F}(E;(\boldsymbol{v};\lambda)).$$

Proof. The claim follows if we can guarantee that

(i) $\mathcal{V}$ is a convex set;
(ii) $\mathcal{F}(E;\cdot)$ is concave for fixed $E\in\mathcal{E}$;
(iii) $\mathcal{E}\subset L^\infty(\Omega)$ is convex and weak*-compact;
(iv) $\mathcal{F}(\cdot;(\boldsymbol{v};\lambda))$ is convex and lower semicontinous on $\mathcal{E}$ (equipped with the weak*-topology of $L^\infty(\Omega)$) for fixed $(\boldsymbol{v};\lambda)\in\mathcal{V}$.

Conditions (i) and (ii) were already discussed above and the convexity in (iii) and (iv) is obvious. The lower-semicontinuity in (iv) follows by assumption on the data and it remains to prove the weak*-compactness in (iii). From $E = E^T \succeq 0$ and $tr(E(x)) \le r^+(x)$ for each $x\in\Omega$ it is easily seen that $E\in\mathcal{E}$ lies in a norm ball of $L^\infty(\Omega)$ which implies the weak*-compactness of $\mathcal{E}$. □

Note that (2.12), (2.15) and (2.16) yield the same objective values but that we work in (2.16) with a restricted domain of definition (Λ is replaced by Λ^0). Obviously, we can extend Λ^0 in (2.16) to Λ for the price of working with an extended-valued variant of $\mathcal{F}$. We avoid these technicalities here since it is the design function E, we are really interested in; and for such E we dispose of an existence result with Theorem 2.1. Further we will see that, after discretizing (2.16), we almost automatically fall back to a problem with $\lambda\in\Lambda$.

3. Discretization and Semidefinite reformulation. Given the existence of an optimal elasticity matrix E^* for (2.16), we ask how to "compute" this E^*. The results of this section supply the key to this question; it is shown that after a finite-element discretization of (2.16), the question can be reduced to a *semidefinite program*, for which efficient computational tools are available.

3.1. The discretized problem. To simplify the notation, we use the same symbols for the discrete objects (vectors) as for the "continuum" ones (functions). Assume that Ω is partitioned into M polygonal elements Ω_m of volumes ω_m. Let N be the number of nodes (vertices of the elements). Assume that E is approximated by a function that is constant on each element Ω_m, i.e., it is fully characterized by a collection $E = (E_1, \dots, E_M)$ of $d \times d$ matrices E_m – the values of E on the elements. The feasible set $\mathcal{E}$ is replaced by its discrete counterpart

$$\mathcal{E} := \left\{ E \in \mathbb{R}^{d \times dM} \mid \begin{array}{l} E_m = E_m^T \succeq 0 \text{ and } r_m^- \le tr(E_m) \le r_m^+ \text{ for } m = 1, \dots, M \\ \sum\limits_{m=1}^{M} tr(E_m)\omega_m \le \alpha \end{array} \right\}.$$

Further assume that the displacement vector u^ℓ corresponding to the load-case ℓ is approximated by a continuous function that is tri/bi-linear (linear in each coordinate) on every element. Such a function can be written as

$$u^\ell(x) = \sum_{n=1}^{N} u_n^\ell \vartheta_n(x)$$

where u_n^ℓ is the value of u^ℓ at n^{th} node and ϑ_n is the basis function associated with n^{th} node (for details, see [4]). Recall that, at each node, the displacement has dim components, hence $u \in \mathbb{R}^D$, $D \le dim \cdot N$ (D could be less than $dim \cdot N$ because of boundary conditions which enforce the displacements of certain nodes to lie in given subspaces of $\mathbb{R}^{dim}$).

Further we define the discrete version of the set U^ℓ of admissible displacements. We assume that the set is given by unilateral contact conditions. The introduction of these conditions is quite technical and the details can be found in [7]. Here we only introduce vectors $\delta^\ell \in \mathbb{R}^r$ (representing the gaps between the contact surfaces and the rigid obstacles) and $r \times M$ matrices C^ℓ (defining the nodes of the contact surface and the direction to the obstacle). The set of admissible displacements for the discretized problem takes the form

$$U^\ell := \{u^\ell \in \mathbb{R}^D \mid C^\ell u^\ell \le \delta^\ell\}. \tag{3.1}$$

For basis functions $\vartheta_n, n = 1, \dots, N$, we define matrices (which are again functions of x)

$$B_n = \begin{pmatrix} \frac{\partial \vartheta_n}{\partial x_1} & 0 \\ 0 & \frac{\partial \vartheta_n}{\partial x_2} \\ \frac{1}{2}\frac{\partial \vartheta_n}{\partial x_2} & \frac{1}{2}\frac{\partial \vartheta_n}{\partial x_1} \end{pmatrix}.$$

for $dim = 2$ and

$$B_n = \begin{pmatrix} \frac{\partial \vartheta_n}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial \vartheta_n}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial \vartheta_n}{\partial x_3} \\ \frac{1}{2}\frac{\partial \vartheta_n}{\partial x_2} & \frac{1}{2}\frac{\partial \vartheta_n}{\partial x_1} & 0 \\ 0 & \frac{1}{2}\frac{\partial \vartheta_n}{\partial x_3} & \frac{1}{2}\frac{\partial \vartheta_n}{\partial x_2} \\ \frac{1}{2}\frac{\partial \vartheta_n}{\partial x_3} & 0 & \frac{1}{2}\frac{\partial \vartheta_n}{\partial x_1} \end{pmatrix}$$

for $dim = 3$. Now, for an element Ω_m, let $\mathcal{D}_m$ be an index set of nodes belonging to this element. The value of the approximate strain tensor e on element Ω_m is then (we add the variable x as a subscript)

$$e_x(u^\ell) = \sum_{n\in\mathcal{D}_m} B_n(x)u_n^\ell \qquad \text{on } \Omega_m;$$

recall that u_n^ℓ has *dim* components.

Finally, the discretized linear functional $F^\ell(u^\ell)$ is $(f^\ell)^T u^\ell$ with $f^\ell \in \mathbb{R}^D$.

As discretized version of the original problem we thus obtain

$$(3.2)\qquad \begin{array}{c} \min\limits_{E=\{E_m\}_{m=1}^M} \phi(E), \\ \text{s.t.}\quad \phi(E) \equiv \sup\limits_{\ell=1,\dots,L}\sup\limits_{u\in U^\ell}\left[-\sum_{m=1}^M tr \quad E_m\int_{\Omega_m} e_x(u)e_x^T(u)dx \quad + 2(f^\ell)^T u\right], \\ E_m \in \Sigma_+^d,\ m=1,...,M; \\ [0\le]\quad r_m^- \le tr(E_m) \le r_m^+ \quad [<\infty],\ m=1,...,M; \\ \sum_{m=1}^M \omega_m tr(E_m) \le \alpha; \end{array}$$

from now on, Σ^p denotes the space of symmetric $p\times p$ matrices, and Σ_+^p is the cone of positive semidefinite matrices from Σ^p.

Now, for each cell Ω_m there exists a finite set of points x_{ms} and positive weights χ_{ms}^2, $s=1,...,S$, such that

$$\int_{\Omega_m} e_x(u)e_x^T(u)dx = \sum_{s=1}^S \chi_{ms}^2 e_{x_{ms}}(u)e_{x_{ms}}^T(u);$$

for all $u\in\mathbb{R}^D$; e.g., one can take $S=4$ for $\dim = 2$ and for linear $B_n(\cdot)$.

Let us define linear matrix-valued functions

$$\zeta_m(u) = \omega_m^{-1/2}[\chi_{m1}e_{x_{m1}}(u); \chi_{m2}e_{x_{m2}}(u); ...; \chi_{mS}e_{x_{mS}}(u)],\ m=1,...,M,$$

taking values in the space of $d\times S$ matrices; then the objective function in (3.2) can be rewritten equivalently as

$$(3.3)\qquad \begin{array}{c} \phi(E) \equiv \sup\limits_{\ell=1,\dots,L}\sup\limits_{u\in U^\ell}\left[-\sum_{m=1}^M \omega_m tr(E_m\zeta_m(u)\zeta_m^T(u)) + 2(f^\ell)^T u\right] \to \min \\ E=\{E_m\}_{m=1}^M, \\ \text{s.t.} \\ E_m\in\Sigma_+^d,\ m=1,...,M; \\ r_m^- \le tr(E_m) \le r_m^+,\ m=1,...,M; \\ \sum_{m=1}^M \omega_m tr(E_m) \le \alpha; \end{array}$$

From now on we assume that

- **A.** The linear inequalities defining the polyhedral sets U^ℓ, $\ell=1,...,L$, satisfy the Slater condition: for every ℓ, there exists u_0^ℓ such that $C^\ell u_0^\ell < \delta^\ell$;
- **B.** The mapping $u\mapsto\{\zeta_m(u)\}_{m=1}^M$ has trivial kernel on $\mathbb{R}^D$ (this is actually the assumption which excludes rigid body motion of the construction);
- **C.** $r_m^- < r_m^+$, $m=1,...,M$, and $\sum_{m=1}^M r_m^-\omega_m < \alpha$.

3.2. The main results. We are about to formulate two main results related to the discretized problem (3.2) (for proofs, see Section 6).

THEOREM 3.1. *Under assumptions* **A**, **B**, **C** *the semidefinite program*

$$
\begin{array}{rcl}
\textit{maximize} & & \\
\psi(v,\nu,\rho^+,\rho^-) & = & -\alpha\nu + 2\sum_{\ell=1}^{L}(f^\ell)^T v^\ell \\
 & & +\sum_{m=1}^{M}(s_m^-\rho_m^- - s_m^+\rho_m^+) \\
\textit{s.t.} & & \\
\mathcal{A}_m(v,\nu,\rho^+,\rho^-) & \equiv & \begin{pmatrix} (\nu+\rho_m^+-\rho_m^-)I_d & \zeta_m(v^1) & \zeta_m(v^2) & \dots & \zeta_m(v^L) \\ \zeta_m^T(v^1) & \lambda_1 I_S & & & \\ \zeta_m^T(v^2) & & \lambda_2 I_S & & \\ \dots & & & \ddots & \\ \zeta_m^T(v^L) & & & & \lambda_L I_S \end{pmatrix} \\
 & \succeq & 0,\ m=1,\dots,M, \\
\mathrm{Diag}(\lambda_\ell\delta^\ell - C^\ell v^\ell) & \succeq & 0,\ \ell=1,\dots,k, \\
\mathrm{Diag}(\rho^+) & \succeq & 0, \\
\mathrm{Diag}(\rho^-) & \succeq & 0, \\
\nu & \geq & 0, \\
\sum_{\ell=1}^{L}\lambda_\ell & = & 1.
\end{array}
\tag{3.4}
$$

(C^ℓ, δ^ℓ are given by (3.1)) with the design variables

$$v = (v^1,\dots,v^L;\lambda) \in (\mathbb{R}^D)^L \times \mathbb{R}^L, \rho^\pm \in \mathbb{R}^M, \nu \in \mathbb{R}$$

and constants

$$s_m^\pm = \omega_m r_m^\pm$$

is dual to the problem of interest (3.2) *in the sense that the optimal value* ϕ^* *of* (3.2) *is equal to the optimal value* ψ^* *of* (3.4).

Theorem 3.1 deals with optimal values of (3.2), (3.4) but does not answer the crucial question of how to recover a (nearly) optimal solution to the original (primal) problem from a (nearly) optimal solution to its dual problem. In order to derive such a recovering routine, recall the notion of a central approximate solution to a semidefinite program. Problem (3.4) is of the generic form

$$\max\{c^T x \mid \mathcal{A}x \succeq 0, e^T x = 1\} \qquad \text{(SDP)},$$

where the design vector x varies in some $\mathbb{R}^n$ and $x \mapsto \mathcal{A}x$ is an affine mapping of $\mathbb{R}^n$ into space Σ of symmetric matrices of a given block-diagonal structure. Assuming the problem (SDP) to be strictly feasible (there exists x with $e^T x = 1$ and positive definite $\mathcal{A}x$), one can equip the relative interior $\mathcal{X}'$ of the feasible set $\mathcal{X}$ of the problem with the standard barrier

$$\mathcal{B}(x) = -\ln \mathrm{Det}(\mathcal{A}x).$$

Now let $t > 0$. A point $x(t) \in \mathcal{X}'$ is called *central approximate solution* to (SDP) *associated with the value t of the penalty parameter*, if $x(t)$ minimizes the aggregate

$$-tc^T x + \mathcal{B}(x) \tag{3.5}$$

over $\mathcal{X}'$.

We are about to establish the following

THEOREM 3.2. *Under assumptions* **A**, **B**, **C**

(i) *Central approximate solutions to* (3.4) *exist for every value* $t > 0$ *of the penalty parameter*

(ii) *A central approximate solution*

$$x(t) = ((v^1(t), ..., v^L(t); \lambda(t)), \nu(t), \rho^+(t), \rho^-(t))$$

to (3.4) *associated with a large value of the penalty parameter can be explicitly converted to a good approximate solution to* (3.2) *as follows. Let*

$$W_m \equiv t^{-1}\mathcal{A}_m^{-1}(x(t)) = \begin{pmatrix} \Xi_m & Q_m^T \\ Q_m & R_m \end{pmatrix}, \; m = 1, ..., M,$$

Ξ_m *being* $d \times d$ *block, and let*

$$E_m^+ = \omega_m^{-1}\Xi_m, \; m = 1, ..., M.$$

Then $E^+ = \{E_m^+\}_{m=1}^M$ *is a feasible solution to* (3.2), *and the value of the objective of the latter problem at* E^+ *is larger than the optimal value* ϕ^* *of* (3.2) *by at most* $\Delta(t)$, *where*

$$\Delta(t) = t^{-1}[N(kS + D + 2) + \sum_{\ell=1}^{L} \dim(\delta^\ell) + 1].$$

4. Computational issues and numerical results. The semidefinite problem (3.4) can be efficiently solved by modern interior point polynomial time methods; the most attractive seem to be the path-following algorithms, since they automatically generate (nearly) central approximate solutions with the value of the penalty parameter growing linearly at the rate $(1 + O(\vartheta^{-2}))$, where

$$\vartheta = M(kS + d) + 2M + \sum_{\ell=1}^{L} \dim(\delta^\ell) + 1$$

is the total row size of matrices from Σ. The computational effort per iteration (i.e., per increasing the penalty parameter in the aforementioned ratio) is dominated by the necessity to assemble and to solve (with respect to d) the Newton system

$$[\nabla^2\mathcal{B}(x)]d = b,$$

$x \in \mathcal{V}'$ and b being given. It is easily seen that for (3.4) the latter task requires $O(L^3D^3)$ arithmetic operations. Theoretical upper bound on the number of iterations required to recover, via the scheme of Theorem 3.2, an ϵ-optimal solution to the original problem (3.2) (i.e., a feasible solution to (3.2) with the value of the objective greater than the optimal one by at most ϵ) is

$$\sqrt{\vartheta}\ln(\vartheta\epsilon^{-1}V),$$

where the *scale factor* V depends on the numerical values of the data. The practical behaviour of a good interior point method as applied to (3.4) is even better than the one predicted by the theoretical complexity bound, and the typical number of iterations required to solve (3.2) to a reasonably high accuracy is few tens.

The illustrative numerical results reported below were obtained with the aid of the Projective method [5] implemented in the LMI Toolbox for use with MATLAB

– the only interior point solver for semidefinite programs we had in our disposal. Unfortunately, this method is *not* a path-following one, this is why we were enforced to combine it with an additional (and relatively cheap computationally) interior point routine, based on Theorem 3.2, which, given a good feasible solution to (3.4), updates it into a central solution of the same quality and uses this "refined" solution to recover a nearly optimal solution to the problem of interest.

5. Examples. Results of three numerical examples are presented in this section. The values of the "density" function ρ are depicted by gradations of grey: full black corresponds to high density, white to zero density (no material), etc.
Example 1. We consider a typical example of structural design: The two forces (or force and fixed boundary) are opposite to each other and there is a hole in between because of technological reasons. The geometry of domain Ω and the forces are depicted in Fig. 5.1. The forces are considered as a single load. Because of symmetry, we could

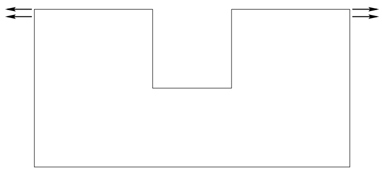

FIG. 5.1. *Example 1*

only compute one half of the original domain. The geometry, forces and boundary conditions for the computational domain are shown in Fig. 5.2. The resulting values

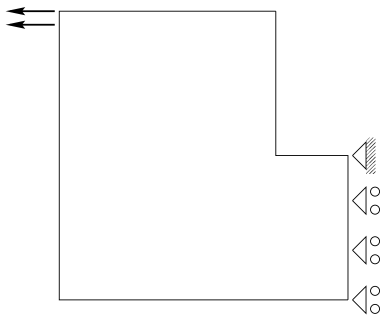

FIG. 5.2. *Example 1*

of the 'density' function ρ for 29×29 mesh are presented in Fig. 5.3; the figure is composed from two computational domains to get the original body.
Example 2. Let us now generalize Example 1 to a symmetric two-sided body shown in Fig. 5.4. The body can be loaded either by the forces on the left or on the right-hand side. Therefore this example has to be considered as MLD (two-load case). Because of symmetry, we can again compute only one half of the original domain. The geometry, forces and boundary conditions for the computational domain are shown in Fig. 5.5. Note that the boundary conditions are different for the different loading scenarios, as indicated in the figure. The resulting values of the 'density' function ρ for 37×25 mesh are presented in Fig. 5.6. Again, the figure is composed from two computational domains to get the full body.

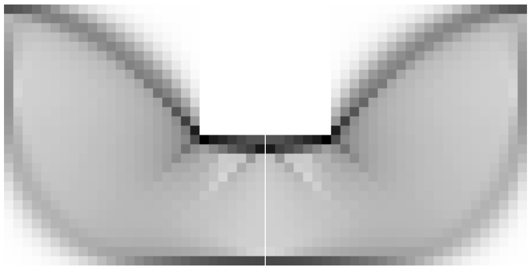

Fig. 5.3. *Example 1*

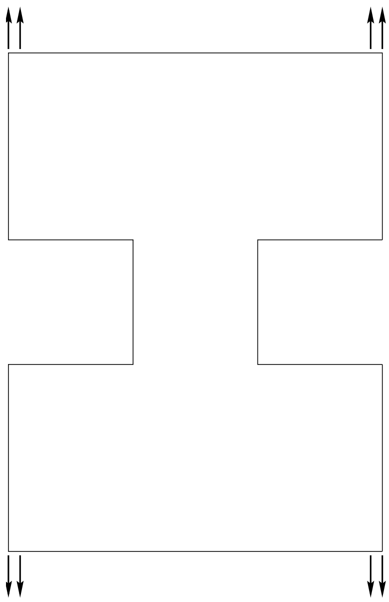

Fig. 5.4. *Example 2*

Example 3. In this example we try to model a spanner. The geometry of domain Ω is depicted in Fig. 5.7. The nut (or the bolt head) (depicted in full black in Fig. 5.7) is considered to present a rigid obstacle for the spanner. Hence the spanner is in unilateral contact with the nut and there are no other boundary conditions. The loads are also shown in Fig. 5.7. Note that the problem is nonlinear because of the unilateral contact conditions and that for positive vertical force we get a different design than for a negative one; hence we have to consider these two forces as two independent loads. The resulting values of the 'density' function ρ for 37×22 discretization are shown in Fig. 5.8. We also performed a more detailed analysis of the most interesting part around the nut: Fig. 5.9 shows the values of ρ for 31×31 discretization of this part.

6. Proofs of Theorems 3.1 and 3.2.

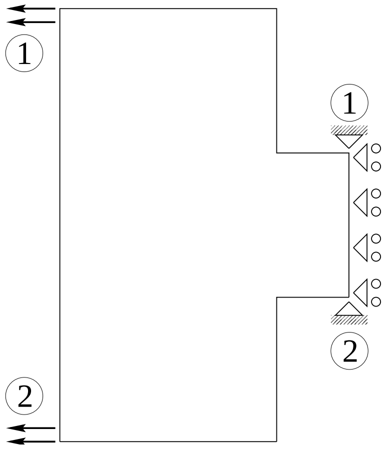

FIG. 5.5. *Example 2*

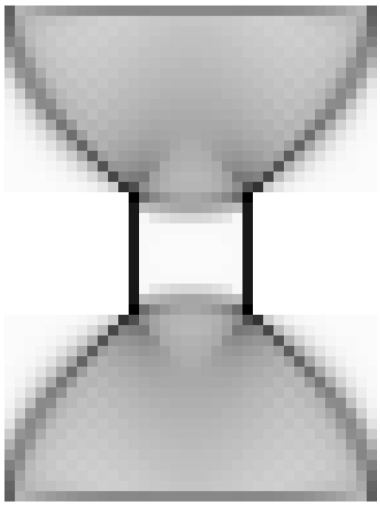

FIG. 5.6. *Example 2*

6.1. From (3.2) to (3.4). Let, similarly to Section 2,

$$\begin{array}{rcl} \mathcal{V}' &=& \{v=(v^1,\ldots,v^L,\lambda)\in(\mathbb{R}^d)^L\times\mathbb{R}^L : C^\ell v^l < \lambda_\ell\delta^\ell, \lambda_\ell>0, \ell=1,...,L; \\ && \sum_{\ell=1}^L \lambda_\ell = 1\} \\ \mathcal{V} &=& \mathrm{cl}\,\mathcal{V}' = \{(v^1,...;v^L;\lambda) : C^\ell v^\ell \le \lambda_\ell\delta^\ell, \lambda_\ell\ge 0,\ \ell=1,...,L; \sum_\ell \lambda_\ell = 1\}, \end{array}$$

(the concluding equality is given by **A**).

Same as in Section 2, we can rewrite the function $\phi(\cdot)$ as

$$\begin{array}{rcl} \phi(E) &=& \sup\limits_{\substack{(u^1,\lambda_1;\ldots;u^L,\lambda_L):\\ \lambda_\ell>0,\sum_\ell\lambda_\ell=1,u^\ell\in U^\ell}} \sum_{\ell=1}^L \left[2\lambda_\ell (f^\ell)^T u^\ell - \lambda_\ell \sum_{m=1}^M \omega_m tr(E_m\zeta_m(u^\ell)\zeta_m^T(u^\ell))\right] \\ &=& \sup\limits_{v=(v^1,\ldots,v^L;\lambda)\in\mathcal{V}'} \left[2\sum_{\ell=1}^L (f^\ell)^T v^\ell - \sum_{m=1}^M\sum_{\ell=1}^L \omega_m\lambda_\ell^{-1} tr(E_m\zeta_m(v^\ell)\zeta_m^T(v^\ell))\right] \end{array}$$

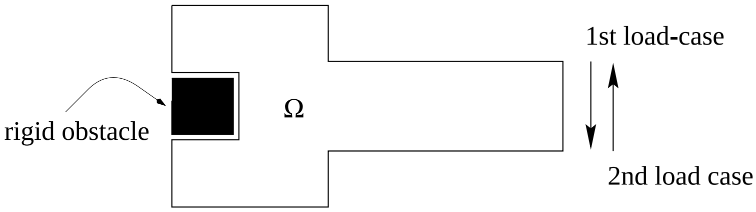

FIG. 5.7. *Example 3*

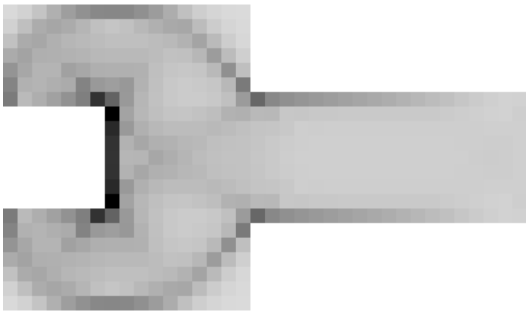

FIG. 5.8. *Example 1*

so that (3.3) is nothing but the problem

$$\begin{array}{c}\min\limits_{E\in\mathcal{E}}\sup\limits_{v\in\mathcal{V}'} T(E;v),\\ T(E,v)=2\sum_{\ell=1}^{L}(f^\ell)^Tv^\ell-\sum_{\ell=1}^{L}\sum_{m=1}^{M}\omega_m\lambda_\ell^{-1}tr(E_m\zeta_m(v^\ell)\zeta_m^T(v^\ell))\\ [v=(v^1,...,v^L;\lambda)]\\ \mathcal{E} \;=\; \{\{E_m\}_{m=1}^M \mid E_m\in\Sigma_+^d, r_m^-\le tr(E_m)\le r_m^+, m=1,...,M,\\ \sum_{m=1}^M\omega_m tr(E_m)\le\alpha\}.\end{array}$$

By penalizing the linear inequalities in the inner sup and taking sup with respect to the penalty coefficients, we can rewrite the latter problem equivalently as

$$\begin{array}{rcl} & \min\limits_{E\in\mathcal{P}} \sup\limits_{\substack{v\in\mathcal{V}',\\ \nu\ge 0,\rho^+,\rho^-\in\mathbb{R}_+^M}} & T(E;v,\nu,\rho^+,\rho^-),\\ T(E;v,\nu,\rho^+,\rho^-) & = & 2\sum_{\ell=1}^L(f^\ell)^Tv^\ell-\sum_{\ell=1}^L\sum_{m=1}^M\omega_m\lambda_\ell^{-1}tr(E_m\zeta_m(v^\ell)\zeta_m^T(v^\ell))\\ & & -\nu[\alpha-\sum_{m=1}^M\omega_m tr(E_m)]\\ & & -\sum_{m=1}^M[\rho_m^-(tr(E_m)-r_m^-)+\rho_m^+(r_m^+-tr(E_m))]\\ & & \mathcal{P}=\{\{E_m\}_{m=1}^M\mid E_m\in\Sigma_+^d, m=1,...,M\}\end{array} \tag{6.1}$$

The optimal value in (6.1), due to the origin of the problem, is exactly the optimal value ϕ^* of (3.2). Now let us pass from (3.2) to the problem with swapped inf and sup:

$$\sup_{(v,\nu,\rho^+,\rho^-)\in\mathcal{V}'\times\mathbb{R}_+\times\mathbb{R}_+^M\times\mathbb{R}_+^M}\ \inf_{E\in\mathcal{P}} T(E;v,\nu,\rho^+,\rho^-) \tag{6.2}$$

and let ϕ^{**} be the optimal value in the latter problem. Note that by weak duality inequality

$$\phi^*\ge\phi^{**}. \tag{6.3}$$

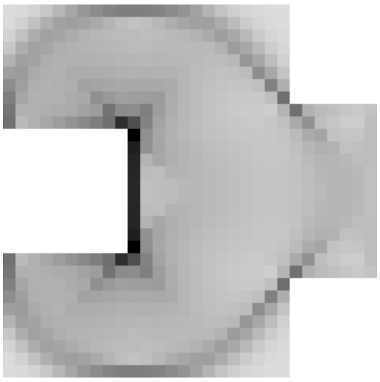

FIG. 5.9. *Example 1*

6.2. Equivalent transformation of (6.2). By passing from E_m to new variables $F_m = \omega_m E_m$ and setting

$$s_m^{\pm} = \omega_m r_m^{\pm},\ m = 1, ..., M$$

(cf. Theorem 3.1), we can rewrite the objective

$$\psi(u, \nu, \rho^+, \rho^-) \equiv \inf_{E \in \mathcal{P}} T(E; u, \nu, \rho^+, \rho^-)$$

of problem (6.2) as

$$\begin{array}{rcl}\psi(v, \nu, \rho^+, \rho^-) & = & \inf\limits_{F \in \mathcal{P}} \left\{ -\alpha\nu + 2\sum\limits_{\ell=1}^{L} (f^\ell)^T v^\ell + \sum\limits_{m=1}^{M} (s_m^- \rho_m^- - s_m^+ \rho_m^+) \right. \\ & & \left. - \sum_{m=1}^{M} \left[\sum_{\ell=1}^{L} \lambda_\ell^{-1} tr(F_m \zeta_m(v^\ell)\zeta_m^T(v^\ell)) + (\rho_m^- - \rho_m^+ - \nu) tr(F_m) \right] \right\}.\end{array} \tag{6.4}$$

Now, denoting by $\mu_{\max}(A)$ the largest eigenvalue of a symmetric matrix A and taking into account the evident relation

$$\max_{B \in \Sigma_+^d, tr(B) = r \geq 0} tr(BC) = r\mu_{\max}(C)$$

which is valid for an arbitrary symmetric $d \times d$ matrix C, we can easily continue the above computation:

$$\begin{array}{rcl}\psi(v, \nu, \rho^+, \rho^-) & = & 2\sum_{\ell=1}^{L} (f^\ell)^T v^\ell + \sum_{m=1}^{M} (s_m^- \rho_m^- - s_m^+ \rho_m^+) - \alpha\nu, \\ & & \text{if } \mu_{\max} \ \sum_{\ell=1}^{L} \lambda_\ell^{-1} \zeta_m(v^\ell)\zeta_m^T(v^\ell) \ \leq \nu + \rho_m^+ - \rho_m^-,\ m = 1, ..., M,\ \&\ \nu \geq 0, \\ \psi(v, \nu, \rho^+, \rho^-) & = & -\infty, \\ & & \text{otherwise}\end{array}$$

Thus, the problem (6.2) becomes the optimization problem

$$\begin{array}{rcl}\text{maximize} & & \\ \psi(v, \nu, \rho^+, \rho^-) & = & -\alpha\nu + 2\sum_{\ell=1}^{L} (f^\ell)^T v^\ell \\ & & + \sum_{m=1}^{M} (s_m^- \rho_m^- - s_m^+ \rho_m^+) \\ \text{s.t.} & & \\ \mu_{\max}(\sum_{\ell=1}^{L} \lambda_\ell^{-1} \zeta_m(v^\ell)\zeta_m^T(v^\ell)) & \leq & \nu + \rho_m^+ - \rho_m^-,\ m = 1, ..., M, \\ v & \in & \mathcal{V}', \\ \rho^{\pm} & \in & \mathbb{R}_+^M, \\ \nu & \geq & 0.\end{array} \tag{6.5}$$

Let I_p denote the unit $p\times p$ matrix, and let us write $A\succeq B$ whenever A,B are symmetric matrices of the same size with $A-B\succeq 0$. For positive λ_ℓ and rectangular $q\times p$ matrices Z_ℓ one clearly has

$$\sum_{\ell=1}^{L}\lambda_\ell^{-1}Z_\ell Z_\ell^T=[Z_1;Z_2;\dots;Z_L][\mathrm{Diag}(\lambda_1 I_p,\lambda_2 I_p,\dots,\lambda_L I_p)]^{-1}[Z_1;Z_2;\dots;Z_L]^T$$

and therefore

$$\begin{array}{c} a\geq \mu_{\max}(\sum_{\ell=1}^{L}\lambda_\ell^{-1}Z_\ell Z_\ell^T)\\ \Updownarrow\\ aI_q\succeq [Z_1;Z_2;\dots;Z_L][\mathrm{Diag}(\lambda_1 I_p,\lambda_2 I_p,\dots,\lambda_L I_p)]^{-1}[Z_1;Z_2;\dots;Z_L]^T\\ \Updownarrow\\ \begin{pmatrix} aI_q & [Z_1;Z_2;\dots;Z_L]\\ [Z_1;Z_2;\dots;Z_L]^T & \mathrm{Diag}(\lambda_1 I_p,\lambda_2 I_p,\dots,\lambda_L I_p)\end{pmatrix}\succeq 0,\end{array}$$

the concluding equivalence being given by the standard results on Schur's complement. We conclude that (6.5) is equivalent to the problem

$$\begin{array}{rcl} \text{maximize} & & \\ \psi(v,\nu,\rho^+,\rho^-) & = & -\alpha\nu+2\sum_{\ell=1}^{L}(f^\ell)^T v^\ell+\sum_{m=1}^{M}(s_m^-\rho_m^- - s_m^+\rho_m^+)\\ \text{s.t.} & & \\ \mathcal{A}_m(v,\nu,\rho^+,\rho^-) & \equiv & \begin{pmatrix} (\nu+\rho_m^+-\rho_m^-)I_d & \zeta_m(v^1) & \zeta_m(v^2) & \dots & \zeta_m(v^L)\\ \zeta_m^T(v^1) & \lambda_1 I_S & & & \\ \zeta_m^T(v^2) & & \lambda_2 I_S & & \\ \dots & & & \ddots & \\ \zeta_m^T(v^L) & & & & \lambda_L I_S \end{pmatrix}\\ & \succeq & 0,\\ v & = & (v^1,\dots,v^L;\lambda)\in\mathcal{V}'=\{(v^1,\dots,v^L;\lambda)\mid \lambda_\ell>0,\\ C^\ell v^\ell<\lambda_\ell\delta^\ell,\sum_{\ell=1}^{L}\lambda_\ell=1\}, & & \\ \rho^\pm & \in & \mathbb{R}_+^M,\\ \nu & \geq & 0. \end{array} \tag{6.6}$$

Problem (6.6) is "almost" the problem (3.4); the only difference is that the "unclosed" inequalities $v\in\mathcal{V}'$, i.e.,

$$C^\ell v^\ell<\lambda_\ell\delta^\ell,\lambda_\ell>0,\sum_\ell\lambda_\ell=1$$

of (6.6) in (3.4) are replaced with their closed versions $v\in\mathcal{V}$, i.e.,

$$C^\ell v^\ell\leq\lambda_\ell\delta^\ell,\lambda_\ell\geq 0,\sum_\ell\lambda_\ell=1.$$

It is immediately seen that this modification does not vary the optimal value. Indeed, (6.6) clearly is feasible (in fact - even strictly feasible: there exists a feasible solution to the problem which makes all its inequalities strict. To get such a solution, it suffices to choose arbitrary $v\in\mathcal{V}'$ and positive vectors $\rho^\pm$ and then to extend this collection by large enough positive ν). Due to feasibility of the problem, the standard approximation arguments demonstrate that its optimal value clearly remains unchanged when we pass from "unclosed" constraint $v\in\mathcal{V}'$ to its "closed" form $v\in\mathcal{V}$, thus coming to the program (3.4). Consequently (see (6.3)),

$$\phi^*\geq\psi^*, \tag{6.7}$$

ψ^* being the optimal value in (3.4).

6.3. Proof of Theorem 3.2.(i). Problem (3.4) is of the form (SDP); from the general theory of interior point methods (see [8]) it is known that existence of central approximate solutions to (SDP) is guaranteed by strict feasibility of the program (which indeed is the case for (3.4)) along with boundedness of the level sets of the objective

$$X(a) = \{x \mid \mathcal{A}x \succeq 0, e^T x = 1, c^T x \geq a\}$$

for every real a. Thus, all we need in order to prove (i) is to verify the boundedness of the level sets $X(a)$.

Consider a sequence

$$\{y_j = ((v^{1,j}, ..., v^{L,j}; \lambda_{1,j}, ..., \lambda_{L,j}), \nu_j, \rho^{+,j}, \rho^{-,j})\}_{j=1}^{\infty}$$

of points from $X(a)$, and let us prove that the sequence is bounded. Let $\pi_j = \max_{m=1}^{M}[\nu_j + \rho_m^{+,j}]$. Since the matrices $\mathcal{A}_m(y_j)$ are positive semidefinite and $0 \leq \lambda_{\ell,j}$, $\sum_\ell \lambda_{\ell,j} = 1$, we have $\|\zeta_m(v^{i,j})\| \leq C\sqrt{\pi_j}$ for some constant C and all m, i, j. By **B**, this observation yields that

$$\|v^{i,j}\| \leq C' \sqrt{\pi_j} \tag{6.8}$$

for all i, j. It follows that the objective of (3.4) at y_j is at most

$$\begin{aligned} \theta_j &= -\alpha\nu_j + O(\sqrt{\pi_j}) + \textstyle\sum_{m=1}^{M}(s_m^- \rho_m^{-,j} - s_m^+ \rho_m^{+,j}) \\ &= O(\sqrt{\pi_j}) - \left\{\textstyle\sum_{m=1}^{M} s_m^-(\nu_j + \rho_m^{+,j} - \rho_m^{-,j})\right\}_1 \\ &\quad - \left\{(\alpha - \textstyle\sum_{m=1}^{M} s_m^-)\nu_j\right\}_2 - \left\{\textstyle\sum_{m=1}^{M}(s_m^+ - s_m^-)\rho_m^{+,j}\right\}_3. \end{aligned}$$

Now, the quantities $\nu_j + \rho_m^{+,j} - \rho_m^{-,j}$ are nonnegative (they are diagonal entries of positive semidefinite matrices $\mathcal{A}_m(y_j)$), so that $\{\cdot\}_1 \geq 0$ and

$$0 \leq \rho_m^{-,j} \leq \pi_j. \tag{6.9}$$

By **C**, we have $\{\cdot\}_2 + \{\cdot\}_3 \geq \kappa\pi_j$ with some positive κ, so that $\theta_j \leq O(\sqrt{\pi_j}) - \kappa\pi_j$. On the other hand, θ_j is an upper bound on $\psi(y_j)$, and therefore the sequence $\{\theta_j\}$ is below bounded; thus, the sequence π_j is bounded, which, in view of (6.9) and (6.8), implies boundedness of $\{y_j\}$.

6.4. Proof of Theorem 3.2.(ii) and Theorem 3.1. As we remember,

(I) *For every feasible solution E to the problem of interest* (3.2), *the value of the objective at the solution is equal to*

$$\sup_{v=\in\mathcal{V}', \nu\geq 0, \rho^{\pm}\in\mathbb{R}_+^M} T(E; v, \nu, \rho^+, \rho^-), \tag{6.10}$$

with T given by (6.1).

Now let $x(t) = (\{v^1(t), \dots, v^L(t); \lambda(t)\}, \nu(t), \rho^+(t), \rho^-(t))$ be a central approximate solution to (3.4), and let $W = t^{-1}[\mathcal{A}x(t)]^{-1}$, where $(\mathcal{A}, e)$ are the data from the representation of (3.4) in the generic form (SDP). Note that W is a block-diagonal positive definite matrix, and that its first N diagonal blocks are the matrices $W_m = \begin{pmatrix} \Xi_m & Q_m^T \\ Q_m & R_m \end{pmatrix}$, $m = 1, ..., M$, mentioned in (ii). Due to the structure of constraints in (3.4), the remaining diagonal blocks in W are k diagonal matrices W_{M+i} of the row sizes $\dim(\delta^\ell)$ associated with the constraints $\mathrm{Diag}(\lambda_\ell\delta^\ell - C^\ell v^\ell) \succeq 0$, $\ell = 1, ..., L$, two more diagonal $N \times N$ matrices W_{M+k+1}, W_{M+k+2} associated with the constraints $\mathrm{Diag}(\rho^+) \succeq 0$, $\mathrm{Diag}(\rho^-) \succeq 0$, respectively, and 1×1 matrix W_{M+k+3} associated with the constraint $\nu \geq 0$.

The fact that $x(t)$ minimizes the aggregate (3.5) over $\mathcal{X}'$ means exactly that the vector

$$\mathcal{A}^* W + c$$

is proportional to the vector e defining, via the equality constraint $e^T x = 1$, the affine span of $\mathcal{X}$; here $\mathcal{A}^*$ is the operator conjugate to $\mathcal{A}$, i.e., $tr(y[\mathcal{A}x]) = (\mathcal{A}^* y)^T x$ for all $x \in \mathbb{R}^N, y \in \Sigma$. Now, the only nonzero component of vector e for the problem (3.4) is the λ-component, and this latter component is comprised of ones. Substituting in the relation

$$\mathcal{A}^* W + c = \theta e, \tag{6.11}$$

the particular data of (3.4), we end up with the following system of relations (where $\mathrm{diag}(Q)$ denotes the diagonal of a square matrix, and $\mathrm{diag}_i(Q)$ is i-th diagonal entry of the matrix):

$$\begin{array}{llcl}
(a.1) & \sum_{m=1}^{M} tr(\Xi_m) + W_{M+k+3} & = & \alpha; \\
(a.2) & tr(\Xi_m) + \mathrm{diag}_m(W_{M+k+1}) & = & s_m^+, m = 1, ..., M; \\
(a.3) & tr(\Xi_m) - \mathrm{diag}_m(W_{M+k+2}) & = & s_m^-, m = 1, ..., M; \\
(b) & 2\sum_{m=1}^{M} tr(Z_m^T(w) Q_m) & & \\
 & -\sum_{\ell=1}^{L} tr(W_{M+i}\mathrm{Diag}(C^\ell w^\ell)) & = & -2\sum_{\ell=1}^{L} (f^\ell)^T w^\ell \quad \forall w = (w^1, ..., w^L), \\
 & & & Z_m^T(w) = [\zeta_m(w^1); ...; \zeta_m(w^L)]; \\
(c) & \sum_{m=1}^{M} tr(R_m \pi(\lambda)) & & \\
 & +\sum_{\ell=1}^{L} tr(W_{M+i}\mathrm{Diag}(\delta^\ell \lambda_\ell)) & = & \theta \sum_{\ell=1}^{L} \lambda_\ell \quad \forall \lambda \in \mathbb{R}^L,
\end{array} \tag{6.12}$$

where $\pi(\lambda)$, $\lambda \in \mathbb{R}^L$, is the $kS \times kS$ diagonal matrix where the first S diagonal entries are equal to λ_1, the next S entries are equal to λ_2, and so on.

Note that (6.12(a)) along with evident positive definiteness of all W_m (and, consequently, of all E_m^+) demonstrate that E^+ is a feasible solution to (3.2).

We have

$$\begin{array}{rcl}
-\psi^* & \leq & -c^T x(t) \\
 & = & (\mathcal{A}^* W - \theta e)^T x(t) \\
 & & [\text{see } (6.11)] \\
 & = & tr(W[\mathcal{A}x]) - \theta \\
 & & [\text{since } e^T x(t) = 1] \\
 & = & t^{-1}[N(kS + D + 2) + \sum_{\ell=1}^{L} \dim(\delta^\ell) + 1] - \theta \\
 & & [\text{since } W = t^{-1}[\mathcal{A}x(t)]^{-1}] \\
 & = & \Delta(t) - \theta.
\end{array} \tag{6.13}$$

According to (6.13), we have

$$\theta \leq \psi^* + \Delta(t). \tag{6.14}$$

Now let $v = (v^1, \ldots, v^L; \lambda) \in \mathcal{V}'$, $\nu \geq 0$, $\rho^{\pm} \in \mathbb{R}^M_+$. Let us evaluate from above the quantity $T(E^+; v, \nu, \rho^+, \rho^-)$. The matrices

$$\begin{array}{rcl} A_m & = & \begin{pmatrix} \sum_{\ell=1}^L \lambda_\ell^{-1} \zeta_m(v^\ell)\zeta_m^T(v^\ell) & Z_m^T(v^1, \ldots, v^l) \\ Z_m(v^1, \ldots, v^l) & \pi(\lambda) \end{pmatrix}, \ m = 1, ..., M, \\ A_m & = & \mathrm{Diag}(\lambda_\ell \delta^\ell - C^\ell v^\ell), \ l = M + i, \ell = 1, ..., k \\ A_{M+k+1} & = & \mathrm{Diag}(\{\omega_m^{-1} \rho_m^+\}_{m=1}^M), \\ A_{M+k+2} & = & \mathrm{Diag}(\{\omega_m^{-1} \rho^- l\}_{m=1}^M), \\ A_{M+k+3} & = & \nu \end{array}$$

clearly are positive semidefinite, so that

$$\begin{array}{rcl} 0 & \leq & \sum_{m=1}^{M+k+3} tr(W_m A_m) \\ & = & \sum_{m=1}^M tr \quad \Xi_m \sum_{\ell=1}^L \lambda_\ell^{-1} \zeta_m(v^\ell)\zeta_m^T(v^\ell) \\ & & +2\sum_{m=1}^M tr(Z_m^T(v^, \ldots, v^L)Q_m) + \sum_{m=1}^M tr(R_m \pi(\lambda)) \\ & & + \sum_{\ell=1}^L tr(\mathrm{Diag}(\lambda_\ell \delta^\ell - C^\ell v^\ell) W_{M+i}) \\ & & + \sum_{m=1}^M \omega_m^{-1}[\rho_m^+ \mathrm{diag}_m(W_{M+k+1}) + \rho_m^- \mathrm{diag}_m(W_{M+k+2})] \\ & & +\nu W_{M+k+3} \\ & = & \sum_{m=1}^M tr \quad \Xi_m \sum_{\ell=1}^L \lambda_\ell^{-1} \zeta_m(v^\ell)\zeta_m^T(v^\ell) \quad - 2\sum_{\ell=1}^L (f^\ell)^T v^\ell \\ & & +\theta \sum_{\ell=1}^L \lambda_\ell \\ & & + \sum_{m=1}^M \omega_m^{-1}[\rho_m^+(s_m^+ - tr(\Xi_m)) + \rho_m^-(tr(\Xi_m) - s_m^-)] \\ & & +\nu W_{M+k+3} \\ & & [\text{we have used } (6.12(a.2, a.3, b, c))] \\ & = & \sum_{m=1}^M tr \quad \Xi_m^+ \sum_{\ell=1}^L \lambda_\ell^{-1} \zeta_m(v^\ell)\zeta_m^T(v^\ell) \quad - 2\sum_{\ell=1}^L (f^\ell)^T v^\ell \\ & & + \sum_{m=1}^M \omega_m^{-1}[\rho_m^+(s_m^+ - tr(\Xi_m)) + \rho_m^-(tr(\Xi_m) - s_m^-)] \\ & & +\nu(\alpha - \sum_{m=1}^M tr(\Xi_m)) + \theta \\ & & [\text{we have used } (6.12(a.1))] \\ & = & -T(E^+; v, \nu, \rho^+, \rho^-) + \theta \\ & & [\text{see } (6.1)] \\ & & \Rightarrow \\ T(E^+; u, \nu, \rho^+, \rho^-) & \leq & \theta \leq \psi^* + \Delta(t), \end{array}$$

the concluding inequality being given by (6.14). Applying (I), we conclude that the value of the objective of (3.2) at E^+ is at most by $\Delta(t)$ greater than ψ^*. Since the optimal value in (3.2) is $\phi^* \geq \psi^*$ (see (6.7)), this observation completes the proof.

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