# Structural Design via Semidefinite Programming [1)]

**Aharon Ben-Tal** and **Arkadi Nemirovski**

morbt@ie.technion.ac.il nemirovs@ie.technion.ac.il

Faculty of Industrial Engineering and Management
Technion – Israel Institute of Technology,
Technion City, Haifa 32000, Israel

**Abstract**

We consider a general problem of Structural Design and demonstrate that under mild assumptions it can be posed as a Semidefinite program. The approach in question allows for unified treatment of single-load/multi-load Truss and Shape design with and without obstacles, same as for obstacle-free robust Truss and Shape design. We present three equivalent Semidefinite formulations of the Structural Design problem, discuss possibilities of their computational processing by interior point methods, same as the techniques for recovering optimal structures from the solutions to the corresponding SDP's.

## 1 Structural Design: general setting

The problem we are interested in is, in its general setting, as follows. Consider a mechanical construction $\mathcal{S}$ with finitely many degrees of freedom $M$; thus, virtual displacements of the system are specified by vectors $w \in \mathbf{R}^M$. The potential energy capacitated by the construction under a displacement $w$ is assumed to be a nonnegative quadratic function $\frac{1}{2}w^T\mathcal{Q}w$ of the displacement, $\mathcal{Q}$ being a symmetric positive semidefinite matrix characterizing the construction; this matrix is assumed to depend linearly on the vector $t$ of design parameters of the construction:

$$\mathcal{Q} = \mathcal{Q}(t).$$

We are given also a set $W \subset \mathbf{R}^M$ of *kinematically admissible* displacements.

The construction can be subject to an external *load*; mathematically, such a load is a vector $f \in \mathbf{R}^M$. The displacement caused by the load maximizes the function

$$2f^Tw - w^T\mathcal{Q}(t)w$$

over $w \in W$; the corresponding optimal value

$$\mathrm{compl}(f;t) = \sup_{w \in W}\left[2f^Tw - w^T\mathcal{Q}(t)w\right]$$

is twice the *compliance* of the construction under load $f$; the less is this compliance, the better are the rigidity properties of the construction with respect to the load.

The problem of *optimal structural design* in its general setting is as follows: given a set $\mathcal{F} \subset \mathbf{R}^M$ of tentative loads and a set $T$ of admissible values of the design vector $t$, we are

[1)] The research was partly supported by the Israel Science Foundation grant # 306/94-3, by the Fund for the Promotion of Research at Technion and by the G.I.F. contract No. I-0455-214.06/95

interested to find $t \in T$ which minimizes the worst-case, w.r.t. loads from $\mathcal{F}$, compliance of the construction, i.e., to solve the problem

$$(P_{\rm ini}) \qquad \min\left\{ \mathrm{compl}_{\mathcal{F}}(t) \equiv \sup_{f\in\mathcal{F}} \mathrm{compl}(f;t) : t \in T \right\}$$

The above general setting has two particular cases which are of especial interest.

**Truss Design.** A *truss* is a construction, like an electric mast or the Eifel Tower, comprised of thin elastic *bars* linked with each other at *nodes*. In the standard Truss Topology Design (TTD) problem the nodes form a given in advance finite set in $\mathbf{R}^d$, where $d = 2$ for planar and $d = 3$ for spatial constructions, and all pair connections of nodes by bars are allowed. For every node, allowed displacements of it form a given linear subspace of $\mathbf{R}^d$, and the space $\mathbf{R}^M$ of virtual displacements of the construction is just the direct product of these spaces over all nodes. The set $W$ of admissible displacements is cut off $\mathbf{R}^M$ by a number of inequality constraints (typically linear) representing *obstacles* – restrictions on the displacements of the nodes like absolutely rigid partial supports.

The design variables of the truss are volumes $t_l$ of tentative bars, and the corresponding matrix $\mathcal{Q}(t)$ is of the form

$$\bar{Q}(t) = \sum_{l=1}^{N} b_l b_l^T t_l,$$

where $N$ is the number of tentative bars and $b_l \in \mathbf{R}^M$ are given vectors. As about the set $T$ of feasible design vectors, it always is a subset of $\mathbf{R}^N_+$ – bar volumes should be nonnegative – satisfying the *resource constraint*

$$\sum_{l=1}^{N} t_l \le v$$

(upper bound on the weight of the construction), and, possibly, some other (normally linear) constraints. The most important case is the one of

$$\begin{array}{c} T = \{t \in \mathbf{R}^N_+ \mid \underline{\rho}_l \le t_l \le \overline{\rho}_l,\ l = 1, ..., N; \sum_{l=1}^N t_l \le v\}, \\ [0 \le \underline{\rho}_l \le \overline{\rho}_l < \infty,\ \ l = 1, ..., N]. \end{array} \tag{1}$$

**Shape Design.** In the case of Shape Design the construction in question occupies a given 2D or 3D domain $\Omega$ and is comprised of material with continuously varying from point to point mechanical properties; thus, in fact we are speaking about a "distributed" mechanical system with infinitely many degrees of freedom. However, in order to get a computationally tractable model, we from the very beginning apply the finite element method in order to pass from the actual model to its finite-dimensional approximation. Namely, we replace the infinite-dimensional space of displacements of the actual construction (which is the space of vector fields on $\Omega$) by its finite-dimensional subspace $\mathbf{R}^M$. Similarly, we partition $\Omega$ in $N$ cells $C_l$, $l = 1, ..., N$, and assume that the mechanical properties of the material are constant within every cell. With this approximation, the potential energy capacitated by the construction under displacement $w$ is

$$\sum_{l=1}^{N} \nu_l^{-1}\, \mathrm{Tr}(t_l \int_{C_l} e_P(w) e_P^T(w) dP), \tag{2}$$

where

- $\nu_l$ is $d$-dimensional volume of cell $C_l$;
- $e_P(w)$ is the "strain tensor" caused by displacement $w$ at a point $P \in \Omega$; all which is important for us is that $e_P(w)$ is an $L_\infty$ function of $P$ taking values in the Euclidean space $\mathbf{R}^D$ ($D = d(d+1)/2$), and that $e_P(w)$ is linear in $w \in \mathbf{R}^M$;
- $\nu_l^{-1} t_l$ is the "rigidity tensor of the material" in the cell $C_l$; mathematically it means that $t_l$ is a symmetric positive semidefinite $D \times D$ matrix.

The set $T$ of feasible design vectors is always a subset of the space $(\mathbf{S}^D_+)^N$, $\mathbf{S}^D_+$ being the cone of positive semidefinite $D \times D$ matrices – the rigidity tensors should be positive semidefinite. Typical additional restrictions defining $T$ are the "resource constraints" imposed on the quantities $\mathrm{Tr}(t_l)$ – these quantities in a sense measure densities of the material in the cells. The most important case is the one of

$$T = \{t \in (\mathbf{S}^D_+)^N \mid \underline{\rho}_l \le \mathrm{Tr}(t_l) \le \overline{\rho}_l,\ l = 1, ..., N; \sum_{l=1}^N \mathrm{Tr}(t_l) \le v\}, \quad [0 \le \underline{\rho}_l \le \overline{\rho}_l < \infty,\ l = 1, ..., N]. \tag{3}$$

**The "standard" case.** In fact the Truss and the Shape problems can be covered by a single particular case of our general setting – the one where the "design variables" $t_l$ are positive semidefinite symmetric matrices of row dimension $D_l$, the constraints defining $T$ are restrictions on vectors comprised of traces of these matrices, and $\mathcal{Q}(t)$ has the following structure:

$$\mathcal{Q}(t) = \sum_{l=1}^{N} \sum_{s=1}^{S} b_{ls} t_l b_{ls}^T, \tag{4}$$

where $b_{ls}$ are given $M \times D$ matrices.

Indeed, the Truss problem clearly fits the indicated scheme (with $D = S = 1$). To see that the Shape problem also fits it, note that there exists $S$, "quadrature grids" $\{P_{ls}\}_{s=1}^S$ and "quadrature weights" $\{\gamma_{ls}^2\}_{s=1}^S$, $l = 1, ..., N$, such that

$$\frac{1}{\nu_l} \int_{C_l} e_P(w) e_P^T(w) = \sum_{s=1}^{S} \gamma_{ls}^2 e_{P_{ls}}(w) e_{P_{ls}}^T(w)$$

identically in $w \in \mathbf{R}^M$. Defining matrices $b_{ls}$ by the relation

$$b_{ls}^T w = \gamma_{ls} e_{P_{ls}}(w), \quad w \in \mathbf{R}^M,$$

we represent the potential energy (2) in the form of

$$w^T \mathcal{Q}(t) w$$

with $\mathcal{Q}(t)$ being given by (4).

In what follows we refer to our original setting as to the *general case*, and specify its particular *standard case* as the one where

**S.1.** The space $\mathbf{R}^n$ of design vectors is the direct product of $N$ spaces $\mathbf{S}^{D_l}$ of symmetric $D_l \times D_l$ matrices, so that the design vector is

$$t = (t_1, ..., t_N) : t_l \in \mathbf{S}^{D_l},\ l = 1, ..., N;$$

**S.2.** The mapping $t \mapsto \mathcal{Q}(t)$ is

$$\mathcal{Q}(t) = \sum_{l=1}^{N} \sum_{s=1}^{S} b_{ls} t_l b_{ls}^T,$$

$b_{ls}$ being given $M \times D_l$ matrices;

**S.3** The set $W$ of kinematically admissible displacements is a polytope

$$W = \{w \in \mathbf{R}^M \mid Rw \leq r\},$$

and the system of constraints $Rw \leq r$ satisfies the Slater condition;

**S.4.** The set $T$ of admissible design vectors is

$$T = \{t \in (\mathbf{S}_+^D)^N \mid \underline{\rho}_l \leq \mathrm{Tr}(t_l) \leq \overline{\rho}_l,\ l = 1, ..., N; \textstyle\sum_{l=1}^N \mathrm{Tr}(t_l) \leq v\},$$
$$\left[0 \leq \underline{\rho}_l < \overline{\rho}_l < \infty,\ l = 1, ..., N; \textstyle\sum_{l=1}^N \underline{\rho}_l < v.\right]$$

From now on, $\mathbf{S}_+^q$ denotes the cone of positive semidefinite symmetric $q \times q$ matrices.

**S.5.** The matrix

$$\sum_{l=1}^{N} \sum_{s=1}^{S} b_{ls} b_{ls}^T$$

is positive definite.

**The set of loads** $\mathcal{F}$ in the traditional literature on Structural Design (see [1, 2, 3, 5, 6, 7] and references therein) either is a singleton $\{f\}$ ("single-load case") or a finite set $\{f_1, f_2, ..., f_k\}$ ("multi-load case"). Recently, the *robust* setting of the problem was proposed and motivated [4], where $\mathcal{F}$ is an ellipsoid:

$$\mathcal{F} = \{f = Qu \mid u \in \mathbf{R}^k, u^T u \leq 1\}.$$

The rest of the paper is organized as follows. In Section 2 we demonstrate that under reasonable assumptions our general Structural Design problem can be posed as a Semidefinite program; in particular, this is true for the situations we actually are interested in, namely, for

(i) The Standard case of the multi-load Structural design;

(ii) The obstacle-free (i.e., with $W = \mathbf{R}^M$) Standard case of the robust Structural design.

Moreover, we present a Semidefinite program $(P)$ which, on one hand, is enough specific to allow for instructive processing, and on the other hand, is enough general to cover the "problems of interest" (i) – (ii). Sections 3 and 4 are devoted to mathematical processing of $(P)$; namely, in Section 3 we demonstrate that the Fenchel dual to $(P)$ admits eliminating "most" of the variables, so that the resulting problem $(D)$ in some important cases (e.g., (i)) is much better suited for numerical solution than the original problem $(P)$. In Section 4 we build the Fenchel dual to $(D)$, thus coming to an instructive equivalent reformulation $(P^+)$ of $(P)$. Section 5 contains list of explicit formulations of the problems $(P)$, $(D)$, $(P^+)$ for the situations of interest (i) – (ii). The concluding Section 6 is devoted to discussing the relevant computational issues, in particular, to those of recovering the design parameters of the nearly optimal construction from nearly optimal solutions to $(D)$.

## 2 Semidefinite Reformulation of $(P_{\rm ini})$

We are about to demonstrate that problem $(P_{\rm ini})$ normally can be modeled as a semidefinite program.

**Notation and conventions.** In what follows $\mathbf{S}^p$ denotes the space of symmetric $p\times p$ matrices, $\mathbf{S}^p_+$ is the cone of positive semidefinite matrices from $\mathbf{S}^p$, and $\mathbf{L}^{p,q}$ – the space of rectangular $p\times q$ matrices with real entries. The spaces $\mathbf{S}^p$, $\mathbf{L}^{p,q}$ are regarded as Euclidean spaces equipped with the Frobenius inner product

$$\langle x, y\rangle = \mathrm{Tr}(x^T y).$$

Given a linear operator $A : E \to F$ from Euclidean space $(E, \langle\cdot,\cdot\rangle_E)$ to Euclidean space $(F, \langle\cdot,\cdot\rangle)$, we denote by $A^* : F \to E$ the conjugate of $A$:

$$\langle Ae, f\rangle_F = \langle e, A^* f\rangle\,,\ e\in E, f\in F.$$

**The central observation** is as follows:

**Proposition 2.1** *Assume that $W$ is a closed convex set with a nonempty interior, let*

$$W^+ = \{(w,u)\mid\ u>0, u^{-1}w\in W\}$$

*be the conic hull of $W$, and let*

$$W^* = \{(\nu,\mu)\in\mathbf{R}^M\times\mathbf{R} : \nu^T w\le\mu\quad\forall w\in W\}.$$

*Assume that $\mathcal{Q}(t)\ge 0$ for $t\in T$.*

*For $s\in\mathbf{R}$, $t\in T$ and $f\in\mathbf{R}^M$, the relation*

$$\mathrm{compl}(f;t)\le s$$

*is satisfied if and only if there exists $(\nu,\mu)\in W^*$ such that*

$$\mathcal{C}(f,t,s,\nu,\mu)\equiv\begin{pmatrix} s-\mu & f^T+\frac{1}{2}\nu^T\\ f+\frac{1}{2}\nu & \mathcal{Q}(t)\end{pmatrix}\ge 0;$$

*from now on, an inequality between symmetric matrices is understood as positive semidefiniteness of the corresponding difference.*

**Proof.** We have

$$\mathrm{compl}(f;t)\le s \Leftrightarrow g(w)\equiv w^T\mathcal{Q}(t)w-2f^Tw\ge -s\quad\forall w\in W\Leftrightarrow \inf_{w\in\mathbf{R}^M}[g(w)+\delta(w)]\ge -s,$$

where $\delta(w)=\begin{cases}0, & w\in W\\ +\infty, & w\notin W\end{cases}$. Now, for $t\in T$ the function $g(w)$ is convex and $W$ is a closed convex set with a nonempty interior. Thus, we can apply the Fenchel Duality Theorem to get the equivalence

$$\{\mathrm{compl}(f;t)\le s\}_1\Leftrightarrow\{\exists\nu\in\mathbf{R}^M : g^*(\nu)+\delta^*(-\nu)\ge -s\}_2, \tag{5}$$

where

$$h^*(\pi)=\inf_{w\in\mathbf{R}^M}\{h(w)-\pi^T w\}.$$

Now assume that $\mathrm{compl}(f;t) \leq s$. Then in view of the above equivalence, there exists $\nu$ such that

$$g^*(\nu) \geq -s - \delta^*(-\nu). \tag{6}$$

Setting

$$\begin{array}{rcl} -\mu & = & \delta^*(-\nu) \\ & \equiv & \inf_{w \in W} (-\nu)^T w, \end{array},$$

we conclude that $(\nu, \mu) \in W^*$. On the other hand, (6) means exactly that

$$w^T \mathcal{Q}(t) w - 2 f^T w - \nu^T w \geq -s + \mu \quad \forall w \in \mathbf{R}^M, \tag{7}$$

or, which is the same, that the matrix $\mathcal{C}(f, t, s, \nu, \mu)$ is positive semidefinite.

Vice versa, if there exists $(\nu, \mu) \in W^*$ such that the matrix $\mathcal{C}(f, t, s, \nu, \mu)$ is positive semidefinite, then, inverting step by step our reasoning, we get that $(\nu, \mu) \in W^*$ and that the predicate $\{\cdot\}_2$ is true; by (5) we get $\mathrm{compl}(f;t) \leq s$. ■

In view of Proposition (2.1), in order to be able to convert $(P_{\rm ini})$ to an equivalent semidefinite program, it suffices to make the following two assumptions:

**(I).** The set $T$ can be represented via Linear Matrix Inequalities, i.e., there exists a symmetric matrix $\mathcal{T}(t, \tau)$ affinely depending on $t$ and a vector of additional variables $\tau$ such that

$$t \in T \Leftrightarrow \exists \tau : \mathcal{T}(t, \tau) \geq 0;$$

**(II).** The predicate

$$\mathcal{R}(t, s) : \quad \forall (f \in \mathcal{F}) \exists ((\nu, \mu) \in W^*) : \mathcal{C}(f, t, s, \nu, \mu) \geq 0$$

can be represented via Linear Matrix Inequalities, i.e., there exists a symmetric matrix $\mathcal{P}(t, s, \zeta)$ affinely depending on $t, s$ and a vector of additional variables $\zeta$ such that the above relation is satisfied for a given $(t, s)$ if and only if there exists $\zeta$ such that $\mathcal{P}(t, s, \zeta) \geq 0$.

Under these assumptions, the semidefinite reformulation of $(P_{\rm ini})$ is the semidefinite program

$$\begin{array}{lrcl} \text{minimize} & & s & \\ \text{s.t.} & & & \\ & \mathcal{T}(t, \tau) & \geq & 0, \\ & \mathcal{P}(t, s, \zeta) & \geq & 0. \end{array} \tag{8}$$

Let us list several cases where we can easily satisfy the assumptions **(I), (II)**.

**Example I. Single-load Structural Design with polyhedral $W$ and "simple" $T$.** Assume that

$$\mathcal{F} = \{f\}$$

and that

$$\begin{array}{rcl} W & = & \{w \in \mathbf{R}^M \mid Rw \leq r\}, \\ T & = & \{(t_1, ..., t_N) \mid t_l \in \mathbf{S}_+^{D_l}, \, l = 1, ..., N; Pt \leq p\} \end{array} \tag{9}$$

Here one can set

$$\begin{array}{rcl} \mathcal{T}(t) & = & \mathrm{Diag}(t_1, t_2, ..., t_N, \mathrm{Diag}(p - Pt)), \\ \mathcal{P}(t, s, y, \mu) & = & \mathrm{Diag}(\mathcal{C}(f, t, s, R^T y, \mu), \mathrm{Diag}(y), \mu - r^T y); \end{array}$$

Indeed, the LMI $\mathcal{T}(t) \geq 0$ clearly represents the inclusion $t \in T$. Now, the inclusion $(\nu, \mu) \in W^*$ clearly is equivalent to the existence of $y$ such that $y \geq 0$, $\nu = R^T y$ and $\mu \geq r^T y$; consequently,

$$\begin{array}{c}\forall (f \in \mathcal{F}) \exists ((\nu, \mu) \in W^*) : \mathcal{C}(f, t, s, \nu, \mu) \geq 0 \\ \Updownarrow \\ \exists (y, \mu) : \mathcal{C}(f, t, s, R^T y, \mu) \geq 0, y \geq 0, \mu \geq r^T y \\ \Updownarrow \\ \exists (y, \mu) : \mathcal{P}(t, s, y, \mu) \geq 0.\end{array}$$

Note that in the standard case the resulting problem (8) can be written down in the following form:

$$\begin{array}{llcl}
(I) & & & \\
\text{minimize} & s & & \\
\text{s.t.} & & & \\
(i) & \begin{pmatrix} s + Dz + d & [Ez + e]^T \\ [Ez + e] & \sum_{l=1}^N \sum_{s=1}^S b_{ls} t_l b_{ls}^T \end{pmatrix} & \geq & 0; \\
(ii) & t_l & \geq & 0, \ l = 1, ..., N; \\
(iii) & \underline{\rho}_l \leq \mathrm{Tr}(t_l) & \leq & \overline{\rho}_l, \ l = 1, ..., N; \\
(iv) & \sum_{l=1}^N \mathrm{Tr}(t_l) & \leq & v; \\
(v) & z & \geq & 0,
\end{array}$$

where

$$\begin{array}{rcl} z & \equiv & (y, \mu - r^T y) \\ Dz + d & = & -\mu, \\ Ez + e & = & f + \frac{1}{2} R^T y \end{array}$$

Note that there exists $\zeta > 0, \epsilon$:

$$D^* \zeta + 2E^* \epsilon < 0;$$

as it is immediately seen, this nothing but the assumption **S.3**.

**Example II. Multi-load Structural Design with polyhedral $W$ and "simple" $T$.** Now assume that

$$\mathcal{F} = \{f_1, ..., f_k\}$$

and that $W, T$ are given by (9). To meet **(I), (II)**, one can set

$$\begin{array}{rcl} \mathcal{T}(t) & = & \mathrm{Diag}(t_1, t_2, ..., t_N, \mathrm{Diag}(p - Pt)), \\ \mathcal{P}(t, s, y_1, ..., y_k, \mu_1, ..., \mu_k) & = & \mathrm{Diag}(\{\mathcal{C}(f_i, t, s, R^T y_i, \mu_i), \mathrm{Diag}(y_i), \mu_i - r^T y_i\}_{i=1}^k). \end{array}$$

Note that in the standard case the resulting problem (8) can be written down in the following form:

$$\begin{array}{llcl}
(II) & & & \\
\text{minimize} & s & & \\
\text{s.t.} & & & \\
(i) & \begin{pmatrix} s + D_i z + d_i & [E_i z + e_i]^T \\ [E_i z + e_i] & \sum_{l=1}^L \sum_{s=1}^S b_{ls} t_l b_{ls}^T \end{pmatrix} & \geq & 0, \ i = 1, ..., k; \\
(ii) & t_l & \geq & 0, \ l = 1, ..., N; \\
(iii) & \underline{\rho}_l \leq \mathrm{Tr}(t_l) & \leq & \overline{\rho}_l, \ l = 1, ..., N; \\
(iv) & \sum_{l=1}^N \mathrm{Tr}(t_l) & \leq & v; \\
(v) & z & \geq & 0,
\end{array}$$

where

$$\begin{array}{rcl} z & \equiv & (y_1, \mu_1 - r^T y_1, y_2, \mu_2 - r^T y_2, ..., \mu_k - r^T y_k) \\ D_i z + d_i & = & -\mu_i, \\ E_i z_i + e_i & = & f_i + \frac{1}{2} R^T y_i \end{array}$$

Note that here again the assumption **S.3** ensures that for every $i = 1, ..., k$ there exists $\zeta_i > 0, \epsilon_i$:

$$D_i^* \zeta_i + 2E_i^* \epsilon_i < 0.$$

**Example III. Robust Structural Design without obstacles and with "simple" $T$.** Now assume that

$$\mathcal{F} = \{f = Qu \mid u \in \mathbf{R}^k, u^T u \leq 1\}$$

and that

$$\begin{array}{rcl} W & = & \mathbf{R}^M, \\ T & = & \{(t_1, ..., t_N) \mid t_l \in \mathbf{S}_+^{D_l}, \, l = 1, ..., N; Pt \leq p\} \end{array} \tag{10}$$

In this case $W^* = \{(0, \mu) \mid \mu \geq 0\}$, and we have

$$\begin{array}{c} \forall (f \in \mathcal{F}) \exists ((\nu, \mu) \in W^*) : \begin{pmatrix} s - \mu & f^T + \frac{1}{2}\nu^T \\ f + \frac{1}{2}\nu & \mathcal{Q}(t) \end{pmatrix} \geq 0 \\ \Updownarrow \\ \forall (f \in \mathcal{F}) : \begin{pmatrix} s & f^T \\ f & \mathcal{Q}(t) \end{pmatrix} \geq 0 \\ \Updownarrow \\ \forall (u : u^T u \leq 1) : \begin{pmatrix} s & u^T Q^T \\ Qu & \mathcal{Q}(t) \end{pmatrix} \geq 0 \\ \Updownarrow \\ \forall (u : u^T u \leq 1, w, p) : sp^2 + 2w^T Q(pu) + w^T \mathcal{Q}(t) w \geq 0 \\ \Updownarrow \\ \forall (u : u^T u = 1, w, p) : sp^2 + 2w^T Q(pu) + w^T \mathcal{Q}(t) w \geq 0 \\ \Updownarrow \\ \forall (u' = pu, u^T u = 1, w) : s(u')^T u' + 2w^T Q u' + w^T \mathcal{Q}(t) w \geq 0 \\ \Updownarrow \\ \begin{pmatrix} sI_k & Q^T \\ Q & \mathcal{Q}(t) \end{pmatrix} \geq 0, \\ \Updownarrow \\ \exists (\mu \geq 0) : \begin{pmatrix} (s - \mu) I_k & Q^T \\ Q & \mathcal{Q}(t) \end{pmatrix} \geq 0, \end{array}$$

$I_q$ being the $q \times q$ unit matrix.

Thus, to meet **(I), (II)** it suffices to set

$$\begin{array}{rcl} \mathcal{T}(t) & = & \mathrm{Diag}(t_1, t_2, ..., t_N, \mathrm{Diag}(p - Pt)), \\ \mathcal{P}(t, s, \mu) & = & \begin{pmatrix} (s - \mu) I_k & Q^T \\ Q & \mathcal{Q}(t) \end{pmatrix}. \end{array}$$

Note that in the standard case the resulting problem (8) can be written down in the following form:

$$
\begin{array}{llcl}
(III) & & & \\
\text{minimize} & s & & \\
\text{s.t.} & & & \\
(i) & \begin{pmatrix} sI_k + Dz + d & [Ez+e]^T \\ [Ez+e] & \sum_{l=1}^{N}\sum_{s=1}^{S} b_{ls} t_l b_{ls}^T \end{pmatrix} & \geq & 0; \\
(ii) & t_l & \geq & 0, \ l = 1, ..., N; \\
(iii) & \underline{\rho}_l \leq \mathrm{Tr}(t_l) & \leq & \overline{\rho}_l, \ l = 1, ..., N; \\
(iv) & \sum_{l=1}^{N} \mathrm{Tr}(t_l) & \leq & v; \\
(v) & z & \geq & 0,
\end{array}
$$

where

$$
\begin{array}{rcl}
z & \equiv & \mu \\
Dz + d & = & -\mu I_k, \\
Ez + e & = & Q
\end{array}
$$

Note that here again exists $\zeta \in \mathbf{S}_+^k$ (e.g., $\zeta = I_k$) and $\epsilon \in \mathbf{L}^{M,k}$ (e.g., $\epsilon = 0$) such that

$$D^*\zeta + 2E^*\epsilon < 0.$$

**Intermediate summary.** We have demonstrated that the Structural Design problem ($P_{\text{ini}}$) in three important situations I, II, III above can be posed as a semidefinite program. Note, however, that some of the resulting semidefinite programs are badly suited for numerical processing: they are of design dimension of order of $N$ and involve "large" LMI's – with row size of order of $M$. We are about to demonstrate that in the standard case, as described above, it is possible to apply semidefinite duality to simplify the SDP's in question. In what follows we focus on the standard case in the situations I, II, III, i.e., on problems (I) – (III). As we just have seen, these three problems are covered by the following single setting:

$$
\begin{array}{llcl}
(P) & & & \\
\text{minimize} & s & & \\
\text{s.t.} & & & \\
(i) & \begin{pmatrix} sI_{p_i} + D_i z + d_i & [E_i z+e_i]^T \\ [E_i z+e_i] & \sum_{l=1}^{N}\sum_{s=1}^{S} b_{ls} t_l b_{ls}^T \end{pmatrix} & \geq & 0, \ i = 1, ..., k; \\
(ii) & t_l & \geq & 0, \ l = 1, ..., N; \\
(iii) & \underline{\rho}_l \leq \mathrm{Tr}(t_l) & \leq & \overline{\rho}_l, \ l = 1, ..., N; \\
(iv) & \sum_{l=1}^{N} \mathrm{Tr}(t_l) & \leq & v; \\
(v) & z & \geq & 0,
\end{array}
$$

the design variables in the problem being $s \in \mathbf{R}$, $\{t_l \in \mathbf{S}^{D_l}\}_{l=1}^{N}$, $z \in \mathbf{R}^q$. From now on we focus on problem $(P)$; the results related to the "problems of actual interest" (I) – (III) will be obtained by specifying the data of $(P)$ accordingly.

When processing $(P)$, we always make the following assumptions:

**A.** The objective of $(P)$ is below bounded on the feasible set of the problem

**B.** $(P)$ is strictly feasible, i.e., there exists a feasible solution at which LMI's $(P).(i) - (v)$ are satisfied as strict

**C.** For every $i = 1, ..., k$ there exists $w_i \in \mathbf{L}^{M,p_i}$ and a positive definite $p_i \times p_i$ matrix $\alpha_i$ such that the $q$-dimensional vector

$$D_i^* \alpha_i + 2E_i^* w_i$$

is negative.

Note that in the cases (I), (II), (III) we actually are interested in the assumptions **A** – **C** indeed are satisfied. The assumption **C** was verified explicitly when formulating the problems (I) – (III). **A** immediately follows from the fact that in the situations (I) – (III) the $s$-component of a feasible solution to $(P)$ is an upper bound on the compliance of certain construction (given by a design vector $t \in T$) w.r.t. a once for ever fixed load $f$; since $T$ is compact by **S.4**, the compliances of all constructions coming from $T$ w.r.t. $f, W$ are uniformly below bounded. To verify **B**, note that by **S.4** there exists $t^* = \{t_l^*\}_{l=1}^N$ satisfying strict versions of the constraints $(ii)$ – $(iv)$. By **S.5** $\mathcal{Q}(t^*)$ is positive definite. Now let us choose somehow positive vector $z^*$; since $\mathcal{Q}(t^*)$ is positive definite, the collection $(s^*, t^*, z^*)$ is, for all large enough values of $s^*$, a strictly feasible solution to $(P)$.

In what follows we refer to $(P)$ as to *primal* form of the Structural Design problem. Sometimes it will be more convenient to use the following generic notation for $(P)$:

$$c^T x \to \min \mid \mathcal{A}x + b \geq 0, \tag{11}$$

where

- $x = (t, s, z) \in \mathbf{R}^n$ is the design vector of the problem;
- $x \mapsto \mathcal{A}x$ is a homogeneous linear mapping from $\mathbf{R}^n$ to the space $\mathbf{S}$ of symmetric block-diagonal matrices of certain fixed block-diagonal structure;
- $b \in \mathbf{S}$ is a fixed vector;
- as always, the inequality in (11) means positive semidefiniteness of the block-diagonal matrix written in the left hand side.

# 3 From primal to dual

Here we derive and transform equivalently the Fenchel dual to $(P)$. According to **A, B**, $(P)$ is below bounded and strictly feasible semidefinite program of the form (11); it is known that for such a program one can built its Fenchel dual, which is of the form

$$\mathrm{Tr}(b\xi) \to \min \mid \mathcal{A}^*\xi = c, \xi \geq 0, \tag{12}$$

$\xi \mapsto \mathcal{A}^*\xi$ being the linear mapping from $\mathbf{S}$ to $\mathbf{R}^n$ conjugate to the mapping $x \mapsto \mathcal{A}x$.

Since $(P)$ is below bounded and strictly feasible, the standard results on conic duality imply the following

**Proposition 3.1** *Problem* (12) *is solvable, and its optimal value $D^*$ and optimal value $P^*$ of $(P)$ satisfy the relation*

$$P^* + D^* = 0. \tag{13}$$

*Besides this, the level sets*

$$\{y \geq 0 \mid \mathcal{A}^* y = c, \mathrm{Tr}(by) \leq a\}$$

*are bounded for every $a \in \mathbf{R}$.*

Taking into account specific structure of $(P)$, we obtain the following explicit form of the dual problem (12):

$$
\begin{array}{lrcl}
(D_{\rm ini}) & & & \\
\text{minimize} & \phi & \equiv & \sum_{i=1}^k \mathrm{Tr}(d_i\alpha_i + 2e_i^T w_i) \\
& & & +\sum_{l=1}^N [\overline{\rho}_l\sigma_l^+ - \underline{\rho}_l\sigma_l^-] \\
& & & +\upsilon\gamma \\
\text{s.t.} & & & \\
& \begin{pmatrix} \alpha_i & w_i^T \\ w_i & \mu_i \end{pmatrix} & \geq & 0,\ i=1,...,k \\
& & & [\alpha_i \in \mathbf{S}^{p_i}, \mu_i \in \mathbf{S}^M, w_i \in \mathbf{R}^{p_i\times M}], \\
& \tau_l & \geq & 0,\ l=1,...,N \\
& & & [\tau_l \in \mathbf{S}^{D_l}], \\
& \sigma_l^+, \sigma_l^- & \geq & 0,\ l=1,...,N \\
& & & [\sigma_l^+, \sigma_l^- \in \mathbf{R}], \\
& \gamma & \geq & 0 \\
& & & [\gamma \in \mathbf{R}], \\
& \eta & \geq & 0 \\
& & & [\eta \in \mathbf{R}^q], \\
& \sum_{i=1}^k \mathrm{Tr}(\alpha_i) & = & 1, \\
& \sum_{i=1}^k [D_i^*\alpha_i + 2E_i^* w_i] + \eta & = & 0, \\
& \sum_{i=1}^k \sum_{s=1}^S b_{ls}^T \mu_i b_{ls} + \tau_l & & \\
& +[\sigma_l^- - \sigma_l^+ - \gamma] I_{D_l} & = & 0,\ \ l=1,...,N.
\end{array}
$$

Our local goal is to convert $(D_{\rm ini})$ into an equivalent simpler form.

**Step 1. Eliminating $\eta$ and $\{\tau_l\}_{l=1}^N$.** $(D_{\rm ini})$ clearly is equivalent to the problem

$$
\begin{array}{lrcl}
(D') & & & \\
\text{minimize} & \phi & \equiv & \sum_{i=1}^k \mathrm{Tr}(d_i\alpha_i + 2e_i^T w_i) \\
& & & +\sum_{l=1}^N [\overline{\rho}_l\sigma_l^+ - \underline{\rho}_l\sigma_l^-] \\
& & & +\upsilon\gamma \\
\text{s.t.} & & & \\
(a) & \begin{pmatrix} \alpha_i & w_i^T \\ w_i & \mu_i \end{pmatrix} & \geq & 0,\ i=1,...,k; \\
(b) & \sigma_l^+, \sigma_l^- & \geq & 0,\ l=1,...,N; \\
(c) & \gamma & \geq & 0; \\
(d) & \sum_{i=1}^k \mathrm{Tr}(\alpha_i) & = & 1; \\
(e) & \sum_{i=1}^k [D_i^*\alpha_i + 2E_i^* w_i] & \leq & 0; \\
(f) & \sum_{i=1}^k \sum_{s=1}^S b_{ls}^T \mu_i b_{ls} & \leq & [\gamma + \sigma_l^+ - \sigma_l^-] I_{D_l}, \\
& & & l=1,...,N.
\end{array}
$$

**Step 2. Eliminating $\{\mu_i\}_{i=1}^k$.** We start with observation that $(D')$ is strictly feasible. Indeed, let us set, say, $\sigma_l^\pm = 1$. By **C** there exist positive definite matrices $\alpha_i$ and rectangular matrices $w_i$ of appropriate sizes satisfying strict version of $(e)$; by normalization, we may enforce these matrices to satisfy $(d)$. Given indicated $\alpha_i, w_i$ and choosing "large enough" $\mu_i$, we enforce validity of strict versions of $(a)$. Finally, choosing large enough $\gamma > 0$, we enforce strict versions of $(c)$ and $(f)$.

Since $(D')$ is strictly feasible, its optimal value (i.e., $D^*$) is the same as in problem $(D'')$ obtained from $(D')$ by adding to the set of constraints the constraints

$$(g) \quad \alpha_i > 0,\ i = 1, ..., k.$$

Now note that if a collection

$$(\alpha = \{\alpha_i\}_{i=1}^k, w = \{w_i\}_{i=1}^k, \mu = \{\mu_i\}_{i=1}^k, \sigma = \{\sigma_l^{\pm}\}_{l=1}^N, \gamma)$$

is a feasible solution to $(D'')$, then the collection

$$(\alpha, w, \mu(\alpha, w) = \{\mu_i(\alpha, w) = w_i \alpha_i^{-1} w_i^T\}_{i=1}^k, \sigma, \gamma)$$

also is a feasible solution to $(D'')$ with the same value of the objective; indeed, from LMI's $(a)$ it follows that $\mu_i(\alpha, w) \le \mu_i$, so that replacing $\mu_i$ with $\mu_i(\alpha, w)$ we preserve validity of the LMI's $(f)$ as well as $(a)$. Consequently, $(D'')$ is equivalent to the problem

$$\begin{array}{lrcl}
(D''') & & & \\
\text{minimize} & \phi & \equiv & \sum_{i=1}^k \mathrm{Tr}(d_i \alpha_i + 2e_i^T w_i) \\
& & & + \sum_{l=1}^N [\overline{\rho}_l \sigma_l^+ - \underline{\rho}_l \sigma_l^-] \\
& & & + v\gamma \\
\text{s.t.} & & & \\
(b) & \sigma_l^+, \sigma_l^- & \ge & 0,\ l = 1, ..., N; \\
(c) & \gamma & \ge & 0; \\
(d) & \sum_{i=1}^k \mathrm{Tr}(\alpha_i) & = & 1; \\
(e) & \sum_{i=1}^k [D_i^* \alpha_i + 2E_i^* w_i] & \le & 0; \\
(f') & \sum_{i=1}^k \sum_{s=1}^S b_{ls}^T w_i \alpha_i^{-1} w_i^T b_{ls} & \le & [\gamma + \sigma_l^+ - \sigma_l^-] I_{D_l},\ l = 1, ..., N, \\
(g) & \alpha_i & > & 0,\ i = 1, ..., k.
\end{array}$$

Now note that $(g)\&(f')$ clearly is equivalent to the system of LMI's

$$\begin{array}{rcl}
\begin{pmatrix} A(\alpha) & B_l^T(w) \\ B_l(w) & (\gamma + \sigma_l^+ - \sigma_l^-) I_{D_l} \end{pmatrix} & \ge & 0,\ l = 1, ..., N, \\
A(\alpha) & > & 0,
\end{array} \tag{14}$$

where

$$\begin{array}{rcl}
\alpha & = & \{\alpha_i\}_{i=1}^k, \\
w & = & \{w_i\}_{i=1}^k, \\
A(\alpha) & = & \mathrm{Diag}(\overbrace{\alpha_1, ..., \alpha_1}^{S \text{ times}}, \overbrace{\alpha_2, ..., \alpha_2}^{S \text{ times}}, ..., \overbrace{\alpha_k, ..., \alpha_k}^{S \text{ times}}), \\
B_l(w) & = & [b_{l1}^T w_1, b_{l2}^T w_1, ..., b_{lS}^T w_1; b_{l1}^T w_2, b_{l2}^T w_2, ..., b_{lS}^T w_2; ...; b_{l1}^T w_k, b_{l2}^T w_k, ..., b_{lS}^T w_k];
\end{array} \tag{15}$$

indeed, the left hand side of $(f')$ is the Schur complement of the South-Eastern block in the left hand side matrix in (14). Consequently, $(D''')$ is equivalent to the problem

$$
\begin{array}{lrcl}
\text{minimize} & \phi & \equiv & \sum_{i=1}^k \mathrm{Tr}(d_i\alpha_i + 2e_i^T w_i) \\
 & & & +\sum_{l=1}^N [\bar{\rho}_l \sigma_l^+ - \underline{\rho}_l \sigma_l^-] \\
 & & & +\upsilon\gamma \\
\text{s.t.} & & & \\
 & \begin{pmatrix} A(\alpha) & B_l^T(w) \\ B_l(w) & (\gamma + \sigma_l^+ - \sigma_l^-) I_{D_l} \end{pmatrix} & \geq & 0,\ l = 1, ..., N, \\
 & \sigma_l^+, \sigma_l^- & \geq & 0,\ l = 1, ..., N; \\
 & \gamma & \geq & 0; \\
 & \sum_{i=1}^k \mathrm{Tr}(\alpha_i) & = & 1; \\
 & \alpha_i & > & 0,\ i = 1, ..., k; \\
 & \sum_{i=1}^k [D_i^* \alpha_i + 2E_i^* w_i] & \leq & 0.
\end{array}
$$

Problem $(D''')$ is strictly feasible along with $(D')$, so that its optimal value remains unchanged when we remove from $(D''')$ the strict inequalities, thus coming to the final form of the problem dual to $(P)$:

$$
\begin{array}{lrcl}
(D) & & & \\
\text{minimize} & \phi & \equiv & \sum_{i=1}^k \mathrm{Tr}(d_i\alpha_i + 2e_i^T w_i) \\
 & & & +\sum_{l=1}^N [\bar{\rho}_l \sigma_l^+ - \underline{\rho}_l \sigma_l^-] \\
 & & & +\upsilon\gamma \\
\text{s.t.} & & & \\
 & \begin{pmatrix} A(\alpha) & B_l^T(w) \\ B_l(w) & (\gamma + \sigma_l^+ - \sigma_l^-) I_{D_l} \end{pmatrix} & \geq & 0,\ l = 1, ..., N, \\
 & \sigma_l^+, \sigma_l^- & \geq & 0,\ l = 1, ..., N; \\
 & \gamma & \geq & 0; \\
 & \sum_{i=1}^k \mathrm{Tr}(\alpha_i) & = & 1; \\
 & \sum_{i=1}^k [D_i^* \alpha_i + 2E_i^* w_i] & \leq & 0,
\end{array}
$$

the design variables of the problem being

$$\alpha = \{\alpha_i \in \mathbf{S}^{p_i}\}_{i=1}^k, w = \{w_i \in \mathbf{L}^{M \times p_i}\}_{i=1}^k, \sigma = \{\sigma_l^\pm \in \mathbf{R}\}_{l=1}^N, \gamma \in \mathbf{R}.$$

Due to its origin and Proposition 3.1, $(D)$ is strictly feasible and the level sets of the objective are bounded (so that $(D)$ is solvable); the optimal value of $(D)$ is the negation of the optimal value of $(P)$.

# 4 From dual to primal

Problem $(D)$ is not the Fenchel dual to $(P)$, it is obtained from this dual by eliminating part of the variables. What happens when we pass from $(D)$ to its Fenchel dual? It turns out that we end up with a nontrivial (and instructive) equivalent reformulation of $(P)$, namely, with the problem

$$
\begin{array}{lrcl}
(P^+) & & & \\
\text{minimize} & s & & \\
\text{s.t.} & & & \\
(i) & \left(\begin{array}{c|ccc|c|ccc}
sI_{p_i}+D_iz+d_i & [q^i_{11}]^T & \cdots & [q^i_{1S}]^T & \cdots & [q^i_{N1}]^T & \cdots & [q^i_{NS}]^T \\ \hline
q^i_{11} & t_1 & & & & & & \\
\cdots & & \ddots & & & & & \\
q^i_{1S} & & & t_1 & & & & \\ \hline
\cdots & & & & \ddots & & & \\ \hline
q^i_{N1} & & & & & t_N & & \\
\cdots & & & & & & \ddots & \\
q^i_{NS} & & & & & & & t_N
\end{array}\right) & \geq & 0,\ i=1,...,k; \\
(ii) & t_l & \geq & 0,\ l=1,...,N; \\
(iii) & \underline{\rho}_l \leq \mathrm{Tr}(t_l) & \leq & \overline{\rho}_l,\ l=1,...,N; \\
(iv) & \sum_{l=1}^N \mathrm{Tr}(t_l) & \leq & v; \\
(v) & \sum_{l=1}^N \sum_{s=1}^S b_{ls} q^i_{ls} & = & e_i + E_i z,\ i=1,...,k; \\
(vi) & z & \geq & 0,
\end{array}
$$

the design variables in the problem being symmetric $D_l \times D_l$ matrices $t_l$, $l = 1, ..., N$, $D_l \times p_i$ matrices $q^i_{ls}$, $i = 1, ..., k, l = 1, ..., N, s = 1, ..., S$, real $s$ and $z \in \mathbf{R}^q$. Problem $(P^+)$ is not the straightforward Fenchel dual of $(D)$; it is obtained from this dual by eliminating part of the variables. Instead of boring derivation of $(P^+)$ via Fenchel duality, we prefer to give a direct proof of equivalence between $(P)$ and $(P^+)$:

**Theorem 4.1** *A collection $(\{t_l\}_{l=1}^N, z, s)$ is a feasible solution to $(P)$ if and only if it can be extended by properly chosen $\{q^i_{ls} | i = 1, ..., k, l = 1, ..., N, s = 1, ..., S\}$ to a feasible solution to $(P^+)$.*

**Proof.** "if" part: let a collection

$$(\{t_l\}_{l=1}^N, z, s, \{q^i_{ls} | i = 1, ..., k, l = 1, ..., N, s = 1, ..., S\})$$

be a feasible solution to $(P^+)$; all we should prove is the validity of the LMI's $(P).(i)$. Let us fix $i \leq k$; we should prove that for every pair $(x, y)$ of vectors of appropriate dimensions we have

$$x^T[sI_{p_i} + D_i z + d_i]x + 2x^T[e_i + E_i z]^T y + y^T[\sum_{l=1}^N \sum_{s=1}^S b_{ls}^T t_l b_{ls}^T]y \geq 0. \tag{16}$$

Indeed, in view of $(P^+).(v)$ the left hand side of (16) is equal to

$$
\begin{array}{rcl}
x^T[sI_{p_i} + D_i z + d_i]x & & \\
+2x^T[\sum_{l=1}^N \sum_{s=1}^S b_{ls} q^i_{ls}]^T y + y^T[\sum_{l=1}^N \sum_{s=1}^S b_{ls} t_l b_{ls}^T]y & = & x^T[SI_{p_i} + D_i z + d_i]x \\
& & +2\sum_{l=1}^N \sum_{s=1}^S x^T[q^i_{ls}]^T y_{ls} \\
& & +\sum_{l=1}^N \sum_{s=1}^s y_{ls}^T t_l y_{ls}, \\
& & y_{ls} = b_{ls}^T y.
\end{array}
$$

The resulting expression is nothing but the value of the quadratic form with the matrix involved into the left hand side of the corresponding LMI $(P^+).(i)$ at the vector comprised of $x$ and $\{y_{ls}\}_{l,s}$, and therefore it is nonnegative, as claimed.

"only if" part: let

$$(\{t_l\}_{l=1}^N, z, s)$$

be a feasible solution to $(P)$. Let us fix $i$, $1 \le i \le k$, and let us set $f_i = E_i z + e_i$. For every $x \in \mathbf{R}^{p_i}$ the quadratic form of $y \in \mathbf{R}^M$:

$$x^T[sI_{p_i} + D_i z + d_i] + 2x^T f_i^T y + y^T \mathcal{Q}(t) y$$

is nonnegative, i.e., the equation

$$\mathcal{Q}(t) y = f_i x$$

is solvable for every $x$; of course, we can choose its solution to be linear in $x$:

$$y = R_i x;$$

note that then

$$\mathcal{Q}(t) R_i x = f_i x \quad \forall x \Leftrightarrow \mathcal{Q}(t) R_i = f_i.$$

Let us now set

$$[q_{ls}^i]^T = R_i^T b_{ls} t_l; \tag{17}$$

then

$$\sum_{l=1}^{N} \sum_{s=1}^{S} b_{ls} q_{ls}^i = \sum_{l=1}^{N} \sum_{s=1}^{S} b_{ls} t_l b_{ls}^T R_i = f_i;$$

thus, extending $(\{t_l\}, z, s)$ by $\{q_{ls}^i\}$, we ensure the validity of $(P^+).(v)$. It remains to verify that the indicated extensions ensures the validity of LMI's $(P^+).(i)$ as well. What we should verify is that for every collection $\{y_{ls}\}$ of vectors of appropriate dimension and for every $x \in \mathbf{R}^{p_i}$ we have

$$F(x, \{y_{ls}\}) \equiv x^T[sI_{p_i} + D_i z + d_i]x + 2x^T \sum_{l=1}^{N} \sum_{s=1}^{S} [q_{ls}^i]^T y_{ls} + \sum_{l=1}^{N} \sum_{s=1}^{S} y_{ls}^T t_l y_{ls} \ge 0. \tag{18}$$

Given $x$, let us set

$$y_{ls}^* = -b_{ls}^T R_i x,$$

and let us prove that the collection $\{y_{ls}^*\}$ minimizes $F(x, \cdot)$, which is immediate: $F(x, \cdot)$ is convex quadratic form, and its partial derivative w.r.t. $y_{ls}$ at the point $\{y_{ls}^*\}$ is equal to

$$2q_{ls}^i x + 2t_l y_{ls}^* = 2[t_l b_{ls}^T R_i x - t_l b_{ls}^T R_i x] = 0$$

for all $l, s$. It remains to note that

$$\begin{array}{rcl}
F(x, \{y_{ls}^*\}) & = & x^T[sI_{p_i} + D_i z + d_i]x - 2x^T \sum_{l=1}^{N} \sum_{s=1}^{S} [q_{ls}^i]^T b_{ls}^T R_i x \\
& & + \sum_{l=1}^{N} \sum_{s=1}^{S} x^T R_i^T b_{ls} t_l b_{ls}^T R_i x \\
& = & x^T[sI_{p_i} + D_i z + d_i]x - 2x^T [\sum_{l=1}^{N} \sum_{s=1}^{S} b_{ls} q_{ls}^i]^T R_i x \\
& & + x^T R_i^T \mathcal{Q}(t) R_i x \\
& = & x^T[sI_{p_i} + D_i z + d_i]x - 2x^T [E_i z + e_i]^T R_i x + x^T R_i^T \mathcal{Q}(t) R_i x \\
& & \text{[due to already proved } (P^+).(v)] \\
& = & (x^T; -x^T R_i^T) \begin{pmatrix} sI_{p_i} + D_i z + d_i & [E_i z + e_i]^T \\ E_i z + e_i & \mathcal{Q}(t) \end{pmatrix} \begin{pmatrix} x \\ -R_i x \end{pmatrix} \\
& \ge & 0 \\
& & \text{[since } (\{t_l\}, z, s) \text{ is feasible for } (P)]
\end{array}$$

Thus, the minimum of $F(x, \{y_{ls}\})$ in $\{y_{ls}\}$ is nonnegative, and therefore (18) indeed is valid. ■

## 5 Explicit forms of the standard Truss and Shape problems

Let us list the explicit forms of problems $(P), (D), (P^+)$ for the standard cases of multi-load and robust Truss/Shape design (the case of single-load design is obtained from the multi-load one by specifying the number of loads as 1). Note that in the case of "simple bounds" on the material densities ($\underline{\rho}_l = 0, \overline{\rho}_l > v$) the formulations in question can be slightly simplified, and below we point out explicitly also these simplified settings.

**Multi-Load Truss Design.** Here

$$\begin{array}{rcl} S &=& 1, \\ D_l &=& 1,\ l = 1, ..., N, \\ \mathcal{F} &=& \{f_1, ..., f_k\}, \\ W &=& \{w \in \mathbf{R}^M \mid Rw \le r\} \quad [\dim(r) = q] \\ T &=& \{t \in \mathbf{R}^N \mid [0 \le]\underline{\rho}_l \le t_l \le \overline{\rho}_l, 1 \le l \le N, \sum_{l=1}^N t_l \le v\} \end{array}$$

The settings are

- $(P)$, General bounds:

$$\begin{array}{rcl} s &\to& \min \\ \begin{pmatrix} s + r^T y_i - \mu_i & f_i^T + \frac{1}{2} y_i^T R \\ f_i + \frac{1}{2} R^T y_i & \sum_{l=1}^N b_l b_l^T t_l \end{pmatrix} &\succeq& 0,\ i = 1, ..., k; \\ \underline{\rho}_l \le t_l &\le& \overline{\rho}_l, l = 1, ..., N; \\ \sum_{l=1}^N t_l &\le& v; \\ y_i &\ge& 0,\ i = 1, ..., k; \\ \mu_i &\ge& r^T y_i,\ i = 1, ..., k. \\ [s, t_l, \mu_i \in \mathbf{R}, y_i \in \mathbf{R}^q] && \end{array}$$

- $(P)$, Simple bounds:

$$\begin{array}{rcl} s &\to& \min \\ \begin{pmatrix} s + r^T y_i - \mu_i & f_i^T + \frac{1}{2} y_i^T R \\ f_i + \frac{1}{2} R^T y_i & \sum_{l=1}^N b_l b_l^T t_l \end{pmatrix} &\succeq& 0,\ i = 1, ..., k; \\ t_l &\ge& 0,\ l = 1, ..., N; \\ \sum_{l=1}^N t_l &\le& v; \\ y_i &\ge& 0,\ i = 1, ..., k; \\ \mu_i &\ge& r^T y_i,\ i = 1, ..., k. \\ [s, t_l, \mu_i \in \mathbf{R}, y_i \in \mathbf{R}^q] && \end{array}$$

• $(D)$, General bounds:

$$
\begin{array}{rcl}
-2\sum_{i=1}^k f_i^T w_i + \sum_{l=1}^N [\overline{\rho}_l \sigma_l^+ - \underline{\rho}_l \sigma_l^-] + v\gamma & \to & \min \\
Z_l \equiv \left(\begin{array}{ccc|c} \alpha_1 & & & b_l^T w_1 \\ & \ddots & & \cdots \\ & & \alpha_k & b_l^T w_k \\ \hline b_l^T w_1 & \cdots & b_l^T w_k & \gamma + \sigma_l^+ - \sigma_l^- \end{array}\right) & \geq & 0,\ l = 1, ..., N; \\
\sigma_l^\pm & \geq & 0,\ l = 1, ..., N; \\
\gamma & \geq & 0; \\
Rw_i & \leq & \alpha_i r_i,\ i = 1, ..., k; \\
\sum_{i=1}^k \alpha_i & = & 1. \\
{}[\alpha_i, \sigma_l^\pm, \gamma \in \mathbf{R}, w_i \in \mathbf{R}^M] & &
\end{array}
$$

• $(D)$, Simple bounds:

$$
\begin{array}{rcl}
-2\sum_{i=1}^k f_i^T w_i + v\gamma & \to & \min \\
Z_l \equiv \left(\begin{array}{ccc|c} \alpha_1 & & & b_l^T w_1 \\ & \ddots & & \cdots \\ & & \alpha_k & b_l^T w_k \\ \hline b_l^T w_1 & \cdots & b_l^T w_k & \gamma \end{array}\right) & \geq & 0,\ l = 1, ..., N; \\
Rw_i & \leq & \alpha_i r_i,\ i = 1, ..., k; \\
\sum_{i=1}^k \alpha_i & = & 1. \\
{}[\alpha_i, \gamma \in \mathbf{R}, w_i \in \mathbf{R}^M] & &
\end{array}
$$

• $(P^+)$, General bounds:

$$
\begin{array}{rcl}
s & \to & \min \\
\left(\begin{array}{c|ccc} s + r^T y_i - \mu_i & q_1^i & \cdots & q_N^i \\ \hline q_1^i & t_1 & & \\ \cdots & & \ddots & \\ q_N^i & & & t_N \end{array}\right) & \geq & 0,\ i = 1, ..., k; \\
\underline{\rho}_l \leq t_l & \leq & \overline{\rho}_l,\ l = 1, ..., N; \\
\sum_{l=1}^N t_l & \leq & v; \\
\sum_{l=1}^N q_l^i b_l & = & f_i + \frac{1}{2} R^T y_i,\ i = 1, ..., k; \\
y_i & \geq & 0,\ i = 1, ..., k; \\
\mu_i & \geq & r^T y_i,\ i = 1, ..., k. \\
{}[s, t_l, q_l^i, \mu_i \in \mathbf{R}, y_i \in \mathbf{R}^q] & &
\end{array}
$$

• $(P^+)$, Simple bounds:

$$
\begin{array}{rcl}
s & \to & \min \\
\left(\begin{array}{c|ccc} s + r^T y_i - \mu_i & q_1^i & \cdots & q_N^i \\ \hline q_1^i & t_1 & & \\ \cdots & & \ddots & \\ q_N^i & & & t_N \end{array}\right) & \geq & 0,\ \ i = 1, ..., k; \\
\sum_{l=1}^N t_l & \leq & v; \\
\sum_{l=1}^N q_l^i b_l & = & f_i + \frac{1}{2} R^T y_i,\ \ i = 1, ..., k; \\
y_i & \geq & 0,\ \ i = 1, ..., k; \\
\mu_i & \geq & r^T y_i,\ \ i = 1, ..., k. \\
{[s, t_l, q_l^i, \mu_i \in \mathbf{R}, y_i \in \mathbf{R}^q]} & &
\end{array}
$$

**Robust Truss Design without obstacles.** Here

$$
\begin{array}{rcl}
S & = & 1, \\
D_l & = & 1,\ \ l = 1, ..., N, \\
\mathcal{F} & = & \{f = Qu \mid\ u \in \mathbf{R}^k, u^T u \leq 1\}, \\
W & = & \mathbf{R}^M, \\
T & = & \{t \in \mathbf{R}^N \mid\ [0 \leq] \underline{\rho}_l \leq t_l \leq \overline{\rho}_l, 1 \leq l \leq N, \sum_{l=1}^N t_l \leq v\}
\end{array}
$$

The settings are

• $(P)$, General bounds:

$$
\begin{array}{rcl}
s & \to & \min \\
\begin{pmatrix} sI_k & Q^T \\ Q & \sum_{l=1}^N b_l b_l^T t_l \end{pmatrix} & \geq & 0; \\
\underline{\rho}_l \leq t_l & \leq & \overline{\rho}_l, l = 1, ..., N; \\
\sum_{l=1}^N t_l & \leq & v. \\
{[s, t_l \in \mathbf{R}]} & &
\end{array}
$$

• $(P)$, Simple bounds:

$$
\begin{array}{rcl}
s & \to & \min \\
\begin{pmatrix} sI_k & Q^T \\ Q & \sum_{l=1}^N b_l b_l^T t_l \end{pmatrix} & \geq & 0; \\
t_l & \geq & 0,\ \ l = 1, ..., N; \\
\sum_{l=1}^N t_l & \leq & v. \\
{[s, t_l \in \mathbf{R}]} & &
\end{array}
$$

- $(D)$, General bounds:

$$
\begin{array}{rcl}
-2\,\mathrm{Tr}(Q^T w) + \sum_{l=1}^{N}[\overline{\rho}_l \sigma_l^{+} - \underline{\rho}_l \sigma_l^{-}] + v\gamma & \rightarrow & \min \\
Z_l \equiv \begin{pmatrix} \alpha & w^T b_l \\ b_l^T w & \gamma + \sigma_l^{+} - \sigma_l^{-} \end{pmatrix} & \geq & 0,\ l = 1, ..., N; \\
\sigma_l^{\pm} & \geq & 0,\ l = 1, ..., N; \\
\gamma & \geq & 0; \\
\mathrm{Tr}(\alpha) & = & 1. \\
[\alpha \in \mathbf{S}^k, \sigma_l^{\pm}, \gamma \in \mathbf{R}, w \in \mathbf{L}^{M,k}] & &
\end{array}
$$

- $(D)$, Simple bounds:

$$
\begin{array}{rcl}
-2\,\mathrm{Tr}(Q^T w) + v\gamma & \rightarrow & \min \\
Z_l \equiv \begin{pmatrix} \alpha & w^T b_l \\ b_l^T w & \gamma \end{pmatrix} & \geq & 0,\ l = 1, ..., N; \\
\mathrm{Tr}(\alpha) & = & 1. \\
\sum_{i=1}^{k} \alpha_i & = & 1. \\
[\alpha \in \mathbf{S}^k, \gamma \in \mathbf{R}, w \in \mathbf{L}^{M,k}] & &
\end{array}
$$

- $(P^{+})$, General bounds:

$$
\begin{array}{rcl}
s & \rightarrow & \min \\
\left(\begin{array}{c|ccc} sI_k & q_1^T & \cdots & q_N^T \\ \hline q_1 & t_1 & & \\ \cdots & & \ddots & \\ q_N & & & t_N \end{array}\right) & \geq & 0; \\
\underline{\rho}_l \leq t_l & \leq & \overline{\rho}_l,\ l = 1, ..., N; \\
\sum_{l=1}^{N} t_l & \leq & v; \\
\sum_{l=1}^{N} b_l q_l & = & Q; \\
[t_l \in \mathbf{R}, q_l^T \in \mathbf{R}^k] & &
\end{array}
$$

• $(P^+)$, Simple bounds:

$$\begin{array}{rcl}
s & \to & \min \\
\left(\begin{array}{c|ccc} sI_k & q_1^T & \cdots & q_N^T \\ \hline q_1 & t_1 & & \\ \cdots & & \ddots & \\ q_N & & & t_N \end{array}\right) & \succeq & 0; \\
\sum_{l=1}^N t_l & \leq & v; \\
\sum_{l=1}^N b_l q_l & = & Q; \\
{[t_l \in \mathbf{R}, q_l^T \in \mathbf{R}^k]} & &
\end{array}$$

**Multi-Load Shape Design.** Here

$$\begin{array}{rcl}
D_l & = & D,\ l = 1, ..., N\ [D = 3 \text{ in the planar and } D = 6 \text{ in the spatial case}] \\
\mathcal{F} & = & \{f_1, ..., f_k\}, \\
W & = & \{w \in \mathbf{R}^M \mid Rw \leq r\} \quad [\dim(r) = q] \\
T & = & \{t \in (\mathbf{S}_+^D)^N \mid [0 \leq]\underline{\rho}_l \leq \mathrm{Tr}(t_l) \leq \overline{\rho}_l, 1 \leq l \leq N, \sum_{l=1}^N \mathrm{Tr}(t_l) \leq v\}
\end{array}$$

The settings are

• $(P)$, General bounds:

$$\begin{array}{rcl}
s & \to & \min \\
\begin{pmatrix} s + r^T y_i - \mu_i & f_i^T + \frac{1}{2} y_i^T R \\ f_i + \frac{1}{2} R^T y_i & \sum_{l=1}^N \sum_{s=1}^S b_{ls} t_l b_{ls}^T \end{pmatrix} & \succeq & 0,\ i = 1, ..., k; \\
t_l & \succeq & 0,\ l = 1, ..., N; \\
\underline{\rho}_l \leq \mathrm{Tr}(t_l) & \leq & \overline{\rho}_l, l = 1, ..., N; \\
\sum_{l=1}^N \mathrm{Tr}(t_l) & \leq & v; \\
y_i & \geq & 0,\ i = 1, ..., k; \\
\mu_i & \geq & r^T y_i,\ i = 1, ..., k. \\
{[s, \mu_i \in \mathbf{R}, t_l \in \mathbf{S}^D, y_i \in \mathbf{R}^q]} & &
\end{array}$$

• $(P)$, Simple bounds:

$$\begin{array}{rcl}
s & \to & \min \\
\begin{pmatrix} s + r^T y_i - \mu_i & f_i^T + \frac{1}{2} y_i^T R \\ f_i + \frac{1}{2} R^T y_i & \sum_{l=1}^N \sum_{s=1}^S b_{ls} t_l b_{ls}^T \end{pmatrix} & \succeq & 0,\ i = 1, ..., k; \\
t_l & \succeq & 0,\ l = 1, ..., N; \\
\sum_{l=1}^N \mathrm{Tr}(t_l) & \leq & v; \\
y_i & \geq & 0,\ i = 1, ..., k; \\
\mu_i & \geq & r^T y_i,\ i = 1, ..., k. \\
{[s, \mu_i \in \mathbf{R}, t_l \in \mathbf{S}^D, y_i \in \mathbf{R}^q]} & &
\end{array}$$

• $(D)$, General bounds:

$$\begin{array}{rcl}
-2 \sum_{i=1}^k f_i^T w_i + \sum_{l=1}^N [\overline{\rho}_l \sigma_l^+ - \underline{\rho}_l \sigma_l^-] + v\gamma & \to & \min \\
Z_l \equiv \left(\begin{array}{ccc|c|ccc|c}
\alpha_1 & & & & & & & w_1^T b_{l1} \\
& \ddots & & & & & & \cdots \\
& & \alpha_1 & & & & & w_1^T b_{lS} \\ \hline
& & & \ddots & & & & \cdots \\ \hline
& & & & \alpha_k & & & w_k^T b_{l1} \\
& & & & & \ddots & & \cdots \\
& & & & & & \alpha_k & w_k^T b_{lS} \\
b_{l1}^T w_1 & \cdots & b_{lS}^T w_1 & \cdots & b_{l1}^T w_k & \cdots & b_{lS}^T w_k & (\gamma + \sigma_l^+ - \sigma_l^-) I_D
\end{array}\right) & \succeq & 0,\ l = 1, ..., N; \\
\sigma_l^{\pm} & \geq & 0,\ l = 1, ..., N; \\
\gamma & \geq & 0; \\
R w_i & \leq & \alpha_i r_i,\ i = 1, ..., k; \\
\sum_{i=1}^k \alpha_i & = & 1. \\
{[\alpha_i, \sigma_l^{\pm}, \gamma \in \mathbf{R}, w_i \in \mathbf{R}^M]} & &
\end{array}$$

- $(D)$, Simple bounds:

$$
\begin{array}{rcl}
-2\sum_{i=1}^k f_i^T w_i + v\gamma & \to & \min \\
Z_l \equiv \left(\begin{array}{ccc|c|ccc|c}
\alpha_1 & & & & & & & w_1^T b_{l1} \\
 & \ddots & & & & & & \cdots \\
 & & \alpha_1 & & & & & w_1^T b_{lS} \\ \hline
 & & & \ddots & & & & \cdots \\ \hline
 & & & & \alpha_k & & & w_k^T b_{l1} \\
 & & & & & \ddots & & \cdots \\
 & & & & & & \alpha_k & w_k^T b_{lS} \\
b_{l1}^T w_1 & \cdots & b_{lS}^T w_1 & \cdots & b_{l1}^T w_k & \cdots & b_{lS}^T w_k & \gamma I_D
\end{array}\right) & \geq & 0,\ l = 1, ..., N; \\
R w_i & \leq & \alpha_i r_i,\ i = 1, ..., k; \\
\sum_{i=1}^k \alpha_i & = & 1. \\
[\alpha_i, \gamma \in \mathbf{R}, w_i \in \mathbf{R}^M] & &
\end{array}
$$

- $(P^+)$, General bounds:

$$
\begin{array}{rcl}
s & \to & \min \\
\left(\begin{array}{c|ccc|c|ccc}
s + r^T y_i - \mu_i & [q_{11}^i]^T & \cdots & [q_{1S}^i]^T & \cdots & [q_{N1}^i]^T & \cdots & [q_{NS}^i]^T \\ \hline
q_{11}^i & t_1 & & & & & & \\
\cdots & & \ddots & & & & & \\
q_{1S}^i & & & t_1 & & & & \\ \hline
\cdots & & & & \ddots & & & \\ \hline
q_{N1}^i & & & & & t_N & & \\
\cdots & & & & & & \ddots & \\
q_{NS}^i & & & & & & & t_N
\end{array}\right) & \geq & 0,\ i = 1, ..., k; \\
\underline{\rho}_l \leq \mathrm{Tr}(t_l) & \leq & \overline{\rho}_l,\ l = 1, ..., N; \\
\sum_{l=1}^N \mathrm{Tr}(t_l) & \leq & v; \\
\sum_{l=1}^N \sum_{s=1}^S b_{ls} q_{ls}^i & = & f_i + \frac{1}{2} R^T y_i,\ i = 1, ..., k; \\
y_i & \geq & 0,\ i = 1, ..., k; \\
\mu_i & \geq & r^T y_i,\ i = 1, ..., k. \\
[s, \mu_i \in \mathbf{R}, t_l \in \mathbf{S}^D, q_{ls}^i \in \mathbf{R}^D, y_i \in \mathbf{R}^q] & &
\end{array}
$$

• $(P^+)$, Simple bounds:

$$
\begin{array}{rcl}
s & \to & \min \\
\left(\begin{array}{c|ccc|c|ccc}
s + r^T y_i - \mu_i & [q^i_{11}]^T & \cdots & [q^i_{1S}]^T & \cdots & [q^i_{N1}]^T & \cdots & [q^i_{NS}]^T \\
\hline
q^i_{11} & t_1 & & & & & & \\
\cdots & & \ddots & & & & & \\
q^i_{1S} & & & t_1 & & & & \\
\hline
\cdots & & & & \ddots & & & \\
\hline
q^i_{N1} & & & & & t_N & & \\
\cdots & & & & & & \ddots & \\
q^i_{NS} & & & & & & & t_N
\end{array}\right) & \succeq & 0,\ i = 1, ..., k; \\
\sum_{l=1}^N t_l & \preceq & v; \\
\sum_{l=1}^N b_{ls} q^i_{ls} & = & f_i + \frac{1}{2} R^T y_i,\ i = 1, ..., k; \\
y_i & \geq & 0,\ i = 1, ..., k; \\
\mu_i & \geq & r^T y_i,\ i = 1, ..., k. \\
{[s, \mu_i \in \mathbf{R}, t_l \in \mathbf{S}^D, q^i_{ls} \in \mathbf{R}^D, y_i \in \mathbf{R}^q]} & &
\end{array}
$$

**Robust Shape Design without obstacles.** Here

$$
\begin{array}{rcl}
D_l & = & D,\ l = 1, ..., N,\ [D = 3 \text{ in the planar and } D = 6 \text{ in the spatial case}] \\
\mathcal{F} & = & \{f = Qu \mid u \in \mathbf{R}^k, u^T u \leq 1\}, \\
W & = & \mathbf{R}^M, \\
T & = & \{t \in (\mathbf{S}^D)^N \mid [0 \leq]\underline{\rho}_l \leq \mathrm{Tr}(t_l) \leq \overline{\rho}_l, 1 \leq l \leq N, \sum_{l=1}^N \mathrm{Tr}(t_l) \leq v\}
\end{array}
$$

The settings are

• $(P)$, General bounds:

$$
\begin{array}{rcl}
s & \to & \min \\
\begin{pmatrix} sI_k & Q^T \\ Q & \sum_{l=1}^N \sum_{s=1}^S b_{ls} t_l b_{ls}^T \end{pmatrix} & \succeq & 0; \\
t_l & \succeq & 0,\ l = 1, ..., N; \\
\underline{\rho}_l \leq \mathrm{Tr}(t_l) & \leq & \overline{\rho}_l, l = 1, ..., N; \\
\sum_{l=1}^N \mathrm{Tr}(t_l) & \leq & v. \\
{[s \in \mathbf{R}, t_l \in \mathbf{S}^D]} & &
\end{array}
$$

• $(P)$, Simple bounds:

$$\begin{array}{rcl} s & \to & \min \\ \begin{pmatrix} sI_k & Q^T \\ Q & \sum_{l=1}^N \sum_{s=1}^S b_{ls} t_l b_{ls}^T \end{pmatrix} & \geq & 0; \\ t_l & \geq & 0,\ l=1,...,N; \\ \sum_{l=1}^N \mathrm{Tr}(t_l) & \leq & v. \\ {[s \in \mathbf{R}, t_l \in \mathbf{S}^D]} & & \end{array}$$

• $(D)$, General bounds:

$$\begin{array}{rcl} -2\,\mathrm{Tr}(Q^T w) + \sum_{l=1}^N [\overline{\rho}_l \sigma_l^+ - \underline{\rho}_l \sigma_l^-] + v\gamma & \to & \min \\ Z_l \equiv \left(\begin{array}{ccc|c} \alpha & & & w^T b_{l1} \\ & \ddots & & \cdots \\ & & \alpha & w^T b_{lS} \\ \hline b_{l1}^T w & \cdots & b_{lS}^T w & (\gamma + \sigma_l^+ - \sigma_l^-) I_D \end{array}\right) & \geq & 0,\ l=1,...,N; \\ \sigma_l^\pm & \geq & 0,\ l=1,...,N; \\ \gamma & \geq & 0; \\ \mathrm{Tr}(\alpha) & = & 1. \\ {[\alpha \in \mathbf{S}^k, \sigma_l^\pm, \gamma \in \mathbf{R}, w \in \mathbf{L}^{M,k}]} & & \end{array}$$

• $(D)$, Simple bounds:

$$\begin{array}{rcl} -2\,\mathrm{Tr}(Q^T w) + v\gamma & \to & \min \\ Z_l \equiv \left(\begin{array}{ccc|c} \alpha & & & w^T b_{l1} \\ & \ddots & & \cdots \\ & & \alpha & w^T b_{lS} \\ \hline b_{l1}^T w & \cdots & b_{lS}^T w & \gamma I_D \end{array}\right) & \geq & 0,\ l=1,...,N; \\ \mathrm{Tr}(\alpha) & = & 1. \\ {[\alpha \in \mathbf{S}^k, \gamma \in \mathbf{R}, w \in \mathbf{L}^{M,k}]} & & \end{array}$$

• $(P^+)$, General bounds:

$$
\begin{array}{rcl}
s & \to & \min \\
\left(\begin{array}{c|ccc|c|ccc}
sI_k & [q_{11}]^T & \cdots & [q_{1S}]^T & \cdots & [q_{N1}]^T & \cdots & [q_{NS}]^T \\ \hline
q_{11} & t_1 & & & & & & \\
\cdots & & \ddots & & & & & \\
q_{1S} & & & t_1 & & & & \\ \hline
\cdots & & & & \ddots & & & \\ \hline
q_{N1} & & & & & t_N & & \\
\cdots & & & & & & \ddots & \\
q_{NS} & & & & & & & t_N
\end{array}\right) & \succeq & 0; \\
\underline{\rho}_l \le \mathrm{Tr}(t_l) & \le & \bar{\rho}_l,\ l = 1, ..., N; \\
\sum_{l=1}^N \mathrm{Tr}(t_l) & \le & v; \\
\sum_{l=1}^N \sum_{s=1}^S b_{ls} q_{ls} & = & Q; \\
{[s \in \mathbf{R}, t_l \in \mathbf{S}^D, q_{ls} \in \mathbf{L}^{D,k}]} & &
\end{array}
$$

• $(P^+)$, Simple bounds:

$$
\begin{array}{rcl}
s & \to & \min \\
\left(\begin{array}{c|ccc|c|ccc}
sI_k & [q_{11}]^T & \cdots & [q_{1S}]^T & \cdots & [q_{N1}]^T & \cdots & [q_{NS}]^T \\ \hline
q_{11} & t_1 & & & & & & \\
\cdots & & \ddots & & & & & \\
q_{1S} & & & t_1 & & & & \\ \hline
\cdots & & & & \ddots & & & \\ \hline
q_{N1} & & & & & t_N & & \\
\cdots & & & & & & \ddots & \\
q_{NS} & & & & & & & t_N
\end{array}\right) & \succeq & 0; \\
\sum_{l=1}^N \mathrm{Tr}(t_l) & \le & v; \\
\sum_{l=1}^N \sum_{s=1}^S b_{ls} q_{ls} & = & Q; \\
{[s \in \mathbf{R}, t_l \in \mathbf{S}^D, q_{ls} \in \mathbf{L}^{D,k}]} & &
\end{array}
$$

# 6 Computational issues

**Which formulation to choose?** We have presented several equivalent SDP settings of the Standard case of the Structural Design problem. A natural question is which formulation is better suited for straightforward numerical processing. Since all settings are large-scale, the only practical possibility to solve them is to use modern polynomial-time interior point methods. As applied to a semidefinite program of the form

$$
\text{(SDP)} \qquad c^T x \to \min \mid \mathcal{A}x + b \succeq 0 \qquad [x \mapsto \mathcal{A}x : \mathbf{R}^n \to \mathbf{S} = \mathbf{S}^{m_1} \times ... \times \mathbf{S}^{m_p}]
$$

such a method normally finds high-accuracy solution to the problem in a moderate number of iterations; theoretical complexity bound on this number (called *Newton complexity* of an interior point method) for the best known so far methods is proportional to $\sqrt{\sum_{i=1}^{p} m_i}\ln(1/\epsilon)$, $\epsilon$ being properly defined relative accuracy; in practice, the number of iterations is 20-50-80. The computational effort at a single iteration is dominated by the necessity, given a strictly feasible solution $x$ and a vector $h$, to assemble and to solve the Newton system

$$He = h, \quad H = \nabla_x[-\ln \mathrm{Det}\,(Ax+b)]. \tag{19}$$

Thus, the computational effort to solve the problem mainly depends on the arithmetic cost of assembling and solving a Newton system. In what follows we list these quantities for the aforementioned problems, assuming that

1. $N$ and $M$ are large parameters; formally speaking, we will be interested in principal terms of the complexities in question as $N, M \to \infty$. Note that in practical design $N$ and $M$ are at least hundreds.

2. $N \approx \theta M^2$ for truss problems and $N \approx \theta M$ for shape problems; here and in what follows $\approx$ means "equal up to a factor tending to 1 as $N, M \to \infty$".

3. The number $k$ (# of loading scenarios in multi-load settings or dimension of loading ellipsoid in the robust settings) is $o(1)M$, i.e., the ratio $k/M$ tends to 0 as $N, M \to \infty$. Note that in multi-load design $k$ normally is $\leq 5$.

4. The number $q$ of linear inequality constraints defining the set $W$ of admissible displacements is $o(1)M$.

5. $S = O(1)$, i.e., $S$ remains bounded as $N, M \to \infty$.

6. The maximal, over matrices $b_{l,s}$, number of nonzero entries in the matrix is $O(1)$. The maximal, over the rows of the constraint matrix $R$ defining the set $W$ of admissible displacements, number of nonzeros is $O(1)$.

These assumptions indeed are satisfied in typical truss/shape design problems, as it is seen from the following table:

**Table 1.** Parameters of typical "ground structures"

| Problem's type | $\theta$ | $D$ | $S$ |
|---|---|---|---|
| Planar truss design | $\frac{1}{8}$ | 1 | 1 |
| Spatial truss design | $\frac{1}{18}$ | 1 | 1 |
| Planar shape design, cells-triangles | 1 | 3 | 1[1)] |
| Planar shape design, cells-rectangles | 2 | 3 | 4[2)] |
| Spatial shape design, cells-tetrahedrons | $\frac{5}{3}$ | 6 | 1[1)] |
| Spatial shape design, cells-boxes | $\frac{1}{3}$ | 6 | 36[3)] |

[1)] provided basic strain fields $e_P(\cdot)$ are constant within cells
[2)] provided basic strain fields $e_P(\cdot)$ are linear in $P$ within cells
[3)] provided basic strain fields $e_P(\cdot)$ are quadratic in $P$ within cells

In Table 2, we present two major "size" characteristics of the SDP's we are interested in – the *design dimension* $\dim(x)$ of the problem, i.e., the total number of decision variables, and the *row dimension* $m$ of the problem, i.e., the total row size of the corresponding LMI's; as it was already mentioned, the theoretical Newton complexity of a (good) interior point method is proportional to $\sqrt{m}$. [2)].

**Table 2.** Sizes of SDP settings of Structural Design problems

| Problem's type | Setting $(P)$ | | Setting $(D)$ | | Setting $(P^+)$ | |
|---|---|---|---|---|---|---|
| | $\dim(x) \approx$ | $m \approx$ | $\dim(x) \approx$ | $m \approx$ | $\dim(x) \approx$ | $m \approx$ |
| Multi-load truss, general bounds | $\theta M^2$ | $2\theta M^2$ | $2\theta M^2$ | $\theta(k+3)M^2$ | $\theta(k+1)M^2$ | $\theta(k+2)M^2$ |
| Multi-load truss, simple bounds | $\theta M^2$ | $\theta M^2$ | $kM$ | $\theta(k+1)M^2$ | $\theta(k+1)M^2$ | $\theta k M^2$ |
| Robust truss, general bounds | $\theta M^2$ | $2\theta M^2$ | $2\theta M^2$ | $\theta(k+3)M^2$ | $\theta(k+1)M^2$ | $3\theta M^2$ |
| Robust truss, simple bounds | $\theta M^2$ | $\theta M^2$ | $kM$ | $\theta(k+1)M^2$ | $\theta(k+1)M^2$ | $\theta M^2$ |
| Multi-load planar shape, general bounds | $6\theta M$ | $(5\theta+k)M$ | $(k+2\theta)M$ | $\theta(kS+5)M$ | $\theta(3kS+6)M$ | $\theta(3kS+2)M$ |
| Multi-load planar shape, simple bounds | $6\theta M$ | $(3\theta+k)M$ | $kM$ | $\theta(kS+3)M$ | $\theta(3kS+6)M$ | $3\theta kSM$ |
| Robust planar shape, general bounds | $6\theta M$ | $(5\theta+1)M$ | $(k+2\theta)M$ | $\theta(kS+3)M$ | $\theta(3kS+6)M$ | $\theta(3S+2)M$ |
| Robust planar shape, simple bounds | $6\theta M$ | $(3\theta+1)M$ | $kM$ | $\theta(kS+1)M$ | $\theta(3kS+6)M$ | $3\theta SM$ |
| Multi-load spatial shape, general bounds | $21\theta M$ | $(8\theta+k)M$ | $(k+2\theta)M$ | $\theta(kS+8)M$ | $\theta(6kS+21)M$ | $\theta(6kS+2)M$ |
| Multi-load spatial shape, simple bounds | $21\theta M$ | $(6\theta+k)M$ | $kM$ | $\theta(kS+6)M$ | $\theta(6kS+21)M$ | $6\theta kSM$ |
| Robust spatial shape, general bounds | $21\theta M$ | $(8\theta+1)M$ | $(k+2\theta)M$ | $\theta(kS+8)M$ | $\theta(6kS+21)M$ | $\theta(6S+2)M$ |
| Robust spatial shape, simple bounds | $21\theta M$ | $(6\theta+1)M$ | $kM$ | $\theta(kS+6)M$ | $\theta(6kS+21)M$ | $6\theta SM$ |

From the presented data it immediately follows that at least in the case of *truss* problems with *simple* bounds the dual form $(D)$ of the problem is by far more preferable than all other forms of it. Indeed, in the case in question the design dimension of $(D)$ is, for large problems, incomparably less than the design dimension of $(P)$ and $(P^+)$. As a result, the part of expenses related to solving the Newton system (which, for the Standard Linear Algebra routines, is proportional to the cube of the design dimension) for $(D)$ is just $O(k^3M^3)$ per iteration, while for $(P)$ and $(P^+)$ these expenses are $O(M^6)$; since $k = o(M)$, the second quantity is by orders of magnitudes larger than the first one. To the moment we did not take into account another part of the expenses per iteration – the cost of assembling the Newton system. Here the comparison also is in favour of $(D)$: the $O(N)$ LMI's constituting $(D)$ are of small – just $k+1$ – row size; therefore,

[2)] when computing $m$, linear equalities are ignored, since they do not influence the theoretical Newton complexity of a method as it is easily seen, the cost of assembling the Newton system for $(D)$ is $O(k^3N) = O(k^3M^2)$. Thus, the expenses for assembling the Newton system for $(D)$ are, for large problems, negligible as compared to the cost of solving the Newton system, so that these expenses do not influence the results of the comparison.

The outlined comparison dealt with the case of truss problem with simple bounds, but in fact it is valid for truss problems with general bounds as well, in spite of the fact that in the latter case the "formal" design dimension of $(D)$ is of the same order as the design dimension of $(P)$ and $(P^+)$. The reason is that the design dimension of $(D)$ mainly comes from $2N$ scalar dual variables $\sigma_l^\pm$, and, as it is easily seen, the corresponding to these variables block of $H$ is extremely simple:

$$H = \begin{pmatrix} \Delta & D^T \\ D & C \end{pmatrix},$$

where $D$ is block-diagonal matrix of the row size $2N$ with $N$ $2\times 2$ diagonal blocks, and $C$ is a "small" – of the size $O(kM) \times O(kM)$ – matrix. As a result, one can solve the Newton system at the effort $O(k^3M^3)$, same as in the case of simple bounds; as in this latter case, the cost of assembling the Newton system is negligible as compared to the cost of solving it. Thus, we see that in the case of truss problems the dual setting is much better suited for the numerical processing than the primal one.

In the case of *shape* problems, it hardly can be said once for ever which one of the settings is better suited for numerical processing. Indeed, here the design dimensions of the primal and the dual forms of the problem are of the same order, and which form is better, it depends on the values of $\theta$, $k$, $S$, on how tricky is the technique for assembling the Newton system and what is the cost of this assembling, etc. E.g., for Robust Shape Design with not very small dimension $k$ of the loading ellipsoid the best setting definitely is $(P)$ (note that here both the cost of assembling and the cost of solving the Newton system are $O(N^3)$).

**Recovering primal solutions from the dual ones.** We have seen that at least in some important cases the dual form of the Structural Design problem is much better suited for direct numerical processing. There is, however, an evident drawback of the dual form: the actual design variables $t_l$ do not appear in $(D)$ at all. Thus, we come to the question of how one can recover a "good" approximate solution to the problem of interest $(P)$ from a good approximate solution to $(D)$. This question can be easily resolved, when the dual approximate solution in question is a *central* one. The latter notion is defined as follows. $(D)$ is of the generic form

$$\text{(SDP)} \qquad d^T\xi \to \min \mid \mathcal{G}\xi + g \geq 0,$$

where $\xi$ is design vector varying in some Euclidean space $X$, and $\xi \mapsto \mathcal{G}\xi$ is a linear mapping from $X$ to space $\mathbf{S}$ of block-diagonal symmetric matrices of certain block-diagonal structure [3)]. We say that a strictly feasible solution $\xi(\rho)$ to the problem is the central solution corresponding to a value $\rho > 0$ of the penalty parameter, if $\xi(\rho)$ minimizes the aggregate

$$\rho d^T\xi - \ln \mathrm{Det}\,(\mathcal{G}\xi + g)$$

over the set of strictly feasible solutions to the problem. From the general theory of interior point polynomial time methods it is known that

[3)] Note that in order to represent $(D)$ in this form, one should include the linear equality constraints of the formulation as given above into the definition of $X$

**P.1.** If $(D)$ is strictly feasible and the level sets

$$\{\xi \in X \mid \mathcal{G}\xi + g \geq 0, d^T\xi \leq a\}$$

are bounded for every $a$, then central solutions exist for every value $\rho > 0$ of the penalty parameter

**P.2** If $\xi(\rho)$ is a central solution, then the point

$$\Xi(\rho) = \rho^{-1}[\mathcal{G}\xi(\rho) + g]^{-1}$$

is a strictly feasible solution to the Fenchel dual of (SDP) – to the problem

$$(\mathrm{SDP}^*) \qquad \mathrm{Tr}(g\Xi) \to \min \mid \mathcal{G}^*\Xi = d, \ \Xi \geq 0,$$

and this solution is $O(\rho^{-1})$-optimal:

$$\mathrm{Tr}(g\Xi) - \mathrm{Opt}(\mathrm{SDP}^*) \leq \frac{m}{\rho},$$

where $\mathrm{Opt}(\mathrm{SDP}^*)$ is the optimal value in $(\mathrm{SDP}^*)$, and $m$ is the total row size of matrices from $\mathbf{S}$.

Now, in our context, as we have already seen, that $(D)$ is strictly feasible; boundedness of the level sets of the objective is stated by Proposition 3.1. Furthermore, the Fenchel dual to $(D)$, after eliminating part of the variables, is exactly the problem $(P^+)$. Combining P.2 and Theorem 4.1, we conclude that a central solution $\xi(\rho)$ to $(D)$ cam be immediately converted to a $(m\rho^{-1})$-optimal solution to $(P)$. Note also that when $(D)$ is solved by path-following interior point methods, the iterates can be easily converted to central solutions to $(D)$, and the corresponding values of the penalty parameter grow linearly with the rate $(1 + O(1)m^{-1/2})$.

It remains to present explicit scheme for recovering the values of $t_l$ from a central solution $\xi(\rho)$ to $(D)$, which is immediate: $\rho t_l$ is just the South-Eastern $D \times D$ block in the matrix $Z_l^{-1}$ (see explicit forms of $(D)$ above), $Z_l$ being evaluated at the point $\xi(\rho)$.

# References


[1] Achtziger, A., Ben-Tal, Bendsœ, M., and Zowe, J., "Equivalent displacement-based formulations for maximum strength truss topology design", *IMPACT of Computing in Science and Engineering* **4** (1992), 315-345.

[2] Ben-Tal, A. and Bendsœ, M.P., "A new method for optimal Truss Topology Design", *SIAM Journal of Optimization* **3** (1993), 322-358.

[3] Ben-Tal, A. and Nemirovski, A., "Potential reduction polynomial time method for Truss Topology Design", *SIAM Journal of Optimization* **4** (1994), 596-612.

[4] Ben-Tal, A., and Nemirovski, A., "Stable Truss Topology Design via Semidefinite Programming" – to appear in 1997 in *SIAM Journal of Optimization*

[5] Bendsœ, M.P., Ben-Tal, A., and Zowe, J., "Optimization methods for Truss Geometry and Topology Design", *Structural Optimization* (to appear)

[6] Bendsœ, M.P., Guedes, J.M., Haber, R.B., Redersen, P. and Taylor, J.E. (1992) "An analytical model to predict optimal material properties in the context of Optimal Structural Design", Research Report # 453, Technical University of Denmark, Lyngby

[7] Bendsœ, M.P., *Optimization of Structural Topology, Shape and Material* – Springer, Heidelberg, 1995.