Course: **Interior Point Polynomial Time Algorithms in Convex Programming**
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Lecturer: **Prof. Arkadi Nemirovski**, on sabbatical leave from the Technion – Israel Institute of Technology

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♣ <u>Polynomial time method $\mathcal{B}$:</u> a code for Real Arithmetic Computer with the following properties:

• When loaded by $\mathcal{B}$ and given on input

$\diamondsuit$ the *data* Data(p) of an instance p of a generic optimization problem $\mathcal{P}$,

$\diamondsuit$ required accuracy $\epsilon > 0$,

the computer eventually terminates and outputs an ϵ-solution to p (or a correct claim "p has no solutions");

• The number of Real Arithmetic operations in course of this computation does not exceed a polynomial in the *size* of the instance and the *number of required accuracy digits*.

♣ Here:

• *A generic optimization problem* $\mathcal{P}$ is a family of *instances*

$$(p): \qquad \min_x \left\{ p_0(x) : x \in X(p) \subset \mathbf{R}^{n(p)} \right\}$$

such that

♢ every instance $p \in \mathcal{P}$ is specified in $\mathcal{P}$ by finite-dimensional *data vector* Data(p);

♢ $\mathcal{P}$ is equipped with *infeasibility measure* – a nonnegative function $\text{Infeas}_{\mathcal{P}}(p,x)$ of instance and candidate solution which measures to which extent x is feasible for p;

Example: LP

• $\mathcal{P}$ is the family of all LPs of the form $\min\limits_x \left\{ c^T x : Ax \leq b, \|x\|_\infty \leq R \right\}$ $[A : m \times n]$;

• $\text{Data}(p) = \{m, n, R, c, b, A\}$

• $\text{Infeas}_{\mathcal{LP}}(p,x) = \max\left[0, \max\limits_i [Ax - b]_i, \|x\|_\infty - R\right]$

• $x \in \mathbf{R}^{n(p)}$ is an *ϵ-solution to $p \in \mathcal{P}$* iff

$$p(x) \leq \inf_{u \in X(p)} p_0(u) + \epsilon \ \& \ \text{Infeas}_{\mathcal{P}}(p,x) \leq \epsilon$$

• $\text{Size}(p) = \dim \text{Data}(p)$ and # of accuracy digits in ϵ-solution is

$$\ln\left(\frac{\dim \text{Data}(p) + \|\text{Data}(p)\|_1 + \epsilon^2}{\epsilon}\right)$$

♣ Polynomial time algorithm for solving $\mathcal{P}$: ϵ-solution is built in

$$\boxed{\mathcal{N}(p,\epsilon) \leq \chi\,[\mathrm{dim\,Data}(p)]^{\alpha}\,[\mathrm{Digits}(p,\epsilon)]^{\beta}}$$

operations.

Interpretation: 10-fold growth in computer power allows to increase *by constant factor*

– the size of instances which can be solved within a given accuracy in a given time,
– the # of accuracy digits to which instances of a given size can be solved in a given time

♣ For the majority of polynomial time algorithms,

$$\mathcal{N}(p,\epsilon) \leq \chi\,[\mathrm{dim\,Data}(p)]^{\alpha}\,\mathrm{Digits}(p,\epsilon)$$

– *the price of accuracy digit is independent of the position of the digit and is polynomial in* Size(p).

♣ Polynomiality $\Leftrightarrow$ Computational Efficiency

♣ History of polynomial time algorithms

• A generic optimization problem $\mathcal{P}$ is called *convex*, if both $p_0(x)$ and $\text{Infeas}(p, x)$ are convex in x for every $p \in \mathcal{P}$.

♢ Under mild computability and boundedness assumptions, a convex generic problem is polynomially solvable.

♢ For typical nonconvex generic optimization problems, *no* polynomial time solution algorithms are known, and there are strong reasons to believe that no algorithms of this type exist.

♣ "Universal" polynomial time algorithms for Convex Programming are based on the *Ellipsoid method* (1976) capable to minimize within accuracy ϵ a convex continuous function f over a ball $\{x \in \mathbf{R}^n : \|x\|_2 \le R\}$ in

$$O(1)n^2 \ln\left(\frac{\max\limits_{\|x\|_2 \le R} f(x) - \min\limits_{\|x\|_2 \le R} f(x)}{\epsilon} + 1\right)$$

steps, with a single computation of $f(x)$, $f'(x)$ plus $O(n^2)$ arithmetic operations per step.

♣ When solving convex program with n variables by an universal polynomial time algorithm, the price of accuracy digit is *at least* $O(n^4)$. This price, although polynomial, becomes impractically large when n is few hundreds or more.

♣ In final analysis, practical drawbacks of universal polynomial time methods originate from their universality: such a method uses complete a priori knowledge of problem's structure (and data) to the only purpose of computing the values and the gradients of the objective and the constraints at a point, while *for a practical method, adjusting to problem's structure is a must*.

♣ Interior Point Polynomial Time methods is the first (and, essentially, the only known) theoretically efficient technique allowing to adjust a solution method to problem's structure.

Milestones in history of polynomial time IPM's:

• 1978 – L. Khachiyan uses Ellipsoid method (1976) to prove polynomial solvability of LP (the price of accuracy digit for $O(n) \times O(n)$ LP's is $O(n^4)$. This theoretical breakthrough has no direct computational consequences...

• 1984 – N. Karmarkar proposes polynomial time Projective algorithm for LP, thus discovering the very first polynomial time interior point method for LP and reducing the price of accuracy digit to $O(n^{3.5})$. Much more importantly, Karmarkar's algorithm demonstrates clear practical potential...

• 1986 – J. Renegar and C. Gonzaga reduce the price of accuracy digit in LP to $O(n^3)$. Much more importantly, their algorithms, in contrast to "exotic" algorithm of Karmarkar, look quite traditional. As a result, understanding of intrinsic nature of interior point algorithms for LP and extending these algorithms beyond LP becomes just a matter of time...

• 1987 – Yu. Nesterov discovers general approach to developing interior point polynomial time methods for Convex Programming. The general theory of IPMs developed in 1988-94 (Yu. Nesterov & A. Nem.) allows to explain and to extend to general convex case basically all IPM constructions and results known for LP.
• Mid-90's: discovery of a specialized version of general theory for the "self-scaled" case covering Linear, Conic Quadratic and Semidefinite Programming (Yu. Nesterov & M. Todd).

Outline of IPMs – Path-Following Scheme

♣ "Common wisdom" in Optimization says that

The simplest convex program is to minimize a C^3-smooth convex objective $f(x)$ when it is nondegenerate ($f''(x) \succ 0$ for $x \in \mathrm{Dom} f$) and the minimum is "unconstrained": $\mathrm{Dom} f$ is open and all level sets $\{x \in \mathrm{Dom} f : f(x) \leq a\}$ of f are compact.

Equivalently: $\mathrm{Dom} f$ is open, and $f(x_i) \to \infty$ for every sequence $x_i \in \mathrm{Dom} f$ such that either $\|x_i\| \to \infty$, or $x_i \to x \in \partial \mathrm{Dom} f$ as $i \to \infty$.

The same common wisdom says that

In the unconstrained case, Newton method is *the* method of choice.

♣ The Newton method in the case of smooth nondegenerate convex objective with unconstrained minimum is:

♢ Given an iterate $x_t \in \mathrm{Dom} f$, one approximates f in a neighbourhood of x_t by the second-order Taylor expansion

$$\begin{array}{rcl} f(y) & \approx & \widetilde{f}_{x_t}(y) = f(x_t) + (y - x_t)^T \nabla f(x_t) \\ & & + \frac{1}{2}(y - x_t)^T [f''(x_t)](y - x_t) \end{array}$$

and passes from x_t to the minimizer of this expansion:

$$\begin{array}{rcl} x_t \mapsto x_{t+1} & = & \operatorname*{argmin}_y \widetilde{f}_{x_t}(y) \\ & = & x_t - [f''(x_t)]^{-1} \nabla f(x_t) \end{array} \qquad (*)$$

• The *basic Newton method* $(*)$ is known to converge *locally quadratically*:

$$\begin{array}{l} \exists (r > 0, C < \infty) : \|x_t - x_*\| \le r \Rightarrow \\ \|x_{t+1} - x_*\| \le C\|x_t - x_*\|^2 \le \frac{1}{2}\|x_t - x_*\| \end{array}$$

• When started "far" from x_*, the basic Newton method can leave $\mathrm{Dom} f$ or diverge; however, incorporating appropriate line search, one can enforce global asymptotically quadratic convergence.

♣ Consider a convex program in the form

$$\min_{x \in X} c^T x, \qquad (P)$$

($X \subset \mathbf{R}^n$: convex compact set, int$X \neq \emptyset$). Equipping (P) with *interior penalty* $F(\cdot)$ – a C^3 strongly convex function, Dom$F$ = int$X$, (P) is the "limit", as $t \to \infty$, of problems

$$\min_{x \in \text{int}X} F_t(x) \equiv tc^T x + F(x) \qquad (P_t)$$

• (P_t) has a unique solution $x(t)$ $\forall t > 0$, and the *path* $x(t)$ converges to the optimal set X_* of (P) as $t \to \infty$. Thus, we can approach X_* by *tracing the path*:

Given tight approximation x_{s-1} of $x(t_{s-1})$, we update t_{s-1} into $t_s > t_{s-1}$ and minimize F_{t_s} by the Newton method started at x_{s-1} until tight approximation x_s to the new target $x(t_s)$ is built.

• The path $x(t)$ is smooth

$\Rightarrow$ With t_s/t_{s-1} not too large, x_{s-1} will be in the domain of local quadratic convergence of the Newton method as applied to F_{t_s}, and updating $x_{s-1} \mapsto x_s$ will take a small number of Newton steps.

♣ The above *classical path-following scheme* [Fiacco & McCormic, 1967] worked reasonably well in practice, but was *not* known admit polynomial time implementation until the breakthrough results of J. Renegar and C. Gonzaga. The results, based on highly technical ad hoc analysis, state that when

♢ X is a polytope: $X = \{x : a_i^T x \leq b_i, i \leq m\}$,

♢ F is the *standard logarithmic barrier* for polytope: $F(x) = -\sum\limits_{i=1}^{m} \ln(b_i - a_i^T x)$,

♢ penalty updates are $t_s = \left(1 + \frac{0.1}{\sqrt{m}}\right) t_{s-1}$,

♢ "closeness of x_s to $x(t_s)$" is defined as

$$F_{t_s}(x_s) - \min_x F_{t_s}(x) \leq 0.1,$$

then a *single* Newton step per each penalty update $t_{s-1} \mapsto t_s$ maintains closeness of x_s to $x(t_s)$ and ensures polynomial time convergence of the process:

$$c^T x_s - \min_{x \in X} c^T x \leq 2m\, e^{-\frac{s}{10\sqrt{m}}} [c^T x_0 - \min_{x \in X} c^T x]$$

♣ Facts:

♢ I. With the standard logarithmic barrier, the classical path-following scheme admits polynomial time implementation

♢ II. Standard results on the Newton method do *not* suggest polynomiality of the path-following scheme. Indeed, they state that

(i) When minimizing a smooth nondegenerate convex objective ϕ, the Newton method, *started close to the minimizer* x_*, converges quadratically

$$\exists (r > 0, C < \infty) : \|x_s - x_*\| \leq r \Rightarrow \\ \|x_{s+1} - x_*\| \leq C\|x_s - x_*\|^2 \leq \frac{1}{2}\|x_s - x_*\|.$$

(ii) The domain of quadratic convergence is given by

$$r = \min\left[r_0, \frac{L_0^3}{32L_1^2 L_2}\right] \qquad (*)$$

where

♢ $r_0 > 0, L_2$ are such that

$$B \equiv \{x : \|x - x_*\| \le r_0\} \subset \mathrm{Dom}\,\phi$$
$$x', x'' \in B \Rightarrow \|\phi''(x') - \phi''(x'')\| \le L_2\|x' - x''\|$$

♢ L_0, L_1, $0 < L_0 \le L_1$, are the minimal and the maximal eigenvalues of $\phi''(x_*)$

♠*The less are r_0, L_0 and the larger are L_1, L_2, the smaller is r – the smaller is the domain of quadratic convergence as given by $(*)$.*

♠ *As path tracing proceeds, the parameters r_0, L_0, L_1, L_2 become worse and worse, which, from the classical viewpoint, makes minimization problems to be solved by the Newton method more and more complicated and therefore slows the overall process down, thus preventing it from being polynomial.*

♣ Facts:

◇ I. With the standard logarithmic barrier, the classical path-following scheme admits polynomial time implementation

◇ II. Standard results on the Newton method do *not* suggest polynomiality of the path-following scheme.

Facts I and II contradict each other; thus, something is wrong with the "common wisdom" on Newton method. What is wrong?

Observation: The classical description of the very fact of quadratic convergence:

$$\exists (r > 0, C < \infty) : \|x_s - x_*\| \leq r \Rightarrow \\ \|x_{s+1} - x_*\| \leq C\|x_s - x_*\|^2 \leq \frac{1}{2}\|x_s - x_*\|.$$

and the quantitative description of this domain

$$r = \min\left[r_0, \frac{L_0^3}{32L_1^2 L_2}\right] \qquad (*)$$

are *frame-dependent* – they depend on how we define the Euclidean norm $\|\cdot\|$, i.e., on how we choose the underlying Euclidean structure.

$$r = \min\left[r_0, \frac{L_0^3}{32L_1^2L_2}\right] \qquad (*)$$

Equivalently: given by $(*)$ domains of quadratic convergence of the Newton method as applied to a pair of *equivalent* problems

$$\min_x \phi(x) \Leftrightarrow \min_y \psi(y) \equiv \phi(Qy)$$

do *not* correspond to each other, in sharp contrast with the fact that *the Newton method is affine invariant: the trajectories* $\{x_s\}$ *and* $\{y_s\}$ *of the method as applied to these problems do correspond to each other:*

$$x_1 = Qy_1 \Rightarrow x_s = Qy_s \ \forall s.$$

♣ Frame-dependent description of *frame-independent* behaviour is incomplete, to say the least, and can be pretty misleading: the conclusions depend on ad hoc choice of the frame!
Cf.: Measuring distances in miles and gas in gallons, John concludes that he can reach Savannah without visiting a gas station; switching to kilometers and liters, John comes to the opposite conclusion. The technique underlying John's reasoning is bad, isn't it?
♣ When investigating the Newton method, "a frame" is a Euclidean structure on the space of variables. What we need is to extract this structure from the problem $\min_x f(x)$ we are solving rather than to use an "ad hoc" Euclidean structure.

Crucial observation: Smooth convex nondegenerate objective $f(\cdot)$ itself defines, at every point $x \in \mathrm{Dom} f$, Euclidean structure with the inner product and the norm given as

$$\langle u,v\rangle_x = D^2 f(x)[u,v] = u^T[f''(x)]v,$$
$$\|u\|_x = \sqrt{u^T[f''(x)]u}$$

Why not to use this Euclidean structure as our (local) frame?

♣ Developing the outlined idea, one arrives at the families of *self-concordant functions* and *self-concordant barriers*.

♢ *Self-concordant functions* are perfectly well suited for Newton minimization: fairly strong *global* convergence properties of the method as applied to such a function admit complete "frame-independent" description

♢ *Self-concordant barriers* are specific self-concordant functions perfectly well suited to serve as interior penalties in the path-following scheme. Given such a penalty, one can equip the path-following scheme with explicit and simple penalty updating rules and proximity measures quantifying "closeness to the path" in a way which ensures polynomiality of the resulting path-following method.

♣ It turns out that

♢ *All polynomial time interior point constructions and results known for Linear Programming stem from the fact that the underlying standard logarithmic barrier for a polytope is a self-concordant barrier.*

♢ *Replacing logarithmic barrier for a polytope with a self-concordant barrier for a given convex domain X* (such a barrier always exists, and usually can be pointed out explicitly), *one can extend interior point constructions and results from Linear to Convex Programming.*

♣ What is ahead:

- Developing tools, I: investigating basic properties of self-concordant functions and barriers

- First fruits: developing basic *primal path-following* polynomial-time interior point method for Convex Programming

- Exploiting duality: passing to convex problems in *conic form* and developing *primal-dual* potential reduction and path-following polynomial time methods

- Developing tools, II: developing "calculus of self-concordant barriers" allowing to build those barriers (and thus – to apply the associated interior point methods) to a wide variety of convex programs

- Applications to Conic Quadratic, Semidefinite and Geometric Programming.

Self-Concordant Functions

♣ For a C^k function $f(x)$ defined in an open domain $\mathrm{Dom} f \subset \mathbf{R}^n$, let

$$\begin{array}{rl} & D^k f(x)[h_1,...,h_k] \\ = & \left.\frac{\partial^k}{\partial t_1 \partial t_2 ... \partial t_k}\right|_{t=0} f(x + t_1 h_1 + ... + t_k h_k) \end{array}$$

be k-th differential of f taken at a point $x \in \mathrm{Dom} f$ along the directions $h_1,...,h_k \in \mathbf{R}^n$. Recall that

♢ $D^k f(x)[h_1,...,h_k]$ is k-linear symmetric form of $h_1,...,h_k$

♢ $Df(x)[h] \equiv D^1 f(x)[h] = h^T f'(x)$

♢ $D^2 f(x)[h_1,h_2] = h_1^T [f''(x)] h_2$

♣ <u>Simple and important fact:</u> Let Q be a positive definite symmetric $n \times n$ matrix and $\|x\|_Q = \sqrt{x^T Q x}$ be the corresponding Euclidean norm on $\mathbf{R}^n$. Let also $M[h_1,...,h_k]$ be a symmetric k-linear form on $\mathbf{R}^n$. Then

$$\max_{\substack{h_1,...,h_k \\ \|h_\ell\|_Q \leq 1, 1 \leq \ell \leq k}} |M[h_1,...,h_k]| = \underbrace{\max_{h: \|h\|_Q \leq 1} |M[h,h,...,h]|}_{\|M\|_{Q,\infty}}.$$

♣ Corollary: Let Q be a positive definite symmetric $n \times n$ matrix and $\|x\|_Q = \sqrt{x^T Q x}$ be the corresponding Euclidean norm on $\mathbf{R}^n$. Let also $M[h_1, ..., h_k]$ be a symmetric k-linear form on $\mathbf{R}^n$. Then

$$\forall h_1, ..., h_k \in \mathbf{R}^n : \\ |M[h_1, ..., h_k]| \leq \|M\|_{Q,\infty} \|h_1\|_Q \|h_2\|_Q ... \|h_k\|_Q$$

♣ Self-concordant function: definition. Let X be a nonempty open convex subset of $\mathbf{R}^n$ and f be a function with Dom$f = X$. Function f is called *self-concordant*,, if

♢ f is convex and 3 times continuously differentiable on X

♢ X is a natural domain of f: for every sequence of points $x_i \in X$ converging to a point $x \notin X$, one has

$$\lim_{i\to\infty} f(x_i) = +\infty$$

♢ f satisfies differential inequality

$$\begin{array}{l} \forall (x \in \mathrm{Dom} f, h \in \mathbf{R}^n) : \\ |D^3 f(x)[h,h,h]| \leq 2 \left[D^2 f(x)[h,h]\right]^{3/2} \end{array} \qquad (*)$$

Example 1. The convex quadratic function $f(x) = \frac{1}{2}x^T Ax + 2b^T x + C$ is self-concordant on $\mathbf{R}^n$

$$\forall (x \in \mathrm{Dom} f, h \in \mathbf{R}^n):$$
$$|D^3 f(x)[h,h,h]| \leq 2\left[D^2 f(x)[h,h]\right]^{3/2} \qquad (*)$$

Example 2. The function

$$f(x) = -\ln x : \{x > 0\} \to \mathbf{R}$$

is self-concordant on its domain; for this function, $(*)$ is identity:

$$\begin{array}{rcl} |D^3 f(x)[h,h,h]| & \equiv & |f'''(x)h^3| = 2|h/x|^3 \\ & = & 2\left[f''(x)h^2\right]^{3/2} \\ & = & 2\left[D^2 f(x)[h,h]\right]^{3/2}. \end{array}$$

$$\forall (x \in \mathrm{Dom} f, h \in \mathbf{R}^n):$$
$$|D^3 f(x)[h,h,h]| \leq 2\left[D^2 f(x)[h,h]\right]^{3/2} \qquad (*)$$

♣ <u>Comments.</u> **1.** Recalling definition of differentials, $(*)$ is of the generic form

$$\forall (x \in \mathrm{Dom} f, h \in \mathbf{R}^n):$$
$$\left|\frac{d^3}{dt^3}\Big|_{t=0} f(x+th)\right| \leq \mathrm{const}\left[\frac{d^2}{dt^2}\Big|_{t=0} f(x+th)\right]^{\beta} \qquad (!)$$

As functions of h, the LHS is of homogeneity degree 3, and the RHS is of the homogeneity degree 2β; thus, *$\beta = 3/2$ is the only option to make (!) possible.*

$$\forall (x \in \mathrm{Dom} f, h \in \mathbf{R}^n):$$
$$\left|\frac{d^3}{dt^3}\Big|_{t=0} f(x+th)\right| \leq \mathrm{const} \left[\frac{d^2}{dt^2}\Big|_{t=0} f(x+th)\right]^{3/2} \quad (!)$$

2. Let $\|h\|_x = \sqrt{h^T[f''(x)]h}$ be the Euclidean norm generated by f at x. By Simple Fact, (!) is equivalent to

$$\begin{array}{l}\left|\frac{d}{dt}\Big|_{t=0} D^2 f(x+t\delta)[h,h]\right| \equiv |D^3 f(x)[\delta,h,h]| \\ \quad \leq \mathrm{const}\, \|\delta\|_x \|h\|_x^2 = \mathrm{const}\, \|\delta\|_x D^2 f(x)[h,h]\end{array}$$

that is, the second derivative of f is Lipschitz continuous, with the constant const, *w.r.t. the Euclidean norm defined by this derivative itself.*

$$\forall (x \in \mathrm{Dom} f, h \in \mathbf{R}^n) :$$
$$\left| \frac{d^3}{dt^3}\Big|_{t=0} f(x+th) \right| \leq \mathrm{const} \left[\frac{d^2}{dt^2}\Big|_{t=0} f(x+th) \right]^{3/2} \quad (!)$$

3. Both sides of (!) are of the same homogeneity degree w.r.t. h, but *are of different homogeneity degrees w.r.t.* f – multiplying f by $\lambda > 0$ *and* updating const according to

$$\mathrm{const} \mapsto \mathrm{const}\, \lambda^{-1/2}.$$

It follows that *if f satisfies (!) with a constant const > 0, then, after proper scaling, f satisfies similar inequality with a once for ever fixed value of the constant*, e.g., with const $= 2$.

$$\forall (x \in \mathrm{Dom} f, h \in \mathbf{R}^n) :$$
$$\left|\frac{d^3}{dt^3}\Big|_{t=0} f(x+th)\right| \leq \mathrm{const}\left[\frac{d^2}{dt^2}\Big|_{t=0} f(x+th)\right]^{3/2} \qquad (!)$$

<u>Conclusion:</u> Inequality (!) says, essentially, that the second derivative of f is Lipschitz continuous w.r.t. the local Euclidean norm defined by this derivative itself.
The Self-concordance inequality

$$\forall (x \in \mathrm{Dom} f, h \in \mathbf{R}^n) :$$
$$|D^3 f(x)[h,h,h]| \leq 2\left[D^2 f(x)[h,h]\right]^{3/2} \qquad (*)$$

is a normalized version of this requirement where the Lipschitz constant equals to 2.
<u>Note:</u> The normalization constant is set to 2 with the only purpose: to make the function $-\ln x$ self-concordant "as it is", without rescaling.

Basic Inequality

Proposition [Basic Inequality] Let f be s.c.f. Then

$$|D^3 f(x)[h_1, h_2, h_3]| \leq 2\|h_1\|_x \|h_2\|_x \|h_3\|_x$$

for all $x \in \mathrm{Dom} f$ and for all $h_1, h_2, h_3 \in \mathbf{R}^n$.

Proof: Simple Fact combined with Self-concordance inequality.

Elementary Calculus of Self-concordant Functions

Proposition Self-concordance is preserved by the following basic operations:

♢ Affine substitution of argument: If $f(x)$ is s.c. on $\mathrm{Dom} f \subset \mathbf{R}^n$ and $x = Ay + b : \mathbf{R}^m \to \mathbf{R}^n$ is an affine mapping with the image intersecting $\mathrm{Dom} f$, then the function

$$g(y) = f(Ay + b)$$

is s.c. on its domain (that is, on $\{y : Ay + b \in \mathrm{Dom} f\}$)

♢ Summation with coefficients ≥ 1: If $f_\ell(x)$, $\ell = 1, ..., L$, are s.c.f.'s with $\bigcap\limits_{\ell=1}^{L} \mathrm{Dom} f_\ell \neq \emptyset$, and $\lambda_\ell \geq 1$, then the function

$$\sum_{\ell=1}^{L} \lambda_\ell f_\ell(x) : \bigcap_{\ell=1}^{L} \mathrm{Dom} f_\ell \to \mathbf{R}$$

is s.c.

$\diamondsuit$ Direct summation: If $f_\ell(x^\ell)$, $\ell = 1, ..., L$, are s.c.f.'s with domains $\mathrm{Dom} f_\ell \subset \mathbf{R}^{n_\ell}$, then the function

$$\sum_{\ell=1}^{L} f_\ell(x^\ell) : \mathrm{Dom} f_1 \times ... \times \mathrm{Dom} f_L \to \mathbf{R}$$

is s.c.

Proof. All statements are nearly trivial. Let us focus on the less trivial of them related to summation.

♢ Smoothness, convexity and barrier properties of

$$f(x) = \sum_{\ell=1}^{L} \lambda_\ell f_\ell(x) \qquad [\lambda_\ell \geq 1]$$

are evident. Let us verify Self-concordance inequality:

$$\begin{array}{rcl}
|D^3 f(x)[h,h,h]| & \leq & \sum\limits_\ell \lambda_\ell |D^3 f_\ell[h,h,h]| \\
& \leq & 2\sum\limits_\ell \lambda_\ell (D^2 f_\ell(x)[h,h])^{3/2} \\
& \leq & 2\sum\limits_\ell (\lambda_\ell D^2 f_\ell(x)[h,h])^{3/2} \\
& & \qquad [\text{due to } \lambda_\ell \geq 1] \\
& \leq & 2\left[\sum\limits_\ell \lambda_\ell D^2 f_\ell(x)[h,h]\right]^{3/2} \\
\multicolumn{3}{l}{[\text{due to } \sum\limits_\ell t_\ell^\alpha \leq \left[\sum\limits_\ell t_\ell\right]^\alpha \text{ for } t_\ell \geq 0 \text{ and } \alpha \geq 1]} \\
& = & 2(D^2 f(x)[h,h])^{3/2}.
\end{array}$$

<u>Corollary:</u> Let $X = \left\{x : a_i^T x \le b_i,\ i = 1, ..., m\right\}$ be a polyhedral set such that $X^o \equiv \{x : a_i^T x < b_i, 1 \le i \le m\} \neq \emptyset$. Then the associated standard logarithmic barrier

$$F(x) = -\sum_{i=1}^{m} \ln(b_i - a_i^T x)$$

is s.c. on its domain X^o.

Indeed, F is obtained from the s.c.f. $-\ln t$ by affine substitutions of arguments and summation, and these operations preserve self-concordance.

Properties of Self-Concordant Functions
I. Behaviour in the Dikin Ellipsoid

Theorem. Let f, $\mathrm{Dom}f = X$, be s.c.f., and let $x \in X$. We define the *Dikin ellipsoid of f with radius r centered at x* as the set

$$W_r(x) = \{y : \|y - x\|_x \leq r\}.$$

Then the unit Dikin ellipsoid $W_1(x)$ is contained in X, and in this ellipsoid
(a) the Hessians of f are "nearly proportional":

$$r \equiv \|h\|_x < 1 \Rightarrow \left\{ \begin{array}{l} (1-r)^2 f''(x) \preceq f''(x+h) \\ f''(x+h) \preceq (1-r)^{-2} f''(x) \end{array} \right.$$

(b) the gradients of f satisfy the following Lipschitz-type condition:

$$\begin{array}{l} r \equiv \|h\|_x < 1 \Rightarrow \\ \quad |z^T[f'(x+h) - f'(x)]| \leq \frac{r}{1-r}\|z\|_x \ \forall z \end{array}$$

(c) we have the following upper and lower bounds on f:

$$r \equiv \|h\|_x < 1 \Rightarrow \left\{ \begin{array}{l} f(x+h) \leq f(x) + h^T f'(x) + \rho(r) \\ f(x+h) \geq f(x) + h^T f'(x) + \rho(-r) \end{array} \right. ,$$

where

$$\rho(r) = -\ln(1-r) - r = \frac{r^2}{2} + \frac{r^3}{3} + \frac{r^4}{4} + ...$$

Lower bound in (c) remains valid for all h such that $x + h \in X$, and not only for h with $\|h\|_x < 1$.

<u>Remark to (c):</u> Taking into account that $\|h\|_x^2 = h^T f''(x) h$, the bounds on f in (c) can be rewritten as

$$\begin{array}{l} r \equiv \|h\|_x < 1 \Rightarrow \\ \quad \begin{array}{rcl} \sigma(-r) & \leq & f(x+h) - [f(x) + h^T f'(x) + \frac{h^T f''(x) h}{2}] \\ & \leq & \sigma(r), \end{array} \end{array}$$

where $\sigma(r) = \rho(r) - \frac{r^2}{2} = \frac{r^3}{3} + \frac{r^4}{4} + ...$

Thus, in the unit Dikin ellipsoid, we get a *universal* bound on the difference between $f(\cdot)$ and the second-order Taylor expansion of f, taken at the center of the ellipsoid.

Sketch of the proof. The results more or less straightforwardly follow from

Lemma: Let f be s.c.f. on $X = \mathrm{Dom} f$, let $x \in X$, and let h be such that $r \equiv \|h\|_x < 1$ and $x + h \in X$. Then, for every vector v, one has

$$\begin{array}{rcll} v^T f''(x+h)v & \leq & (1-r)^{-2} v^T f''(x)v & (i) \\ v^T f''(x+h)v & \geq & (1-r)^{2} v^T f''(x)v & (ii) \end{array}$$

Proof: 1^0. Let $\phi(t) = h^T f''(x+th)h \equiv \|h\|^2_{x+th}$, so that ϕ is nonnegative C^1 function on $[0,1]$, and let $\phi_\epsilon(t) = \phi(t) + \epsilon$. For $0 \leq t \leq 1$ we have

$$|\phi'(t)| = |D^3 f(x+th)[h,h,h]| \leq 2\phi^{3/2}(t),$$

so that for $0 \leq t \leq 1$ one has

$$\begin{array}{ll} & |\phi'_\epsilon(t)| \leq 2\phi_\epsilon^{3/2}(t) \\ \Rightarrow & |\frac{d}{dt}\phi_\epsilon^{-1/2}(t)| \leq 1 \\ \Rightarrow & \psi_\epsilon^{-1/2}(0) - t \leq \phi_\epsilon^{-1/2}(t) \leq \psi_\epsilon^{-1/2}(0) - t \\ \Rightarrow & \frac{\phi_\epsilon(0)}{(1+t\phi_\epsilon^{1/2}(0))^2} \leq \phi_\epsilon(t) \leq \frac{\phi_\epsilon(0)}{(1-t\phi_\epsilon^{1/2}(0))^2} \\ \Rightarrow & \frac{\phi(0)}{(1+t\phi^{1/2}(0))^2} \leq \phi(t) \leq \frac{\phi_\epsilon(0)}{(1-t\phi^{1/2}(0))^2} \\ \Leftrightarrow & \frac{r^2}{(1+tr)^2} \leq \phi(t) \leq \frac{r^2}{(1-tr)^2} \end{array}$$

$\mathbf{2^0}$. Now let $\psi(t) = v^T f''(x+th)v \equiv \|v\|^2_{x+th}$, so that $\psi(t)$ is nonnegative C^1 function on $[0,1]$. For $0 \le t \le 1$, we have

$$\begin{array}{rcl} |\psi'(t)| & = & |D^3 f(x+th)[v,v,h]| \\ & \le & 2\|v\|^2_{x+th}\|h\|_{x+th} \\ & = & 2\psi(t)\phi^{1/2}(t), \end{array}$$

whence, by $\mathbf{1^0}$, $|\psi'(t)| \le 2\psi(t)\frac{r}{1-tr}$. Setting $\psi_\epsilon(t) = \psi(t) + \epsilon$, we get for $0 \le t \le 1$:

$$\begin{array}{rl} & |\psi'_\epsilon(t)| \le 2\psi_\epsilon(t)\frac{r}{1-tr} \\ \Rightarrow & |\frac{d}{dt}\ln\psi_\epsilon(t)| \le \frac{2r}{1-tr} \\ \Rightarrow & (1-tr)^2\psi_\epsilon(0) \le \psi_\epsilon(t) \le (1-tr)^{-2}\psi_\epsilon(0) \\ \Rightarrow & (1-r)^2\psi(0) \le \psi(1) \le (1-r)^{-2}\psi(0) \\ \Leftrightarrow & \begin{array}{rcl} (1-r)^2 v^T f''(x)v & \le & v^T f''(x+h)v \\ & \le & (1-r)^{-2} v^T f''(x)v \end{array} \end{array}$$

Recessive Subspace of a Self-Concordant Function

Proposition Let f be a s.c.f. For $x \in \mathrm{Dom} f$, the kernel of the Hessian of f at x – the subspace $L_x = \{h : f''(x)h = 0\} = \{h : D^2 f(x)[h,h] = 0\}$ is independent of x. This *recessive subspace* E_f of f is such that

$$\mathrm{Dom} f = X + E_f,$$

where X is the intersection of $\mathrm{Dom} f$ with $E_f^\perp$, and

$$f(x) = \phi(Px) + c^T Qx,$$

where ϕ is a self-concordant function on X with nondegenerate Hessian, Px is the ortho-projector of x on $E_f^\perp$, and Qx is the ortho-projector of x on E_f.

In particular, if $f''(x)$ is nondegenerate at some point $x \in \mathrm{Dom} f$, then $f''(x)$ is nondegenerate everywhere on $\mathrm{Dom} f$; in this case f is called nondegenerate s.c.f.

Proof. 1^0. Let $x, y \in \mathrm{Dom} f$ and $f''(x)h = 0$. Setting $\phi(t) = h^T f''(x + t(y - x))h$, we have

$$\begin{aligned} |\phi'(t)| &= |D^3 f(x + t(y - x))[y - x, h, h]| \\ &\leq 2\sqrt{D^2 f(x + t(y - x))[y - x, y - x]}\phi(t). \end{aligned}$$

In other words, for $0 \leq t \leq 1$ $\phi(t)$ satisfies homogeneous linear equation

$$\phi'(t) = \chi(t)\phi(t)$$

with bounded $\chi(\cdot)$. Since $\phi(0) = 0$, we have $\phi(t) \equiv 0$ (Uniqueness Theorem for ordinary differential equations). Thus, $\phi(1) = h^T f''(y)h = 0$, whence $f''(y)h = 0$ due to $f''(y) \succeq 0$. Thus, $f''(x)h = 0$ implies that $f''(y)h = 0$ for all $y \in \mathrm{Dom} f$, that is, the linear subspace $\{h : f''(x)h = 0\}$ is independent of $x \in \mathrm{Dom} f$.

$\mathbf{2}^0$. Let $h \in E_f$ and $x \in \mathrm{Dom}f$. Since $f''(x)h = 0$ by $\mathbf{1}^0$, $\|h\|_x = 0$; thus, $x + h$ belongs to the unit Dikin ellipsoid centered at x, and tghus belongs to $\mathrm{Dom}f$. We conclude that $\mathrm{Dom}f = \mathrm{Dom}f + E_f$. It follows that if X is the orthoprojection of $\mathrm{Dom}f$ onto $E_f^\perp$, then

$$\mathrm{Dom}f = X + E_f.$$

Clearly, the restriction of f onto X is a self-concordant function on X; it is nondegenerate on its domain, since for $0 \neq h \in E_F^\perp$ we clearly have $h^T\phi''(x)h = h^T f''(x)h$, and the latter quantity is positive due to $h \notin E_f$.

$\mathbf{3^0}$. Let us write a point $x \in \mathrm{Dom} f$ as $x = (u,v)$, where u is the projection of x on $E_f^\perp$ and v is the projection of x on E_f. Let $\bar{x} = (\bar{u}, 0) \in \mathrm{Dom} f$, and let $c = \frac{\partial f(\bar{x})}{\partial v}$. Let us prove that

$$f(u,v) = \phi(u) + c^T v \Leftrightarrow \overbrace{f(u,v) - c^T v}^{\tilde{f}(u,v)} \equiv \phi(u).$$

Function $\tilde{f}$ is s.c. along with f and has the same recessive subspace E_f. What we should prove is that $\tilde{f}$ is independent of v, that is, to prove that

$$h \in E_F \Rightarrow h^T \tilde{f}'(x) = 0 \,\forall x \in \mathrm{Dom}\tilde{f}.$$

Indeed, let $x \in \mathrm{Dom}\tilde{f}$ and $h \in E_f$. Let

$$g(t) = h^T \tilde{f}'(\bar{x} + t(x - \bar{x})).$$

We have $g(0) = 0$,

$$\begin{array}{l} g'(t) = h^T \tilde{f}''(\bar{x} + t(x - \bar{x}))(x - \bar{x}) \\ \quad = (x - \bar{x})^T [f''(\bar{x} + t(x - \bar{x}))h] = 0, \end{array}$$

that is, $g(t) \equiv 0$, and thus $g(1) = h^T \tilde{f}'(x) = 0$, Q.E.D.

Newton Decrement of a Self-Concordant Function

♣ Definition. Let f be a nondegenerate s.c.f. and $x \in \mathrm{Dom} f$. The Newton Decrement of f at x is the quantity

$$\lambda(f,x) \equiv \sqrt{(f'(x))^T[f''(x)]^{-1}f'(x)}$$

Remarks **I.** $\frac{1}{2}\lambda^2(f,x)$ is exactly the amount by which the NEwton step, taken at x, decreases the second order Taylor expansion of f:

$$\begin{array}{l} f(x) - \min\limits_h \left\{ f(x) + h^T f'(x) + \frac{1}{2} h^T f''(x) h \right\} \\ = (f'(x))^T [F''(x)]^{-1} f'(x) \\ \quad -\frac{1}{2}\left[[f''(x)]^{-1}f'(x)\right]^T [f''(x)] \left[[f''(x)]^{-1}f'(x)\right] \\ = \frac{1}{2}(f'(x))^T[f''(x)]^{-1}f'(x). \end{array}$$

II. Compute the norm conjugate to a norm $\|\cdot\|_Q$:

$$\|\xi\|_Q^* \equiv \max_h \left\{h^T\xi : \|h\|_Q \leq 1\right\}$$
$$= \max_h \left\{([Q^{1/2}h]^T[Q^{-1/2}\xi] : [Q^{1/2}h]^T[Q^{1/2}h] \leq 1\right\}$$
$$= \|Q^{-1/2}\xi\|_2$$
$$= \sqrt{\xi^T Q^{-1}\xi}$$

Thus, Newton Decrement $\lambda(f,x)$ is nothing but the conjugate to $\|\cdot\|_x$ norm of $f'(x)$:

$$\lambda(f,x) = \max_h \left\{h^T f'(x) : \|h\|_x \leq 1\right\} = \|f'(x)\|_x^*,$$

where

$$\|\xi\|_x^* = \max_h \left\{\xi^T h : \|x\|_x \leq 1\right\} = \sqrt{\xi^T [f''(x)]^{-1}\xi}.$$

III. Let $e(x) = [f''(x)]^{-1}f'(x)$ be the *Newton Displacement* of f at x, so that the Newton iterate of x is

$$x_+ = x - [f''(x)]^{-1}f'(x) = x - e(x).$$

Then

$$\begin{aligned}
\lambda(f,x) &= \sqrt{(f'(x))^T[f''(x)]^{-1}f'(x)} \\
&= \sqrt{(f'(x))^T e(x)} \\
&= \sqrt{\left[[f''(x)]^{-1}f'(x)\right]^T [f''(x)] \left[[f''(x)]^{-1}f'(x)\right]} \\
&= \|e(x)\|_x
\end{aligned}$$

Damped Newton Method

<u>Theorem</u> Let f be a nondegenerate s.c.f. and let $x \in \mathrm{Dom} f$. *Damped Newton Iterate of* x is the point

$$\begin{array}{rcl} x_+ & = & x - \frac{1}{1+\lambda(f,x)} e(x) \\ & = & x - \frac{1}{1+\lambda(f,x)} [f''(x)]^{-1} f'(x). \end{array}$$

This point belongs to $\mathrm{Dom} f$, and

$$\begin{array}{rcl} f(x_+) & \leq & f(x) - \rho(-\lambda(f,x)) \\ & = & f(x) - [\lambda(f,x) - \ln(1+\lambda(f,x))]\,,\ , \\ \lambda(f,x_+) & \leq & 2\lambda^2(f,x). \end{array}$$

where, as always,

$$\rho(r) = -\ln(1-r) - r.$$

Proof. Let us set $\lambda = \lambda(f, x)$. Then

$$\|x_+ - x\|_x = \frac{1}{1+\lambda}\|e(x)\|_x = \frac{\lambda}{1+\lambda} < 1,$$

whence $x_+ \in \mathrm{Dom} f$. We now have

$$\begin{array}{rcl} f(x_+) & \leq & f(x) + (x_+ - x)^T f'(x) + \rho(\|x_+ - x\|_x) \\ & = & f(x) - \frac{1}{1+\lambda} e^T(x) f'(x) + \rho(\lambda/(1+\lambda)) \\ & = & f(x) - \frac{\lambda^2}{1+\lambda} + \rho(\lambda/(1+\lambda)) \\ & = & f(x) - \frac{\lambda^2}{1+\lambda} - \frac{\lambda}{1+\lambda} - \ln\left(1 - \frac{\lambda}{1+\lambda}\right) \\ & = & f(x) - \lambda + \ln(1+\lambda) = f(x) - \rho(-\lambda) \end{array}$$

Further, let us set $d = x_+ - x$, so that $\|d\|_x = \theta \equiv \frac{\lambda}{1+\lambda}$. Given z, let $\phi(t) = z^T f'(x+td)$. We have

$$
\begin{array}{rcl}
\multicolumn{3}{l}{(1+t\|d\|_x)^2 f''(x) \preceq f''(x+td) \preceq \frac{f''(x)}{(1-t\|d\|_x)^2}} \\
\Rightarrow \quad \phi'(t) & = & z^T f''(x+td)d \\
& \leq & z^T f''(x)d + \|z\|_x \|d\|_x \left[\frac{1}{(1-t\theta)^2} - 1\right] \\
& \leq & z^T f''(x)d + \|z\|_x \theta \left[\frac{1}{(1-t\theta)^2} - 1\right] \\
\Rightarrow \quad \phi(1) & \leq & \phi(0) + z^T f''(x)d + \|z\|_x \frac{\theta^2}{1-\theta} \\
& = & z^T f'(x) - \frac{1}{1+\lambda} z^T f'(x) + \|z\|_x \frac{\theta^2}{1-\theta} \\
& = & \theta z^T f'(x) + \|z\|_x \frac{\theta^2}{1-\theta} \\
& \leq & \theta \|z\|_x \|f'(x)\|_x^* + \|z\|_x \frac{\lambda^2}{1+\lambda} \\
& = & \|z\|_x \left[\theta\lambda + \frac{\lambda^2}{1+\lambda}\right] \\
& \leq & \frac{\|z\|_{x_+}}{1-\|d\|_x} \frac{2\lambda^2}{1+\lambda} = 2\lambda^2 \|z\|_{x_+}.
\end{array}
$$

Thus,

$$z^T f'(x_+) \leq 2\lambda^2 \|z\|_{x_+} \ \forall z,$$

whence

$$\lambda(f, x_+) \equiv \|f'(x_+)\|_{x_+}^* \leq 2\lambda^2.$$

Minimizers of Self-Concordant Functions

Theorem Let f be a nondegenerate s.c.f. f attains its minimum on Dom$f$ iff f is below bounded, and in this case the minimizer x_* is unique.

Proof. All we need to prove is that if f is below bounded, then f attains its minimum on $\mathrm{Dom} f$. Consider the Damped Newton method

$$x_{t+1} = x_t - \frac{1}{1+\lambda_t} e(x_t),\ \lambda_t = \lambda(f, x_t).$$

Step t decreases f by at least $\rho(-\lambda_t) = \lambda_t - \ln(1+\lambda_t)$; since f is below bounded, $\lambda_t \to 0$ as $t \to \infty$.

We have

$$\begin{array}{l} \|d\|_{x_t} = \frac{1}{2} \Rightarrow f(x_t+d) \geq f(x_t) + d^T f'(x_t) + \rho(-\frac{1}{2}) \\ \qquad \geq f(x_t) - \|d\|_{x_t} \|f'(x_t)\|_{x_t}^* + \rho(-\frac{1}{2}) \\ \qquad = f(x_t) - \frac{1}{2}\lambda_t + \rho(-\frac{1}{2}) \end{array}$$

If follows that for t large enough one has

$$\|d\|_{x_t} = \frac{1}{2} f(x_t + d) > f(x_t),$$

that is, the minimizer of $f(x)$ on the set $\{y : \|y - x_t\|_{x_t} \leq \frac{1}{2}\} \subset \mathrm{Dom} f$ (which definitely exists) is an interior point of the set. By convexity, this minimizer minimizes f on $\mathrm{Dom} f$ as well. The uniqueness of the minimizer follows from the strong convexity of f.

Self-Concordance and Legendre Transformation

♣ Recall that the Legendre transformation of a convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is the function

$$f_*(\xi) = \sup_x \left[\xi^T x - f(x)\right].$$

The Legendre transformation always is convex, and twice taken Legendre transformation of convex function f with *closed epigraph* is f itself.

Theorem Let f be a nondegenerate s.c.f. and f_* be the Legendre transformation of f. Then

♢ f_* is nondegenerate s.c.f. with the domain $\{\xi = f'(x) : x \in \mathrm{Dom} f\}$,

♢ $(f_*)_* = f$,

♢ the mapping $\xi \mapsto f'_*(\xi)$ is one-to-one C^2 mapping of $\mathrm{Dom} f_*$ onto $\mathrm{Dom} f$, and the inverse of this mapping is exactly the mapping $x \mapsto f'(x)$.

Proof. 1^0. The function

$$f_* = \sup_x \left[\xi^T x - f(x)\right] = -\inf_x \left[f(x) - \xi^T x\right]$$

is convex function with the domain

$$\left\{\xi : \inf_x \left[f(x) - \xi^T x\right] > -\infty\right\}$$

Since the function $f(x) - \xi^T x$ is s.c. along with f, it is below bounded iff it attains its minimum, which is the case iff $\xi = f'(x)$ for some x. Thus, the domain of f_* (which always is convex) is exactly the set $\{\xi = f'(x) : x \in \mathrm{Dom} f\}$, as claimed. Since f'' is nondegenerate at every point, $\mathrm{Dom} f_* = f'(\mathrm{Dom} f)$ is an open set (Inverse Function Theorem).

2^0. S.c.f. f clearly has closed epigraph, whence $(f_*)_* = f$, as claimed.

3^0. It remains to prove that f_* is s.c.f. and that the mappings $\xi \mapsto f_*'(\xi)$ and $x \mapsto f'(x)$ are inverses of each other.

3^0.1. As we have seen, for every $\xi \in \mathrm{Dom} f_*$ there exists uniquely defined $x = x(\xi) \in \mathrm{Dom} f$ such that $f'(x(\xi)) = \xi$. Since f'' is nondegenerate, by the Implicit Function Theorem the function $x(\xi)$ is C^2 on its domain. Besides this,

$$f'_*(\xi) = x(\xi)$$

Indeed, $f_*(\eta) \geq \eta^T x(\xi) - f(x(\xi))$, and this inequality is equality when $\eta = \xi$. Thus, $x(\xi)$ is a subgradient of f_* at ξ, and since this subgradient continuously depends on ξ, it is the gradient.
Since $f'_*(\xi) = x(\xi)$ is C^2, f_* is C^3.
$\diamondsuit$ Thus, f_* is convex C^3 function with open domain, as required by the definition of a s.c.f.

3^0.2. We have seen that

$$\begin{array}{rcl}\mathrm{Dom} f_* & = & f'(\mathrm{Dom} f)\\ f_*'(f'(x)) & \equiv & x.\end{array} \qquad (*)$$

Since f' is one-to-one (due to strong convexity of f), $(*)$ means that the mappings $f'(\cdot)$ and $f_*'(\cdot)$ are inverses of each other.

3^0.3. Differentiating the identity in $(*)$, we get

$$f_*''(f'(x)) = [f''(x)]^{-1};$$

since $f'(\mathrm{Dom} f) = \mathrm{Dom} f_*$, it follows that f_* is nondegenerate.

3^0.4. Let us prove that f_* is interior penalty for $\mathrm{Dom} f_*$. Assuming opposite, there exists a sequence $\xi_i \in \mathrm{Dom} f_*$ converging to a boundary point ξ of $\mathrm{Dom} f_*$ such that the sequence $f_*(\xi_i)$ is bounded, that is, the functions

$$\xi_i^T x - f(x)$$

are uniformly bounded from above. Then the function $\xi^T x - f(x)$ also is bounded from above, whence $\xi \in \mathrm{Dom} f_*$; since the latter set is open and ξ is its boundary point, we arrive at a contradiction.

3^0.5. It remains to verify Self-concordance inequality. We have

$$D^2 f_*(f'(x))[q,q] = q^T[f_*''(f'(x))]q = q^T[f''(x)]^{-1}q$$

Differentiating this identity in x in a direction h, we get

$$\begin{array}{rl} & D^3 f_*(f'(x))[[f''(x)]h, q, q] \\ = & -q^T[f''(x)]^{-1}\left[\left.\frac{d}{dt}\right|_{t=0} f''(x+th)\right][f''(x)]^{-1}q \\ = & -\left.\frac{d}{dt}\right|_{t=0} D^2 f(x+th)[[f''(x)]^{-1}q, [f''(x)]^{-1}q] \\ = & -D^3 f(x)[h, [f''(x)]^{-1}q, [f''(x)]^{-1}q] \end{array}$$

which with $h = [f''(x)]^{-1}q$ results in

$$\begin{array}{l} D^3 f_*(f'(x))[q,q,q] \\ \quad = -D^3 f(x)[[f''(x)]^{-1}q, [f''(x)]^{-1}q, [f''(x)]^{-1}q], \end{array}$$

whence

$$\begin{array}{rl} & |D^3 f_*(f'(x))[q,q,q]| \\ = & |D^3 f(x)[[f''(x)]^{-1}q, [f''(x)]^{-1}q, [f''(x)]^{-1}q]| \\ \leq & 2\left([[f''(x)]^{-1}q]^T f''(x)[[f''(x)]^{-1}q]\right)^{3/2} \\ = & 2(q^T[f''(x)]^{-1}q)^{3/2} \\ = & 2(q^T f_*''(f'(x))q)^{3/2} \\ = & 2\left(D^2 f_*(f'(x))[q,q]\right)^{3/2} \end{array}$$

Since $f'(\text{Dom} f) = \text{Dom} f_*$ and q is arbitrary, f_* satisfies Self-concordant Inequality.

Minimizers of Self-Concordant Functions (continued)

Theorem: A nondegenerate s.c.f. f attains its minimum iff $\lambda(f,x) < 1$ for some $x \in \mathrm{Dom} f$, and for every x with this property one has

$$f(x) - \min f \le \rho(\lambda(f,x))$$
$$[\rho(r) = -r - \ln(1-r)]$$

and

$$\max\left[\|x - x_*\|_{x_*}, \|x - x_*\|_x\right] \le \frac{\lambda(f,x)}{1-\lambda(f,x)},$$

where x_* is the minimizer of f.

Proof. If f attains its minimum at a point x_*, then $\lambda(f,x_*) = 0 < 1$. Now assume that $\lambda(f,x) < 1$ for certain x.

♢ Let f_* be the Legendre transformation of f, and let $y = f'(x)$. We have

$$1 > \lambda^2(f,x) = y^T[f''(x)]^{-1}y = y^T[f_*''(y)]y,$$

that is, *0 belongs to the unit Dikin ellipsoid, centered at $y = f'(x)$, of the s.c.f. f_**, whence $0 \in \mathrm{Dom} f_*$, or $\exists x_* : f'(x_*) = 0$.

$$\boxed{\begin{array}{c} f'_*(f'(u)) \equiv u \\ f_*(f'(u)) \equiv u^T f'(u) - f(u) \\ f''_*(f'(u)) \equiv [f''(u)]^{-1} \end{array}}$$

$$\lambda \equiv \lambda(f, x),\ y = f'(x) \Rightarrow y^T [f''_*(y)] y = \lambda^2$$

In the case of $\lambda < 1$ we have

$$\begin{array}{rcl} f_*(0) & = & f_*(y + (-y)) \leq f_*(y) - y^T \underbrace{f'_*(y)}_{x} + \rho(\lambda) \\ & = & f_*(y) - y^T x + \rho(\lambda) \end{array} \qquad (1)$$

Since $y = f'(x)$, we have

$$f_*(y) = x^T y - f(x), \qquad (2),$$

and since $f'(x_*) = 0$, we have

$$f_*(0) = -f(x_*). \qquad (3)$$

Together, (1) − (3) say that

$$-f(x_*) \leq [x^T y - f(x)] - y^T x + \rho(\lambda),$$

and we arrive at

$$f(x) - \min f \leq \rho(\lambda(f, x)).$$

$$\lambda \equiv \lambda(f,x) < 1 \Rightarrow$$
$$f(x) - \min f \le \rho(\lambda) = -\lambda - \ln(1-\lambda).$$

Let $r = \|x - x_*\|_{x_*}$. Then

$$f(x) \ge f(x_*) + \rho(-r) \equiv \min f + r - \ln(1+r).$$

We arrive at

$$r - \ln(1+r) \le -\lambda - \ln(1-\lambda),$$

whence

$$\|x - x_*\|_{x_*} = r \le \frac{\lambda}{1-\lambda}.$$

$$\lambda \equiv \lambda(f,x),\ y = f'(x) \Rightarrow y^T[f''_*(y)]y = \lambda^2$$

In the case of $\lambda < 1$ we have

$$\begin{array}{rcl}|z^T(x_* - x)| & = & |z^T[f'_*(0) - f'_*(y)]| \\ & \leq & \frac{\|y\|_{f_*,y}}{1-\|y\|_{f_*,y}}\sqrt{z^T[f''_*(y)]z} \\ & = & \frac{\lambda}{1-\lambda}\sqrt{z^T[f''(x)]^{-1}z}.\end{array}$$

Substituting $z = f''(x)(x_* - x)$, we get

$$\|x_* - x\|_x^2 \leq \frac{\lambda}{1-\lambda}\|x_* - x\|_x,$$

whence

$$\|x_* - x\|_x \leq \frac{\lambda}{1-\lambda}$$

Summary: Let f be nondegenerate s.c.f., and let $\lambda(f,x) = \sqrt{[f'(x)]^T[f''(x)]^{-1}f'(x)}$. Then

$\diamondsuit$ f attains its minimum on Dom$f$ iff f is below bounded, and iff there exists x with $\lambda(f,x) < 1$. The minimizer x_*, if exists, is unique.

$\diamondsuit$ The Damped Newton method

$$x_{t+1} = x_t - \frac{1}{1+\underbrace{\lambda(f,x_t)}_{\lambda_t}}[f''(x_t)]^{-1}f'(x_t)$$

is well-defined, provided that $x_0 \in \mathrm{Dom}f$, and for this method

$$\begin{array}{rcl} \lambda_{t+1} & \leq & 2\lambda_t^2 \\ f(x_{t+1}) & \leq & f(x_t) - [\lambda_t - \ln(1+\lambda_t)] \end{array}$$

$\diamondsuit$ For every t such that $\lambda_t < 1$ one has

$$\begin{array}{rcl} f(x_t) - f(x_*) & \leq & -\lambda_t - \ln(1-\lambda_t) \leq \frac{\lambda_t^2}{2(1-\lambda_t)^2} \\ \|x_* - x_t\|_{x_t} & \leq & \frac{\lambda_t}{1-\lambda_t} \end{array}$$

$\diamondsuit$ When f is below bounded, the domain

$$\mathcal{D} = \{x : \lambda(f,x) < \frac{1}{4}\}$$

is nonempty; once entering this domain, the Damped Newton method never leaves it and converges quadratically:

$$\lambda_t < \frac{1}{4} \Rightarrow \left\{ \begin{array}{rcl} \lambda_{t+1} & \leq & 2\lambda_t^2 \leq \frac{1}{2}\lambda_t \\ f(x_{t+1}) - f(x_*) & \leq & \frac{\lambda_{t+1}^2}{2(1-\lambda_{t+1})^2} \\ \|x_* - x_{t+1}\|_{x_*} & \leq & \frac{\lambda_{t+1}}{1-2\lambda_{t+1}} \end{array} \right.$$

Entering $\mathcal{D}$ takes at most

$$O(1)[f(x_0) - \min_x f(x)]$$

damped Newton steps.

Damped Newton Minimization of

$$f(x) = -\sum_{i=1}^{m} \ln\left(b_i - \sum_{j=1}^{n} a_{ij}x_j\right)$$

$$[n = 1000,\ m = 10000]$$

Itr	f	λ	Itr	f	λ
1	0.000000	29	31	−483.936054	3.2
2	−28.407093	29	32	−486.075608	2.4
3	−56.126388	28	33	−487.521929	1.7
⋮	…	…	34	−488.375100	1.0
11	−250.504812	22	35	−488.773190	5e-01
12	−271.074919	21	36	−488.893100	2e-01
13	−290.755934	20	37	−488.908327	3e-02
⋮	…	…	38	−488.908669	7e-04
21	−414.650294	13	39	−488.908669	4e-07
22	−425.818147	12	40	−488.908669	2e-13
23	−436.016865	11			

Self-Concordant Barriers

♣ Definition. Let $\vartheta \geq 0$. A function f on $\mathbb{R}^n$ is called *ϑ-self-concordant barrier* for $G = \mathrm{cl}\mathrm{Dom} f$, if

♢ f is self-concordant

♢ f is Lipschitz continuous, with constant ϑ, w.r.t. the local norm defined by f:

$$|Df(x)[h]| \leq \vartheta \|h\|_x \equiv \vartheta^{1/2}\sqrt{D^2 f(x)[h,h]} \qquad \forall (x \in \mathrm{Dom} f, h \in \mathbb{R}^n)$$

Note: Nondegenerate s.c.f. f is ϑ-s.c.b. iff

$$\lambda(f,x) \equiv \max_{h:\|h\|_x \leq 1} Df(x)[h] \leq \vartheta^{1/2}.$$

Trivial example: $f(x) \equiv \mathrm{const}$ is a 0-s.c.b. for $\mathbb{R}^n$.

Note: A constant is the only s.c.b. for the entire $\mathbb{R}^n$, as well as the only s.c.b. with constant < 1. We will never deal with this trivial barrier, so that in the sequel we always have $\vartheta \geq 1$.

Less trivial example: $f(x) = -\ln x$ is a 1-s.c.b. for $\mathbb{R}_+$.

Further examples are implied by the following

Proposition. (i) Let $f(x)$ be a ϑ-s.c.b., and let $x = Ay + b$ be an affine mapping with the image intersecting $\mathrm{Dom} f$, Then the function

$$g(y) = f(Ay + b)$$

is ϑ-s.c.b.

(ii) Let $f_i(x)$, $i = 1, ..., m$, be ϑ-s.c.b.'s with intersecting domains, and let $\lambda_i \geq 1$. Then the function

$$f(x) = \sum_i \lambda_i f_i(x)$$

is $(\sum_i \lambda_i \vartheta_i)$-s.c.b.

(iii) Let $f_i(x_i)$, $i = 1, ..., m$, be ϑ_i-s.c.b.'s. Then the function

$$f(x_1, ..., x_m) = \sum_i f_i(x_i)$$

is a $\left(\sum_i \vartheta_i\right)$-s.c.b.

Example: Since $\phi(t) = -\ln(t)$ is a 1-s.c.b., (i) and (ii) imply that the function

$$f(x) = -\sum_{i=1}^{m} \ln(b_i - a_i^T x)$$

is m-s.c.b. for the set $\{x : Ax \leq b\}$ given by strictly feasible system of linear inequalities.

Proof of (ii): f is s.c.f. and

$$\begin{array}{rcl} |Df(x)[h]| & \leq & \sum\limits_i \lambda_i |Df_i(x)[h]| \\ & \leq & \sum\limits_i \lambda_i \vartheta_i^{1/2} \sqrt{D^2 f_i(x)[h,h]} \\ & \leq & \sqrt{\lambda_i \vartheta_i} \sqrt{\lambda_i D^2 f_i(x)[h,h]} \\ & \leq & \sqrt{\sum\limits_i \lambda_i \vartheta_i} \sqrt{\sum\limits_i \lambda_i D^2 f_i(x)[h,h]} \\ & = & \sqrt{\sum\limits_i \lambda_i \vartheta_i} \sqrt{D^2 f(x)[h,h]} \end{array}$$

♣ Why s.c.b.'s?

♠ In the path-following scheme, we should apply the Newton minimization to a *sequence* of functions

$$f_{t_i}(x) = t_i c^T x + f(x) \qquad (*)$$

rather than to a single function. What is crucial is to be able both

♢ to minimize well every function from the sequence,

and

♢ to update rapidly the penalty (to have t_{i+1}/t_i not too close to 1) under the restriction that "old" iterate x_i is a good starting point for minimizing "new" function $f_{t_{i+1}}(\cdot)$.

Self-concordant *functions* f fit the first of these requirements, while s.c. *barriers* are designed to fit *both of them.*

♠ Assume that when starting step t we are at the path:

$$\nabla_x f_{t_i}(x_i) = 0 \Leftrightarrow f'(x_i) = -t_i c. \qquad (*)$$

Question: How large should be t_{i+1}/t_i under the restriction that x_i will be in the domain of quadratic convergence of the Damped Newton method as applied to $f_{t_{i+1}}$?
Answer: We want to have $\lambda(f_{t_{i+1}}, x_i) \leq 1/2$. We have

$$\begin{array}{rcl}\lambda(f_{t_{i+1}}, x_i) & = & \|\nabla_x f_{t_{i+1}}(x_i)\|^*_{x_i} \\ & = & \|f'(x_i) + t_{i+1}c\|^*_{x_i} \\ & = & \|(t_{i+1}/t_i - 1)f'(x_i)\|^*_{x_i}\end{array}$$

Thus, we definitely can take

$$\frac{t_{i+1}}{t_i} = 1 + \frac{1}{2\|f'(x_i)\|^*_{x_i}} = 1 + \frac{1}{2\lambda(f, x_i)}.$$

To avoid slowing down we should keep $\frac{t_{i+1}}{t_i}$ bounded away from 1
$\Rightarrow$ $\lambda(f, x)$ should be bounded from above

Properties of Self-Concordant Barriers

♣ Preliminaries: Minkowski function. Let G be a closed convex domain, and $x \in \mathrm{int}G$. The Minkowski function of G with the pole at x is the function

$$\pi_x(y) = \inf_t \left\{ t : x + t^{-1}(y - x) \in G \right\}.$$

Geometrically: $\pi_x(y)$ is the ratio of $\|y - x\|$ and the distance from x to ∂G, the distance being taken in the direction $y - x$.

♣ "Explosion": *Let f be ϑ-s.c.b. for $G \subset \mathbb{R}^n$. Then*

$$x \in \mathrm{int}G, y \in \mathbb{R}^n \Rightarrow Df(x)[y - x] \leq \vartheta \pi_x(y).$$

Thus, if $DF(x)[y - x] > 0$, then the point

$$x + \frac{\vartheta}{Df(x)[y - x]} \notin \mathrm{int}G.$$

$$x \in \text{int}G, y \in \mathbb{R}^n \Rightarrow Df(x)[y-x] \leq \vartheta \pi_x(y).$$

Proof. There is nothing to prove when $Df(x)[y-x] \leq 0$. Thus, let $Df(x)[y-x] > 0$, and let

$$\begin{array}{rcl} \phi(t) & = & f(x + t(y-x)),\ 0 \leq t < T \equiv \frac{1}{\pi_x(y)} \\ \psi(t) & = & \phi'(t) \end{array}$$

Since $\phi'(0) = Df(x)[y-x] > 0$, we have for $0 \leq t < T$:

$$\begin{array}{rl} & 0 < \phi'(0) \leq \phi'(t) \leq \sqrt{\vartheta}\sqrt{\phi''(t)} \\ \Rightarrow & 0 < \psi(0) \leq \psi(t) \leq \sqrt{\vartheta}\sqrt{\psi'(t)} \\ \Rightarrow & \frac{d}{dt}(\psi^{-1}(t)) \leq -\vartheta^{-1/2} \\ \Rightarrow & \psi^{-1}(t) \leq \psi^{-1}(0) - \vartheta^{-1/2}t \\ \Rightarrow & \psi(t) \geq \vartheta\psi(0)\vartheta - t\psi(0) \end{array}$$

Since $\psi(\cdot)$ is continuous on $[0, T)$, we get

$$\pi_x^{-1}(y) = T \leq \vartheta\psi^{-1}(0) = \frac{\vartheta}{Df(x)[y-x]}.$$

♣ Semiboundedness: Let f be ϑ-s.c.b. for G. Then

$$x \in \mathrm{int} G, y \in G \Rightarrow Df(x)[y - x] \leq \vartheta$$

Indeed, $Df(x)[y-x] \leq \vartheta \pi_x(y)$ and $\pi_x(y) \leq 1$ when $y \in G$.

Corollary. At a point $x = x(t)$ of the path one has

$$c^T x - \min_{y \in G} c^T y \leq \frac{\vartheta}{t}$$

Indeed, we have

$$f'(x) + tc = 0 \Rightarrow c = -t^{-1} f'(x),$$

whence

$$y \in G \Rightarrow c^T x - c^T y = t^{-1} Df(x)[y - x] \leq \frac{\vartheta}{t}.$$

Bounds: Let f be ϑ-s.c.b. for G, and let $x, y \in \text{int}G$. Then

$$\begin{array}{lrcl} (a) & f(y) & \leq & f(x) + \vartheta \ln \frac{1}{1-\pi_x(y)} \\ (b) & f(y) & \geq & f(x) + Df(x)[y-x] + \rho(\pi_x(y)) \\ & & & [\rho(s) = -s - \ln(1-s)] \end{array}$$

Proof. (a): Let $x_t = x + t(y-x)$. Then for $0 \leq t \leq 1$:

$$\pi_{x_t}(y) = \frac{(1-t)\pi_x(y)}{1-t\pi_x(y)} \ \& \ Df(x_t)[y - x_t] \leq \vartheta \pi_{x_t}(y)$$

$$\Rightarrow (1-t)Df(x_t)[y-x] \leq \vartheta \frac{(1-t)\pi_x(y)}{1-t\pi_x(y)}$$

$$\Rightarrow Df(x_t)[y-x] \leq \vartheta \frac{\pi_x(y)}{1-t\pi_x(y)}$$

$$\Rightarrow f(y) - f(x) \leq \int_0^1 Df(x_t)[y-x]dt$$

$$\leq \int_0^1 \frac{\vartheta \pi_x(y)}{1-t\pi_x(y)} dt = \vartheta \ln \frac{1}{1-\pi_x(y)}$$

(b): Let $\phi(t) = f(x_t)$, $-T_- < t < T = \pi_x^{-1}(y)$. ϕ is ϑ-s.c.b. for $[-T_-, T]$, whence $t + [\phi''(t)]^{-1/2} \leq T$, or

$$\phi''(t) \geq (T-t)^{-2} \text{ for } 0 \leq t < T$$

$$\begin{array}{rl} \Rightarrow & f(y) - f(x) - Df(x)[y-x] \\ = & \phi(1) - \phi(0) - \phi'(0) = \int_0^1 dt \int_0^t (T-\tau)^{-2} d\tau \\ = & \ln(T/(T-1)) - 1/T = \rho(1/T) = \rho(\pi_x(y)). \end{array}$$

Bound on the norm of the first derivative: Let f be ϑ-s.c.f. for G and $x, y \in \mathrm{int}G$. Then

$$|Df(y)[h]| \leq \frac{\vartheta}{1 - \pi_x(y)} \|h\|_x. \qquad (*)$$

Proof. Let $\pi \in (\pi_x(y), 1)$. Then

$$\exists w \in G : y = x + \pi[w - x].$$

Since $w \in G$ and the unit Dikin ellipsoid, centered at x, belongs to G, we have

$$V = \{y + h : \|h\|_x \leq (1 - \pi)\} = (1 - \pi)\{x + g : \|g\|_x \leq 1$$

By Semiboundedness,

$$Df(y)[z - v] \leq \vartheta \; \forall z \in V \Rightarrow |Df(y)[h]| \leq \frac{\vartheta}{1 - \pi} \|h\|_x \, \forall h.$$

Since $\pi \in (\pi_x(y), 1)$ is arbitrary, $(*)$ follows.

Minimizer and Centering property: Let f be a ϑ-s.c.b. for G. f is nondegenerate iff G does not contain lines. If G does not contain lines, F attains its minimum on G iff G is bounded. In the latter case, the minimizer x_* of f is unique and possesses the following Centering property:

$$\{x : \|x - x_*\|_{x_*} \leq 1\} \subset G \subset \{x : \|x - x_*\|_{x_*} \leq \vartheta + 2\sqrt{\vartheta}. \tag{$*$}$$

Proof. 1^0. If a s.c.f. is degenerate, its domain contains lines. Let us prove that for an s.c.b. f the opposite is true: if Dom$f$ contains lines, then f is degenerate. Indeed, if $h \neq 0$ is a direction of line in G and $x \in \text{int}G$, then $x \pm th \in \text{int}G$ for all t. Therefore, by Semiboundedness, $Df(x + th)[sh] \leq \vartheta$ for all t, s, which is possible only if $Df(x+th)[h] = 0$, Since this is true for all t, $D^2 f(x)[h, h] = 0$, and f is degenerate.

2^0. All which remains to prove is that if f attains its minimum on int$G$, then the minimizer x_* is unique (which is evident, since f is nondegenerate) and Centering holds true. The left inclusion in $(*)$ merely says that the closed unit Dikin ellipsoid cantered at x is contained in $G = \mathrm{clDom} f$; this is indeed true. The right inclusion in $(*)$ is given by the following

Lemma: Let f be a nondegenerate ϑ-s.c.b. and $x \in \mathrm{int}G$, h be such that $\|h\|_x = 1$ and $Df(x)[h] \geq 0$. Then

$$x + (\vartheta + 2\sqrt{\vartheta})h \notin \mathrm{int}G.$$

Proof. Let $\phi(t) = D^2 f(x+th)[h,h]$ and $T = \sup\{t : x+th \in G\}$. By self-concordance,

$$
\begin{array}{l}
\quad\phi'(t) \geq -2\phi^{3/2}(t),\ 0 \leq t < T \\
\Rightarrow \frac{d}{dt}(\phi^{-1/2}(t)) \leq 1,\ 0 \leq t \leq T \\
\Rightarrow \phi^{-1/2}(t) - \underbrace{\phi^{-1/2}(0)}_{=\|h\|_x^{-1}=1} \leq t,\ 0 \leq t < T \\
\Rightarrow \phi(t) \geq (1+t)^{-2},\ 0 \leq t < T, \\
\Rightarrow Df(x+rh)[h] \geq \int\limits_0^r \phi(t)dt \geq \frac{r}{1+r},\ 0 \leq r < T. \\
\hfill [\text{since } Df(x)[h] \geq 0]
\end{array}
$$

With this in mind, by Seminboundedness,

$$
\begin{array}{l}
0 \leq r < t < T \Rightarrow \vartheta \geq Df(x+rh)[(t-r)h] \geq \frac{(t-r)r}{1+r} \\
\Rightarrow t \leq r + \frac{(1+r)\vartheta}{r} \Rightarrow T \leq r + \frac{(1+r)\vartheta}{r}
\end{array}
$$

Assuming $T > \sqrt{\theta}$ and setting $r = \sqrt{\theta}$, we get

$$T = \sup\{t : x + rh \in G\} \leq \vartheta + 2\sqrt{\vartheta}.$$

Corollary. Let f be a ϑ-s.c.b. for G and h be a recessive direction of G: $x+th \in G$ whenever $x \in G$ and $t \geq 0$. Then

$$\forall x \in \mathrm{int}G : Df(x)[h] \leq -\|h\|_x \qquad (*)$$

Proof. Let $x \in \mathrm{int}G$. Consider the function $\phi(t) = f(x+th)$; it is a ϑ-s.c.b. for the set $\Delta = [-a, \infty) = \{t : x+th \in G\}$. In the case of $\|h\|_x = 0$ we have $|Df(x)[h]| \leq \sqrt{\vartheta}\|h\|_x = 0$, and $(*)$ hods true. Now let $\|h\|_x > 0$; then $\phi''(0) = D^2f(x)[h,h] = \|h\|_x^2 > 0$, i.e., ϕ is a nondegenerate ϑ-s.c.b. for Δ. Since Δ is not bounded, ϕ is not bounded below.

By Lemma, the case of $\phi'(0) \geq 0$ is impossible. Thus, $\phi'(0) = Df(x)[h] < 0$. Finally, since ϕ is not bounded below, we have

$$1 \leq \lambda^2(\phi, 0) = \frac{(\phi'(0))^2}{\phi''(0)} = \frac{(Df(x)[h,h])^2}{D^2f(x)[h,h]},$$

and $(*)$ follows.

Approximating properties of Dikin's ellipsoid.

Let f be a ϑ-s.c.b. for G. For $x \in \mathrm{int}G$, let

$$p_x(h) = \inf\{r \geq 0 : x \pm r^{-1}h \in G\}$$

be the norm with the unit ball $G_x = \{y : x \pm y \in G\}$. Then

$$p_x(h) \leq \|h\|_x \leq (\vartheta + 2\sqrt{\vartheta})p_x(h)$$

Proof: The inequality $p_x(h) \leq \|h\|_x$ follows from the fact that the set $x + \{h : \|h\|_x \leq 1\}$ is contained in G, whence the unit ball of $\|\cdot\|_x$ is contained in G_x.
To prove that $\|h\|_x \leq (\vartheta + 2\sqrt{\vartheta})p_x(h)$ is the same as to verify that if $\|h\|_x = 1$, then $p_x((\vartheta + 2\sqrt{\vartheta})h) \geq 1$, that is, at least one of the vectors $x \pm (\vartheta + 2\sqrt{\vartheta})h$ does not belong to int$G$. This is given by Lemma.

<u>Existence of s.c.b.'s</u> Let G be a closed convex domain in $\mathbb{R}^n$. Then G admits an $O(n)$-s.c.b. If G does not contain lines, this barrier can be chosen as

$$f(x) = O(1)\ln\left(\mathrm{mes}_n\left\{\xi : \xi^T(y-x) \le 1 \forall y \in G\right\}\right)$$

Basic Path-Following Method

Problem of interest:

$$c_* = \min_{x \in G} c^T x$$

G - closed and bounded convex domain in $\mathbb{R}^n$ *given by* ϑ*-s.c.b.* F*.*

Path: Since G is bounded, the following *path* is well-defined:

$$x_*(t) = \operatorname*{argmin}_{x \in \mathrm{int} G} F_t(x), \qquad F_t(x) = tc^T x + F(x)$$

$$0 < t < \infty$$

$$c_* = \min_x \left\{ c^T x : x \in \mathrm{clDom}F \right\} \qquad [F : \vartheta-\text{s.c.b.}]$$

$$\Downarrow$$

$$x_*(t) = \operatorname*{argmin}_{x \in \mathrm{Dom}F} \underbrace{tc^T x + F(x)}_{F_t(x)},\ 0 < t < \infty$$

<u>Basic Path-Following method</u> traces the path $x_*(t)$ as $t \to \infty$, that is, generates sequence of pairs (x_i, t_i) with $x_i \in \mathrm{int}G$ "close" to $x_*(t_i)$ and $t_i \to \infty$. Specifically,

♢ "Closeness" of x to $x_*(t)$ is understood as

$$\lambda(F_t, x) \le \kappa,$$

where $\kappa \in (0,1)$ is a parameter;

♢ Penalty t_i is updated as

$$t_{i+1} = \left(1 + \frac{\gamma}{\sqrt{\vartheta}}\right) t_i$$

♢ x_{i+1} is given by the Damped Newton method as applied to $F_{t_{i+1}}$. The method is started at x_i and is run until a close to $x_*(t_{i+1})$ iterate is produced; this iterate is taken as x_{i+1}.

<u>Note:</u> The initial pair (x_0, t_0) with $t_0 > 0$ and x_0 close to $x_*(t_0)$ is given in advance.

Convergence and Complexity

Proposition: In Basic Path-Following method, one has

$$
\begin{array}{c}
c^T x_i - c_* \leq \frac{\chi}{t_i} = \frac{\chi}{t_0}\left(1+\gamma\vartheta^{-1/2}\right)^{-i} \\
\chi = \vartheta + \frac{\sqrt{\vartheta}\kappa}{1-\kappa},
\end{array} \qquad (*)
$$

.

Proof. We have $\|x_i - x_*(t_i)\|_{x_*(t_i)} \leq \frac{\kappa}{1-\kappa}$ and $F'(x_*(t_i)) = -t_i c$, whence

$$
\begin{array}{ll}
& c^T x_i - c^T x_*(t_i) \\
\leq & \|c\|^*_{x_*(t_i)} \|x_i - x_*(t_i)\|_{x_*(t_i)} \\
= & t_i^{-1} \|F'(x_*(t_i))\|^*_{x_*(t_i)} \|x_i - x_*(t_i)\|_{x_*(t_i)} \\
\leq & \frac{\sqrt{\vartheta}\kappa}{(1-\kappa)t_i}
\end{array}
$$

and $c^T x_*(t_i) - c_* \leq \vartheta / t_i$, and we arrive at $(*)$.

<u>Proposition:</u> Let x be close to $x_*(t)$. Then

$$F_\tau(x) - \min_u F_\tau(u)$$
$$\leq \rho(\kappa) + \frac{\kappa}{1-\kappa}\left|1 - \frac{\tau}{t}\right|\sqrt{\vartheta} + \vartheta\rho\left(1 - \frac{\tau}{t}\right)$$
$$\rho(s) = -s - \ln(1-s)$$

Proof. We have

$$\begin{array}{rcl} F'(x_*(\tau)) + \tau c &=& 0 \\ \Downarrow && \\ x_*'(\tau) &=& -[F''(x_*(\tau))]^{-1}c \\ &=& \tau^{-1}[F''(x_*(\tau))]^{-1}F'(x_*(\tau)) \end{array}$$

Setting

$$\phi(\tau) = [\tau c^T x_*(t) + F(x_*(t))] - [\tau c^T x_*(\tau) + F(x_*(\tau))],$$

we have

$$\begin{array}{rcl} \phi'(\tau) &=& c^T x_*(t) - c^T x_*(\tau) \\ && -[x_*'(\tau)]^T \underbrace{[\tau c + F'(x_*(\tau))]}_{0} \\ \Rightarrow \phi''(\tau) &=& c^T [F''(x_*(\tau))]^{-1} c \\ &=& \frac{1}{\tau^2}[F'(x_*(\tau))]^T [F''(x_*(\tau))]^{-1} F'(x_*(\tau)) \end{array}$$

whence

$$\phi(t) = \phi'(t) = 0,\ 0 \leq \phi''(\tau) \leq \tau^{-2}\vartheta$$
$$\Downarrow$$
$$\phi(\tau) \leq \vartheta\rho(1 - \tau/t)$$

$$
\begin{array}{rl}
& [\tau c^T x_*(t) + F(x_*(t))] - [\tau c^T x_*(\tau) + F(x_*(\tau))] \\
\leq & \vartheta\rho(1 - \frac{\tau}{t})
\end{array}
\qquad (*)
$$

Further,

$$
\begin{array}{rl}
& [\tau c^T x + F(x)] - [\tau c^T x_*(t) + F(x_*(t))] \\
= & \Big[t c^T x + F(x)] - [t c^T x_*(t) + F(x_*(t))] \Big] \\
& + [\tau - t] c^T (x - x_*(t)) \\
\leq & \rho(\kappa) + |\tau - t| \|c\|^*_{x_*(t)} \|x - x_*(t)\|_{x_*(t)} \\
= & \rho(\kappa) + |1 - \tau/t| \|F'(x_*(t))\|^*_{x_*(t)} \|x - x_*(t)\|_{x_*(t)} \\
= & \rho(\kappa) + |1 - \tau/t| \sqrt{\vartheta} \frac{\kappa}{1-\kappa}
\end{array}
$$

which combines with $(*)$ to yield

$$
\begin{array}{rl}
& F_\tau(x) - \min\limits_u F_\tau(u) \\
\leq & \rho(\kappa) + \frac{\kappa}{1-\kappa} \left| 1 - \frac{\tau}{t} \right| \sqrt{\vartheta} + \vartheta\rho \left(1 - \frac{\tau}{t} \right)
\end{array}
$$

Corollary: Newton complexity of a step in the Basic Path-Following method does not exceed

$$\mathcal{N} = O(1)\left[\rho(\kappa) + \frac{\kappa\gamma}{1-\kappa} + \vartheta\rho\left(\frac{\gamma}{\sqrt{\vartheta}}\right)\right]$$
$$\rho(s) = -s - \ln(1-s)$$

♣ How to start the Basic Path-Following Method?

♣ Assume we are given a point $x_{\rm ini} \in \mathrm{Dom}F$, and let

$$f = -F'(x_{\rm ini})$$

The paths

$$\begin{array}{rcl} y_*(t) &=& \operatorname*{argmin}_x \left[tf^T x + F(x)\right] \\ x_*(t) &=& \operatorname*{argmin}_x \left[tc^T x + F(x)\right] \end{array}$$

as $t \to +0$, converge to the minimizer x_F of F and thus approach each other. At the same time, $y_*(1) = x_{\rm ini}$.

Consequently, *we can trace the auxiliary path $y_*(t)$, starting with $x = x_{\rm ini}, t = 1$, as $t \to 0$, until we approach x_F, e.g., until a point*

$$x_0 : \lambda(F, x_0) \leq \frac{\kappa}{2}$$

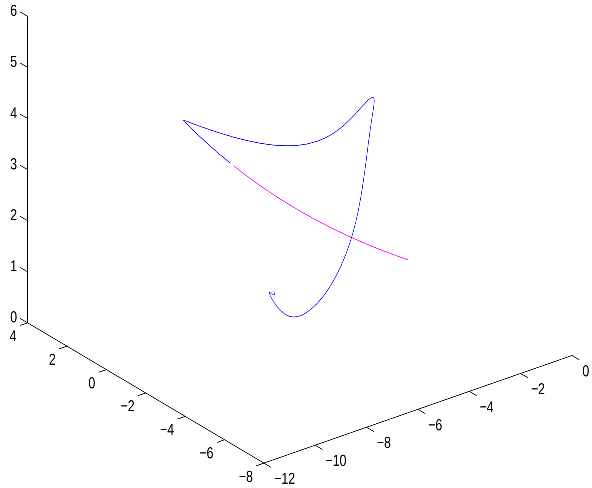
is met. We can then choose $t_0 > 0$ in such a way that

$$\lambda(F_{t_0}, x_0) < \kappa,$$

and start tracing the path of interest $x_*(t)$ with the pair x_0, t_0.

m=10000 n=1000

Step	Penalty	Objective	Newton steps
1	2.96e-1	-3.1089918	3
2	5.92e-1	-6.4744761	3
3	1.18e+0	-12.9940893	4
4	2.37e+0	-26.3332802	7
5	4.74e+0	-50.7247307	11
6	9.47e+0	-88.6103727	17
7	1.89e+1	-129.7037505	23
8	3.79e+1	-160.4897883	28
9	7.58e+1	-178.1641436	29
10	1.52e+2	-187.2479755	30
11	3.03e+2	-191.5784964	29
12	6.06e+2	-193.6780300	28
13	1.21e+3	-194.5293250	27
14	2.43e+3	-194.9707960	26
15	4.85e+3	-195.1894802	26
16	9.70e+3	-195.2984255	26
17	1.94e+4	-195.3525217	26
18	3.88e+4	-195.3789643	26
19	7.76e+4	-195.3920128	26
20	1.55e+5	-195.3985293	26
21	3.10e+5	-195.4017881	26
22	6.21e+5	-195.4034183	26
23	1.24e+6	-195.4042317	26
24	2.48e+6	-195.4046366	26
25	4.97e+6	-195.4048382	26
26	9.93e+6	-195.4049388	26
27	1.99e+7	-195.4049891	26



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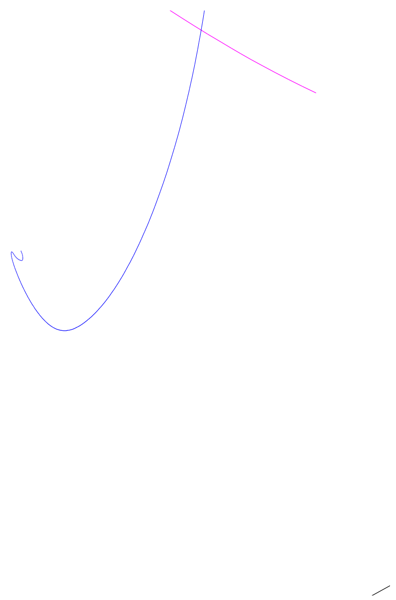


The Method of Karmarkar

♣ Problem of interest is a conic program

$$c_* = \min_x \left\{ c^T x : x \in [\mathcal{L} + b] \cap \mathbf{K} \right\} \qquad (P)$$

♣ Assumptions:

♢ *the feasible set*

$$K_f = [\mathcal{L} + b] \cap \mathbf{K}$$

is bounded and intersects int$\mathbf{K}$.

♢ *a strictly feasible solution $\bar{x}$ is given*

♢ *the optimal value c_* of the problem is known*

♣ The method of Karmarkar is associated with a ϑ-l.-h. s.c.f. $F(\cdot)$ for $\mathbf{K}$.

$$c_* = \min_x \left\{ c^T x : x \in [\mathcal{L} + b] \cap \mathbf{K} \right\} \qquad (P)$$

♣ Since the feasible set intersects int$\mathbf{K}$ and is bounded, it does not contain the origin. Therefore the feasible plane $[\mathcal{L} + b]$ of the problem can be represented as

$$\mathcal{L} + b = \{x : Ax = 0, e^T x = 1\}$$

Passing from the objective $c^T x$ to the objective $(c - c_* e)^T x$, we arrive at an equivalent problem

$$0 = \min_x \left\{ \sigma^T x : x \in M \cap \mathbf{K}, e^T x = 1 \right\} \qquad (P')$$

where $M = \{x : Ax = 0\}$ is a linear subspace which intersects the interior of $\mathbf{K}$.

♣ There is nothing to solve when the objective $\sigma^T x$ is constant on the feasible set; in this case (which is easy to recognize) our initial strictly feasible solution $\bar{x}$ is optimal. Assume that $\sigma^T x$ is nonconstant on the feasible set, whence $\sigma^T x$ is positive at every strictly feasible solution.

♣ Karmarkar Potential. Let us set

$$v(x) = F(x) + \vartheta \ln(\sigma^T x)$$

♣ Observation I: If x is a strictly feasible solution to (P') with "very negative" $v(x)$, then x is nearly optimal:

$$\begin{array}{c}\sigma^T x \equiv c^T x - c^* \leq \mathcal{V} \exp\{-\frac{v(\bar{x})-v(x)}{\vartheta}\}, \\ \mathcal{V} = (c^T\bar{x} - c^*) \exp\{\frac{F(\bar{x})-\min\limits_{K_f} F}{\vartheta}\}\end{array} \qquad (*)$$

Proof:

$$\begin{array}{rl} & v(\bar{x}) - v(x) \\ = & \vartheta[\ln(\sigma^T\bar{x}) - \ln(\sigma^T x)] + F(\bar{x}) - F(x) \\ \leq & \vartheta[\ln(\sigma^T\bar{x}) - \ln(\sigma^T x)] + F(\bar{x}) - \min\limits_{K_f} F, \end{array}$$

and $(*)$ follows.

$$(P'): \qquad 0 = \min_x \left\{ \sigma^T x : x \in M \cap \mathbf{K}, e^T x = 1 \right\}$$
$$\Downarrow$$
$$v(x) = F(x) + \vartheta \ln(\sigma^T x)$$

♣ Observation II: The potential $v(x)$ is constant along rays:

$$v(tx) = v(x) \ \forall t > 0.$$

In particular, when x is strictly feasible for (P'), the point $\widehat{x} = (\sigma^T x)^{-1} x$ satisfies

$$\widehat{x} \in N \equiv \{x \in M \cap \mathrm{int}\mathbf{K} : \sigma^T x = 1\}$$
$$v(x) = v(\widehat{x}) = F(\widehat{x})$$

Observation III: When $y \in \mathrm{rint}\, N = M \cap \mathrm{int} K$, one has $e^T y > 0$. Thus, every $y \in \mathrm{rint}\, N$ is $\widehat{x}$ for $x = (e^T y)^{-1} y \in \mathrm{rint}\, K_f$.
Indeed, there exists $y' \in \mathrm{rint}\, N$ with $e^T y' > 0$. If there were $y_+ \in \mathrm{rint}\, N$ with $e^T y_+ \le 0$, $\mathrm{rint}\, N$ would contain a segment $[y', y'']$ such that $e^T y'' = 0$. Let $y_i \in [y', y'')$ tend to y'' as $i \to \infty$. Since $\sigma^T y_i = 1$, we have $\|y_i\| \ge c > 0$, whence $x_i = (e^T y_i)^{-1} y_i$ form an unbounded sequence of feasible solutions to the original problem, which is impossible.

Conclusion: In order to get a sequence of strictly feasible solutions x_i to (P') with the value of the objective approaching 0, it suffices to ensure strict feasibility of x_i and the relation $v(x_i) \to -\infty$. To this end, in turn, it suffices to build a sequence $\widehat{x}_i \in \mathrm{rint}\, N$ such that $F(\widehat{x}_i) \to -\infty$ and to set

$$x_i = (e^T \widehat{x}_i)^{-1} \widehat{x}_i$$

♣ Whether it is possible to push the barrier F to $-\infty$ while staying in $\mathrm{rint}\, N$?

◇ The restriction of F onto $\mathrm{rint}\, N$ is a nondegenerate ϑ-self-concordant barrier for N. Such a barrier is below unbounded iff N is unbounded. Is it the case? – Yes!

Indeed, since $\inf\limits_{x \in M \cap \mathrm{int}\mathbf{K}, e^T x = 1} \sigma^T x = 0$, there exists a sequence $x_i \in M \cap \mathrm{int}\mathbf{K}$ with $e^T x_i = 1$ such that $0 < \sigma^T x_i \to 0$. Since $e^T x_i = 1$, we have $\|x_i\| \geq c > 0$, whence the points $\widehat{x}_i = (\sigma^T x_i)^{-1} x_i \in \mathrm{rint}\, N$ satisfy $\|\widehat{x}_i\| \to \infty$.

♣ How to push F to $-\infty$, staying in rint N?

♢ In order to push a nondegenerate barrier $\Phi = F\Big|_{\mathrm{rint}\, N}$ for the *unbounded* set N to $-\infty$, it suffices to subject Φ to damped Newton minimization. Indeed,

$$\begin{array}{c}
\Phi \text{ is below unbounded} \\
\Downarrow \\
\lambda(\Phi, x) \geq 1 \ \forall x \in \mathrm{rint}\, N \\
\Downarrow
\end{array}$$

The process

$$\widehat{x}_{i+1} = \widehat{x}_i - \frac{\lambda_i}{1+\lambda_i} f_i$$

with

$$\begin{array}{rcl}
f_i & = & \underset{h \in M, h^T F''(\widehat{x}_i) h \leq 1}{\mathrm{argmax}} \quad h^T F'(x_i) \\
\lambda_i & = & \lambda(\Phi, \widehat{x}_i) = f_i^T F'(\widehat{x}_i) \\
\widehat{x}_0 & = & (\sigma^T \bar{x})^{-1} \bar{x}
\end{array}$$

ensures that

$\widehat{x}_i \in \mathrm{rint}\, N, F(\widehat{x}_{i+1}) \leq F(\widehat{x}_i) - [1 - \ln 2]$

Setting $x_i = (e^T \widehat{x}_i)^{-1} \widehat{x}_i$, we get a sequence of feasible solutions to (P) such that

$$c^T x_i - c_* \leq (c^T \bar{x} - c^*) \exp\left\{\beta - \frac{\chi}{\vartheta} i\right\}$$

$$\beta = \frac{F(\bar{x}) - \min_{K_f} F}{\vartheta}, \ \chi = 1 - \ln 2 > 0.$$

Primal-Dual Potential Reduction Methods

♣ Problem of interest: primal-dual pair of conic problems

$$\begin{array}{ll}\min\limits_x \left\{c^T x : x \in [\mathcal{L} + b] \cap \mathbf{K}\right\} & (P)\\ \min\limits_s \left\{b^T s : s \in [\mathcal{L}^\perp + c] \cap \mathbf{K}_*\right\} & (D)\end{array}$$

$\mathbf{K}$ is equipped with ϑ-l.-h. s.c.b. F, $\mathbf{K}_*$ – with the ϑ-l.h. s.c.b.

$$F_+(s) = F_*(-s),$$

where F_+ is the Legendre transformation of F.

♣ Assumption: $(P - D)$ is primal-dual strictly feasible.

<u>Note:</u> problems are written in purely symmetric form. In this form, the duality gap at a primal-dual feasible pair (x, s) is

$$\begin{array}{ll} & \text{DualityGap}(x,s)\\ \equiv & [c^T x - \text{Opt}(P)] + [b^T s - \text{Opt}(D)]\\ = & x^T s\\ = & c^T x + b^T s - c^T b\end{array}$$

Indeed, for $x \in [\mathcal{L} + b]$, $s \in \mathcal{L}^\perp + c]$ we have

$$(x - b)^T (s - c) = 0 \Leftrightarrow x^T s = c^T x + b^T s - c^T b$$

$$\min_x \left\{ c^T x : x \in [\mathcal{L} + b] \cap \mathbf{K} \right\} \quad (P)$$
$$\min_s \left\{ b^T s : s \in [\mathcal{L}^\perp + c] \cap \mathbf{K}_* \right\} \quad (D)$$

♣ The idea. Assume we have parameterized the feasible set of (P): $\mathcal{L} + b = \{x = Au + b\}$, where A is a matrix with trivial kernel. Then

$$(P) \Leftrightarrow \min_u \left\{ (A^T c)^T u : u \in \text{clDom}\Phi \right\}$$
$$\Phi(u) = F(Au + b)$$

Φ is a nondegenerate ϑ-s.c.b. for clDomΦ, and we could solve the problem

$$\min_u \left\{ (A^T c)^T u : u \in \text{clDom}\Phi \right\}$$

by tracing the path

$$u_*(t) = \text{argmin}[t(A^T c)^T u + \Phi(u)]$$

Question: *What is the image of $u_*(\cdot)$ under the mapping $u \mapsto Au + b$?*
Answer: $u_*(t)$ minimizes the function $tc^T(Au + b) + F(Au + b)$, so that

$$x_*(t) \equiv Au_*(t) + b = \underset{x \in [\mathcal{L}+b] \cap \text{int}\mathbf{K}}{\text{argmin}} \left[tc^T x + F(x) \right]$$

$x_*(t)$ is called the *primal central path* of $(P - D)$.

$$\min_x \left\{c^T x : x \in [\mathcal{L}+b] \cap \mathbf{K}\right\} \quad (P)$$
$$\min_s \left\{b^T s : s \in [\mathcal{L}^\perp + c] \cap \mathbf{K}_*\right\} \quad (D)$$

♣ [Existence of central paths] If $(P-D)$ is primal-dual strictly feasible, then, for every $t>0$, the points

$$\begin{array}{rcl} x_*(t) &=& \operatorname*{argmin}_{x\in[\mathcal{L}+b]\cap \mathrm{int}\mathbf{K}} \left[tc^T x + F(x)\right] \\ s_*(t) &=& \operatorname*{argmin}_{s\in[\mathcal{L}^\perp+c]\cap \mathrm{int}\mathbf{K}_*} \left[tb^T s + F_+(s)\right] \end{array}$$

do exist.

(P) is strictly feasible $\Rightarrow K_P \equiv [\mathcal{L}+b] \cap \mathrm{int}\mathbf{K} \neq \emptyset$. (D) is strictly feasible $\Rightarrow$ there exists $\widehat{c} \in \mathrm{int}\mathbf{K}_*$ such that $\widehat{c} - c \in \mathcal{L}^\perp$. *On* K_P, the self-concordant functions $t\widehat{c}^T x + F(x)$ and $tc^T x + F(x)$ differ by a constant and therefore the second attains its minimum on K_P if the first does. This indeed is the case:

$$x \in \mathrm{int}\mathbf{K} \Rightarrow t\widehat{c}^T x + F(x) \geq t\alpha\|x\| + \beta - \gamma \ln(\|x\|)$$

with $\alpha > 0$ (since $\widehat{c} \in \mathrm{int}\mathbf{K}_*$), whence $t\widehat{c}^T x + F(x)$ is below bounded on K_P and therefore, being s.c.f. on K_P, attains its minimum on this set.

♣ Primal-Dual pair of conic problems

$$\min_x \left\{ c^T x : x \in [\mathcal{L} + b] \cap \mathbf{K} \right\} \quad (P)$$
$$\min_s \left\{ b^T s : s \in [\mathcal{L}^\perp + c] \cap \mathbf{K}_* \right\} \quad (D)$$

generates the *pair of central paths*

$$x_*(t) = \operatorname*{argmin}_{x \in [\mathcal{L}+b] \cap \text{int}\mathbf{K}} \left[tc^T x + F(x) \right] \qquad \text{(primal)}$$
$$s_*(t) = \operatorname*{argmin}_{s \in [\mathcal{L}^\perp + c] \cap \text{int}\mathbf{K}_*} \left[tb^T s + F_+(s) \right] \qquad \text{(dual)}$$

<u>Note:</u> ♢ $x = x_*(t)$ iff

• x is strictly dual feasible

$tc + F'(x) \in \mathcal{L}^\perp$, or, which is the same, $-t^{-1}F'(x) \in \mathcal{L}^\perp + c$. Since $-F'(\cdot) \in \text{int}\mathbf{K}_*$, this is the same as to say that

• $-t^{-1}F'(x)$ is strictly dual feasible.

Similarly,

♢ $s = s_*(t)$ iff

• s is strictly dual feasible

• $-t^{-1}F'_+(s)$ is strictly primal feasible

$\diamondsuit$ $x_*(t)$ is strictly primal feasible and such that $-t^{-1}F'(x_*(t))$ is strictly dual feasible

$\diamondsuit$ $s_*(t)$ is strictly dual feasible and such that $-t^{-1}F'_+(s_*(t))$ is strictly primal feasible

<u>Note:</u> $F'(rx) = r^{-1}F'(x)$ and $F'_+(-F'(x)) \equiv x$, and similarly when swapping F, F_+. Thus,

$\diamondsuit$ $s_*(t) = -t^{-1}F'(x_*(t))$; $x_*(t) = -t^{-1}F'_+(s_*(t))$.

Besides this,

$$\begin{array}{rl} & \text{DualityGap}(x_*(t), s_*(t)) \\ = & s_*^T(t)x_*(t) = -t^{-1}(F'(x_*(t)))^T x_*(t) \\ = & \vartheta t^{-1} \end{array}$$

whence

$\diamondsuit$ If $s = s_*(t)$, $x = x_*(t)$, then

- (x, s) is primal-dual strictly feasible,
- s is proportional to $-F'(x)$, x is proportional to $-F'_+(s)$,
- $s^T x = \frac{\vartheta}{t}$

Vice versa,

$\diamondsuit$ If (x, s) is primal-dual strictly feasible and s is proportional to $-F'(x)$ (or, which is the same, x is proportional to $-F'_+(s)$), then $x = x_*(t)$, $s = s_*(t)$ with $t = \frac{\vartheta}{s^T x}$.

♣ Recall that we have a "variational" description of the set

$\mathcal{K} = \{(x,s) \in \mathrm{int}\mathbf{K} \times \mathrm{int}\mathbf{K}_* : s \in -\mathbb{R}_{++} \cdot F'(x)\}$

This is exactly the set of minimizers of the function

$$\Phi(x,s) = F(x) + F_+(s) + \vartheta \ln(s^T x),$$

and on this set this function is equal to const $= \vartheta \ln(\vartheta) - \vartheta$. Thus,

Geometrically, the primal-dual central path is the intersection of the set of minimizers $\Phi(x,s)$ on the primal-dual feasible set.

Now let $\mu > 0$. Consider the *primal-dual potential*

$$\begin{aligned} \Psi(x,s) &= \Phi(x,s) + \mu \ln(s^T x) \\ &= F(x) + F_+(s) + (\vartheta + \mu) \ln(s^T x) \end{aligned}$$

On the path $x_*(t), s_*(t)$ this potential is equal to

$$\Phi(x_*(t), s_*(t)) + \mu \ln(s_*^T(t) x_*(t)) = \text{const} + \mu \ln\left(\frac{\vartheta}{t}\right)$$

and thus it goes to $-\infty$ as $t \to \infty$.

$$
\begin{array}{rcl}
\Psi(x,s) & = & \Phi(x,s) + \mu \ln(s^T x) \\
 & = & F(x) + F_+(s) + (\vartheta + \mu) \ln(s^T x)
\end{array}
$$

Observation: If (x,s) is primal-dual feasible and $\Psi(x,s)$ is very negative, then (x,s) is nearly optimal:

$$
\mathrm{DualityGap}(x,s) \leq \exp\{\frac{\Psi(x,s) - \mathrm{const}}{\mu}\}.
$$

Indeed,

$$
\begin{array}{rrcl}
 & \Psi(x,s) & = & \Phi(x,s) + \mu \ln(s^T x) \\
 & & \geq & \mathrm{const} + \mu \ln(s^T x) \\
\Rightarrow & \ln(s^T x) & \leq & \frac{\Psi(x,s) - \mathrm{const}}{\mu} \\
\Rightarrow & \mathrm{DualityGap}(x,s) & = & s^T x \\
 & & \leq & \exp\{\frac{\Psi(x,s) - \mathrm{const}}{\mu}\}.
\end{array}
$$

$$\begin{array}{rcl}\Psi(x,s) & = & \Phi(x,s)+\mu\ln(s^Tx)\\ & = & F(x)+F_+(s)+(\vartheta+\mu)\ln(s^Tx)\end{array}$$

Conclusion: Given a mechanism which allows to update strictly feasible pair (x,s) into another strictly feasible pair (x_+,s_+) with the value of the potential less by at least $\kappa>0$ (with κ independent of (x,s)), then, iterating the updating, we get a sequence of strictly feasible primal-dual feasible solutions (x_i,s_i) with

$$\begin{array}{c}\text{DualityGap}(x_i,s_i)\leq\mathcal{V}\exp\{-\frac{\kappa}{\mu}i\},\\ \mathcal{V}=\exp\{\frac{\Psi(x_0,s_0)-\text{const}}{\mu}\}.\end{array}$$

Fact: The required mechanism does exist. Its "quality" $\frac{\kappa}{\mu}$ depends on μ and is optimized when $\mu=O(\sqrt{\vartheta})$. In this case, $\kappa=O(1)$, which results in the $\sqrt{\vartheta}$-rate of convergence

$$\text{DualityGap}(x_i,s_i)\leq\mathcal{V}\exp\{-O(1)\frac{i}{\sqrt{\vartheta}}\}.$$

For the sake of definiteness, from now on we set $\mu=\sqrt{\vartheta}$.

$$\min_x \left\{ c^T x : x \in [\mathcal{L} + b] \cap \mathbf{K} \right\} \quad (P)$$
$$\min_s \left\{ b^T s : s \in [\mathcal{L}^\perp + c] \cap \mathbf{K}_* \right\} \quad (D)$$
$$\Downarrow$$
$$\Psi(x,s) = F(x) + F_+(s) + (\vartheta + \sqrt{\vartheta}) \ln(s^T x)$$

♣ Situation: We are given a strictly feasible solution (x,s) to $(P-D)$

♣ Goal: To update (x,s) into strictly feasible solution (x_+, s_+) in such a way that

$$\Psi(x_+, s_+) \le \Psi(x,s) - O(1).$$

♣ We are about to present two updating rules which *always* convert (x,s) into strictly feasible pairs, and at least one of them definitely reduces the potential by at least $O(1)$.

$$\Psi(y,s)=F(y)+F_+(s)+(\vartheta+\sqrt{\vartheta})\ln(s^Ty)$$

Rule 1. $\mathbf{1}^0$. *Treat the potential as a function of y only and linearize the log-term at $y=x$, thus arriving at linearized potential*

$$\psi(y)=F(y)+(\vartheta+\sqrt{\vartheta})\tfrac{s^Ty}{s^Tx}+\mathrm{const}(x,s),$$
$$\mathrm{const}(x,s)=F_+(s)+(\vartheta+\sqrt{\vartheta})[\ln(s^Tx)-1]$$

Note: Due to concavity of ln, we have

$$\psi(y)\geq\Psi(y,s),\quad \psi(x)=\Psi(x,s).\qquad (*)$$

2^0. *Treat $\psi(y)$ as a function on the domain $K_P=[\mathcal{L}+b]\cap\mathrm{int}\mathbf{K}$ if strictly primal feasible solutions. $\psi(y)$ is self-concordant on this domain; let x_+ be the damped Newton iterate of x:*

$$x_+=x-\tfrac{\lambda}{1+\lambda}f,$$
$$\left[\begin{array}{l} f=\operatorname*{argmax}_h\{h^Tv'(x):h\in\mathcal{L},\|h\|_{F''(x)}\leq 1\},\\ \lambda=f^Tv'(x)\end{array}\right]$$

By $(*)$ and basic facts on s.c.f.'s:

$$\Psi(x,s)-\Psi(x_+,s)\geq\psi(x)-\psi(x_+)\geq\lambda-\ln(1+\lambda).$$

Thus, *Rule 1 decreases Φ by $O(1)$, unless λ is small.*

$$\Psi(y,s) = F(y) + F_+(s) + \overbrace{(\vartheta+\sqrt{\vartheta})\,\mathrm{ln}(s^T y)}^{\gamma(s^T y)}$$
$$\Rightarrow \phi(y) = F(y) + (\vartheta+\sqrt{\vartheta})\tfrac{s^T y}{s^T x} + \mathrm{const}(x,s)$$
$$x_+ = x - \tfrac{\lambda}{1+\lambda} f,$$
$$\Rightarrow \left[\begin{array}{l} f = \underset{h}{\mathrm{argmax}}\{h^T v'(x) : h \in \mathcal{L}, \|h\|_{F''(x)} \le 1\}, \\ \lambda = f^T v'(x) \end{array}\right]$$

What happens when Rule 1 does not work, that is, when λ is small? In the extreme case $\lambda = 0$, $v(y)$ attains its minimum on the set of strictly primal feasible solutions at the point x, that is,

$$\begin{array}{rl} & F'(x) + (\vartheta+\sqrt{\vartheta})\tfrac{s}{s^T x} \in \mathcal{L}^\perp \\ \Rightarrow & \tfrac{s^T x}{\vartheta+\sqrt{\vartheta}} F'(x) + s \in \mathcal{L}^\perp \\ \Rightarrow & -\underbrace{\dfrac{s^T x}{\vartheta+\sqrt{\vartheta}}}_{t^{-1}} F'(x) \in \mathcal{L}^\perp + s = \mathcal{L}^\perp + c \\ \Rightarrow & x = x_*(t_+), s_*(t_+) = -\tfrac{s^T x}{\vartheta+\sqrt{\vartheta}} F'(x) \end{array}$$

Setting $s_+ = -\frac{s^T x}{\vartheta+\sqrt{\vartheta}} F'(x) = s_*(t)$, we have

$$\begin{array}{rcl}
\Psi(x,s) & = & \Phi(x,s) + \sqrt{\vartheta}\ln s^T x \\
& \geq & \text{const} + \sqrt{\vartheta}\ln s^T x \\
\Psi(x,s_+) & = & \Psi(x_*(t), s_*(t)) \\
& = & \Phi(x_*(t), s_*(t)) + \sqrt{\vartheta}\ln s_+^T x \\
& = & \text{const} + \sqrt{\vartheta}\ln s_+^T x
\end{array}$$

whence

$$\begin{array}{rl}
& \Psi(x,s) - \Psi(x,s_+) \\
\geq & \sqrt{\vartheta}\ln \frac{s^T x}{s_+^T x} = \sqrt{\vartheta}\ln \frac{s^T x}{\frac{s^T x}{\vartheta+\sqrt{\vartheta}}(-x^T F'(x))} \\
= & \sqrt{\vartheta}\ln \frac{\vartheta+\sqrt{\vartheta}}{\vartheta} = O(1)
\end{array}$$

Thus, *When Rule 1 completely does not work* ($\lambda = 0$), *the updating*

$$(x,s) \mapsto (x, -\frac{s^T x}{\vartheta+\sqrt{\vartheta}} F'(x)) \qquad (*)$$

results in a strictly feasible primal-dual pair and decreases the potential by $O(1)$.

It is natural to guess (and is indeed true) that when $\lambda > 0$ is small, then appropriate version of $(*)$ still decreases the potential by $O(1)$, thus yielding a "complement" to Rule 1.

$$(x,s) \mapsto (x, -\frac{s^T x}{\vartheta + \sqrt{\vartheta}} F'(x)) \qquad (*)$$

When x is *not* on the primal path, $(*)$ does not work, since s_+ is in $\mathrm{int}\mathbf{K}_*$, but does not belong to the dual plane $\mathcal{L}^\perp + c$. To overcome this difficulty, we correct the updating as

$$s_+ = -\frac{s^T x}{\vartheta + \sqrt{\vartheta}} [F'(x) - \lambda F''(x) f] \qquad (+)$$

Facts: ♢ $s_+ \in \mathcal{L}^\perp + c$

♢ $\lambda < 1 \Rightarrow s_+ \in \mathrm{int}\mathbf{K}_*$, and in this case

$$\Psi(x,s) - \Psi(x,s_+) \geq \sqrt{\vartheta} \ln\left(\frac{\vartheta+\sqrt{\vartheta}}{\vartheta+\lambda\sqrt{\vartheta}}\right) - \rho(\lambda)$$

and the latter quantity is $> O(1)$, provided that $\lambda < \bar{\lambda}$ with appropriate absolute constant $\bar{\lambda}$.

Computation shows that Rule 2 reduces Ψ by ≥ 0.09, provided $\lambda \leq 0.5$, while Rule 1 reduces Ψ by ≥ 0.09, provided $\lambda \geq 0.5$.

Proof of facts. Let

$$\mathcal{L} = \mathrm{Im}A\ [\mathrm{Ker}A = \{0\}], x = Au + b,$$
$$F' = F'(x),\ F'' = F''(x)$$
$$t = \frac{\vartheta+\sqrt{\vartheta}}{s^T x},\ \widehat{s} = -t^{-1}F'$$

Then

$$\begin{array}{rcl}
\mathcal{L} + b & = & \{y = Av + b\} \\
\mathcal{L}^{\perp} + c & = & \{\sigma : A^T\sigma = e \equiv A^T c\} \\
\lambda f & = & A[A^T F'' A]^{-1}[te + A^T F'] \\
\lambda & = & \sqrt{[te + A^T F']^T [A^T F'' A]^{-1}[te + A^T F']} \\
s_+ & = & \widehat{s} + t^{-1}F''[\lambda f] = -t^{-1}\left(F' - F''[\lambda f]\right)
\end{array}$$

Consequently,

$$\begin{array}{rl}
& A^T s_+ \\
= & A^T \left[-t^{-1}\left(F' - F''A[A^T F'' A]^{-1}[te + A^T F']\right)\right] \\
= & e
\end{array}$$

Thus, s_+ belongs to the dual feasible plane.

Let $\lambda < 1$. To prove that then $s_+ \in \mathrm{int}\mathbf{K}_*$, set

$$v = -F',$$

so that

$$s_+ = t^{-1}\Bigg(v - \underbrace{F''A[A^TF''A]^{-1}[te + A^TF']}_{d}\Bigg)$$

We have

$$\begin{array}{rl} & \|d\|^2_{F''_+(v)} \\ = & d^T[F'']^{-1}d \\ = & [te + A^TF']^T[A^TF''A]^{-1}A^TF''[F'']^{-1}F''A \\ & \times[A^TF''A]^{-1}[tE + A^TF'] \\ = & [te + A^TF']^T[A^TF''A]^{-1}[tE + A^TF'] \\ = & \lambda^2 < 1. \end{array}$$

We see that $v - d \in \mathrm{int}\mathbf{K}_*$, whence $s_+ = t^{-1}(v - d) \in \mathbf{K}_*$.

Assuming $\lambda < 1$, let us evaluate progress in potential

$$\begin{array}{rcl}\alpha & = & \Psi(x,s) - \Psi(x,s_+) \\ & = & \Phi(x,s) - \Phi(x,s_+) - \sqrt{\vartheta}\ln\frac{s_+^T x}{s^T x}.\end{array}$$

We have

$$\begin{array}{rl} & \Phi(x,s_+) \\ = & F(x) + F_+(s_+) + \vartheta \ln s_+^T x \\ = & \Big[F(x) + F_+(\widehat{s}) + \vartheta \ln \widehat{s}^T x\Big]_1 \\ & + \left[F_+(s_+) - F_+(\widehat{s}) + \vartheta \ln \frac{s_+^T x}{\widehat{s}^T x}\right]_2 \\ = & \text{const} + \left[F_+(s_+) - F_+(\widehat{s}) + \vartheta \ln \frac{s_+^T x}{\widehat{s}^T x}\right]_2 \end{array}$$

We have seen that $\|ts_+ - v\|_v = \lambda$, where $v = -F' = t\widehat{s}$, whence

$$\|s_+ - \widehat{s}\|_{\widehat{s}} = \lambda.$$

so that

$$\begin{array}{rcl} F_+(s_+) - F_+(\widehat{s}) & \leq & (s_+ - \widehat{s})^T F_+'(\widehat{s}) + \rho(\lambda) \\ & = & -t(s_+ - \widehat{s})^T x + \rho(\lambda) \\ & = & -x^T F''[\lambda f] + \rho(\lambda) \end{array}$$

whence

$$[\cdot]_2 \leq -x^T F''[\lambda f] + \rho(\lambda) + \vartheta \ln \frac{s_+^T x}{(\widehat{s})^T x}$$

$$
\begin{array}{rcl}
t & = & \frac{\vartheta+\sqrt{\vartheta}}{s^T x} \\
\widehat{s} & = & -t^{-1}F' \\
s_+ & = & \widehat{s} + t^{-1}F''[\lambda f] \\
\Phi(x, s_+) & \leq & \text{const} - x^T F''[\lambda f] + \rho(\lambda) + \vartheta \ln \frac{s_+^T x}{\widehat{s}^T x}
\end{array}
$$

We have

$$
\begin{array}{rcl}
\alpha & = & \Phi(x, s) - \Phi(x, s_+) - \sqrt{\vartheta} \ln \frac{s_+^T x}{s^T x} \\
& \geq & [\Phi(x, s) - \text{const}] + x^T F''[\lambda f] - \rho(\lambda) \\
& & -\vartheta \ln \frac{s_+^T x}{\widehat{s}^T x} - \sqrt{\vartheta} \ln \frac{s_+^T x}{s^T x} \\
& \geq & \vartheta \ln \frac{\widehat{s}^T x}{s_+^T x} + \sqrt{\vartheta} \ln \frac{s^T x}{s_+^T x} - \rho(\lambda) + \underbrace{x^T F''[\lambda f]}_{\pi}
\end{array}
$$

Note that

$$
\begin{array}{rcl}
\widehat{s}^T x & = & t^{-1}\vartheta \\
s_+^T x & = & t^{-1}\vartheta + t^{-1}\pi \\
s^T x & = & \frac{\vartheta+\sqrt{\vartheta}}{t}
\end{array}
$$

whence

$$
\alpha \geq \vartheta \ln \frac{\vartheta}{\vartheta+\pi} + \sqrt{\vartheta} \ln \frac{\vartheta+\sqrt{\vartheta}}{\vartheta+\pi} - \rho(\lambda) + \pi
$$

When $\lambda < 1$, Rule 2 reduces the potential by

$$\begin{array}{rcl}\alpha & \geq & \left[\pi - \vartheta \ln\left(1 + \frac{\pi}{\vartheta}\right)\right] + \sqrt{\vartheta} \ln \frac{\vartheta + \sqrt{\vartheta}}{\vartheta + \pi} - \rho(\lambda) \\ & \geq & \sqrt{\vartheta} \ln \frac{\vartheta + \sqrt{\vartheta}}{\vartheta + \pi} - \rho(\lambda)\end{array}$$

where

$$\pi = x^T F''[\lambda f] = \lambda(-F')^T f.$$

By origin of f we have $\|f\|_x \leq 1$, while $\|F'\|_x^* \leq \sqrt{\vartheta}$. We arrive at

$$|\pi| \leq \lambda\sqrt{\vartheta}$$

whence

$$\alpha \geq \sqrt{\vartheta} \ln \frac{\vartheta + \sqrt{\vartheta}}{\vartheta + \lambda\sqrt{\vartheta}} - \rho(\lambda)$$

Long-Step Path-Following Methods

♣ Problem of interest:

$$\min_x \left\{ c^T x : x \in G \right\}$$

$G = \mathrm{clDom} F$, F − nondegenerate ϑ-s.c.b.
Assumption: The sets

$$\{x \in G : c^T x \le a\}$$

are bounded.
♣ Under Assumption, the central path

$$x_*(t) = \mathrm{Argmin}[tc^T x + F(x)]$$

is well-defined.

$$\min_x \left\{c^T x : x \in G\right\}$$
$$\Downarrow$$
$$x_*(t) = \text{Argmin}[\underbrace{tc^T x + F(x)}_{F_t(x)}]$$

♣ Let us fix a *path tolerance* $\kappa \in (0, 1)$ and call a pair (x, t) *κ-close to the path* if

$$t > 0 \;\&\; x \in \text{int}G \;\&\; \lambda(F_t, x) \le \kappa$$

♣ Predictor-corrector scheme iterates the updatings as follows:

Given κ-close to the path pair (x, t),

♢ *replace t with $t_+ = t + dt > t$*

♢ [predictor step] *form a predictor $x^f \in \text{int}G$ of $x_*(t_+)$*

♢ [corrector step] *Apply to $F_{t_+}(\cdot)$ Damped Newton minimization, starting with x^f, until a point x_+ such that (x_+, t_+) is κ-close to the path is built.*

♣ The predictor x^f can be defined from the linearized path equation:

$$\begin{array}{rl} & \nabla\left[(t+dt)c^T y + F(y)\right] \\ \approx & \left[\nabla F_{t+dt}(x) + F''(x)(y-x)\right] = 0 \\ & \Downarrow \\ & x^f = x^f(dt) = x - [F''(x)]^{-1}\nabla F_{t+dt}(x) \end{array}$$

The PDS becomes as follows:
To update a given κ-close to the path pair (t,x) into a new pair (t^+,x^+) of the same type, act as follows:

♢ [predictor step] *Form the primal search line*

$$P = \{(t+dt, x+dx(dt)) : dt \in \mathbb{R}\},$$

$$dx(dt) = -[F''(x)]^{-1}\nabla F_{t+dt}(x)$$

choose a stepsize $\delta t > 0$ and form the fore-cast

$$t_+ = t+\delta t,\ x^f = x+dx(\delta t)$$

♢ [corrector step] *Starting from $y = x^f$, iter-ate the damped Newton method*

$$y \mapsto y - \frac{1}{1+\lambda(F_{t_+},y)}[F''(y)]^{-1}\nabla F_{t_+}(y)$$

until $(x_+ = y, t_+)$ is κ-close to the path.

♣ To ensure bound on Newton complexity of a step, we impose on δt the requirement

$$F_{t+\delta t}(x+dx(\delta t)) - \min_y F_{t+\delta t}(y) \leq \widehat{\kappa} \qquad (*)$$

where $\widehat{\kappa} = O(1)$ is method's parameter.

$$F_{t+\delta t}(x + dx(\delta t)) - \min_y F_{t+\delta t}(y) \;\leq\; \widehat{\kappa} \quad (*)$$

♣ Question: How to check $(*)$ in a "cheap" fashion?

♣ Structural Assumption: *The barrier F is given as*

$$F(x) = \Phi(\pi x + p),$$

where Φ is a nondegenerate ϑ-s.c.b. for a domain G^+ with known Legendre transformation

$$\Phi_*(s) = \max_z \left[s^T z - \Phi(z)\right]$$

Example 1: Problem in conic form:

$$\min_x \left\{c^T x : Ax - b \in \mathbf{K}\right\}$$

where $\mathbf{K}$ is given by ϑ-l.-h. s.c.b. with known Legendre transformation Φ_*.

This is the case where the potential reduction primal-dual methods are applicable.

Example 2: Geometric Programming. A GP program is

$$
\begin{array}{rl}
& \min\limits_x \left\{ c^T x : \sum\limits_\ell \exp\{a_{i\ell}^T x + b_{i\ell}\} \le 1, Rx \le r \right\} \\
\Leftrightarrow & \min\limits_x \left\{ c^T x : \begin{array}{l} \sum\limits_\ell u_{i\ell} \le 0, \\ \exp\{a_{i\ell}^T x + b_{i\ell}\} \le u_{i\ell} \\ Rx \le r \end{array} \right\}
\end{array}
$$

Clearly, the problem can be rewritten as

$$
\begin{array}{c}
\min\limits_{x,u} \left\{ c^T x : Bx + Cu + d \in G^+ \right\} \\
G^+ = \left\{ (t, s, y) : \begin{array}{l} t_i \ge 0, i = 1, ..., p, \\ \exp\{s_j\} \le y_j, j = 1, ..., q \end{array} \right\}.
\end{array}
$$

The domain G^+ admits $(p + 2q)$-s.c.b.

$$
\Phi(t, s, y) = -\sum_i \ln t_i - \sum_j [\ln(\ln y_j - s_j) + \ln(y_j)]
$$

with explicit Legendre transformation

$$
\begin{array}{rl}
& \Phi_*(\tau, \eta, \sigma) \\
= & -(p + 2q) - \sum\limits_{i=1}^p \ln(-\tau_i) \\
& - \sum\limits_{j=1}^q \left[(\eta_j + 1) \ln \left(\frac{-\sigma_j}{\eta_j + 1} \right) + \ln \eta_j + \eta_j \right]
\end{array}
$$

Thus, Structural Assumption is valid for GP.

$$\min_x \left\{ c^T x : \Phi(\pi x + p) < \infty \right\}$$
$$\Phi_*(s) = \max_z \left\{ s^T z - \Phi(z) \right]$$

<u>Observation:</u> Let (τ, s) be such that

$$\pi^T s + \tau c = 0 \qquad (*)$$

and $s \in \mathrm{Dom}\Phi_*$. Then

$$f_s \equiv p^T s - \Phi_*(s) \le \min F_\tau(\cdot)$$

and consequently the quantity

$$\begin{array}{rcl} V_s(\tau, y) & = & F_\tau(y) - f_s \\ & = & \tau c^T y + F(y) + \Phi_*(s) - p^T s \end{array}$$

is an upper bound on

$$F_\tau(y) - \min F_\tau(\cdot).$$

Proof: By definition of Legendre transformation, for every $u \in \mathrm{Dom}F$ we have

$$\begin{array}{rcl} 0 & \le & \Phi_*(s) + \Phi(\pi u + p) - (\pi u + p)^T s \\ & = & \Phi_*(s) + F(u) - p^T s + \tau c^T u \\ & = & -f_s + F_\tau(u) \\ \Rightarrow & & f_s \le \min\limits_u F_\tau(u) \end{array}$$

Observation: *There exist a systematic way to generate "dual feasible" pairs* (τ, s), *that is, pairs with* $s \in \mathrm{Dom}\Phi_*$ *and with*

$$\pi^T s + \tau c = 0. \qquad (*)$$

Specifically, given $x \in \mathrm{Dom}F$ *and* $t > 0$, *let us set*

$$\begin{array}{rcl} u & = & \pi x + p \\ du(dt) & = & \pi dx(dt) \\ s & = & \Phi'(u) \\ ds(dt) & = & \Phi''(u) du(dt) \end{array}$$

♢ *Every pair* $S(dt)$ *on the Dual search line*

$$D = \{S(dt) = (t+dt, s^f(dt) = s+ds(dt)) : dt \in \mathbb{R}\}$$

satisfies $(*)$.

♢ *If* (t, x) *is* κ*-close to the path, then* $s^f(0)$, *and therefore every* $s^f(dt)$ *with small enough* $|dt|$, *is in* $\mathrm{Dom}\Phi^*$.

$$
\begin{array}{rclrcl}
u & = & \pi x + p, & du(dt) & = & \pi dx(dt), \\
s & = & \Phi'(u), & ds(dt) & = & \Phi''(u) du(dt).
\end{array}
\tag{1}
$$

Proof. The first claim is immediate:

$$
\begin{array}{cl}
& (t+dt)c + \pi^T(s + ds(dt)) \\
= & (t+dt)c + \pi^T \left[\Phi'(u) + \Phi''(u)\pi dx(dt)\right] \\
= & (t+dt)c + F'(x) + F''(x)dx(dt) \\
= & 0
\end{array}
$$

To prove the second claim, observe first that

$$
\begin{array}{lcl}
(a) & & \|ds(dt)\|_s^2 = \|du(dt)\|_u^2 = \|dx(dt)\|_x^2 \\
& = & [du(dt)]^T ds(dt) \\
(b) & & \|ds(0)\|_s = \|dx(0)\|_x = \lambda^2(F_t, x).
\end{array}
$$

which is immediate due to (1) and $\Phi_*'\prime(s) = [\Phi''(u)]^{-1}$.

It follows that if $\lambda(F_t, x) \leq \kappa$, then $s^f(0) = s + ds(0)$ is in the Dikin ellipsoid of Φ_* centered at $s = \Phi'(u) \in \mathrm{Dom}\Phi_*$, thus, $s^f(0) \in \mathrm{Dom}\Phi_*$.

♣ Now we can equip the Predictor step with *Acceptability Test*:

Given (x,t) κ-close to the path and a candidate stepsize dt, build

$$\begin{array}{rcl} x^f(dt) & = & x + dx(dt) \\ & = & x - [F''(x)]^{-1}[(t+dt)c + F'(x)] \\ s^f(dt) & = & s + ds(dt) \\ & = & \Phi'(\underbrace{\pi x + p}_{u}) + \Phi''(u)\pi dx(dt) \end{array}$$

check whether

$$\begin{array}{rcl} v(dt) & \equiv & (t+dt)c^T x^f(dt) + F(x^f(dt)) \\ & & +\Phi_*(s^f(dt)) - p^T s^f(dt) \leq \widehat{\kappa} \end{array}$$

If it is the case, accept dt – in this case

$$F_{t+dt}(x^f(t)) - \min_y F_{t+dt}(y) \leq \widehat{\kappa},$$

otherwise reject dt.

<u>Note:</u> It makes sense to choose, as the actual stepsize δt, the largest stepsize dt which passes the Acceptability Test.

♣ To justify the scheme theoretically, we should prove that the "short step" $dt = O(1)t/\sqrt{\vartheta}$ passes the Acceptability Test.

♣ Given a point $u \in \mathrm{Dom}\Phi \subset \mathbb{R}^m$ and a direction $\delta u \in \mathbb{R}^m$, let use define the *conjugate* point and direction as

$$s = \Phi'(u), \quad \delta s = \Phi''(u)\delta u,$$

and let

$$\begin{array}{rcl}
\rho_u^*[\delta u] & = & \Phi(u+\delta u) + \Phi^*(s+\delta s) \\
 & & -\left[\Phi(u) + \Phi^*(s)\right] \\
 & & -\left[[\delta u]^T\Phi'(u) + [\delta s]^T\Phi'_*(s)\right] \\
 & & -\left[\frac{[\delta u]^T\Phi''(u)\delta u}{2} + \frac{[\delta s]^T(\Phi_*)''(s)\delta s}{2}\right]
\end{array}$$

Fact: One has

$$\begin{array}{lrcl}
(a) & \zeta & \equiv & \|\delta u\|_u = \|\delta s\|_s = \sqrt{[\delta u]^T\delta s}; \\
(b) & \zeta < 1 & \Rightarrow & \\
 & \rho_u^*[\delta u] & \leq & 2\rho(\zeta) - \zeta^2 = -2\ln(1-\zeta) - 2\zeta - \zeta^2 \\
 & & = & \frac{2}{3}\zeta^3 + \frac{2}{4}\zeta^4 + \frac{2}{5}\zeta^5 + \ldots
\end{array}$$

Proposition: Let (t, x) be κ-close to the path, and let dt, $|dt| < t$, be a stepsize. Let also

$$\begin{array}{rclrcl} u & = & \pi x + p, & du(dt) & = & \pi dx(dt), \\ s & = & \Phi'(u), & ds(dt) & = & \Phi''(u) du(dt). \end{array}$$

♢ The quantity $v(dt)$ to be compared with $\widehat{\kappa}$ in the Acceptability Test admits the bound

$$v(dt) \leq \rho_u^*[du(dt)]$$

while

$$\begin{array}{rcl} \|du(dt)\|_u \leq \omega & \equiv & \lambda(F_t, x) + \frac{|dt|}{t}[\lambda(F_t, x) + \lambda(F, x)] \\ & \leq & \kappa + \frac{|dt|}{t}[\kappa + \sqrt{\vartheta}]. \end{array}$$

♢ Thus, $\omega < 1 \Rightarrow v(dt) \leq 2\rho(\omega) - \omega^2$. Consequently, if $2\rho(\kappa) - \kappa^2 < \widehat{\kappa}$, then all stepsizes dt satisfying the inequality

$$\frac{|dt|}{t} \leq \frac{\kappa_+ - \kappa}{\kappa + \lambda(F, x)},$$

where κ_+ is the root of the equation

$$2\rho(z) - z^2 = \widehat{\kappa},$$

pass the Acceptability Test.

♣ *How long are long steps?*

For the sake of simplicity, assume that $x = x_*(t)$. It turns out that for a well-defined family of s.c.b.'s Φ which includes, e.g.,

♢ the barrier $-\sum\limits_i \ln x_i$ for $\mathbb{R}^m_+$

♢ the barrier $-\ln \mathrm{Det} X$ for the semidefinite cone $\mathbf{S}^m_+$

the step dt is such that the "forecast shift"

$$\delta = \|dt x_*'(t)\|_{x_*(t)} = \|x^f(dt) - x\|_x$$

is of order of $\Delta^{1/2}$, where

$$\Delta = \min_p \left\{ \|p x_*'(t)\|_{x_*(t)} : x_*(t) + p x_*'(t) \notin G \right\}.$$

<u>Note:</u> The standard short step $dt = O(1)t/\sqrt{\vartheta}$ corresponds to $\delta = 1$, so that

$$\frac{dt_{\text{long-step}}}{dt_{\text{short-step}}} \geq O(\sqrt{\Delta}).$$

Note that Δ is at least 1 and can be as large as ϑ.

How to Build S.C.B.'s?

♣ With the outlined IPMs, solving a convex program requires

♢ reformulating problem in the standard form

$$\min_x \left\{ c^T x : x \in G \right\} \qquad (P)$$

with convex set G

Usually this is straightforward; typical description of G is like

$$G = \{g_j(x) \le 0, \; j = 1, ..., m\} \qquad (*)$$

where g_j are explicitly given convex functions.

♢ Equipping G with a "computable" s.c.b. F.

♣ For the time being, we know just 3 "basic" s.c.b.'s:

♢ 1-s.c.b. $-\ln x$ for nonnegative ray

♢ m-s.c.b. $-\ln \mathrm{Det} X$ for the cone $\mathbf{S}^m$ of $m \times m$ symmetric positive semidefinite matrices

♢ 2-s.c.b. $-\ln(t^2 - x^T x)$ for the Lorentz cone $\{(t, x) : t \ge \|x\|_2\}$.

♣ We have a kind of "calculus" of s.c.b.'s allowing to produce new s.c.b.'s from known ones applying the following 3 operations:

♢ Summation: Sum of ϑ_i-s.c.b.'s for domains G_i, $i \leq m$, is a $(\sum\limits_i \vartheta_i)$-s.c.b. for $\bigcap\limits_i G_i$.

This rule allows to "decompose" the problem of building s.c.b. for the domain

$$\{x : g_i(x) \leq 0, ., i = 1, ..., m\}$$

into m similar problems for the simpler domains

$$\{x : g_i(x) \leq 0\},$$

$i = 1, ..., m$.

♢ Affine substitution: If F is a ϑ-s.c.b. for G, then $F(Ax + b)$ is a ϑ-s.c.b. for $G' = \{x : Ax + b \in G\}$, provided that the image of the mapping $Ax + b$ intersects int$G$;

♢ Direct summation: If F_i are ϑ-s.c.b.'s for G_i, $i \leq m$, then $\sum\limits_i F_i(x^i)$ is $(\sum\limits_i \vartheta_i)$-s.c.b. for $G_1 \times ... \times G_m$.

♣ It turns out that a key to dramatic extension of the "power" of barrier's calculus is in answering the following question:

(?) *Given a closed convex domain G^+ equipped with s.c.b. F^+ and a nonlinear mapping $\mathcal{A}(x)$, how to build a s.c.b. for the domain*

$$G^- = \{x : \mathcal{A}(x) \in \mathrm{int}G^+\}.$$

Definition: Let K be a closed convex cone in $\mathbb{R}^N$, G^- be a closed convex domain in $\mathbb{R}^n$, let $\beta \geq 0$ and

$$\mathcal{A}(x) : \text{int}G^- \to \mathbb{R}^N$$

be C^3 mapping. $\mathcal{A}$ is called *β-appropriate* for K, if

♢ $\mathcal{A}$ is K-concave, that is, $\forall(x', x'' \in \text{int}G^-, \lambda \in [0,1])$:

$$\mathcal{A}(\lambda x' + (1-\lambda)x'') \geq_K \lambda\mathcal{A}(x') + (1-\lambda)\mathcal{A}(x'')$$

Equivalently: $D^2\mathcal{A}(x)[h,h] \leq_K 0$ for all $x \in \text{int}G_-$, $h \in \mathbb{R}^n$

♢ $\mathcal{A}$ is *β-compatible* with G^-:

$$x \pm h \in \text{int}G^- \Rightarrow D^3\mathcal{A}(x)[h,h,h] \leq_K -3\beta D^2\mathcal{A}(x)[h,h]$$

Note: Affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^N$, restricted on the interior of a closed convex domain $G^- \subset \mathbb{R}^n$, is 0-compatible with every closed convex cone in $\mathbb{R}^N$.

$$\mathcal{A}(\cdot): \text{int}G^- \to \mathbb{R}^N, \;\; K \subset \mathbb{R}^N: \;\text{ closed cone}$$

Main Theorem: Let 1) G^+ be a closed convex domain in $\mathbb{R}^N$ with a ϑ_+-s.c.b. F_+
2) K be a closed convex cone in $\mathbb{R}^N$ such that $G^+ + K \subset G^+$
3) G^- be a closed convex domain in $\mathbb{R}^n$ with ϑ_--s.c.b. F_-
4) $\mathcal{A}$: int$G^- \to \mathbb{R}^N$ be a β-appropriate for K mapping such that $\mathcal{A}(\text{int}G^-)$ intersects intG^+.
Then the function

$$F(x) = F_+(\mathcal{A}(x)) + \max[1, \beta^2] F_-(x)$$

is a ϑ-self-concordant barrier for the closed convex domain

$$G = \text{cl}\{x \in \text{int}G^- : \mathcal{A}(x) \in \text{int}G^+\},$$

with

$$\vartheta = \vartheta_+ + \max[1, \beta^2]\vartheta_-.$$

Note: When $G^- = \mathbb{R}^n$, $F_- \equiv 0$ and $\mathcal{A}$ is affine, Main Theorem becomes the Substitution rule.

Calculus of appropriate mappings: Let $K \subset \mathbb{R}^N$ be a closed convex cone.

[restriction] If $\mathcal{A} : \text{int}G^- \to \mathbb{R}^N$ is β-appropriate for K, then the restriction on $\mathcal{A}$ on the interior of every closed convex domain $G \subset G^-$ is β-appropriate for K

[conic combination] If $\mathcal{A}_i : \text{int}G_i^- \to \mathbb{R}^N$ are β_i-appropriate for K and $\lambda_i \geq 0$, $i = 1, ..., m$, then the mapping

$$\mathcal{A}(x) = \sum_i \lambda_i \mathcal{A}_i(x) : \text{int}\left(\bigcap_{i=1}^m G_i^-\right) \to \mathbb{R}^N$$

is $[\max_i \beta_i]$-appropriate for $\mathbf{K}$.

[direct product] If $\mathcal{A}_i : \text{int}G_i^- \to \mathbb{R}^{N_i}$ are β_i-appropriate for closed cones $K_i \subset \mathbb{R}^{N_i}$ and $\lambda_i \geq 0$, $i = 1, ..., m$, then the mapping

$$\mathcal{A}(x^1, ..., x^m) = \{\lambda_i \mathcal{A}_i(x^i)\}_{i=1}^m : \text{int}(G_1^- \times ... \times G_m^-) \to \mathbb{R}^{N_1+...+N_m}$$

is $[\max_i \beta_i]$-appropriate for $K_1 \times ... \times K_m$

[direct summation] If $\mathcal{A}_i : \text{int}G_i^- \to \mathbb{R}^N$ are β_i-appropriate for K and $\lambda_i \geq 0$, $i = 1, ..., m$, then the mapping

$$\mathcal{A}(x^1, ..., x^m) = \sum_i \lambda_i \mathcal{A}_i(x^i) : \text{int}(G_1^- \times ... \times G_m^-) \to \mathbb{R}^N$$

is $[\max_i \beta_i]$-appropriate for K

[affine pre-composition] If $\mathcal{A} : \mathrm{int}G^- \to \mathbb{R}^N$ is β-appropriate for K, then

$$\mathcal{A}(Ax+b) : \{x : Ax+b \in \mathrm{int}G^-\} \to \mathbb{R}^N$$

is β-appropriate for K.
[affine post-composition] If $\mathcal{A} : \mathrm{int}G^- \to \mathbb{R}^M$ is β-appropriate for a closed cone N in $\mathbb{R}^M$ and $y \mapsto Ay : \mathbb{R}^M \to \mathbb{R}^N$ is a linear mapping such that $A(N) \subset K$, then the mapping $A \circ \mathcal{A}(\cdot)$ is β-appropriate for K.

Verification for conic combination:

$$
\begin{array}{c}
x \pm h \in \mathrm{int}\left(\bigcap\limits_{i=1}^{m} G_i^-\right)\\
\Downarrow\\
x \pm h \in \mathrm{int} G_i^-,\ i = 1, ..., m\\
\Downarrow\\
D^3\mathcal{A}_i(x)[h,h,h] \leq_K -3\beta_i D^2\mathcal{A}_i(x)[h,h],\ i = 1, ..., m\\
\Downarrow\\
D^3\mathcal{A}_i(x)[h,h,h] \leq_K -3[\max\limits_j \beta_j] D^2\mathcal{A}_i(x)[h,h],\ i = 1, ..., m\\
\Downarrow \sum\limits_i \lambda_i D^3\mathcal{A}_i(x)[h,h,h] \leq_K -3[\max\limits_j \beta_j] \sum\limits_i \lambda_i D^2\mathcal{A}_i(x)[h,h]\\
\Downarrow\\
D^3\mathcal{A}(x)[h,h,h] \leq_K -3[\max\limits_j \beta_j] D^2\mathcal{A}(x)[h,h].
\end{array}
$$

Applications of Main Theorem: Barriers for Epigraphs of Univariate Functions

♣ Let $G^+ = K = \mathbb{R}_+$, $G^- = \{(x,t) \in \mathbb{R}^2 : t \geq 0\}$, and

$$\mathcal{A}(x,t) = f(t) - x : \text{int} G^- \to \mathbb{R}.$$

Observe that by Calculus, $\mathcal{A}$ is β-appropriate for K, if so is the mapping

$$t \mapsto f(t) : \{t > 0\} \to \mathbb{R}.$$

With this observation, in the situation in question Main Theorem as applied with

$$F_+(u) = -\ln u; \quad F_-(x,t) = -\ln t$$

reduces to the following statement

Proposition. I. Let $f(t)$ be a C^3 concave function on $\{t > 0\}$ such that

$$t \pm h > 0 \Rightarrow |f'''(t)h^3| \le -3\beta f''(t)h^2$$
$$\Updownarrow$$
$$t > 0 \Rightarrow |f'''(t)| \le -3\beta t^{-1} f''(t)$$

Then the function

$$F(x,t) = -\ln(f(t) - x) - \max[1, \beta^2] \ln t$$

is $(1 + \max[1, \beta^2])$-s.c.b. for

$$G_f = \mathrm{cl}\{(x,t) \in \mathbf{R}^2 \mid t > 0, x < f(t)\}.$$

II. Let $g(x)$ be C^3 and convex on $\{x > 0\}$ such that

$$x > 0 \Rightarrow |g'''(x)| \le 3\beta x^{-1} G''(x).$$

Then the function

$$G(x,t) = -\ln(t - g(x)) - \max[1, \beta^2] \ln x$$

is $(1 + \max[1, \beta^2])$-s.c.b. for

$$G^g = \mathrm{cl}\{(x,t) \in \mathbf{R}^2 \mid x > 0, t > f(x)\}.$$

Epigraph of increasing power function: Whenever $p \geq 1$, the function

$$-\ln t - \ln(t^{1/p} - x)$$

is 2-s.c.b. for the epigraph

$$\{(x,t) \in \mathbf{R}^2 : t \geq x_+^p \equiv [\max\{0,x\}]^p\}$$

of the power function x_+^p, and the function

$$-2\ln t - \ln(t^{2/p} - x^2)$$

is 4-s.c.b. for the epigraph

$$\{(x,t) \in \mathbf{R}^2 \mid t \geq |x|^p\}$$

of the function $|x|^p$.

Indeed, $f(t) = t^{1/p} : (0,\infty) \to \mathbb{R}$ satisfies

$$|f'''(t)| \leq -3\beta t^{-1} f''(t)$$

with $\beta = \frac{2p-1}{3p} \leq 1$, and the set

$$G_f = \mathrm{cl}\{(x,t) : t > 0, x < f(t)\}$$

is exactly $\mathrm{Epi}\{x_+^p\}$.

Epigraph of decreasing power function: Whenever $p > 0$, the function

$$\begin{cases} -\ln x - \ln(t - x^{-p}), & 0 < p \leq 1 \\ -\ln t - \ln(x - t^{-1/p}), & p > 1 \end{cases}$$

is 2-s.c.b. for the epigraph

$$\text{cl}\{(x,t) \in \mathbf{R}^2 \mid t \geq x^{-p}, x > 0\}$$

of the function x^{-p}.

The case of $0 < p \leq 1$ is covered by Proposition, II applied to $g(x) = x^{-p}$ ($\beta = \frac{2+p}{3}$).

The case of $p > 1$ reduces to the previous one, since for $t, x > 0$ one has

$$t \geq x^{-p} \Leftrightarrow x \geq t^{-1/p}$$

Epigraph of the exponent: The function

$$-\ln t - \ln(\ln t - x)$$

is 2-s.c.b. for the epigraph

$$\{(x,t) \in \mathbf{R}^2 \mid t \geq \exp\{x\}\}$$

of the exponent.
This case is covered by Proposition, I as applied to $f(t) = \ln t$ ($\beta = \frac{2}{3}$)

Epigraph of the entropy function: The function

$$-\ln x - \ln(t - x\ln x)$$

is 2-self-concordant barrier for the epigraph

$$\mathrm{cl}\{(x,t) \in \mathbf{R}^2 \mid t \geq x\ln x, x > 0\}$$

of the entropy function $x\ln x$.
This case is covered by Proposition, II as applied to $f(x) = x\ln x$ $(\beta = \frac{1}{3})$.

Example 1: Geometric Programming. A GP program

$$\min_x \left\{ c^T x : \begin{array}{l} \ln(\sum\limits_\ell a_{i\ell} \exp\{b_{i\ell}^T x\}) + c_i^T x + d_i \leq 0, i \leq m \\ Px \geq p \end{array} \right\}$$

with $a_{i\ell} > 0$ can be rewritten as

$$\min_{x,u} \left\{ c^T x : \begin{array}{l} \exp\{d_i + \ln a_{i\ell} + [c_i + b_{i\ell}]^T x\} \leq u_{i\ell} \\ \sum\limits_\ell u_{i\ell} \leq 1 \\ Px \geq p \end{array} \right\}$$

$$\Updownarrow$$

$$\min_{x,u} \left\{ c^T x : Rx + Su + r \in H \right\}$$

$$H = \{t, s, y : t_j \geq 0, j \leq p, \exp\{s_\ell\} \leq y_\ell, \ell = 1, ..., q\}$$

It suffices to equip with a s.c.b. the set H, and here is $(p + 2q)$-barrier for the set:

$$\Phi(t, s, y) = -\sum_j \ln t_j - \sum_\ell [\ln y_\ell + \ln(\ln y_\ell - s_\ell)]$$

Example 2: Entropy Minimization. An EM program

$$\min_x \left\{ c^T x : \begin{array}{l} \sum_\ell \alpha_{i\ell}(a_{i\ell}+b_{i\ell}^T x)\ln\left(a_{i\ell}+b_{i\ell}^T x\right) \\ \quad +c_i^T x + d_i \leq 0, i \leq m \\ Px \geq p \end{array} \right\}$$

with $\alpha_{i\ell} > 0$ can be rewritten as

$$\min_{x,u} \left\{ c^T x : \begin{array}{l} (a_{i\ell}+b_{i\ell}^T x)\ln\left(a_{i\ell}+b_{i\ell}^T x\right) \leq u_{i\ell} \\ \sum_\ell \alpha_{i\ell}u_{i\ell} + c_i^T x + d_i \leq 0 \\ Px \geq p \end{array} \right\}$$

$$\Updownarrow$$

$$\min_{x,u} \left\{ c^T x : Rx + Su + r \in H \right\}$$

$$H = \{t, s, y : t_j \geq 0, j \leq p,\ s_\ell \ln s_\ell \leq y_\ell,\ \ell = 1, ..., q\}$$

It suffices to equip with a s.c.b. the set H, and here is $(p+2q)$-barrier for the set:

$$\Phi(t,s,y) = -\sum_j \ln t_j - \sum_\ell [\ln s_\ell + \ln(y_\ell - s_\ell \ln s_\ell)]$$

Applications of Main Theorem: Barriers for Epigraphs of Some Multivariate Functions

<u>Lemma:</u> Let $\pi_j \geq 0$, $\sum_j \pi_j = 1$. The function

$$f(x) = x^\pi \equiv \prod_{i=1}^m x_j^{\pi_j} : \mathrm{int}\mathbb{R}^m_+ \to \mathbb{R}$$

is 1-appropriate for $K = \mathbb{R}_+$.

Proof: Assume that $x \pm h > 0$. Setting $\delta_j = \frac{h_j}{x_j}$, we have

$$\begin{array}{rcl}
Df(x)[h] & = & f(x)\left[\sum_j \pi_j \frac{h_j}{x_j}\right] \\
D^2 f(x)[h,h] & = & f(x)\left[\sum_j \pi_j \frac{h_j}{x_j}\right]^2 \\
& & -f(x)\left[\sum_j \pi_j \frac{h_j^2}{x_j^2}\right] \\
D^3 f(x)[h,h,h] & = & f(x)\left[\sum_j \pi_j \frac{h_j}{x_j}\right]^3 \\
& & -3f(x)\left[\sum_j \pi_j \frac{h_j^2}{x_j^2}\right]\left[\sum_j \pi_j \frac{h_j}{x_j}\right] \\
& & +2f(x)\left[\sum_j \pi_j \frac{h_j^3}{x_j^3}\right].
\end{array}$$

Setting

$$\begin{array}{rcl} M_\kappa & = & \sum_j \pi_j \delta_j^\kappa,\ \kappa = 1,2,3, \\ \xi_j & = & \delta_j - M_1, \end{array}$$

we get $\sum_j \pi_j \xi_j = 0$, $M_2 = M_1^2 + \overbrace{\sum_j \xi_j^2}^{\sigma^2}$, and

$$\begin{array}{rcl} -D^2 f(x)[h,h] & = & f(x)\left[M_2 - M_1^2\right] = f(x)\sigma^2 \\ D^3 f(x)[h,h,h] & = & f(x)\left[2\sum_j \pi_j (M_1 + \xi_j)^3 - 3M_1(M_1^2 + \sigma^2) \right. \\ & & \left. + M_1^3\right] \\ & = & f(x)\left[3M_1\sigma^2 + 2\sum_j \pi_j \xi_j^3\right]. \end{array}$$

Since $|\delta_j| \le 1$, we have $|M_1| \le 1$ and $\xi_j \le 1 - M_1$, whence

$$\sum_j \pi_j \xi_j^3 \le \sum_j \pi_j \xi_j^2 (1 - M_1) = (1 - M_1)\sigma^2,$$

whence

$$D^3 f(x)[h,h,h] \le f(x)(M_1 + 2)\sigma^2 \le -3D^2 f(x)[h,h].$$

Hypograph of geometric mean: Let $\pi_{ij} \geq 0$ with $\sum\limits_j \pi_{ij} \leq 1$, $i = 1,...,m$, and let $a_i \geq 0$. The function

$$F(t,x) = -\ln(\sum_i a_i \prod_j x_j^{\pi_{ij}} - t) - \sum_j \ln x_j$$

is $(n+1)$-s.c.b. for $\{x \geq 0, t \leq \sum\limits_i a_i \prod\limits_j x_j^{\pi_{ij}}\}$.

Indeed, by Lemma the function $\left(\prod\limits_j x_j^{\pi_{ij}}\right) y^{1-\sum_j \pi_{ij}}$: $\mathrm{int}\mathbb{R}^{n+1}_+ \to \mathbb{R}$ is 1-appropriate for $\mathbb{R}_+$; by Calculus, so is the function

$$f(t,x) = \sum_i a_i \prod_j x_j^{\pi_{ij}} - t : \{(t,x) : x > 0\} \to \mathbb{R}.$$

By Main Theorem as applied with $G^+ = \mathbb{R}^+$, $F_+(z) = -\ln z$, $F_-(t,x) = -\sum\limits_i \ln x_i$, the function

$$-\ln f(t,x) - \sum_i \ln x_i$$

is $(n+1)$-s.c.b. for the set $\mathrm{cl}\{(t,x) : x > 0, t < x^\pi\} = \{x \geq 0, t \leq \sum\limits_i a_i \prod\limits_j x_j^{\pi_{ij}}\}$.

Epigraph of $\prod\limits_i x_i^{-\alpha_i}$: Let $\alpha_i > 0$. The function

$$F(t,x) = -\ln\left([tx^\alpha]^{\frac{1}{1+\sum\limits_i \alpha_i}} - 1\right) - \sum_i \ln x_i$$

is $(n+1)$-s.c.b. for the set $\{x > 0, t \geq x^{-\alpha}\}$.
Indeed,

$$\{x > 0, t \geq x^{-\alpha}\} = \{(x,t) \geq 0 : [tx^\alpha]^{\frac{1}{1+\sum\limits_i \alpha_i}} \geq 1\}.$$

Example 3: $\|\cdot\|_p$-approximations. Consider a constraint

$$\|Ax+b\|_p \leq c^T x + d. \qquad (*)$$

In order to work with this constraint, it suffices to represent the set

$$\{(x,t) : \|x\|_p \leq t\}$$

in the form $\{(x,t) : \exists y : Ax+By+tc+d \in G^+\}$, where G^+ admits explicit s.c.b. Φ. Given such a representation, we can replace $(*)$ with the constraint

$$Ax + By + tc + d \in G^+$$

in the extended space of variables; the new constraint already admits a s.c.b. $\Phi(Ax + By + tc + d)$.

Here is the desired representation for $1 < p < \infty$ (the cases of $p = 1$ and $p = \infty$ are trivial):

$$\begin{array}{rl} & \{(t,x) : \|x\|_p \leq t\} \\ = & \left\{(t,x) : \sum\limits_i t^{-(p-1)}|x_i|^p \leq t\right\} \\ = & \left\{(t,x) : \exists \tau_i : |x_i| \leq t^{\frac{p-1}{p}} \tau_i^{\frac{1}{p}}, \sum\limits_i \tau_i \leq t\right\} \end{array}$$

The set

$$\{(t,x,\tau): |x_i| \leq t^{\frac{p-1}{p}} \tau_i^{\frac{1}{p}}, i \leq m, \sum_i \tau_i \leq t,\} \quad (S)$$

admits $(3m+1)$-s.c.b.

$$F(t,x,\tau) = -\sum_i \left[\ln\left(t^{\frac{2(p-1)}{p}} \tau_i^{\frac{2}{p}} - x_i^2\right) + \ln \tau_i\right] - \ln(t - \sum_i \tau_i). \tag{$*$}$$

Indeed, consider the mapping

$$\mathcal{A}(t,\tau,x) = \left\{(t^{\frac{p-1}{p}} \tau_i^{\frac{1}{p}} - x_i, t_i^{\frac{p-1}{p}} \tau_i^{\frac{1}{p}} + x_i)\right\} : \mathrm{int} G^- \to \mathbb{R}^{2m}$$
$$G^- = \{(t,\tau,x) : t \geq 0, \tau \geq 0, \sum_i \tau_i \leq t\}$$

By Lemma and Calculus, this mapping is 1-appropriate for $\mathbb{R}^{2m}_+$ (as restriction of pre-composition of direct product of 1-appropriate mappings). Setting

$$G^+ = K = \mathbb{R}^{2m}_+,\ F_+(s) = -\sum_{i=1}^{2m} \ln s_i,$$
$$F_-(t,\tau) = -\sum_i \ln \tau_i - \ln(t - \sum_i \tau_i)$$

and applying Main Theorem, we conclude that $(*)$ is a $(3m+1)$-s.c.b. for (S).

Fractional-Quadratic Substitution

<u>Theorem.</u> Let G be a closed convex domain in $\mathbb{R}^n$ and $Y(x):\mathbb{R}^n\to\mathbf{S}^m$ be affine and

$$x\in \mathrm{int} G \Rightarrow Y(x)\succ 0.$$

Then the mapping

$$\mathcal{A}(X,x) = -X^TY^{-1}(x)X : \mathrm{int}\underbrace{(\mathbb{R}^{m\times k}\times G)}_{G^-}\to\mathbf{S}^k$$

is 1-appropriate for $\mathbf{S}^k_+$.

Proof: Let $(X,x)\in\mathrm{int}G^-$ and (dX,dx) are such that $(X,x)\pm(dX,dx)\in\mathrm{int}G$. Let us set $dY=Y(x+h)-Y(x)$. Setting

$$\delta Y = Y^{-1/2}dYY^{-1/2} = (\delta Y)^T,\ \delta X = Y^{-1/2}dX,\ Z = Y^{-1/2}X:$$

$$\begin{array}{rcl}
-d\mathcal{A} &=& dX^TY^{-1}X + X^TY^{-1}dX - X^TY^{-1}dYY^{-1}X\\
-d^2\mathcal{A} &=& 2dX^TY^{-1}dX - 2dX^TY^{-1}dYY^{-1}X\\
&& -2X^TY^{-1}dYY^{-1}dX + 2X^TY^{-1}dYY^{-1}dYY^{-1}X\\
&=& 2\big[(\delta X)^T(\delta X) - (\delta X)^T(\delta Y)Z -\\
&& Z^T(\delta Y)(\delta X) + Z^T(\delta Y)^2Z\big]\\
&=& 2\left[\delta X - (\delta Y)Z\right]^T\left[\delta X - (\delta Y)Z\right]\\
-d^3\mathcal{A} &=& -6dX^TY^{-1}dYY^{-1}dX\\
&& +6dX^TY^{-1}dYY^{-1}dYY^{-1}X\\
&& +6X^TY^{-1}dYY^{-1}dYY^{-1}dX\\
&& -6X^TY^{-1}dYY^{-1}dYY^{-1}dYY^{-1}X\\
&=& -6\big[(\delta X)^T(\delta Y)(\delta X) - (\delta X)^T(\delta Y)^2Z\\
&& -Z^T(\delta Y)^2(\delta X) + Z^T(\delta Y)^3Z\big]\\
&=& -6\left[\delta X - (\delta Y)Z\right]^T(\delta Y)\left[\delta X - (\delta Y)Z\right]
\end{array}$$

$$\begin{array}{rcl} d^2\mathcal{A} & = & -2\,[\delta X - (\delta Y)Z]^T\,[\delta X - (\delta Y)Z] \\ d^3\mathcal{A} & = & 6\,[\delta X - (\delta Y)Z]^T\,(\delta Y)\,[\delta X - (\delta Y)Z] \end{array}$$

Since $x \pm dx \in G$ and $Y(x) \succ 0$ in G, we get $Y \pm dY \succ 0$, or $-I \preceq \delta Y \preceq I$. Consequently,

$$\begin{array}{rcl} d^3\mathcal{A} & = & 6\,[\delta X - (\delta Y)Z]^T\,(\delta Y)\,[\delta X - (\delta Y)Z] \\ & \preceq & 6\,[\delta X - (\delta Y)Z]^T\,[\delta X - (\delta Y)Z] = -3d^2\mathcal{A}. \end{array}$$

Example 4: Epigraph of convex quadratic form.

The function

$$f(t,x) = -\ln(t - a - 2b^T x - x^T Q^T Q x)$$

is 1-s.c.b. for the epigraph of convex quadratic form $x^T Q^T Q x + 2b^T x + c$.

In view of the Affine Substitution rule, it suffices to prove that

$$g(t,y) = -\ln(t - y^T y)$$

is 1-s.c.b. for $\{(t,y) : y^T y \le t\}$. By Calculus and Fractional Quadratic Substitution rule as applied to the data

$$\mathcal{A}(y,u) \equiv y^T Y^{-1}(u) y, \; Y(u) \equiv I,$$

the function

$$f(t,y,u) \equiv t - y^T Y^{-1}(u) y = t - y^T y : \overbrace{\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}}^{G^-} \to \mathbf{S}^1 = \mathbb{R}$$

is 1-appropriate for $\mathbf{S}^1_+ = \mathbb{R}_+$. Setting $G^+ = \mathbb{R}_+$, $F_+(z) = -\ln z$, $F_- \equiv 0$ and applying Main Theorem, we see that

$$-\ln(t - y^T Y^{-1}(u) y) = -\ln(t - y^T y)$$

is 1-s.c.b. for the set $\{(t,y) : y^T y \le t\}$.

Example 5: Lorentz cone. The function

$$f(t,x) = -\ln(t^2 - x^T x)$$

is 2-s.c.b. for the Lorentz cone $\{(t,x) : t \geq \|x\|_2\}$.

By Calculus and Fractional Quadratic Substitution rule as applied to the data

$$\mathcal{A}(x,u) \equiv x^T Y^{-1}(u)x, \;\; Y(u) \equiv uI,$$

the function

$$f(t,x,u) \equiv t - x^T Y^{-1}(u)x : \overbrace{\mathbb{R} \times \mathbb{R}^n \times \{u > 0\}}^{G^-} \to \mathbf{S}^1 = \mathbb{R}$$

is 1-appropriate for $\mathbf{S}^1_+ = \mathbb{R}_+$. Setting $G^+ = \mathbb{R}_+$, $F_+(z) = -\ln z$, $F_-(u) = -\ln u$ and applying Main Theorem, we see that

$$-\ln(t - u^{-1}y^T y) - \ln(u) = -\ln(tu - x^T x)$$

is 2-s.c.b. for the set $\mathrm{cl}\{tu \geq x^T x, u > 0\}$. By Affine Substitution rule,

$$-\ln(t^2 - x^T x)$$

is 2-s.c.b. for $\{(t,x) : t \geq \sqrt{x^T x}\}$.

Example 6: Epigraph of matrix norm. Function

$$f(t,x) = -\ln \mathrm{Det}(t^2 I_n - X^T X) - \ln t \qquad [X \in \mathbf{R}^{m\times n}]$$

is m-s.c.b. for the cone $\{(t,X) : t \geq \|X\|\}$, where $\|X\| = \max\limits_{\|z\|_2=1} \frac{\|Xz\|_2}{\|z\|_2}$ is the matrix norm of X.

By Calculus and Fractional Quadratic Substitution rule as applied to the data

$$\mathcal{A}(X,u) \equiv X^T Y^{-1}(u) X, \; Y(u) \equiv u I_m,$$

the function

$$f(V,X,u) \equiv V - X^T Y^{-1}(u) X : \overbrace{\mathbf{S}^n \times \mathbb{R}^{m\times n} \times \{u > 0\}}^{\mathrm{int} G^-} \to \mathbf{S}^n$$

is 1-appropriate for $\mathbf{S}^n_+$. Setting $G^+ = \mathbf{S}^m_+$, $F_+(Z) = -\ln \mathrm{Det} Z$, $F_-(u) = -\ln u$ and applying Main Theorem, we see that

$$-\ln \mathrm{Det}(V - u^{-1} X^T X) - \ln(u)$$

is $(n+1)$-s.c.b. for the set $\{uV \succeq X^T X, u > 0\}$. Substituting $V = tI_n$, $u = t$, we see that

$$-\ln(tI_n - t^{-1} X^T X) - \ln t$$

is $(n+1)$-s.c.b. for the set $\{(t,X) : t^2 I_n \succeq X^T x\} = \{(t,X) : t \geq \|X\|\}$.

Note: With our Main Theorem, only two s.c.b.'s remain guessed rather than derived – these are s.c.b.'s $-\ln t$ for $\mathbb{R}_+$ and $-\ln \mathrm{Det} X$ for $\mathbf{S}^m_+$. In fact, the second barrier also can be derived.
Indeed, the function

$$f(z,x,X) = z - x^T X^{-1} x : \overbrace{\mathbb{R}\times\mathbb{R}^m\times\mathrm{int}\mathbf{S}^m_+}^{\mathrm{int}G^-} \to \mathbb{R}$$

is 1-appropriate for $\mathbb{R}_+$. From Main Theorem it follows that if $F_m(X)$ is θ_m-s.c.b. for $\mathbf{S}^m_+$, then the function

$$F_{m+1}(z,x,X) = F_m(X) - \ln(z - x^T X^{-1} x)$$

is (ϑ_m+1)-s.c.b. for the set $(z,x,X) = \mathrm{cl}\{z \geq x^T X^{-1} x, X \succ 0\} = \left\{(z,x,X : \begin{bmatrix} z & x^T \\ x & X \end{bmatrix} \succeq 0\right\}$.
Thus, we get a recurrence which converts ϑ_m-s.c.b. for $\mathbf{S}^m_+$ into (ϑ_m+1)-s.c.b. for $\mathbf{S}^{m+1}_+$. Starting recurrence with $F_1(x) = -\ln x$ $(\vartheta_1 = 1)$, it is easy to check that the recurrence results in $F_m(X) = -\ln \mathrm{Det}(X)$, $\vartheta_m = m$.

Proof of Main Theorem

A. Let

$$G^0 = \{x \in \text{int}G^- : \mathcal{A}(x) \in \text{int}G^+\}$$

Let us prove that G^0 is an open convex domain.

Indeed, all we need to prove is that G^0 is convex, which is immediate: with $x', x'' \in G^0$ and $\lambda \in [0,1]$ we have $\lambda x' + (1-\lambda)x'' \in \text{int}G^-$ and

$$\begin{array}{rcl} \mathcal{A}(\lambda x' + (1-\lambda)x'') & \in & \lambda\mathcal{A}(x') + (1-\lambda)\mathcal{A}(x'') \\ & & +K \\ & \in & (\text{int}G^+) + K \subset \text{int}G^+. \end{array}$$

B. Let us prove that

$$F(x) = F_{+}(\mathcal{A}(x)) + \max[1, \beta^2] F_{-}(x)$$

is s.c.f. on G^0.

We fix $x \in G^0$, h and will verify that

$$\begin{array}{ll} (a) & |D^3F(x)[h,h,h]| \leq 2\{D^2F(x)[h,h]\}^{3/2}, \\ (b) & |DF(x)[h]| \leq \vartheta^{1/2}\{D^2F(x)[h,h]\}^{1/2}. \end{array}$$

Setting $\gamma = \max[1, \beta]$ and

$$\begin{array}{ll} a = \mathcal{A}(x), & a' = D\mathcal{A}(x)[h], \\ a'' = D^2\mathcal{A}(x)[h,h], & a''' = D^3\mathcal{A}(x)[h,h,h], \\ DF(x)[h] = dF, & D^2F(x)[hh] = d^2F, \\ \multicolumn{2}{c}{D^3F(x)[h,h,h] = d^3F} \end{array}$$

we have

$$\begin{array}{rcl} dF &=& DF_+(a)[a'] + \gamma^2 DF_-(x)[h] \\ d^2F &=& D^2F_+(a)[a',a'] + DF_+(a)[a''] \\ && +\gamma^2 D^2F_-(x)[h,h], \\ d^3F &=& D^3F_+(a)[a',a',a'] + 3DF_+(a)[a',a''] \\ && +DF_+(a)[a'''] + \gamma^2 D^3F_-(x)[h,h,h]. \end{array}$$

Setting $p = \sqrt{D^2F_+(a)[a',a']}$, $q = \sqrt{D^2F_-(x)[h,h]}$, we have

$$\begin{array}{ll} |DF_+(a)[a']| \le p\sqrt{\vartheta_+}, & |DF_-(x)[h]| \le q\sqrt{\vartheta_-} \\ |D^3F_+[a',a',a']| \le 2p^3, & |D^3F_-[h,h,h]| \le 2q^3 \end{array}$$

Since F_+ is s.c.b. for G^+ and $-a''$ is a recessive direction of G^+,

$$\sqrt{D^2F_+(a)[a'',a'']} \le D^2F_+[a''] \equiv r^2.$$

Lemma. One has

$$3\beta q a'' \leq_K a''' \leq_K -3\beta q a'', \qquad (*)$$

Indeed, let a real t be such that $|t|\|h\|_x \equiv tq \leq 1$, and let $h_t = th$; then $D^2F_-(x)[h_t,h_t] = t^2q^2 \leq 1$, whence $x \pm h_t \in G^-$ and therefore

$$t^3a''' \equiv D^3\mathcal{A}(x)[h_t,h_t,h_t] \leq_K -3\beta D^2\mathcal{A}(x)[h_t,h_t] \equiv -3\beta t^2 a''.$$

This is valid for all t with $|t|q \leq 1$, and $(*)$ follows.

♣ Since $DF_+(x)[\cdot]$ is nonpositive on the recessive cone of G^+ and thus on K, $(*)$ implies

$$|DF_+(a)[a''']| \leq 3\beta q DF_+(a)[a''] = 3\beta q r^2.$$

Finally, by Cauchy's inequality

$$\begin{array}{rcl}|D^2F_+(a)[a',a'']| & \leq & \sqrt{D^2F_+(a)[a',a']}\sqrt{D^2F_+(a)[a'',a'']} \\ & \leq & pr^2\end{array}$$

♣ Putting things together, we get

$$\begin{array}{rcl} |DF(x)[h]| & \leq & p\sqrt{\vartheta_+} + q\gamma^2\sqrt{\vartheta_-} \\ D^2F(x)[h,h] & = & p^2 + r^2 + \gamma^2 q^2 \\ |D^3F(x)[h,h,h]| & \leq & 2[p^3 + \frac{3}{2}pr^2 + \frac{3}{2}\beta q r^2] + 2\gamma^2 q^3. \end{array} \qquad (*)$$

Taking into account that $\gamma = \max[1,\beta]$, it is straightforward to derive from $(*)$ that

$$\begin{array}{ll} (a) & |d^3F| \leq 2\{d^2F\}^{3/2}, \\ (b) & |dF| \leq \vartheta^{1/2}\{d^2F\}^{1/2}. \end{array}$$

Note: Setting

$$H(x) = F_+(\mathcal{A}(x)),$$

the above computations imply that

$$D^2H(x)[h,h] = p^2 + r^2 \geq 0.$$

Thus, $H(\cdot)$ is convex on G^0.

♣ It remains to verify that $F(\cdot)$ is interior penalty for $\mathrm{cl}G^0$. Let $x_i \in G^0$, and $x = \lim\limits_{i\to\infty} x_i \in \partial G^0$; we should prove that

$$F(x_i) \to \infty. \qquad (*)$$

Indeed, let $y_i = \mathcal{A}(x_i)$.
<u>Case 1:</u> $x \in \mathrm{int}G^-$. In this case, $y_i \to \mathcal{A}(x)$ as $i \to \infty$ and $\mathcal{A}(x) \notin \mathrm{int}G^+$, whence

$$H(x_i) = F_+(\mathcal{A}(x_i)) \to +\infty, \, i \to \infty,$$

while $F_-(x_i)$ remain bounded below (since $F_-(\cdot)$ is convex), and $(*)$ follows.
<u>Case 2:</u> $x \notin \mathrm{int}G^-$. In this case,

$$F_-(x_i) \to +\infty, \, i \to \infty,$$

while $H(x_i)$ remain bounded below (since $H(\cdot)$ is convex), and $(*)$ follows.

How to initialize IPMs
Self-Dual Embedding

♣ In the IPMs presented so far, we assumed that

♢ we know in advance a strictly feasible primal solution (or primal-dual solutions)

♢ (in path-following methods) the initial strictly feasible solution(s) are close to the path(s).

♠ How to get rid of these restrictions?

♣ Assume we are interested to solve conic problem

$$\min_x \left\{ c^T x : X \equiv \mathcal{A}x - B \in \mathbf{K} \right\}, \qquad \text{(CP)}$$

($\mathbf{K}$ is a closed convex pointed cone with a nonempty interior in Euclidean space E). The corresponding primal-dual pair is

$$\begin{array}{lr} \min\limits_X \{\langle C, X\rangle : X \in (\mathcal{L} - B) \cap \mathbf{K}\} & \text{(P)} \\ \max\limits_S \left\{\langle B, S\rangle : S \in (\mathcal{L}^\perp + C) \cap \mathbf{K}_*\right\} & \text{(D)} \\ \left[\mathcal{L} = \mathrm{Im}\,\mathcal{A}, \mathcal{L}^\perp = \mathrm{Ker}\,\mathcal{A}^T, \mathcal{A}^T C = c\right] & \end{array}$$

♠ <u>Assumption:</u> (P), (D) is strictly primal-dual feasible.

$$\min_X \{\langle C, X\rangle : X \in (\mathcal{L} - B) \cap \mathbf{K}\} \qquad \text{(P)}$$
$$\max_S \left\{\langle B, S\rangle : S \in (\mathcal{L}^\perp + C) \cap \mathbf{K}_*\right\} \qquad \text{(D)}$$
$$\left[\mathcal{L} = \mathrm{Im}\,\mathcal{A}, \mathcal{L}^\perp = \mathrm{Ker}\,\mathcal{A}^T, \mathcal{A}^T C = c\right]$$

Observation: When shifting B along $\mathcal{L}$, we do not vary (P) and replace (D) with an equivalent problem; thus, we can assume that $B \in \mathcal{L}^\perp$. Similarly, when shifting C along $\mathcal{L}^\perp$, we do not vary (D) and replace (P) with an equivalent problem; thus, we may assume that $C \in \mathcal{L}$. With $B \in \mathcal{L}^\perp$, $C \in \mathcal{L}$ we have

$$\text{DualityGap}(X, S) = \langle X, S\rangle = \langle C, X\rangle - \langle B, S\rangle$$
$$[= \langle C, X\rangle - \langle B, S\rangle + \langle C, B\rangle].$$

♣ Goal: *To develop an IPM for* (P), (D) *which requires neither a priori knowledge of a primal-dual strictly feasible pair of solutions.*

$$\min_X \{\langle C, X\rangle : X \in (\mathcal{L} - B) \cap \mathbf{K}\} \qquad \text{(P)}$$
$$\max_S \left\{\langle B, S\rangle : S \in (\mathcal{L}^\perp + C) \cap \mathbf{K}_*\right\} \qquad \text{(D)}$$
$$\left[C \in \mathcal{L} = \text{Im}\, \mathcal{A}, B \in \mathcal{L}^\perp = \text{Ker}\, \mathcal{A}^T, \mathcal{A}^T C = c\right]$$

The approach: $\mathbf{1}^0$. Write down system of constraints in variables X, S and $\tau, \sigma \in \mathbb{R}$:

$$\begin{array}{lrcl}
(a) & X + \tau B - P & \in & \mathcal{L}; \\
(b) & S - \tau C - D & \in & \mathcal{L}^\perp; \\
(c) & \langle C, X\rangle - \langle B, S\rangle + \sigma - d & = & 0; \\
(e) & X & \in & \mathbf{K}; \\
(f) & S & \in & \mathbf{K}_*; \\
(g) & \sigma & \geq & 0; \\
(h) & \tau & \geq & 0.
\end{array} \qquad \text{(C)}$$

• P, D, d are such that

♢ (I) it is easy to point out a strictly feasible solution $\widehat{Y} = (\widehat{X}, \widehat{S}, \widehat{\sigma}, \widehat{\tau} = 1)$ to (C);

♢ (II) the solution set $\mathcal{Y}$ of (C) is unbounded; moreover, whenever $Y_i = (X_i, S_i, \sigma_i, \tau_i) \in \mathcal{Y}$ is an unbounded sequence, we have $\tau_i \to \infty$.

$$
\begin{array}{lrcl}
(a) & X + \tau B - P & \in & \mathcal{L}; \\
(b) & S - \tau C - D & \in & \mathcal{L}^{\perp}; \\
(c) & \langle C, X\rangle - \langle B, S\rangle + \sigma - d & = & 0; \\
(e) & X & \in & \mathbf{K}; \\
(f) & S & \in & \mathbf{K}_*; \\
(g) & \sigma & \geq & 0; \\
(h) & \tau & \geq & 0.
\end{array}
\qquad (\mathrm{C})
$$

2^0. Assume we have a mechanism allowing to "run away to ∞ along $\mathcal{Y}$" – to generate points $Y_i = (X_i, S_i, \sigma_i, \tau_i) \in \mathcal{Y}$ such that $\|Y_i\| \equiv \sqrt{\|X_i\|_2^2 + \|S_i\|_2^2 + \sigma_i^2 + \tau_i^2} \to \infty$. By (II) $\tau_i \to \infty$, $i \to \infty$. Setting

$$
\widetilde{X}_i = \tau_i^{-1} X_i, \quad \widetilde{S}_i = \tau_i^{-1} S_i.
$$

we get

$$
\begin{array}{ll}
(a) & \widetilde{X}_i \in (\mathcal{L} - B + \tau_i^{-1} P) \cap \mathbf{K}; \\
(b) & \widetilde{S}_i \in (\mathcal{L}^{\perp} + C + \tau_i^{-1} D) \cap \mathbf{K}_*; \\
(c) & \underbrace{\langle C, \widetilde{X}_i\rangle - \langle B, \widetilde{S}_i\rangle}_{\text{DualityGap}(\widetilde{X}_i, \widetilde{S}_i)} \leq \tau_i^{-1} d.
\end{array}
\qquad (\mathrm{C}')
$$

Since $\tau_i \to \infty$, we see that $\widetilde{X}_i, \widetilde{S}_i$ simultaneously approach primal-dual feasibility for (P), (D) (by $(a - b)$) and primal-dual optimality (by (c)).

$$\begin{array}{lrcl}
(a) & X + \tau B - P & \in & \mathcal{L}; \\
(b) & S - \tau C - D & \in & \mathcal{L}^{\perp}; \\
(c) & \langle C, X\rangle - \langle B, S\rangle + \sigma - d & = & 0; \\
(e) & X & \in & \mathbf{K}; \\
(f) & S & \in & \mathbf{K}_*; \\
(g) & \sigma & \geq & 0; \\
(h) & \tau & \geq & 0.
\end{array} \qquad \text{(C)}$$

♣ *How to run to ∞ along the solution set $\mathcal{Y}$ of (C)?*
(C) is of generic form

$$Y \equiv (X, S, \sigma, \tau) \in (\mathcal{M} + R) \cap \widetilde{\mathbf{K}} \qquad \text{(G)}$$

where

$$\begin{array}{l}
\widetilde{\mathbf{K}} = \mathbf{K} \times \mathbf{K}_* \times \mathbb{R}_+ \times \mathbb{R}_+ \subset E \times E \times \mathbb{R} \times \mathbb{R} \\
\mathcal{M} = \left\{ (U, V, s, r) : \begin{array}{c} U + rB \in \mathcal{L}, \\ V - rC \in \mathcal{L}^{\perp}, \\ \langle C, U\rangle - \langle B, V\rangle + s = 0 \end{array} \right\} \\
R = (P, D, d - \langle C, P\rangle + \langle B, D\rangle, 0) \in \widetilde{E}
\end{array}$$

Given a ϑ-l.-h. s.c.b. F for $\mathbf{K}$ with known Legendre transformation, we can convert it into $2(\vartheta + 1)$-l.-h. s.c.b. $\widetilde{F}(\cdot)$, also with known Legendre transformation, for $\widetilde{\mathbf{K}}$.

$$
\begin{array}{lrcll}
(a) & X + \tau B - P & \in & \mathcal{L}; & \\
(b) & S - \tau C - D & \in & \mathcal{L}^{\perp}; & \\
(c) & \langle C, X\rangle - \langle B, S\rangle + \sigma - d & = & 0; & \\
(e) & X & \in & \mathbf{K}; & \text{(C)} \\
(f) & S & \in & \mathbf{K}_*; & \\
(g) & \sigma & \geq & 0; & \\
(h) & \tau & \geq & 0. &
\end{array}
$$

$$\Updownarrow$$

$$Y \equiv (X, S, \sigma, \tau) \in (\mathcal{M} + R) \cap \widetilde{\mathbf{K}} \qquad \text{(G)}$$

♣ By our assumptions, we know a strictly feasible solution

$$\widehat{Y} = (\widehat{X}, \widehat{S}, \widehat{\sigma}, \widehat{\tau} = 1)$$

to (G). Let

$$\widetilde{C} = -\nabla \widetilde{F}(\widehat{Y}).$$

Consider the auxiliary problem

$$\min_Y \left\{ \langle \widetilde{C}, Y\rangle : Y \in (\mathcal{M} + R) \cap \widetilde{\mathbf{K}} \right\}. \qquad \text{(Aux)}$$

By origin of $\widetilde{C}$, $\widehat{Y}$ lies on the primal central path $\widetilde{Y}_*(t)$ of (Aux):

$$\widehat{Y} = \widetilde{Y}_*(1).$$

Summary:

$$\begin{array}{lrcll}
(a) & X+\tau B-P & \in & \mathcal{L}; & \\
(b) & S-\tau C-D & \in & \mathcal{L}^{\perp}; & \\
(c) & \langle C,X\rangle-\langle B,S\rangle+\sigma-d & = & 0; & \\
(e) & X & \in & \mathbf{K}; & \qquad (\mathrm{C}) \\
(f) & S & \in & \mathbf{K}_*; & \\
(g) & \sigma & \geq & 0; & \\
(h) & \tau & \geq & 0. &
\end{array}$$

$$\Updownarrow$$

$$Y\equiv(X,S,\sigma,\tau)\in(\mathcal{M}+R)\cap\widetilde{\mathbf{K}} \qquad (\mathrm{G})$$

$$\Downarrow$$

$$\min_Y\left\{\langle\widetilde{C},Y\rangle : Y\in(\mathcal{M}+R)\cap\widetilde{\mathbf{K}}\right\} \qquad (\mathrm{Aux})$$

We have in our disposal a point $\widehat{Y}=\widetilde{Y}_*(1)$ on the primal central path of (Aux).

♣ *In order to run to ∞ along the solution set $\mathcal{Y}$ of (C), we can trace the primal central path $\widetilde{Y}_*(\cdot)$, decreasing the penalty, thus enforcing the penalty to approach 0.*

Indeed,

$$\widetilde{Y}_*(t)=\mathrm{argmin}\left[t\langle\widetilde{C},Y\rangle+\widetilde{F}(Y)\right]$$

As $t\to+0$, we are "nearly minimizing" $\widetilde{F}$ over *unbounded* set $\mathcal{Y}$, thus $\inf\limits_{\widetilde{Y}}\widetilde{F}=-\infty$, thus thus we should run to ∞!

Implementing the approach

$$\min_X \{\langle C, X\rangle : X \in (\mathcal{L} - B) \cap \mathbf{K}\} \qquad \text{(P)}$$
$$\max_S \left\{\langle B, S\rangle : S \in (\mathcal{L}^\perp + C) \cap \mathbf{K}_*\right\} \qquad \text{(D)}$$
$$\left[C \in \mathcal{L} = \operatorname{Im} \mathcal{A}, B \in \mathcal{L}^\perp = \operatorname{Ker} \mathcal{A}^T, \mathcal{A}^T C = c\right]$$

♣ Specifying P,D,d. Let us choose somehow $P >_{\mathbf{K}} B$, $D >_{\mathbf{K}_*} -C$, $\widehat{\sigma} > 0$ and set

$$d = \langle C, P - B\rangle - \langle B, D + C\rangle + \widehat{\sigma}.$$

It is immediately seen that with this approach the point

$$\widehat{Y} = (\widehat{X} = P - B, \widehat{S} = C + D, \widehat{\sigma}, \widehat{\tau} = 1)$$

is a strictly feasible solution to

$$\min_Y \left\{\langle \widetilde{C}, Y\rangle : Y \in (\mathcal{M} + R) \cap \widetilde{\mathbf{K}}\right\} \qquad \text{(Aux)}$$

where $\mathcal{M} + R$ is given by

$$\begin{array}{llcl}
(a) & X + \tau B - P & \in & \mathcal{L}; \\
(b) & S - \tau C - D & \in & \mathcal{L}^\perp; \\
(c) & \langle C, X\rangle - \langle B, S\rangle + \sigma - d & = & 0; \\
(e) & X & \in & \mathbf{K}; \\
(f) & S & \in & \mathbf{K}_*; \\
(g) & \sigma & \geq & 0; \\
(h) & \tau & \geq & 0.
\end{array} \qquad \text{(C)}$$

$$\min_Y \left\{ \langle \widetilde{C}, Y\rangle : Y \in (\mathcal{M}+R)\cap \widetilde{\mathbf{K}} \right\} \qquad \text{(Aux)}$$

where $\mathcal{M}+R$ is given by

$$\begin{array}{lrcl}
(a) & X+\tau B-P & \in & \mathcal{L}; \\
(b) & S-\tau C-D & \in & \mathcal{L}^\perp; \\
(c) & \langle C,X\rangle - \langle B,S\rangle + \sigma - d & = & 0; \\
(e) & X & \in & \mathbf{K}; \\
(f) & S & \in & \mathbf{K}_*; \\
(g) & \sigma & \geq & 0; \\
(h) & \tau & \geq & 0.
\end{array} \qquad \text{(C)}$$

♣ <u>Crucial claim:</u> The solution set $\mathcal{Y}$ of (Aux) is unbounded, and $\tau_i \to \infty$ along every sequence $Y_i \in \mathcal{Y}$ with $\|Y_i\| \to \infty$.

Proof: Problem (Aux) is strictly feasible by construction and admits a central path; therefore the conic dual (Aux′) of (Aux) also is strictly feasible (in fact, by construction $\widetilde{C}$ is strictly feasible solution of (Aux′)).

To see that $\mathcal{Y}$ is unbounded, it suffices to verify that $\mathcal{M}^\perp$ does *not* intersect int$\widetilde{\mathbf{K}}_*$

Claim: $\mathcal{M}^{\perp}$ does not intersect int$\widetilde{\mathbf{K}}_*$.
A collection (ξ, η, s, r) is in $\mathcal{M}^{\perp}$ iff linear equation

$$\langle X, \xi \rangle_E + \langle S, \eta \rangle + \sigma s + \tau r = 0$$

in variables X, S, σ, τ is a corollary of the system

$$X + \tau B \in \mathcal{L}, S - \tau C \in \mathcal{L}^{\perp}, \langle X, C \rangle - \langle S, B \rangle + \sigma = 0.$$

By Linear Algebra, this is the case iff there exist $U \in \mathcal{L}^{\perp}, V \in \mathcal{L}, \lambda \in \mathbb{R}$ such that

$$\begin{array}{lcll} (a) & \xi & = & U + \lambda C, \\ (b) & \eta & = & V - \lambda B, \\ (c) & s & = & \lambda, \\ (d) & r & = & \langle U, B \rangle - \langle V, C \rangle. \end{array} \qquad \text{(Pr)}$$

♣ We have seen that $(\xi, \eta, s, r) \in \mathcal{M}^\perp$ iff there exist $U \in \mathcal{L}^\perp, v \in \mathcal{L}, \lambda \in \mathbb{R}$ such that

$$\begin{array}{lrcl} (a) & \xi & = & U + \lambda C, \\ (b) & \eta & = & V - \lambda B, \\ (c) & s & = & \lambda, \\ (d) & r & = & \langle U, B \rangle - \langle V, C \rangle. \end{array} \qquad \text{(Pr)}$$

Assuming that $\mathcal{M}^\perp \cap \mathrm{int}\widetilde{\mathbf{K}}_* \neq \emptyset$, there exist $U \in \mathcal{L}^\perp, V \in \mathcal{L}, \lambda \in \mathbb{R}$ such that (Pr) results in $\xi \in \mathrm{int}\mathbf{K}_*$, $\eta \in \mathrm{int}\mathbf{K}$, $s > 0$, $r > 0$. By (c), $\lambda > 0$; by normalization ((Pr) is homogeneous!), we can make $\lambda = 1$. Since $U, B \in \mathcal{L}^\perp$, $V, C \in \mathcal{L}$, we get from $(a - b)$:

$$\langle \xi, \eta \rangle = \langle C, V \rangle - \langle B, U \rangle;$$

adding this to (d), we get

$$\langle \xi, \eta \rangle + r = 0,$$

which is impossible, since $r > 0$ and $\langle \xi, \eta \rangle > 0$ due to $\xi \in \mathrm{int}\mathbf{K}_*, \eta \in \mathrm{int}\mathbf{K}$.

♣ We have seen that the set $\mathcal{Y}$ of solutions $Y = (X, S, \sigma, \tau)$ to the system

$$\begin{array}{lrcl} (a) & X + \tau B - P & \in & \mathcal{L}; \\ (b) & S - \tau C - D & \in & \mathcal{L}^{\perp}; \\ (c) & \langle C, X\rangle - \langle B, S\rangle + \sigma - d & = & 0; \\ (e) & X & \in & \mathbf{K}; \\ (f) & S & \in & \mathbf{K}_*; \\ (g) & \sigma & \geq & 0; \\ (h) & \tau & \geq & 0. \end{array} \quad \text{(C)}$$

is unbounded. Now let us prove that if $Y_i \in \mathcal{Y}$ are such that $\|Y_i\| \to \infty$, then $\tau_i \to \infty$.
Let $\bar{X}, \bar{S}$ be a strictly feasible pair of primal-dual solutions to (P), (D):

$$\bar{X} >_{\mathbf{K}} 0, X + B \in \mathcal{L}; \bar{S} >_{\mathbf{K}_*} 0, \bar{S} - C \in \mathcal{L}^{\perp}.$$

There exists $\gamma \in (0, 1]$ such that

$$\begin{array}{rcl} \gamma\|X\| & \leq & \langle \bar{S}, X\rangle \;\forall X \in \mathbf{K}, \\ \gamma\|S\| & \leq & \langle \bar{X}, S\rangle \;\forall S \in \mathbf{K}. \end{array}$$

Claim: Whenever $Y = (X, S, \sigma, \tau) \in \mathcal{M} + R$, one has $\|Y\| \leq \alpha\tau + \beta$ with

$$\begin{array}{rcl} \alpha & = & \gamma^{-1}\left[\langle \bar{X}, C\rangle - \langle \bar{S}, B\rangle\right] + 1, \\ \beta & = & \gamma^{-1}\left[\langle \bar{X} + B, D\rangle + \langle \bar{S} - C, P\rangle + d\right]. \end{array}$$

In particular, if $Y_i \in \mathcal{Y}$ and $\|Y_i\| \to \infty$, then $\tau_i \to \infty$.

$$
\begin{array}{lrcl}
(a) & X+\tau B-P & \in & \mathcal{L}; \\
(b) & S-\tau C-D & \in & \mathcal{L}^{\perp}; \\
(c) & \langle C,X\rangle-\langle B,S\rangle+\sigma-d & = & 0; \\
(e) & X & \in & \mathbf{K}; \\
(f) & S & \in & \mathbf{K}_*; \\
(g) & \sigma & \geq & 0; \\
(h) & \tau & \geq & 0.
\end{array}
\qquad \text{(C)}
$$

Proof of Claim: We have $\bar U\equiv \bar X-B\in\mathcal{L}$, $\bar V\equiv\bar S-C\in\mathcal{L}^{\perp}$. Taking into account the constraints (C) defining $\mathcal{Y}$, we get

$$
\begin{array}{rcl}
\langle \bar U, S-\tau C-D\rangle & = & 0\Rightarrow \\
\langle \bar U, S\rangle & = & \langle \bar U,\tau C+D\rangle \Rightarrow \\
\langle \bar X, S\rangle & = & -\langle B,S\rangle+\langle \bar U,\tau C+D\rangle, \\
\langle \bar V, X+\tau B-P\rangle & = & 0\Rightarrow \\
\langle \bar V, X\rangle & = & \langle \bar V,-\tau B+P\rangle \Rightarrow \\
\langle \bar S, X\rangle & = & \langle C,X\rangle+\langle \bar V,-\tau B+P\rangle, \\
\Rightarrow \langle \bar X,S\rangle+\langle \bar S,X\rangle & = & [\langle C,X\rangle-\langle B,S\rangle] \\
 & & +\tau\left[\langle \bar U,C\rangle-\langle \bar V,B\rangle\right] \\
 & & +\left[\langle \bar U,D\rangle+\langle \bar V,P\rangle\right] \\
 & = & d-\sigma+\tau\left[\langle \bar U,C\rangle-\langle \bar V,B\rangle\right] \\
 & & +\left[\langle \bar U,D\rangle+\langle \bar V,P\rangle\right] \\
\Rightarrow \gamma\left[\|X\|+\|S\|\right]+\sigma & \leq & \tau\left[\langle \bar U,C\rangle-\langle \bar V,B\rangle\right] \\
 & & +\left[\langle \bar U,D\rangle+\langle \bar V,P\rangle+d\right]
\end{array}
$$

$$\min_Y \left\{ \langle \widetilde{C}, Y \rangle : Y \in (\mathcal{M} + R) \cap \widetilde{\mathbf{K}} \right\} \qquad \text{(Aux)}$$

♣ Problem (Aux) is strictly primal-dual feasible with known primal-dual *central* solutions $\widehat{Y} = \widetilde{Y}_*(1)$, $\widetilde{C} = \widetilde{S}_*(1)$. The l.-h. s.c.b. $\widetilde{F}$ for $\widetilde{\mathbf{K}}$ has parameter $2(\vartheta+1)$ and known Legendre transformation.

Applying long-step path-following method (where we decrease penalty instead of increasing it), we obtain a sequence of iterates (t_i, Y_i), where $t_i > 0$ and Y_i are strictly feasible solutions to (Aux) which are 0.1-close to the primal central path. In this process, the Newton complexity of building the iterate with $t_i \leq t \in (0, 1)$ does not exceed

$$\mathcal{N}(t) = O(1)\sqrt{\vartheta} \ln \frac{1}{t}.$$

Every iterate (t_i, Y_i) induces a pair

$$\widehat{X}_i = \tau_i^{-1} X_i,\ \widehat{S}_i = \tau_i^{-1} S_i$$

of approximate solutions to (P), (D). These solutions belong to the interiors of the respective cones $\mathbf{K}$, $\mathbf{K}_*$ and are "$O(1/\tau_i)$ primal-dual feasible and primal-dual optimal":

$$\begin{array}{lrcl}
(a) & \widehat{X}_i + B & \in & \mathcal{L} + \tau_i^{-1} P; \\
(b) & \widehat{S}_i - C & \in & \mathcal{L}^\perp + \tau_i^{-1} D; \\
(c) & \underbrace{\langle C, \widehat{X}_i \rangle - \langle B, \widehat{S}_i \rangle}_{\text{DualityGap}(\widehat{X}_i, \widehat{S}_i)} & \leq & \tau_i^{-1} d.
\end{array}$$

Question: *How good are* $\widehat{X}_i$, $\widehat{S}_i$*?*

♢ We do know that t_i are "rapidly" approaching 0 and therefore τ_i go to $+\infty$; we, however, do *not* know at which *rate* $\tau_i \to \infty$.

♢ We do *not* know whether $O(\tau_i^{-1})$-duality gap, even for large τ_i, implies that $\widehat{X}_i$, $\widehat{S}_i$ are nearly optimal for the respective problems – the duality gap bounds non-optimality for *feasible* primal-dual pairs, while $\widehat{X}_i, \widehat{S}_i$ is only nearly feasible.

$$\min_Y \left\{ \langle \widetilde{C}, Y \rangle : Y \in (\mathcal{M} + R) \cap \widetilde{\mathbf{K}} \right\} \qquad \text{(Aux)}$$

Fact: • There exist constant $\Theta > 0$ (depending on the data of (Aux)) such that for every triple (t, Y, U) ($t > 0$, $Y = (X, S, \sigma, \tau)$ is a strictly feasible solution to (Aux), U is a strictly feasible solution to the problem (Aux′) dual to (Aux)) which is 0.1-close to the primal-dual central path of (Aux) − (Aux′), one has

$$\tau \geq \frac{1}{\Theta t} - \Theta.$$

In particular, infeasibility and duality gap evaluated at $(\bar{X}_i, \bar{S}_i)$ admit upper bounds $O(t_i)$, with data-dependent constant factor in $O(\cdot)$.

Besides this, non-optimalities of $\bar{X}_i$, $\bar{S}_i$ for the respective problems do not exceed $O(t_i)$, with data-dependent constant factor in $O(\cdot)$.

♣ Since the number of Newton steps required to get $t_i < t \in (0,1)$ is $\leq O(1)\sqrt{\vartheta}\ln(1/t)$, we conclude that *In order to get ϵ-feasible ϵ-optimal primal and dual solutions, it suffices to make no more than*

$$N(\epsilon) = O(1)\sqrt{\vartheta}\ln\left(\frac{V}{\epsilon}\right)$$

Newton steps, where V is a data-dependent constant.

Simple fact: Let the primal-dual pair of problems

$$\min_X \{\langle C, X\rangle : X \in (\mathcal{L} - B) \cap \mathbf{K}\} \qquad \text{(P)}$$
$$\max_S \left\{\langle B, S\rangle : S \in (\mathcal{L}^\perp + C) \cap \mathbf{K}_*\right\} \qquad \text{(D)}$$
$$\left[\mathcal{L} = \operatorname{Im}\mathcal{A}, \mathcal{L}^\perp = \operatorname{Ker}\mathcal{A}^T, \mathcal{A}^T C = c\right]$$

be strictly primal-dual feasible and be normalized by $\langle C, B\rangle = 0$, let (X_*, S_*) be a primal-dual optimal solution to the pair, and let X, S "ϵ-satisfy" the feasibility and optimality conditions for (P), (D), i.e.,

$$\begin{array}{ll}(a) & X \in \mathbf{K} \cap (\mathcal{L} - B + \Delta X),\ \|\Delta X\|_2 \le \epsilon,\\ (b) & S \in \mathbf{K}_* \cap (\mathcal{L}^\perp + C + \Delta S),\ \|\Delta S\|_2 \le \epsilon,\\ (c) & \langle C, X\rangle - \langle B, S\rangle \le \epsilon.\end{array}$$

Then

$$\begin{array}{rcl}\langle C, X\rangle - \mathrm{Opt(P)} & \le & \epsilon(1 + \|X_* + B\|_2),\\ \mathrm{Opt(D)} - \langle B, S\rangle & \le & \epsilon(1 + \|S_* - C\|_2).\end{array}$$

$$\begin{array}{ll}(a) & X \in \mathbf{K} \cap (\mathcal{L} - B + \Delta X),\ \|\Delta X\|_2 \leq \epsilon,\\ (b) & S \in \mathbf{K}_* \cap (\mathcal{L}^\perp + C + \Delta S),\ \|\Delta S\|_2 \leq \epsilon,\\ (c) & \langle C, X\rangle - \langle B, S\rangle \leq \epsilon.\end{array}$$

We have $S - C - \Delta S \in \mathcal{L}^\perp$, $X_* + B \in \mathcal{L}$, whence

$$\begin{array}{rcl} 0 & = & \langle S - C - \Delta S, X_* + B\rangle \\ & = & \underbrace{\langle S, X_*\rangle}_{\geq 0} - \mathrm{Opt(P)} + \langle S, B\rangle \\ & & + \langle -\Delta S, X_* + B\rangle \\ & & \Rightarrow \\ -\mathrm{Opt(P)} & \leq & -\langle S, B\rangle + \langle \Delta S, X_* + B\rangle \\ & \leq & -\langle S, B\rangle + \epsilon\|X_* + B\|_2. \end{array}$$

Adding (c), we get

$$\langle C, X\rangle - \mathrm{Opt}(P) \leq \epsilon + \epsilon\|X_* + B\|_2.$$

The inequality

$$\mathrm{Opt}(D) - \langle B, S\rangle \leq \epsilon + \epsilon\|S_* - C\|_2$$

is given by “symmetric” reasoning.

Structural Design and Semidefinite Duality

♣ Structural design is an engineering area which has to do with mechanical constructions like *trusses*, comprised of thin elastic bars, and *shapes*, where the material is continuously distributed over 2D/3D domain.
♠ A typical SD problem is "given the type of material, a *resource* (an upper bound on the amount of material to be used) and a set of *loading scenarios* – external loads operating on the construction, find an optimal truss/shape – the one capable to withstand best of all the loads in question".

♣ Building the model.

I. A (linearly elastic) construction C is characterized by

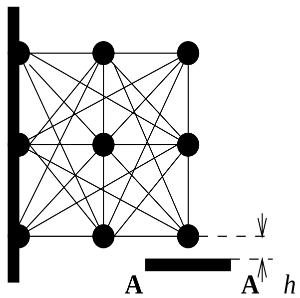
I.1) A linear space $\mathbf{V} = \mathbf{R}^m$ of *virtual displacements* of C;

I.2) A positive semidefinite quadratic form

$$E_C(v) = \frac{1}{2} v^T A_C v$$

on $\mathbf{V}$. $E_C(v)$ is the potential energy stored by the construction as a result of displacement v. The $m \times m$ matrix $A_C \succeq 0$ of this form is called the *stiffness matrix* of C;

I.3) A closed convex subset $\mathcal{V} \subset \mathbf{R}^m$ of *kinematically admissible displacements*.



II. An external load applied to the construction C is represented by a vector $f \in \mathbf{R}^m$; the static equilibrium of C loaded by f is given by the following

Variational Principle: *A construction C is capable to carry an external load f if and only if the quadratic form*

$$E_C^f(v) = \frac{1}{2} v^T A_C v - f^T v$$

of displacements v attains its minimum on the set $\mathcal{V}$ of kinematically admissible displacements, and a displacement yielding (static) equilibrium is a minimizer of $E_C^f(\cdot)$ on $\mathcal{V}$.

♠ The minus minimum value of E_C^f on $\mathcal{V}$ is called the *compliance* of the construction C with respect to the load f:

$$\mathrm{Compl}_f(C) = \sup_{v \in \mathcal{V}} \left[f^T v - \frac{1}{2} v^T A_C v \right]$$

III. The stiffness matrix A_C depends on mechanical characteristics $t_1, ..., t_n$ of "elements" $E_1, ..., E_n$ comprising the construction, and t_i are *positive semidefinite symmetric* $d \times d$ *matrices*, with d given by the type of the construction. Specifically,

$$A_C = \sum_{i=1}^{n} \sum_{s=1}^{S} b_{is} t_i b_{is}^T,$$

where b_{is} are given $m \times d$ matrices.

♢ For trusses, $d = 1$,
♢ For planar shapes, $d = 3$,
♢ For spatial shapes, $d = 6$.

IV. The amount of material "consumed" by a construction C is completely characterized by the vector $(\text{Tr}(t_1), ..., \text{Tr}(t_n))$ of the traces of the matrices $t_1, ..., t_n$.

♣ Static Structural Design problem: Given

♢ "ground structure", i.e.,

• the space $\mathbf{R}^m$ of virtual displacements along with its closed convex subset $\mathcal{V}$ of kinematically admissible displacements,

• a collection $\{b_{is}\}_{\substack{i=1,...,n\\ s=1,...,S}}$ of $m \times d$ matrices,

♢ A set $T = \{(t_1, ..., t_n) \mid t_i \in \mathbf{S}^d_+ \quad \forall i\}$ of "admissible designs",

♢ A set $\mathcal{F} \subset \mathbf{R}^m$ of "loading scenarios",

find an admissible construction which is stiffest, with respect to $\mathcal{F}$, i.e., solve

$$\min_{t\in T} \underbrace{\sup_{f\in\mathcal{F}} \sup_{v\in\mathcal{V}} \left[f^T v - \frac{1}{2} v^T \left[\sum_{i=1}^{n} \sum_{s=1}^{S} b_{is} t_i b_{is}^T \right] v \right]}_{\mathrm{Compl}_{\mathcal{F}}(t)}$$

♣ Standard case:

♢ The set $\mathcal{V}$ of kinematically admissible displacements is polyhedral:

$$\mathcal{V} = \{v \in \mathbf{R}^m \mid Rv \leq r\} \qquad [R \in \mathbf{M}^{q,m}]$$

contains the origin, and the system of linear inequalities $Rv \leq r$ satisfies the Slater condition;

♢ The set T of admissible designs is given by

$$T = \left\{ t = (t_1, ..., t_n) : \begin{array}{l} t_i \in \mathbf{S}^d_+ \\ \underline{\rho}_i \leq \mathrm{Tr}(t_i) \leq \overline{\rho}_i \\ \sum\limits_{i=1}^{n} \mathrm{Tr}(t_i) \leq w \end{array} \right\}$$

and possesses a nonempty interior

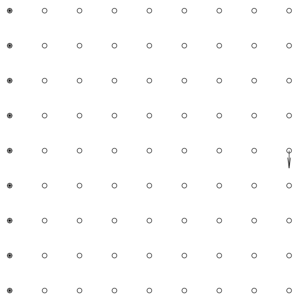
♢ The set $\mathcal{F}$ of loading scenarios is either a finite set:

$$\mathcal{F} = \{f_1, ..., f_k\},$$

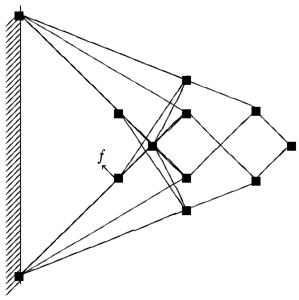
(*multi-load design*), or an ellipsoid:

$$\mathcal{F} = \{f = Qu \mid u^T u \leq 1\} \qquad [Q \in \mathbf{M}^{m,k}]$$

(*robust design*).



9×9 ground structure and load of interest f_*



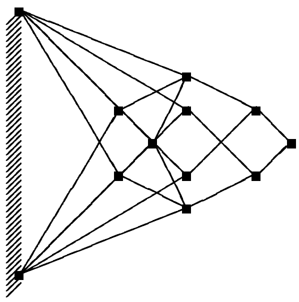
Optimal cantilever; the compliance is 1.000

♠ *Compliance of the nominal design w.r.t. a small load f (just 0.5% of the load of interest!) is 8.4!*

♠ In order to make the design more stable, one can use a two-stage scheme:

♢ At the first stage, we take into consideration only the loads of interest only and solve the corresponding single/multi-load problem, thus getting certain preliminary truss.

♢ At the second stage, we treat the set of nodes actually used by the preliminary truss as our new nodal set, and take as $\mathcal{F}$ the ellipsoidal envelope of the loads of interest and the ball comprised of all "small" occasional loads distributed along this reduced nodal set.



"Robust" cantilever

The maximum, over the ellipsoid of loads $\mathcal{F}$, compliance of the "robust" cantilever is 1.03, and its compliance with respect to the load of interest f^* is 1.0024 – only by 0.24% larger than the optimal compliance given by the single-load design!

♣ In what follows, we focus on the Standard SSD problem and assume that

$$t_i \succ 0,\, i \leq n \Rightarrow \sum_{i=1}^{n} \sum_{s=1}^{S} b_{is} t_i b_{is}^T \succ 0$$

Standard SSD problem as SDP

Proposition: Let $t = (t_1, ..., t_n) \in \left(\mathbf{S}^d_+\right)^n$ and $f \in \mathbf{R}^m$. Then

$$\mathrm{Compl}_f(t) \leq \tau$$

iff $\exists \mu \geq 0$:

$$\mathcal{A}(t, f, \tau, \mu) = \begin{bmatrix} 2\tau - 2r^T\mu & -f^T + \mu^T R \\ -f + R^T\mu & \sum\limits_{i=1}^{n} \sum\limits_{s=1}^{S} b_{is} t_i b_{is}^T \end{bmatrix} \succeq 0.$$

Proof. We have

$$\mathrm{Compl}_f(t) \leq \tau$$

$$\begin{array}{l}
\Leftrightarrow \sup\limits_{v: Rv \leq r} \left[f^T v - \frac{1}{2} v^T \left[\sum\limits_{i=1}^{n} \sum\limits_{s=1}^{S} b_{is} t_i b_{is}^T \right] v \right] \leq \tau \\
\Leftrightarrow \exists \mu \geq 0 : \\
\sup\limits_{v} \left[f^T v - \frac{1}{2} v^T \left[\sum\limits_{i=1}^{n} \sum\limits_{s=1}^{S} b_{is} t_i b_{is}^T \right] v + \mu^T [r - Rv] \right] \leq \tau \\
\Leftrightarrow \exists \mu \geq 0 : \\
\qquad \begin{bmatrix} 2\tau - 2r^T\mu & -f^T + \mu^T R \\ -f + R^T\mu & \sum\limits_{i=1}^{n} \sum\limits_{s=1}^{S} b_{is} t_i b_{is}^T \end{bmatrix} \succeq 0
\end{array}$$

♣ Corollary: A multi-load SSD program can be posed as the SDP

$$
\begin{array}{c}
\text{minimize } \tau \\
\text{s.t.} \\
\begin{bmatrix} 2\tau - 2r^T\mu^\ell & -f_\ell^T + [\mu^\ell]^T R \\ -f_\ell + R^T\mu^\ell & \sum\limits_{i=1}^{n}\sum\limits_{s=1}^{S} b_{is}t_i b_{is}^T \end{bmatrix} \succeq 0,\ \ell = 1,...,k; \\
t_i \succeq 0,\ i = 1,...,n; \\
\sum\limits_{i=1}^{n} \mathrm{Tr}(t_i) \le w; \\
\underline{\rho}_i \le \mathrm{Tr}(t_i) \le \overline{\rho}_i,\ i = 1,...,n; \\
\mu^\ell \ge 0,\ \ell = 1,...,k,
\end{array}
$$

where the design variables are $t_i \in \mathbf{S}^d$, μ^ℓ, and $\tau \in \mathbf{R}$.

♣ *Robust* Standard SSD problem.

Proposition: Let the set of kinematically admissible displacements coincide with the space $\mathbf{R}^m$ of all virtual displacements: $\mathcal{V} = \mathbf{R}^m$, and let $\mathcal{F}$ be an ellipsoid:

$$\mathcal{F} = \{f = Qu \mid u^T u \le 1\} \qquad [Q \in \mathbf{M}^{m,k}]$$

Then for $t = (t_1, ..., t_n) \in (\mathbf{S}^d_+)^n$ one has

$$\mathrm{Compl}_{\mathcal{F}}(t) \le \tau$$

iff

$$\begin{bmatrix} 2\tau I_k & Q^T \\ Q & \sum_{i=1}^n \sum_{s=1}^S b_{is} t_i b_{is}^T \end{bmatrix} \succeq 0.$$

Consequently, the robust obstacle-free Standard SSD problem can be posed as the SDP

$$\begin{array}{c} \text{minimize } \tau \\ \text{s.t.} \\ \begin{bmatrix} 2\tau I_k & Q^T \\ Q & \sum_{i=1}^n \sum_{s=1}^S b_{is} t_i b_{is}^T \end{bmatrix} \succeq 0 \\ t_i \succeq 0,\ i = 1, ..., n \\ \sum_{i=1}^n \mathrm{Tr}(t_i) \le w \\ \underline{\rho}_i \le \mathrm{Tr}(t_i) \le \overline{\rho}_i,\ i = 1, ..., n. \end{array}$$

Proof. Let $A(t)=\sum\limits_{i=1}^{n}\sum\limits_{s=1}^{S}b_{is}t_ib_{is}^T$. We have

$$
\begin{array}{l}
\qquad\qquad\qquad \mathrm{Compl}_{\mathcal{F}}(t)\le\tau\\
\Leftrightarrow \forall v\forall(u:u^Tu\le 1):\\
\qquad\qquad (Qu)^Tv-\frac{1}{2}v^TA(t)v\le\tau\\
\Leftrightarrow \forall v\forall(u:u^Tu=1):\\
\qquad\qquad (Qu)^Tv-\frac{1}{2}v^TA(t)v\le\tau\\
\Leftrightarrow \forall v\forall(w\neq 0):\\
\qquad\quad (Q\|w\|_2^{-1}w)^Tv-\frac{1}{2}v^TA(t)v\le\tau\\
\Leftrightarrow \forall v\forall(w\neq 0):\\
(Qw)^T\underbrace{(\|w\|_2v)}_{-y}-\frac{1}{2}(\|w\|_2v)^TA(t)(\|w\|_2v)\le\tau w^Tw\\
\Leftrightarrow \forall y\forall(w\neq 0):\\
\qquad 2\tau w^Tw+2w^TQ^Ty+y^TA(t)y\ge 0\\
\Leftrightarrow \forall y\in\mathbf{R}^m, w\in\mathbf{R}^k:\\
\qquad 2\tau w^Tw+2w^TQ^Ty+y^TA(t)y\ge 0\\
\qquad\qquad\Leftrightarrow \left[\begin{array}{cc}2\tau I_k & Q^T\\ Q & A(t)\end{array}\right]\succeq 0.
\end{array}
$$

♣ <u>"Universal" form of the Standard SSD.</u> Both the multi-load Standard SSD problem and the robust obstacle-free problem are particular cases of the following generic semidefinite program:

$$\min_{\tau, t_i, z} \left\{ \tau : \begin{array}{c} \left[\begin{array}{cc} 2\tau I_p + \mathcal{D}_\ell z + D_\ell & [\mathcal{E}_\ell z + E_\ell]^T \\ [\mathcal{E}_\ell z + E_\ell] & \sum\limits_{i=1}^n \sum\limits_{s=1}^S b_{is} t_i b_{is}^T \end{array} \right] \succeq 0, \\ \ell = 1, ..., K \\ t_i \succeq 0, \; i = 1, ..., n \\ \sum\limits_{i=1}^n \mathrm{Tr}(t_i) \le w \\ \underline{\rho}_i \le \mathrm{Tr}(t_i) \le \overline{\rho}_i, \; i = 1, ..., n \\ z \ge 0 \end{array} \right\}$$

♢ The multi-load problem corresponds to

$$\begin{array}{l} z = (\mu^1, ..., \mu^k) \in \mathbf{R}^q \times ... \times \mathbf{R}^q \\ \mathcal{D}_\ell z + D_\ell = -2r^T \mu^\ell \\ \mathcal{E}_\ell z + E_\ell = -f_\ell + R^T \mu^\ell \end{array}$$

♢ The robust problem corresponds to the case where z-variable in empty.

<u>Note:</u> *For every* $\ell = 1, ..., K$, *there exists* $\alpha_\ell \succ 0$ *and* $V_\ell \in \mathbf{M}^{m,p}$ *such that*

$$\sum_{\ell=1}^K [\mathcal{D}_\ell^* \alpha_\ell + 2\mathcal{E}_\ell^* V_\ell] < 0.$$

$$
\min_{\tau, t_i, z} \left\{ \tau : \begin{array}{c} \left[\begin{array}{cc} 2\tau I_p + \mathcal{D}_\ell z + D_\ell & [\mathcal{E}_\ell z + E_\ell]^T \\ [\mathcal{E}_\ell z + E_\ell] & \sum\limits_{i=1}^{n} \sum\limits_{s=1}^{S} b_{is} t_i b_{is}^T \end{array} \right] \succeq 0, \\ \ell = 1, ..., K \\ t_i \succeq 0, \ i = 1, ..., n \\ \sum\limits_{i=1}^{n} \mathrm{Tr}(t_i) \leq w \\ \underline{\rho}_i \leq \mathrm{Tr}(t_i) \leq \overline{\rho}_i, \ i = 1, ..., n \\ z \geq 0 \end{array} \right\}
$$

Note: The primal problem is strictly feasible and bounded.

♣ Passing to the dual.

$$\min_{\tau,t_i,z}\left\{\tau:\begin{array}{c}\left[\begin{array}{cc}2\tau I_p+\mathcal{D}_\ell z+D_\ell & [\mathcal{E}_\ell z+E_\ell]^T\\ [\mathcal{E}_\ell z+E_\ell] & \sum\limits_{i=1}^n\sum\limits_{s=1}^S b_{is}t_ib_{is}^T\end{array}\right]\succeq 0,\\ \Rightarrow \mathrm{Tr}\Bigg(\underbrace{\left[\begin{array}{cc}\alpha_\ell & V_\ell^T\\ V_\ell & \beta_\ell\end{array}\right]}_{\succeq 0}\times\ldots\Bigg)\geq 0\\ t_i\succeq 0\Rightarrow \mathrm{Tr}(\underbrace{\tau_i}_{\succeq 0}\times\ldots)\geq 0\\ -\sum\limits_{i=1}^n\mathrm{Tr}(t_i)\geq -w\Rightarrow \underbrace{\gamma}_{\geq 0}\times\geq -w\gamma\\ \mathrm{Tr}(t_i)\geq \underline{\rho}_i\Rightarrow \underbrace{\sigma_i^-}_{\geq 0}\times\ldots\geq \underline{\rho}_i\sigma_i^-\\ -\mathrm{Tr}(t_i)\geq -\overline{\rho}_i\Rightarrow \underbrace{\sigma_i^+}_{\geq 0}\times\ldots\geq -\overline{\rho}_i\sigma_i^+\\ z\geq 0\Rightarrow \underbrace{\eta^T}_{\geq 0}\times\ldots\geq 0\end{array}\right\}$$

$$\sum_\ell\Big\{\mathrm{Tr}\left([2\tau I_p+\mathcal{D}_\ell z+D_\ell]\alpha_\ell\right)+2\mathrm{Tr}\left([\mathcal{E}_\ell z+E_\ell]^TV_\ell\right)$$
$$+\mathrm{Tr}\left(\left(\sum_{i=1}^n\sum_{s=1}^S b_{is}t_ib_{is}^T\right)\beta_\ell\right)\Big\}$$
$$+\sum_i\mathrm{Tr}\left([\tau_i+(\sigma_i^--\sigma_i^+-\gamma)I_d]t_i\right)+z^T\eta$$
$$\geq\sum_i(\sigma_i^-\underline{\rho}_i-\sigma_i^+\overline{\rho}_i)-\gamma w$$

$$\sum_\ell \Big\{ \mathrm{Tr}\left([2\tau I_p + \mathcal{D}_\ell z + D_\ell]\alpha_\ell\right) + 2\mathrm{Tr}\left([\mathcal{E}_\ell z + E_\ell]^T V_\ell\right)$$
$$+\mathrm{Tr}\left(\left(\sum_{i=1}^n \sum_{s=1}^S b_{is} t_i b_{is}^T\right)\beta_\ell\right)\Big\}$$
$$+\sum_i \mathrm{Tr}\left([\tau_i + (\sigma_i^- - \sigma_i^+ - \gamma) I_d] t_i\right) + z^T\eta$$
$$\geq \sum_i (\sigma_i^- \underline{\rho}_i - \sigma_i^+ \overline{\rho}_i) - \gamma w$$

To the dual problem, we impose on the weights $\alpha_\ell, \ldots, \eta$, along with positive semidefiniteness/nonnegativity constraints, the requirement to produce, as the "variable part" of $(*)$, the primal objective τ:

$$2\sum_\ell \mathrm{Tr}(\alpha_\ell) = 1$$
$$\sum_{s,\ell} b_{is}^T \beta_\ell b_{is} + \tau_i + (\sigma_i^- - \sigma_i^+) I_d = 0\,\forall i$$
$$\sum_\ell \left[\mathcal{D}_\ell^* \alpha_\ell + 2\mathcal{E}_\ell^* V_\ell\right] + \eta = 0$$

Under these constraints, $(*)$ produces a lower bound on the primal objective in the primal feasible domain:

$$\tau \geq -\sum_\ell \left[\mathrm{Tr}(D_\ell \alpha_\ell) + 2\mathrm{Tr}(E_\ell V_\ell^T)\right]$$
$$-\sum_i (\overline{\rho}_i \sigma_i^+ - \underline{\rho}_i \sigma_i^-) - \gamma w$$

♣ We arrive at the dual problem

$$\begin{array}{rcl}\text{maximize} \ -\phi & \equiv & -\sum_\ell \left[\mathrm{Tr}(D_\ell\alpha_\ell) + 2\mathrm{Tr}(E_\ell^T V_\ell)\right] \\ & & -\sum_i [\overline{\rho}_i\sigma_i^+ - \underline{\rho}_i\sigma_i^-] - w\gamma \end{array}$$

s.t.

$$\begin{array}{c} 2\sum_\ell \mathrm{Tr}(\alpha_\ell) = 1, \sum_\ell [\mathcal{D}_\ell^*\alpha_\ell + 2\mathcal{E}_\ell^* V_\ell] + \eta = 0 \\ \sum_{\ell,s} b_{is}^T\beta_\ell b_{is} + \tau_i + [\sigma_i^- - \sigma_i^+ - \gamma]I_d = 0 \\ \left[\begin{array}{cc}\alpha_\ell & V_\ell^T \\ V_\ell & \beta_\ell\end{array}\right] \succeq 0, \tau_i \succeq 0, \sigma_i^+, \sigma_i^- \geq 0, \gamma \geq 0, \eta \geq 0 \end{array}$$

with design variables $\{\alpha_\ell, \beta_\ell, V_\ell\}_{\ell=1}^K$, $\{\sigma_i^+, \sigma_i^-, \tau_i\}_{i=1}^n$, γ, η.

♠ We can immediately eliminate the τ_i and the η-variables, arriving at

$$\begin{array}{rcl}\text{minimize} \ \phi & \equiv & \sum_\ell \left[\mathrm{Tr}(D_\ell\alpha_\ell) + 2\mathrm{Tr}(E_\ell^T V_\ell)\right] \\ & & +\sum_i [\overline{\rho}_i\sigma_i^+ - \underline{\rho}_i\sigma_i^-] + w\gamma \end{array}$$

s.t.

$$\begin{array}{c} 2\sum_\ell \mathrm{Tr}(\alpha_\ell) = 1, \sum_\ell [\mathcal{D}_\ell^*\alpha_\ell + 2\mathcal{E}_\ell^* V_\ell] \leq 0 \\ \sum_{\ell,s} b_{is}^T\beta_\ell b_{is} \preceq [\gamma + \sigma_i^+ - \sigma_i^-]I_d \\ \left[\begin{array}{cc}\alpha_\ell & V_\ell^T \\ V_\ell & \beta_\ell\end{array}\right] \succeq 0, \sigma_i^+, \sigma_i^- \geq 0, \gamma \geq 0 \end{array}$$

$$
\begin{array}{rcl}
\text{minimize } \phi & \equiv & \sum\limits_{\ell} \left[\mathrm{Tr}(D_\ell \alpha_\ell) + 2\mathrm{Tr}(E_\ell^T V_\ell) \right] \\
& & + \sum\limits_{i} [\overline{\rho}_i \sigma_i^+ - \underline{\rho}_i \sigma_i^-] + w\gamma
\end{array}
$$

s.t.

$$
\begin{array}{c}
2\sum\limits_{\ell} \mathrm{Tr}(\alpha_\ell) = 1, \sum\limits_{\ell} [\mathcal{D}_\ell^* \alpha_\ell + 2\mathcal{E}_\ell^* V_\ell] \le 0 \\
\sum\limits_{\ell,s} b_{is}^T \beta_\ell b_{is} \preceq [\gamma + \sigma_i^+ - \sigma_i^-] I_d \\
\left[\begin{array}{cc} \alpha_\ell & V_\ell^T \\ V_\ell & \beta_\ell \end{array} \right] \succeq 0, \sigma_i^+, \sigma_i^- \ge 0, \gamma \ge 0
\end{array}
$$

♠ We have seen that the constraints

$$
\sum_{\ell} [\mathcal{D}_\ell^* \alpha_\ell + 2\mathcal{E}_\ell^* V_\ell] \le 0, \ , \alpha_\ell \succeq 0
$$

are strictly feasible. It immediately follows that the dual problem is strictly feasible, and therefore its optimal value remains unchanged when the (implicit) constraints $\alpha_\ell \succeq 0$ are strengthen to $\alpha_\ell \succ 0$. We now can eliminate the β_ℓ-variables: their optimal values are

$$
\beta_\ell = V_\ell \alpha_\ell^{-1} V_\ell^T.
$$

$$\begin{aligned}\text{minimize } \phi \quad \equiv \quad & \sum_\ell \left[\text{Tr}(D_\ell\alpha_\ell) + 2\text{Tr}(E_\ell^T V_\ell)\right] \\ & + \sum_i [\overline{\rho}_i\sigma_i^+ - \underline{\rho}_i\sigma_i^-] + w\gamma\end{aligned}$$

s.t.

$$2\sum_\ell \text{Tr}(\alpha_\ell) = 1, \sum_\ell [\mathcal{D}_\ell^*\alpha_\ell + 2\mathcal{E}_\ell^* V_\ell] \leq 0$$

$$\sum_{\ell,s} b_{is}^T \beta_\ell b_{is} \preceq [\gamma + \sigma_i^+ - \sigma_i^-] I_d$$

$$\alpha_\ell \succ 0, \beta_\ell = V_\ell \alpha_\ell^{-1} V_\ell^T, \sigma_i^+, \sigma_i^- \geq 0, \gamma \geq 0$$

Eliminating β_ℓ-variables, we arrive at the problem

$$\begin{aligned}\text{minimize } \phi \quad \equiv \quad & \sum_\ell \left[\text{Tr}(D_\ell\alpha_\ell) + 2\text{Tr}(E_\ell^T V_\ell)\right] \\ & + \sum_i [\overline{\rho}_i\sigma_i^+ - \underline{\rho}_i\sigma_i^-] + w\gamma\end{aligned}$$

s.t.

$$2\sum_\ell \text{Tr}(\alpha_\ell) = 1, \sum_\ell [\mathcal{D}_\ell^*\alpha_\ell + 2\mathcal{E}_\ell^* V_\ell] \leq 0$$

$$\sum_{\ell,s} b_{is}^T V_\ell \alpha_\ell^{-1} V_\ell^T b_{is} \preceq [\gamma + \sigma_i^+ - \sigma_i^-] I_d$$

$$\alpha_\ell \succ 0, \sigma_i^+, \sigma_i^- \geq 0, \gamma \geq 0$$

By Lemma on the Schur Complement, with $\alpha_\ell \succ 0$ relation

$$\sum_{\ell,s} b_{is}^T V_\ell \alpha_\ell^{-1} V_\ell^T b_{is} \preceq \underbrace{[\gamma + \sigma_i^+ - \sigma_i^-]}_{\gamma_i} I_d$$

is equivalent to

$$\left[\begin{array}{ccc|c|ccc|c}
\alpha_1 & & & & & & & V_1^T b_{i1} \\
& \ddots & & & & & & \vdots \\
& & \alpha_1 & & & & & V_1^T b_{iS} \\
\hline
& & & \ddots & & & & \vdots \\
\hline
& & & & \alpha_K & & & V_K^T b_{i1} \\
& & & & & \ddots & & \vdots \\
& & & & & & \alpha_K & V_K^T b_{iS}^T \\
\hline
b_{i1}^T V_1 & \cdots & b_{iS}^T V_1 & \cdots & b_{i1}^T V_K & \cdots & b_{iS}^T V_K & \gamma_i I_d
\end{array}\right] \succeq 0$$

We arrive at the final formulation of the dual problem:

$$\begin{aligned}\text{minimize } \phi \;\; \equiv \;\; & \sum_{\ell} \left[\mathrm{Tr}(D_\ell \alpha_\ell) + 2\mathrm{Tr}(E_\ell^T V_\ell) \right] \\ & + \sum_i [\overline{\rho}_i \sigma_i^+ - \underline{\rho}_i \sigma_i^-] + w\gamma \end{aligned}$$

s.t.

$$2\sum_{\ell} \mathrm{Tr}(\alpha_\ell) = 1, \sum_{\ell} [\mathcal{D}_\ell^* \alpha_\ell + 2\mathcal{E}_\ell^* V_\ell] \leq 0$$

$$\left[\begin{array}{ccc|c|ccc|c} \alpha_1 & & & & & & & V_1^T b_{i1} \\ & \ddots & & & & & & \vdots \\ & & \alpha_1 & & & & & V_1^T b_{iS} \\ \hline & & & \ddots & & & & \vdots \\ \hline & & & & \alpha_K & & & V_K^T b_{i1} \\ & & & & & \ddots & & \vdots \\ & & & & & & \alpha_K & V_K^T b_{iS}^T \\ \hline b_{i1}^T V_1 & \cdots & b_{iS}^T V_1 & \cdots & b_{i1}^T V_K & \cdots & b_{iS}^T V_K & \gamma_i I_d \end{array}\right] \succeq 0$$

$$\gamma_i = \gamma + \sigma_i^+ - \sigma_i^-, \sigma_i^+, \sigma_i^- \geq 0, \gamma \geq 0$$

Note: In the case of "simple bounds" $\underline{\rho}_i = 0$, $\overline{\rho}_i > w$, it is easy to see that at optimum $\sigma_i^{\pm} = 0$, which allows for further simplification of the dual problem to

$$\text{minimize } \phi \;\equiv\; \sum_{\ell} \left[\mathrm{Tr}(D_\ell \alpha_\ell) + 2\mathrm{Tr}(E_\ell^T V_\ell) \right] + w\gamma$$

s.t.

$$2\sum_{\ell} \mathrm{Tr}(\alpha_\ell) = 1, \sum_{\ell} [\mathcal{D}_\ell^* \alpha_\ell + 2\mathcal{E}_\ell^* V_\ell] \leq 0$$

$$\left[\begin{array}{ccc|c|ccc|c}
\alpha_1 & & & & & & & V_1^T b_{i1} \\
& \ddots & & & & & & \vdots \\
& & \alpha_1 & & & & & V_1^T b_{iS} \\ \hline
& & & \ddots & & & & \vdots \\ \hline
& & & & \alpha_K & & & V_K^T b_{i1} \\
& & & & & \ddots & & \vdots \\
& & & & & & \alpha_K & V_K^T b_{iS}^T \\ \hline
b_{i1}^T V_1 & \cdots & b_{iS}^T V_1 & \cdots & b_{i1}^T V_K & \cdots & b_{iS}^T V_K & \gamma I_d
\end{array}\right] \succeq 0$$

Advantages of the dual: Consider the case of multi-load obstacle-free truss design with simple bounds. In this case, the sizes of the primal and the dual problems are as follows (M is the number of nodes in the ground structure):

Problem	Design dim.	# and sizes of LMIs	# of linear constr.
Primal	$\approx 0.5M^2$	k of $2M \times 2M$ LMIs	$\approx 0.5M^2$
Dual	$2Mk$	$\approx 0.5M^2$ of $(k+1) \times (k+1)$ LMIs	$k+1$

Typical range of values of M is thousands, while k is within 10. With these sizes, the dual problem is much better suited for numerical solution than the primal one...