Appendix B

Solutions to Exercises

B.1 Exercises from Lecture 1

Exercise 1.1. We should prove that x is robust feasible if and only if it can be extended, by properly chosen $u, v \geq 0$ such that $u - v = x$, to a feasible solution to (1.6.1). First, let x be robust feasible, and let $u_i = \max[x_i, 0]$, $v_i = \max[-x_i, 0]$. Then $u, v \geq 0$ and $u - v = x$. Besides this, since x is robust feasible and uncertainty is element-wise, we have for every i

$$\sum_j \max_{\underline{A}_{ij} \leq a_{ij} \leq \overline{A}_{ij}} a_{ij} x_j \leq \underline{b}_i. \qquad (*)$$

With our u, v we clearly have $\max_{\underline{A}_{ij} \leq a_{ij} \leq \overline{A}_{ij}} a_{ij} x_j = [\overline{A}_{ij} u_j - \underline{A}_{ij} v_j]$, so that by $(*)$ we have $\overline{A}u - \underline{A}v \leq \underline{b}$, so that (x, u, v) is a feasible solution of (1.6.1).

Vice versa, let (x, u, v) be feasible for (1.6.1), and let us prove that x is robust feasible for the original uncertain problem, that is, that the relations $(*)$ take place. This is immediate, since from $u, v \geq 0$ and $x = u - v$ it clearly follows that $\max_{\underline{A}_{ij} \leq a_{ij} \leq \overline{A}_{ij}} a_{ij} x_j \leq \overline{A}_{ij} u_j - \underline{A}_{ij} v_j$, so that the validity of $(*)$ is ensured by the constraints of (1.6.1). The respective RCs are (equivalent to)

$$\begin{array}{ll}
[a^{\rm n}; b^{\rm n}]^T [x; -1] + \rho \|P^T [x; -1]\|_q \leq 0, \; q = \frac{p}{p-1} & (a) \\
[a^{\rm n}; b^{\rm n}]^T [x; -1] + \rho \|(P^T [x; -1])_+\|_q \leq 0, \; q = \frac{p}{p-1} & (b) \\
[a^{\rm n}; b^{\rm n}]^T [x; -1] + \rho \|P^T [x; -1]\|_\infty \leq 0 & (c)
\end{array}$$

where for a vector $u = [u_1; ...; u_k]$ the vector $(u)_+$ has the coordinates $\max[u_i, 0]$, $i = 1, ..., k$.
Comment to (c): The uncertainty set in question is nonconvex; since the RC remains intact when a given uncertainty set is replaced with its convex hull, we can replace the restriction $\|\zeta\|_p \leq \rho$ in (c) with the restriction $\zeta \in \text{Conv}\{\zeta : \|\zeta\|_p \leq \rho\} = \{\|\zeta\|_1 \leq \rho\}$, where the concluding equality is due to the following reasons: on one hand, with $p \in (0, 1)$ we have

$$\begin{array}{l}
\|\zeta\|_p \leq \rho \Leftrightarrow \sum_i (|\zeta_i|/\rho)^p \leq 1 \Rightarrow |\zeta_i|/\rho \leq 1 \forall i \Rightarrow |\zeta_i|/\rho \leq (|\zeta_i|/\rho)^p \\
\Rightarrow \sum_i |\zeta_i|/\rho \leq \sum_i (|\zeta_i|/\rho)^p \leq 1,
\end{array}$$

whence $\text{Conv}\{\|\zeta\|_p \leq \rho\} \subset \{\|\zeta\|_1 \leq \rho\}$. To prove the inverse inclusion, note that all extreme points of the latter set (that is, vectors with all but one coordinates equal to 0 and the remaining coordinate equal $\pm\rho$) satisfy $\|\zeta\|_p \leq 1$.

Exercise 1.3: The RC can be represented by the system of conic quadratic constraints

$$\begin{array}{l}[a^{\rm n};b^{\rm n}]^T[x;-1]+\rho\sum_j\|u_j\|_2\leq 0\\ \sum_j Q_j^{1/2}u_j=P^T[x;-1]\end{array}$$

in variables $x, \{u_j\}_{j=1}^J$.

Exercise 1.4: • The RC of i-th problem is

$$\min_x\left\{-x_1-x_2 : 0\leq x_1\leq \min_{b\in\mathcal{U}_i} b_1, 0\leq x_2\leq \min_{b\in\mathcal{U}_i} b_2, x_1+x_2\geq p\right\}$$

We see that both RC's are identical to each other and form the program

$$\min_x\{-x_1-x_2 : 0\leq x_1,x_2\leq 1/3, x_1+x_2\geq p\}$$

• When $p=3/4$, all instances of $\mathcal{P}_1$ are feasible (one can set $x_1=b_1, x_2=b_2$), while the RC of $\mathcal{P}_2$ is not so, so that there is a gap. In contrast to this, $\mathcal{P}_2$ has infeasible instances, and its RC is infeasible; in this case, there is no gap.
• When $p=2/3$, the (common) RC of the two problems $\mathcal{P}_1$, $\mathcal{P}_2$ is feasible with the unique feasible solution $x_1=x_2=1/3$ and the optimal value $-2/3$. Since every instance of $\mathcal{P}_1$ has a feasible solution $x_1=b_1, x_2=b_2$, the optimal value of the instance is $\leq -b_1-b_2\leq -1$, so that there is a gap. In contrast to this, the RC is an instance of $\mathcal{P}_2$, so that in this case there is no gap.

Exercise 1.5: • Problem $\mathcal{P}_2$ has a constraint-wise uncertainty and is the constraint-wise envelope of $\mathcal{P}_1$.
• Proof for item 2: Let us prove first that if all the instances are feasible, then so is the RC. Assume that the RC is infeasible. Then for every $x\in X$ there exists $i=i_x$ and a realization $[a_{i_x,x}^T, b_{i_x,x}]\in\mathcal{U}_{i_x}$ of the uncertain data of i-th constraint such that

$$a_{i_x,x}^T x' - b_{i_x,x} > 0$$

when $x'=x$ and consequently when x' belongs to a small enough neighborhood U_x of x. Since X is a convex compact set, we can find a finite collection of $x_j\in X$ such that the corresponding neighborhoods U_{x_j} cover the entire X. In other words, we can points out finitely many linear forms

$$f_\ell(x)=a_{i_\ell}^T x - b_{i_\ell},\ \ell=1,...,L,$$

such that $[a_{i_\ell}^T, b_{i_\ell}]\in\mathcal{U}_{i_\ell}$ and the maximum of the forms over $\ell=1,...,L$ is positive at every point $x\in X$. By standard facts on convexity it follows that there exists a convex combination of our forms

$$f(x)=\sum_{\ell=1}^L \lambda_\ell[a_{i_\ell}^T x - b_{i_\ell}]$$

which is positive everywhere on X. Now let $I_i=\{\ell : i_\ell=i\}$ and $\mu_i=\sum_{\ell\in I_i}\lambda_\ell$. For i with $\mu_i>0$, let us set

$$[a_i^T, b_i]=\sum_{\ell\in I_i}\frac{\lambda_\ell}{\mu_i}[a_{i_\ell}, b_{i_\ell}],$$

so that $[a_i^T, b_i] \in \mathcal{U}_i$ (since the latter set is convex). For i with $\mu_i = 0$, let $[a_i^T, b_i]$ be a whatever point of $\mathcal{U}_i$. Observe that by construction

$$f(x) \equiv \sum_{i=1}^{m} \mu_i [a_i^T x - b_i].$$

Now, *since the uncertainty is constraint-wise*, the matrix $[A, b]$ with the rows $[a_i^T, b_i]$ belongs to $\mathcal{U}$ and thus corresponds to an instance of $\mathcal{P}$. For this instance, we have

$$\mu^T[Ax - b] = f(x) > 0 \ \forall x \in X,$$

so that no $x \in X$ can be feasible for the instance; due to the origin of X, this means that the instance we have built is infeasible, same as the RC.

Now let us prove that if all instances are feasible, then the optimal value of the RC is the supremum, let it be called τ, of the optimal values of instances (this supremum clearly is achieved and thus is the maximum of optimal values of instances). Consider the uncertain problem $\mathcal{P}'$ which is obtained from $\mathcal{P}$ by adding to every instance the (certain!) constraint $c^T x \leq \tau$. The resulting problem still is with constraint-wise uncertainty, has feasible instances, and feasible solutions of an instance belong to X. By what we have already proved, the RC of $\mathcal{P}'$ is feasible; but a feasible solution to the latter RC is a robust feasible solution of $\mathcal{P}$ with the value of the objective $\leq \tau$, meaning that the optimal value in the RC of $\mathcal{P}$ is $\leq \tau$. Since the strict inequality here is impossible due to the origin of τ, we conclude that the optimal value of the RC of $\mathcal{P}$ is equal to the maximum of optimal values of instances of $\mathcal{P}$, as claimed.

B.2 Exercises from Lecture 2

Exercise 2.1: W.l.o.g., we may assume $t > 0$. Setting $\phi(s) = \cosh(ts) - [\cosh(t) - 1]s^2$, we get an even function such that $\phi(-1) = \phi(0) = \phi(1) = 1$. We claim that $\phi(s) \leq 1$ when $-1 \leq s \leq 1$.

Indeed, otherwise ϕ attains its maximum on $[-1, 1]$ at a point $\bar{s} \in (0, 1)$, and $\phi''(\bar{s}) \leq 0$. The function $g(s) = \phi'(s)$ is convex on $[0, 1]$ and $g(0) = g(\bar{s}) = 0$. The latter, due to $g'(\bar{s}) \leq 0$, implies that $g(s) = 0$, $0 \leq s \leq \bar{s}$. Thus, ϕ is constant on a nontrivial segment, which is not the case.

For a symmetric P supported on $[-1, 1]$ with $\int s^2 dP(s) \equiv \bar{\nu}^2 \leq \nu^2$ we have, due to $\phi(s) \leq 1$, $-1 \leq s \leq 1$:

$$\begin{array}{l}\int \exp\{ts\} dP(s) = \int_{-1}^{1} \cosh(ts) dP(s) \\ = \int_{-1}^{1} [\cosh(ts) - (\cosh(t) - 1)s^2] dP(s) + (\cosh(t) - 1) \int_{-1}^{1} s^2 dP(s) \\ \leq \int_{-1}^{1} dP(s) + (\cosh(t) - 1)\bar{\nu}^2 \leq 1 + (\cosh(t) - 1)\nu^2,\end{array}$$

as claimed in example 8. Setting $h(t) = \ln(\nu^2 \cosh(t) + 1 - \nu^2)$, we have $h(0) = h'(0) = 0$, $h''(t) = \frac{\nu^2(\nu^2 + (1-\nu^2)\cosh(t))}{(\nu^2\cosh(t) + 1 - \nu^2)^2}$, $\max_t h''(t) = \left\{\begin{array}{ll} \nu^2, & \nu^2 \geq \frac{1}{3} \\ \frac{1}{4}\left[1 + \frac{\nu^4}{1 - 2\nu^2}\right] \leq \frac{1}{3}, & \nu^2 \leq \frac{1}{3} \end{array}\right.$, whence $\Sigma_{(3)}(\nu) \leq 1$.

Exercise 2.2: Here are the results:

n	ϵ	$t_{\rm tru}$	$t_{\rm Nrm}$	$t_{\rm Bll}$	$t_{\rm BllBx}$	$t_{\rm Bdg}$
16	5.e-2	3.802	3.799	9.791	9.791	9.791
16	5.e-4	7.406	7.599	15.596	15.596	15.596
16	5.e-6	9.642	10.201	19.764	16.000	16.000
256	5.e-2	15.195	15.195	39.164	39.164	39.164
256	5.e-4	30.350	30.396	62.383	62.383	62.383
256	5.e-6	40.672	40.804	79.054	79.054	79.054

n	ϵ	$t_{\rm tru}$	$t_{\rm E.7}$	$t_{\rm E.8}$	$t_{\rm E.9}$	$t_{\rm Unim}$
16	5.e-2	3.802	6.228	5.653	5.653	10.826
16	5.e-4	7.406	9.920	9.004	9.004	12.502
16	5.e-6	9.642	12.570	11.410	11.410	13.705
256	5.e-2	15.195	24.910	22.611	22.611	139.306
256	5.e-4	30.350	39.678	36.017	36.017	146.009
256	5.e-6	40.672	50.282	45.682	45.682	150.821

Exercise 2.3: Here are the results:

n	ϵ	$t_{\rm tru}$	$t_{\rm Nrm}$	$t_{\rm Bll}$	$t_{\rm BllBx}$	$t_{\rm Bdg}$	$t_{\rm E.7}$	$t_{\rm E.8}$
16	5.e-2	4.000	6.579	9.791	9.791	9.791	9.791	9.791
16	5.e-4	10.000	13.162	15.596	15.596	15.596	15.596	15.596
16	5.e-6	14.000	17.669	19.764	16.000	16.000	19.764	19.764
256	5.e-2	24.000	26.318	39.164	39.164	39.164	39.164	39.164
256	5.e-4	50.000	52.649	63.383	62.383	62.383	62.383	62.383
256	5.e-6	68.000	70.674	79.054	79.054	79.054	79.053	79.053

Exercise 2.4: In the case of (a), the optimal value is $t_a = \sqrt{n}{\rm ErfInv}(\epsilon)$, since for a feasible x we have $\xi^n[x] \sim \mathcal{N}(0,n)$. In the case of (b), the optimal value is $t_b = n{\rm ErfInv}(n\epsilon)$. Indeed, the rows in B_n are of the same Euclidean length and are orthogonal to each other, whence the columns are orthogonal to each other as well. Since the first column of B_n is the all-one vector, the conditional on η distribution of $\xi = \sum_j \widehat{\zeta}_j$ has the mass $1/n$ at the point $n\eta$ and the mass $(n-1)/n$ at the origin. It follows that the distribution of ξ is the convex combination of the Gaussian distribution $\mathcal{N}(0,n^2)$ and the unit mass, sitting at the origin, with the weights $1/n$ and $(n-1)/n$, respectively, and the claim follows.

The numerical results are as follows:

n	ϵ	t_a	t_b	t_b/t_a
10	1.e-2	7.357	12.816	1.74
100	1.e-3	30.902	128.155	4.15
1000	1.e-4	117.606	1281.548	10.90

Exercise 2.5: In the notation of section 2.4.2, we have

$$\begin{array}{rcl}\Phi(w) & \equiv & \ln\left(\mathbf{E}\{\exp\{\sum_\ell w_\ell\zeta_\ell\}\}\right) = \sum_\ell \lambda_\ell(\exp\{w_\ell\}-1)\\ & = & \max_u[w^Tu - \phi(u)],\\ \phi(u) & = & \max_w[u^Tw - \Phi(w)] = \left\{\begin{array}{ll}\sum_\ell[u_\ell\ln(u_\ell/\lambda_\ell) - u_\ell + \lambda_\ell], & u \geq 0\\ +\infty, & \text{otherwise.}\end{array}\right.\end{array}$$

Consequently, the Bernstein approximation is

$$\inf_{\beta>0}\left[z_0 + \beta\sum_\ell \lambda_\ell(\exp\{w_\ell/\beta\}-1) + \beta\ln(1/\epsilon)\right] \leq 0,$$

or, in the RC form,

$$z_0 + \max_u \left\{ w^T u : u \in \mathcal{Z}_\epsilon = \{u \geq 0, \sum_\ell [u_\ell \ln(u_\ell/\lambda_\ell) - u_\ell + \lambda_\ell] \leq \ln(1/\epsilon)\} \right\} \leq 0.$$

Exercise 2.6: $w(\epsilon)$ is the optimal value in the chance constrained optimization problem

$$\min_{w_0} \left\{ w_0 : \mathrm{Prob}\{-w_0 + \sum_{\ell=1}^{L} c_\ell \zeta_\ell \leq 0\} \geq 1 - \epsilon \right\},$$

where ζ_ℓ are independent Poisson random variables with parameters λ_ℓ.

When all c_ℓ are integral in certain scale, the random variable $\zeta^L = \sum_{\ell=1}^{L} c_\ell \zeta_\ell$ is also integral in the same scale, and we can compute its distribution recursively in L:

$$p_0(i) = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0 \end{cases}, p_k(i) = \sum_{j=0}^{\infty} p_{k-1}(i - c_\ell j) \frac{\lambda_k^j}{j!} \exp\{-\lambda_k\};$$

(in computations, $\sum_{j=0}^{\infty}$ should be replaced with $\sum_{j=0}^{N}$ with appropriately large N).
With the numerical data in question, the expected value of per day requested cash is $c^T\lambda = 7,000$, and the remaining requested quantities are listed below:

	ϵ					
	1.e-1	1.e-2	1.e-3	1.e-4	1.e-5	1.e-6
$w(\epsilon)$	8,900	10,800	12,320	13,680	14,900	16,060
CVaR	9,732 +9.3%	11,451 +6.0%	12,897 +4.7%	14,193 +3.7%	15,390 +3.3%	16,516 +2.8%
BCV	9,836 +10.5%	11,578 +7.2%	13,047 +5.9%	14,361 +5.0%	15,572 +4.5%	16,709 +4.0%
B	10,555 +18.6%	12,313 +14.0%	13,770 +11.8%	15,071 +10.2%	16,270 +9.2%	17,397 +8.3%
E	8,900 +0.0%	10,800 +0.0%	12,520 +1.6%	17,100 +25.0%	—	—

"BCV" stands for the bridged Bernstein-CVaR, "B" — for the Bernstein, and "E" — for the $(1-\epsilon)$-reliable empirical bound on $w(\epsilon)$. The BCV bound corresponds to the generating function $\gamma_{16,10}(\cdot)$, see p. 64. The percents represent the relative differences between the bounds and $w(\epsilon)$. All bounds are right-rounded to the closest integers.

Exercise 2.7: The results of computations are as follows (as a benchmark, we display also the results of Exercise 2.6 related to the case of independent $\zeta_1, ..., \zeta_L$):

	ϵ					
	1.e-1	1.e-2	1.e-3	1.e-4	1.e-5	1.e-6
Exer. 2.6	8,900	10,800	12,320	13,680	14,900	16,060
Exer. 2.7, lower bound	11,000 +23.6%	15,680 +45.2%	19,120 +55.2%	21,960 +60.5%	26,140 +75.4%	28,520 +77.6%
Exer. 2.7, upper bound	13,124 +47.5%	17,063 +58.8%	20,507 +66.5%	23,582 +72.4%	26,588 +78.5%	29,173 +81.7%

Percents display relative differences between the bounds and $w(\epsilon)$

Exercise 2.8. Part 1: By Exercise 2.5, the Bernstein upper bound on $w(\epsilon)$ is

$$\begin{array}{rcl} B_\lambda(\epsilon) & = & \inf\{w_0 : \inf_{\beta>0}[-w_0+\beta\sum_\ell \lambda_\ell(\exp\{c_\ell/\beta\}-1)+\beta\ln(1/\epsilon)]\leq 0\} \\ & = & \inf_{\beta>0}[\beta\sum_\ell \lambda_\ell(\exp\{c_\ell/\beta\}-1)+\beta\ln(1/\epsilon)] \end{array}$$

The "ambiguous" Bernstein upper bound on $w(\epsilon)$ is therefore

$$\begin{array}{rcl} B_\Lambda(\epsilon) & = & \max_{\lambda\in\Lambda}\inf_{\beta>0}[\beta\sum_\ell \lambda_\ell(\exp\{c_\ell/\beta\}-1)+\beta\ln(1/\epsilon)] \\ & = & \inf_{\beta>0}\beta\,[\max_{\lambda\in\Lambda}\sum_\ell \lambda_\ell(\exp\{c_\ell/\beta\}-1)+\ln(1/\epsilon)] \end{array} \qquad (*)$$

where the swap of $\inf_{\beta>0}$ and $\max_{\lambda\in\Lambda}$ is justified by the fact that the function $\beta\sum_\ell \lambda_\ell(\exp\{c_\ell/\beta\}-1)+\beta\ln(1/\epsilon)$ is concave in λ, convex in β and by the compactness and convexity of Λ.

Part 2: We should prove that if Λ is a convex compact set in the domain $\lambda\geq 0$ such that for every affine form $f(\lambda)=f_0+e^T\lambda$ one has

$$\max_{\lambda\in\Lambda} f(\lambda)\leq 0 \Rightarrow \mathrm{Prob}_{\lambda\sim P}\{f(\lambda)\leq 0\}\geq 1-\delta, \qquad (!)$$

then, setting $w_0=B_\Lambda(\epsilon)$, one has

$$\mathrm{Prob}_{\lambda\sim P}\left\{\lambda : \mathrm{Prob}_{\zeta\sim P_{\lambda_1}\times\ldots\times P_{\lambda_L}}\left\{\sum_\ell \zeta_\ell c_\ell > w_0\right\} > \epsilon\right\}\leq\delta. \qquad (?)$$

It suffices to prove that under our assumptions on Λ inequality (?) is valid for all $w_0>B_\Lambda(\epsilon)$. Given $w_0>B_\Lambda(\epsilon)$ and invoking the second relation in $(*)$, we can find $\bar\beta>0$ such that

$$\bar\beta\left[\max_{\lambda\in\Lambda}\sum_\ell \lambda_\ell(\exp\{c_\ell/\bar\beta\}-1)+\ln(1/\epsilon)\right]\leq w_0,$$

or, which is the same,

$$[-w_0+\bar\beta\ln(1/\epsilon)]+\max_{\lambda\in\Lambda}\sum_\ell \lambda_\ell[\bar\beta(\exp\{c_\ell/\bar\beta\}-1)]\leq 0,$$

which, by (!) as applied to the affine form

$$f(\lambda)=[-w_0+\bar\beta\ln(1/\epsilon)]+\sum_\ell \lambda_\ell[\bar\beta(\exp\{c_\ell/\bar\beta\}-1)],$$

implies that

$$\mathrm{Prob}_{\lambda\sim P}\{f(\lambda)>0\}\leq\delta. \qquad (**)$$

It remains to note that when $\lambda\geq 0$ is such that $f(\lambda)\leq 0$, the result of Exercise 2.5 states that

$$\mathrm{Prob}_{\zeta\sim P_{\lambda_1}\times\ldots\times P_{\lambda_m}}\left\{-w_0+\sum_\ell \zeta_\ell c_\ell>0\right\}\leq\epsilon.$$

Thus, when $\omega_0>B_\Lambda(\epsilon)$, the set of λ's in the left hand side of (?) is contained in the set $\{\lambda\geq 0 : f(\lambda)>0\}$, and therefore (?) is readily given by $(**)$.

B.3 Exercises from Lecture 3

Exercise 3.1: Let $\mathcal{S}[\cdot]$ be a safe tractable approximation of $(C_{\mathcal{Z}_*}[\cdot])$ tight within the factor ϑ. Let us verify that $\mathcal{S}[\lambda\gamma\rho]$ is a safe tractable approximation of $(C_{\mathcal{Z}}[\rho])$ tight within the factor $\lambda\vartheta$. All we should prove is that (a) if x can be extended to a feasible solution to $\mathcal{S}[\lambda\gamma\rho]$, then x is feasible for $(C_{\mathcal{Z}}[\rho])$, and that (b) if x cannot be extended to a feasible solution to $\mathcal{S}[\lambda\gamma\rho]$, then x is not feasible for $(C_{\mathcal{Z}}[\lambda\vartheta\rho])$. When x can be extended to a feasible solution of $\mathcal{S}[\lambda\gamma\rho]$, x is feasible for $(C_{\mathcal{Z}_*}[\lambda\gamma\rho])$, and since $\rho\mathcal{Z} \subset \lambda\gamma\rho\mathcal{Z}_*$, x is feasible for $(C_{\mathcal{Z}}[\rho])$ as well, as required in (a). Now assume that x cannot be extended to a feasible solution of $\mathcal{S}[\lambda\gamma\rho]$. Then x is not feasible for $(C_{\mathcal{Z}_*}[\vartheta\lambda\gamma\rho])$, and since the set $\vartheta\lambda\gamma\rho\mathcal{Z}_*$ is contained in $\vartheta\lambda\rho\mathcal{Z}$, x is not feasible for $(C_{\mathcal{Z}}[(\vartheta\lambda)\rho])$, as required in (b). □

Exercise 3.2: 1) Consider the ellipsoid

$$\mathcal{Z}_* = \{\zeta : \zeta^T[\sum_i Q_i]\zeta \leq M\}.$$

We clearly have $M^{-1/2}\mathcal{Z}_* \subset \mathcal{Z} \subset \mathcal{Z}_*$; by assumption, $(C_{\mathcal{Z}_*}[\cdot])$ admits a safe tractable approximation tight within the factor ϑ, and it remains to apply the result of Exercise 3.1.

2) This is a particular case of 1) corresponding to $\zeta^T Q_i \zeta = \zeta_i^2$, $1 \leq i \leq M = \dim\zeta$.

3) Let $\mathcal{Z} = \bigcap\limits_{i=1}^{M} E_i$, where E_i are ellipsoids. Since $\mathcal{Z}$ is symmetric w.r.t. the origin, we also have $\mathcal{Z} = \bigcap\limits_{i=1}^{M} [E_i \cap (-E_i)]$. We claim that for every i, the set $E_i \cap (-E_i)$ contains an ellipsoid F_i centered at the origin and such that $E_i \cap (-E_i) \subset \sqrt{2}F_i$, and that this ellipsoid F_i can be easily found. Believing in the claim, we have

$$\mathcal{Z}_* \equiv \bigcap_{i=1}^{M} F_i \subset \mathcal{Z} \subset \sqrt{2}\bigcap_{i=1}^{M} F_i.$$

By 1), $(C_{\mathcal{Z}_*}[\cdot])$ admits a safe tractable approximation with the tightness factor $\vartheta\sqrt{M}$; by Exercise 3.1, $(C_{\mathcal{Z}}[\cdot])$ admits a safe tractable approximation with the tightness factor $\vartheta\sqrt{2M}$.

It remains to support our claim. For a given i, applying nonsingular linear transformation of variables, we can reduce the situation to the one where $E_i = B + e$, where B is the unit Euclidean ball, centered at the origin, and $\|e\|_2 < 1$ (the latter inequality follows from $0 \in \text{int}\mathcal{Z} \subset \text{int}(E_i \cap (-E_i)))$. The intersection $G = E_i \cap (-E_i)$ is a set that is invariant w.r.t. rotations around the axis $\mathbb{R}e$; a 2-D cross-section H of G by a 2D plane Π containing the axis is a 2-D solid symmetric w.r.t. the origin. It is well known that for every symmetric w.r.t. 0 solid Q in $\mathbb{R}^d$ there exists a centered at 0 ellipsoid E such that $E \subset Q \subset \sqrt{d}E$. Therefore there exists (and in fact can easily be found) an ellipsis I, centered at the origin, that is contained in H and is such that $\sqrt{2}I$ contains H. Now, the ellipsis I is the intersection of Π and an ellipsoid F_i that is invariant w.r.t. rotations around the axis $\mathbb{R}e$, and F_i clearly satisfies the required relations $F_i \subset E_i \cap (-E_i) \subset \sqrt{2}F_i$. [1]

[1] In fact, the factor $\sqrt{2}$ in the latter relation can be reduced to $2/\sqrt{3} < \sqrt{2}$, see Solution to Exercise 3.4.

Exercise 3.3: With y given, all we know about x is that there exists $\Delta \in \mathbb{R}^{p\times q}$ with $\|\Delta\|_{2,2} \leq \rho$ such that $y = B_{\rm n}[x;1] + L^T \Delta R[x;1]$, or, denoting $w = \Delta R[x;1]$, that there exists $w \in \mathbb{R}^p$ with $w^T w \leq \rho^2 [x;1]^T R^T R[x;1]$ such that $y = B_{\rm n}[x;1] + L^T w$. Denoting $z = [x;w]$, all we know about the vector z is that it belongs to a given affine plane $\mathcal{A}z = a$ and satisfies the quadratic inequality $z^T \mathcal{C} z + 2c^T z + d \leq 0$, where $\mathcal{A} = [A_{\rm n}, L^T]$, $a = y - b_{\rm n}$, and

$$[\xi;\omega]^T \mathcal{C}[\xi;\omega] + 2c^T[\xi;\omega] + d \equiv \omega^T\omega - \rho^2[\xi;1]^T R^T R[\xi;1],\ [\xi;\omega] \in \mathbb{R}^{n+p}.$$

Using the equations $\mathcal{A}z = a$, we can express the $n+p$ z-variables via $k \leq n+p$ u-variables:

$$\mathcal{A}z = a \Leftrightarrow \exists u \in \mathbb{R}^k : z = Eu + e.$$

Plugging $z = Eu + e$ into the quadratic constraint $z^T \mathcal{C} z + 2c^T z + d \leq 0$, we get a quadratic constraint $u^T F u + 2f^T u + g \leq 0$ on u. Finally, the vector Qx we want to estimate can be represented as Pu with easily computable matrix P. The summary of our developments is as follows:

> (!) *Given y and the data describing $\mathcal{B}$, we can build k, a matrix P and a quadratic form $u^T F u + 2f^T u + g \leq 0$ on $\mathbb{R}^k$ such that the problem of interest becomes the problem of the best, in the worst case, $\|\cdot\|_2$-approximation of Pu, where unknown vector $u \in \mathbb{R}^k$ is known to satisfy the inequality $u^T F u + 2f^T u + g \leq 0$.*

By (!), our goal is to solve the semi-infinite optimization program

$$\min_{t,v} \left\{ t : \|Pu - v\|_2 \leq t \,\forall (u : u^T F u + 2f^T u + g \leq 0) \right\}. \qquad (*)$$

Assuming that $\inf_u \left[u^T F u + 2f^T u + g\right] < 0$ and applying the inhomogeneous version of $\mathcal{S}$-Lemma, the problem becomes

$$\min_{t,v,\lambda} \left\{ t \geq 0 : \left[\begin{array}{c|c} \lambda F - P^T P & \lambda f - P^T v \\ \hline \lambda f^T - v^T P & \lambda g + t^2 - v^T v \end{array}\right] \succeq 0, \lambda \geq 0 \right\}.$$

Passing from minimization of t to minimization of $\tau = t^2$, the latter problem becomes the semidefinite program

$$\min_{\tau,v,\lambda,s} \left\{ \tau : \begin{array}{c} v^T v \leq s, \lambda \geq 0 \\ \left[\begin{array}{c|c} \lambda F - P^T P & \lambda f - P^T v \\ \hline \lambda f^T - v^T P & \lambda g + \tau - s \end{array}\right] \succeq 0 \end{array} \right\}.$$

In fact, the problem of interest can be solved by pure Linear Algebra tools, without Semidefinite optimization. Indeed, assume for a moment that P has trivial kernel. Then $(*)$ is feasible if and only if the solution set S of the quadratic inequality $\phi(u) \equiv u^T F u + 2f^T u + g \leq 0$ in variables u is nonempty and bounded, which is the case if and only if this set is an ellipsoid $(u-c)^T Q (u-c) \leq r^2$ with $Q \succ 0$ and $r \geq 0$; whether this indeed is the case and what are c, Q, r, if any, can be easily found out by Linear Algebra tools. The image PS of S under the mapping P also is an ellipsoid (perhaps "flat") centered at $v_* = Pc$, and the optimal solution to $(*)$ is (t_*, v_*), where t_* is the largest half-axis of the ellipsoid PS. In the case when P has a kernel, let E be the orthogonal complement to $\mathrm{Ker}P$, and $\widehat{P}$ be the restriction of P onto E; this mapping has a trivial kernel. Problem $(*)$ clearly is equivalent to

$$\min_{t,v} \left\{ t : \|\widehat{P}\widehat{u} - v\|_2 \leq t \,\forall (\widehat{u} \in E : \exists w \in \mathrm{Ker}P : \phi(\widehat{u} + w) \leq 0 \right\}.$$

The set
$$\widehat{U} = \{\widehat{u} \in E : \exists w \in \mathrm{Ker} P : \phi(\widehat{u} + w) \le 0\}$$
clearly is given by a single quadratic inequality in variables $\widehat{u} \in E$, and $(*)$ reduces to a similar problem with E in the role of the space where u lives and $\widehat{P}$ in the role of P, and we already know how to solve the resulting problem.

Exercise 3.4: In view of Theorem 3.9, all we need to verify is that $\mathcal{Z}$ can be "safely approximated" within an $O(1)$ factor by an intersection $\widehat{\mathcal{Z}}$ of $O(1)J$ ellipsoids *centered at the origin*: there exists $\widehat{\mathcal{Z}} = \{\eta : \eta^T \widehat{Q}_j \eta \le 1, 1 \le j \le \widehat{J}\}$ with $\widehat{Q}_j \succeq 0$, $\sum_j \widehat{Q}_j \succ 0$ such that
$$\theta^{-1}\widehat{\mathcal{Z}} \subset \mathcal{Z} \subset \widehat{\mathcal{Z}},$$
with an absolute constant θ and $\widehat{J} \le O(1)J$. Let us prove that the just formulated statement holds true with $\widehat{J} = J$ and $\theta = \sqrt{3}/2$. Indeed, since $\mathcal{Z}$ is symmetric w.r.t. the origin, setting $E_j = \{\eta : (\eta - a_j)^T Q_j (\eta - a_j) \le 1\}$, we have
$$\mathcal{Z} = \bigcap_{j=1}^J E_j = \bigcap_{j=1}^J (-E_j) = \bigcap_{j=1}^J (E_j \cap [-E_j]);$$
all we need is to demonstrate that every one of the sets $E_j \cap [-E_j]$ is in between two proportional ellipsoids centered at the origin with the larger one being at most $2/\sqrt{3}$ multiple of the smaller one. After an appropriate linear one-to-one transformation of the space, all we need to prove is that if $E = \{\eta \in \mathbb{R}^d : (\eta_1 - r)^2 + \sum_{j=2}^k \eta_j^2 \le 1\}$ with $0 \le r < 1$, then we can point out the set $F = \{\eta : \eta_1^2/a^2 + \sum_{j=2}^k \eta_j^2/b^2 \le 1\}$ such that
$$\frac{\sqrt{3}}{2} F \subset E \cap [-E] \subset F.$$
When proving the latter statement, we lose nothing when assuming $k = 2$. Renaming η_1 as y, η_2 as x and setting $h = 1 - r \in (0, 1]$ we should prove that the "loop" $\mathcal{L} = \{[x; y] : [|y| + (1-h)]^2 + x^2 \le 1\}$ is in between two proportional ellipses centered at the origin with the ratio of linear sizes $\theta \le 2/\sqrt{3}$. Let us verify that we can take as the smaller of these ellipses the ellipsis
$$\mathcal{E} = \{[x; y] : y^2/h^2 + x^2/(2h - h^2) \le \mu^2\}, \mu = \sqrt{\frac{3-h}{4-2h}},$$
and to choose $\theta = \mu^{-1}$ (so that $\theta \le 2/\sqrt{3}$ due to $0 < h \le 1$). First, let us prove that $\mathcal{E} \subset \mathcal{L}$. This inclusion is evident when $h = 1$, so that we can assume that $0 < h < 1$. Let $[x; y] \in \mathcal{E}$, and let $\lambda = \frac{2(1-h)}{h}$. We have
$$\begin{array}{l}
y^2/h^2 + x^2/(2h - h^2) \le \mu^2 \Rightarrow \left\{\begin{array}{ll} y^2 \le h^2[\mu^2 - x^2/(2h - h^2)] & (a) \\ x^2 \le \mu^2 h(2-h) & (b) \end{array}\right. ; \\
(|y| + (1-h))^2 + x^2 = y^2 + 2|y|(1-h) + (1-h)^2 \le y^2 + \left[\lambda y^2 + \frac{1}{\lambda}(1-h)^2\right] \\
+(1-h)^2 = y^2 \frac{2-h}{h} + \frac{(2-h)(1-h)}{2} + x^2 \\
\le \left[\mu^2 - \frac{x^2}{h(2-h)}\right](2h - h^2) + \frac{(2-h)(1-h)}{2} + x^2 \equiv q(x^2),
\end{array}$$
where the concluding $\le$ is due to (a). Since $0 \le x^2 \le \mu^2(2h - h^2)$ by (b), $q(x^2)$ is in-between its values for $x^2 = 0$ and $x^2 = \mu^2(2h - h^2)$, and both these values with our μ are equal to 1. Thus, $[x; y] \in \mathcal{L}$.

It remains to prove that $\mu^{-1}\mathcal{E} \supset \mathcal{L}$, or, which is the same, that when $[x;y] \in \mathcal{L}$, we have $[\mu x; \mu y] \in \mathcal{E}$. Indeed, we have

$$\begin{array}{l}
[|y| + (1-h)]^2 + x^2 \le 1 \Rightarrow |y| \le h \ \&\ x^2 \le 1 - y^2 - 2|y|(1-h) - (1-h)^2 \\
\Rightarrow x^2 \le 2h - h^2 - y^2 - 2|y|(1-h) \\
\Rightarrow \mu^2\left[\frac{y^2}{h^2} + \frac{x^2}{2h-h^2}\right] = \mu^2\frac{y^2(2-h)+hx^2}{h^2(2-h)} \le \mu^2\frac{h^2(2-h)+2(1-h)\overbrace{[y^2-|y|h]}^{\le 0}}{h^2(2-h)} \le \mu^2 \\
\Rightarrow [x;y] \in \mathcal{E},
\end{array}$$

as claimed.

Exercise 3.5: 1) We have

$$\begin{array}{rl}
\mathrm{EstErr} = & \sup\limits_{v\in V, A\in\mathcal{A}} \sqrt{v^T(GA-I)^T(GA-I)v + \mathrm{Tr}(G^T\Sigma G)} \\
= & \sup\limits_{A\in\mathcal{A}} \sup\limits_{u:u^Tu\le 1} \sqrt{u^TQ^{-1/2}(GA-I)^T(GA-I)Q^{-1/2}u + \mathrm{Tr}(G^T\Sigma G)} \\
& \qquad [\text{substitution } v = Q^{-1/2}u] \\
= & \sqrt{\sup\limits_{A\in\mathcal{A}} \|(GA-I)Q^{-1/2}\|_{2,2}^2 + \mathrm{Tr}(G^T\Sigma G)}.
\end{array}$$

By the Schur Complement Lemma, the relation $\|(GA-I)Q^{-1/2}\|_{2,2} \le \tau$ is equivalent to the LMI $\left[\begin{array}{c|c} \tau I & [(GA-I)Q^{-1/2}]^T \\ \hline (GA-I)Q^{-1/2} & \tau I \end{array}\right]$, and therefore the problem of interest can be posed as the semi-infinite semidefinite program

$$\min_{t,\tau,\delta,G}\left\{ t : \begin{array}{l} \sqrt{\tau^2+\delta^2} \le t,\ \sqrt{\mathrm{Tr}(G^T\Sigma G)} \le \delta \\ \left[\begin{array}{c|c} \tau I & [(GA-I)Q^{-1/2}]^T \\ \hline (GA-I)Q^{-1/2} & \tau I \end{array}\right] \succeq 0\ \forall A \in \mathcal{A} \end{array}\right\},$$

which is nothing but the RC of the uncertain semidefinite program

$$\left\{ \min_{t,\tau,\delta,G}\left\{ t : \begin{array}{l} \sqrt{\tau^2+\delta^2} \le t,\ \sqrt{\mathrm{Tr}(G^T\Sigma G)} \le \delta \\ \left[\begin{array}{c|c} \tau I & [(GA-I)Q^{-1/2}]^T \\ \hline (GA-I)Q^{-1/2} & \tau I \end{array}\right] \succeq 0 \end{array}\right\} : A \in \mathcal{A} \right\}.$$

In order to reformulate the only semi-infinite constraint in the problem in a tractable form, note that with $A = A_{\rm n} + L^T\Delta R$ we have

$$\begin{array}{l}
\mathcal{N}(A) := \left[\begin{array}{c|c} \tau I & [(GA-I)Q^{-1/2}]^T \\ \hline (GA-I)Q^{-1/2} & \tau I \end{array}\right] \\
= \underbrace{\left[\begin{array}{c|c} \tau I & [(GA_{\rm n}-I)Q^{-1/2}]^T \\ \hline (GA_{\rm n}-I)Q^{-1/2} & \tau I \end{array}\right]}_{\mathcal{B}_{\rm n}(G)} + \mathcal{L}^T(G)\Delta\mathcal{R} + \mathcal{R}^T\Delta^T\mathcal{L}(G), \\
\mathcal{L}(G) = \left[0_{p\times n}, LG^T\right], \mathcal{R} = \left[RQ^{-1/2}, 0_{q\times n}\right].
\end{array}$$

Invoking Theorem 3.12, the semi-infinite LMI $\mathcal{N}(A) \succeq 0\ \forall A \in \mathcal{A}$ is equivalent to

$$\exists \lambda : \left[\begin{array}{c|c} \lambda I_p & \rho\mathcal{L}(G) \\ \hline \rho\mathcal{L}^T(G) & \mathcal{B}_{\rm n}(G) - \lambda\mathcal{R}^T\mathcal{R} \end{array}\right] \succeq 0,$$

and thus the RC is equivalent to the semidefinite program

$$\min_{\substack{t,\tau,\\ \delta,\lambda,G}} \left\{ t : \begin{array}{c} \sqrt{\tau^2+\delta^2} \le t,\ \sqrt{\mathrm{Tr}(G^T\Sigma G)} \le \delta \\ \left[\begin{array}{c|c|c} \lambda I_p & & \rho L G^T \\ \hline & \tau I_n - \lambda Q^{-1/2}R^T R Q^{-1/2} & Q^{-1/2}(A_{\mathrm{n}}^T G^T - I_n) \\ \hline \rho G L^T & (GA_{\mathrm{n}} - I_n)Q^{-1/2} & \tau I_n \end{array}\right] \succeq 0 \end{array} \right\}.$$

2): Setting $v = U^T\widehat{v}$, $\widehat{y} = W^T y$, $\widehat{\xi} = W^T\xi$, our estimation problem reduces to the exactly the same problem, but with $\mathrm{Diag}\{a\}$ in the role of A_{n} and the diagonal matrix $\mathrm{Diag}\{q\}$ in the role of Q; a linear estimate $\widehat{G}\widehat{y}$ of $\widehat{v}$ in the new problem corresponds to the linear estimate $U^T\widehat{G}W^T y$, of exactly the same quality, in the original problem. In other words, the situation reduces to the one where A_{n} and Q are diagonal positive semidefinite, respectively, positive definite matrices; all we need is to prove that in this special case we lose nothing when restricting G to be diagonal. Indeed, in the case in question the RC reads

$$\min_{\substack{t,\tau,\\ \delta,\lambda,G}} \left\{ t : \begin{array}{c} \sqrt{\tau^2+\delta^2} \le t,\ \sigma\sqrt{\mathrm{Tr}(G^T G)} \le \delta \\ \left[\begin{array}{c|c|c} \lambda I_n & & \rho G^T \\ \hline & \tau I_n - \lambda \mathrm{Diag}\{\mu\} & \mathrm{Diag}\{\nu\}G^T - \mathrm{Diag}\{\eta\} \\ \hline \rho G & G\mathrm{Diag}\{\nu\} - \mathrm{Diag}\{\eta\} & \tau I_n \end{array}\right] \succeq 0 \end{array} \right\} \qquad (*)$$

where $\mu_i = q_i^{-1}$, $\nu_i = a_i/\sqrt{q_i}$ and $\eta_i = 1/\sqrt{q_i}$. Replacing the G-component in a feasible solution with EGE, where E is a diagonal matrix with diagonal entries ± 1, we preserve feasibility (look what happens when you multiply the matrix in the LMI from the left and from the right by $\mathrm{Diag}\{I, I, E\}$). Since the problem is convex, it follows that whenever a collection $(t, \tau, \delta, \lambda, G)$ is feasible for the RC, so is the collection obtained by replacing the original G with the average of the matrices $E^T GE$ taken over all 2^n diagonal $n \times n$ matrices with diagonal entries ± 1, and this average is the diagonal matrix with the same diagonal as the one of G. Thus, when A_{n} and Q are diagonal and $L = R = I_n$ (or, which is the same in our situation, L and R are orthogonal), we lose nothing when restricting G to be diagonal.

Restricted to diagonal matrices $G = \mathrm{Diag}\{g\}$, the LMI constraint in $(*)$ becomes a bunch of 3×3 LMIs

$$\left[\begin{array}{c|c|c} \lambda & 0 & \rho g_i \\ \hline 0 & \tau - \lambda\mu_i & \nu_i g_i - \eta_i \\ \hline \rho g_i & \nu_i g_i - \eta_i & \tau \end{array}\right] \succeq 0,\ i = 1, ..., n,$$

in variables λ, τ, g_i. Assuming w.l.o.g. that $\lambda > 0$ and applying the Schur Complement Lemma, these 3×3 LMIs reduce to 2×2 matrix inequalities

$$\left[\begin{array}{c|c} \tau - \lambda\mu_i & \nu_i g_i - \eta_i \\ \hline \nu_i g_i - \eta_i & \tau - \rho^2 g_i^2/\lambda \end{array}\right] \succeq 0,\ i = 1, ..., n.$$

For given τ, λ, every one of these inequalities specifies a segment $\Delta_i(\tau, \lambda)$ of possible value of g_i, and the best choice of g_i in this segment is the point $g_i(\tau, \lambda)$ of the segment closest to 0 (when the segment is empty, we set $g_i(\tau, \lambda) = \infty$). Note that $g_i(\tau, \lambda) \ge 0$ (why?). It follows that $(*)$ reduces to the convex (due to its origin) problem

$$\min_{\tau,\lambda \ge 0} \left\{ \sqrt{\tau^2 + \sigma^2 \sum_i g_i^2(\tau, \lambda)} \right\}$$

with easily computable convex nonnegative functions $g_i(\tau, \lambda)$.

Exercise 3.6: 1) Let $\lambda > 0$. For every $\xi \in \mathbb{R}^n$ we have $\xi^T[pq^T + qp^T]\xi = 2(\xi^T p)(\xi^T q) \leq \lambda(\xi^T p)^2 + \frac{1}{\lambda}(\xi^T q)^2 = \xi^T[\lambda pp^T + \frac{1}{\lambda}qq^T]\xi$, whence $pq^T + qp^T \preceq \lambda pp^T + \frac{1}{\lambda}qq^T$. By similar argument, $-[pq^T + qp^T] \preceq \lambda pp^T + \frac{1}{\lambda}qq^T$. 1) is proved.

2) Observe, first, that if $\lambda(A)$ is the vector of eigenvalues of a symmetric matrix A, then $\|\lambda(pq^T + qp^T)\|_1 = 2\|p\|_2\|q\|_2$. Indeed, there is nothing to verify when $p = 0$ or $q = 0$; when $p, q \neq 0$, we can normalize the situation to make p a unit vector and then to choose the orthogonal coordinates in $\mathbb{R}^n$ in such a way that p is the first basic orth, and q is in the linear span of the first two basic orths. With this normalization, the nonzero eigenvalues of A are exactly the same as the eigenvalues of the 2×2 matrix $\begin{bmatrix} 2\alpha & \beta \\ \beta & 0 \end{bmatrix}$, where α and β are the first two coordinates of q in our new orthonormal basis. The eigenvalues of the 2×2 matrix in question are $\alpha \pm \sqrt{\alpha^2 + \beta^2}$, and the sum of their absolute values is $2\sqrt{\alpha^2 + \beta^2} = 2\|q\|_2 = 2\|p\|_2\|q\|_2$, as claimed.

To prove 2), let us lead to a contradiction the assumption that $Y, p, q \neq 0$ are such that $Y \succeq \pm[pq^T + qp^T]$ and there is no $\lambda > 0$ such that $Y - \lambda pp^T - \frac{1}{\lambda}qq^T \succeq 0$, or, which is the same by the Schur Complement Lemma, the LMI

$$\begin{bmatrix} Y - \lambda pp^T & q \\ q^T & \lambda \end{bmatrix} \succeq 0$$

in variable λ has no solution, or, equivalently, the optimal value in the (clearly strictly feasible) SDO program

$$\min_{t,\lambda} \left\{ t : \begin{bmatrix} tI + Y - \lambda pp^T & q \\ q^T & \lambda \end{bmatrix} \succeq 0 \right\}$$

is positive. By semidefinite duality, the latter is equivalent to the dual problem possessing a feasible solution with a positive value of the dual objective. Looking at the dual, this is equivalent to the existence of a matrix $Z \in \mathbf{S}^n$ and a vector $z \in \mathbb{R}^n$ such that

$$\begin{bmatrix} Z & z \\ z^T & p^T Z p \end{bmatrix} \succeq 0, \quad \mathrm{Tr}(ZY) < 2q^T z.$$

Adding, if necessary, to Z a small positive multiple of the unit matrix, we can assume w.l.o.g. that $Z \succ 0$. Setting $\bar{Y} = Z^{1/2} Y Z^{1/2}$, $\bar{p} = Z^{1/2}p$, $\bar{q} = Z^{1/2}q$, $\bar{z} = Z^{-1/2}z$, the above relations become

$$\begin{bmatrix} I & \bar{z} \\ \bar{z}^T & \bar{p}^T\bar{p} \end{bmatrix} \succeq 0, \mathrm{Tr}(\bar{Y}) < 2\bar{q}^T\bar{z}. \qquad (*)$$

Observe that from $Y \succeq \pm[pq^T + qp^T]$ it follows that $\bar{Y} \succeq \pm[\bar{p}\bar{q}^T + \bar{q}\bar{p}^T]$. Looking at what happens in the eigenbasis of the matrix $[\bar{p}\bar{q}^T + \bar{q}\bar{p}^T]$, we conclude from this relation that $\mathrm{Tr}(\bar{Y}) \geq \|\lambda(\bar{p}\bar{q}^T + \bar{q}\bar{p}^T)\|_1 = 2\|\bar{p}\|_2\|\bar{q}\|_2$. On the other hand, the matrix inequality in $(*)$ implies that $\|\bar{z}\|_2 \leq \|\bar{p}\|_2$, and thus $\mathrm{Tr}(\bar{Y}) < 2\|\bar{p}\|_2\|\bar{q}\|_2$ by the second inequality in $(*)$. We have arrived at a desired contradiction.

3) Assume that x is such that all $L_\ell(x)$ are nonzero. Assume that x can be extended to a feasible solution $Y_1, ..., Y_L, x$ of (3.7.2). Invoking 2), we can find $\lambda_\ell > 0$ such that $Y_\ell \succeq \lambda_\ell R_\ell^T R_\ell + \frac{1}{\lambda_\ell} L_\ell^T(x) L_\ell(x)$. Since $\mathcal{A}_{\mathrm{n}}(x) - \rho\sum_\ell Y_\ell \succeq 0$, we have $[\mathcal{A}_{\mathrm{n}}(x) - \rho\sum_\ell \lambda_\ell R_\ell^T R_\ell] - \sum_\ell \frac{\rho}{\lambda_\ell} L_\ell^T(x) L_\ell(x) \succeq 0$, whence, by the Schur Complement Lemma, $\lambda_1, ..., \lambda_L, x$ are feasible for (3.7.3). Vice versa, if $\lambda_1, ..., \lambda_L, x$ are feasible for (3.7.3), then $\lambda_\ell > 0$ for all ℓ due to $L_\ell(x) \neq 0$, and, by the same Schur Complement Lemma, setting $Y_\ell = \lambda_\ell R_\ell^T R_\ell + \frac{1}{\lambda_\ell} L_\ell^T(x) L_\ell(x)$, we have

$$\mathcal{A}_{\mathrm{n}}(x) - \rho \sum_\ell Y_\ell \succeq 0,$$

while $Y_\ell \succeq \pm\left[L_\ell^T(x)R_\ell + R_\ell^T L_\ell(x)\right]$, that is, $Y_1, ..., Y_L, x$ are feasible for (3.7.2).

We have proved the equivalence of (3.7.2) and (3.7.3) in the case when $L_\ell(x) \neq 0$ for all ℓ. The case when some of $L_\ell(x)$ vanish is left to the reader.

Exercise 3.7: *A* solution *might* be as follows. The problem of interest is

$$\min_{G,t}\{t : t \geq \|(GA-I)v + G\xi\|_2 \,\forall (v \in V, \xi \in \Xi, A \in \mathcal{A})\}$$
$$\Updownarrow$$
$$\min_{G,t}\left\{t : u^T(GA-I)v + u^TG\xi \leq t \forall \left(u, v, \xi : \begin{array}{c} u^Tu \leq 1 \\ v^TP_iv \leq 1, \\ 1 \leq i \leq I \\ \xi^TQ_j\xi \leq \rho_\xi^2, \\ 1 \leq j \leq J \end{array}\right) \forall A \in \mathcal{A}\right\}. \qquad (*)$$

Observing that

$$u^T[GA-I]v + u^TG\xi = [u;v;\xi]^T \left[\begin{array}{c|c|c} & \frac{1}{2}[GA-I] & \frac{1}{2}G \\ \hline \frac{1}{2}[GA-I]^T & & \\ \hline \frac{1}{2}G^T & & \end{array}\right][u;v;\xi],$$

for A fixed, a sufficient condition for the validity of the semi-infinite constraint in $(*)$ is the existence of nonnegative μ, ν_i, ω_j such that

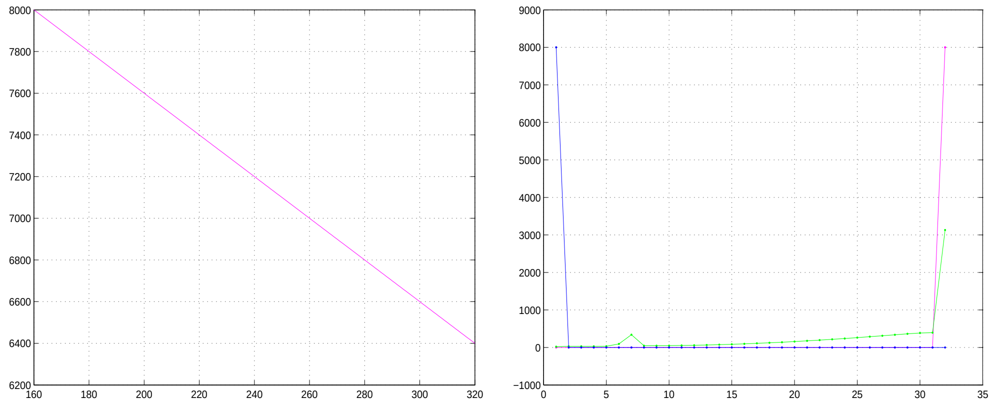
$$\left[\begin{array}{c|c|c} \mu I & & \\ \hline & \sum_i \nu_i P_i & \\ \hline & & \sum_j \omega_j Q_j \end{array}\right] \succeq \left[\begin{array}{c|c|c} & \frac{1}{2}[GA-I] & \frac{1}{2}G \\ \hline \frac{1}{2}[GA-I]^T & & \\ \hline \frac{1}{2}G^T & & \end{array}\right]$$

and $\mu + \sum_i \nu_i + \rho_\xi^2 \sum_j \omega_j \leq t$. It follows that the validity of the semi-infinite system of constraints

$$\begin{array}{l} \mu + \sum_i \nu_i + \rho_\xi^2 \sum_j \omega_j \leq t, \;\; \mu \geq 0, \nu_i \geq 0, \omega_j \geq 0 \\ \left[\begin{array}{c|c|c} \mu I & & \\ \hline & \sum_i \nu_i P_i & \\ \hline & & \sum_j \omega_j Q_j \end{array}\right] \succeq \left[\begin{array}{c|c|c} & \frac{1}{2}[GA-I] & \frac{1}{2}G \\ \hline \frac{1}{2}[GA-I]^T & & \\ \hline \frac{1}{2}G^T & & \end{array}\right] \\ \hfill \forall A \in \mathcal{A} \end{array} \qquad (!)$$

in variables $t, G, \mu, \nu_i, \omega_j$ is a sufficient condition for (G, t) to be feasible for $(*)$. The only semi-infinite constraint in (!) is in fact an LMI with structured norm-bounded uncertainty:

$$\begin{array}{l} \left[\begin{array}{c|c|c} \mu I & & \\ \hline & \sum_i \nu_i P_i & \\ \hline & & \sum_j \omega_j Q_j \end{array}\right] \\ \qquad - \left[\begin{array}{c|c|c} & \frac{1}{2}[GA-I] & \frac{1}{2}G \\ \hline \frac{1}{2}[GA-I]^T & & \\ \hline \frac{1}{2}G^T & & \end{array}\right] \succeq 0 \;\forall A \in \mathcal{A} \\ \Updownarrow \\ \underbrace{\left[\begin{array}{c|c|c} \mu I & -\frac{1}{2}[GA_n - I] & -\frac{1}{2}G \\ \hline -\frac{1}{2}[GA_n - I]^T & \sum_i \nu_i P_i & \\ \hline -\frac{1}{2}G^T & & \sum_j \omega_j Q_i \end{array}\right]}_{\mathcal{B}(\mu,\nu,\omega,G)} \\ \qquad + \sum_{\ell=1}^L [\mathcal{L}_\ell(G)^T \Delta_\ell \mathcal{R}_\ell + \mathcal{R}_\ell^T \Delta_\ell^T \mathcal{L}_\ell(G)] \succeq 0 \\ \hfill \forall (\|\Delta_\ell\|_{2,2} \leq \rho_A, 1 \leq \ell \leq L), \\ \mathcal{L}_\ell(G) = \frac{1}{2}\left[L_\ell G^T, 0_{p_\ell \times n}, 0_{p_\ell \times m}\right], \;\; \mathcal{R}_\ell = [0_{q_\ell \times n}, R_\ell, 0_{q_\ell \times m}]. \end{array}$$



Opt(α) vs. α solutions for $\alpha = 160, 180, 320$

Figure B.1: Results for Exercise 4.1.

Invoking Theorem 3.13, we end up with the following safe tractable approximation of ($*$):

$$\begin{array}{rl}\min\limits_{t,G,\mu,\nu_i,\omega_j,\lambda_\ell,Y_\ell} & t \\ \text{s.t.} & \\ & \mu + \sum_i \nu_i + \rho_\xi^2 \sum_j \omega_j \leq t,\ \ \mu \geq 0, \nu_i \geq 0, \omega_j \geq 0 \\ & \left[\begin{array}{c|c} \lambda_\ell I & \mathcal{L}_\ell(G) \\ \hline \mathcal{L}_\ell^T(G) & Y_\ell - \lambda_\ell \mathcal{R}_\ell^T \mathcal{R}_\ell \end{array}\right] \succeq 0,\ 1 \leq \ell \leq L \\ & \mathcal{B}(\mu,\nu,\omega,G) - \rho_A \sum_{\ell=1}^L Y_\ell \succeq 0. \end{array}$$

B.4 Exercises from Lecture 4

Exercise 4.1: A solution *might be* as follows. We define the normal range of the uncertain cost vector as the box $\mathcal{Z} = \{c' : 0 \leq c' \leq c\}$, where c is the current cost, the cone $\mathcal{L}$ as

$$\mathcal{L} = \{\zeta \in \mathbb{R}^n : \zeta \geq 0, \zeta_j = 0 \text{ whenever } v_j = 0\}$$

and equip $\mathbb{R}^n$ with the norm

$$\|\zeta\|_v = \max_j |\zeta|_j / \bar{v}_j,\ \bar{v}_j = \left\{\begin{array}{ll} v_j, & v_j > 0 \\ 1, & v_j = 0 \end{array}\right.$$

With this setup, the model becomes

$$\mathrm{Opt}(\alpha) = \min_x \left\{c^T x : Px \geq b,\ x \geq 0, v^T x \leq \alpha\right\}.$$

With the data of the Exercise, computation says that the minimal value of α for which the problem is feasible is $\underline{\alpha} = 160$ and that the bound on the sensitivity becomes redundant when $\alpha \geq \overline{\alpha} = 320$. The tradeoff between $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ is shown on the left plot in figure B.1; the right plot depicts the solutions for $\alpha = \underline{\alpha} = 160$ (magenta), $\alpha = \overline{\alpha} = 320$ (blue) and $\alpha = 180$ (green).

Exercise 4.2: 1) With $\phi(\rho) = tau + \alpha\rho$ problem (!) does not make sense, meaning that it is always infeasible, unless $E = \{0\}$. Indeed, otherwise g contains a nonzero vector g, and assuming (!) feasible, we should have for certain τ and α

$$\left[\begin{array}{c|c} 2(\tau + \alpha\rho) & f^T + \rho g^T \\ \hline f + \rho g & A(t) \end{array}\right] \succeq 0 \ \forall \rho > 0$$

or, which is the same,

$$\left[\begin{array}{c|c} 2\rho^{-1}\tau + 2\rho^{-1}\alpha & \rho^{-1}f^T + g^T \\ \hline \rho^{-1}f + g & A(t) \end{array}\right] \succeq 0 \ \forall \rho > 0.$$

passing to limit as $\rho \to +\infty$, the matrix $\left[\begin{array}{c|c} 0 & g^T \\ \hline g & A(t) \end{array}\right]$ should be $\succeq 0$, which is not the case when $g \neq 0$.

The reason why the GRC methodology does not work in our case is pretty simple: we are *not* applying this methodology, we are doing something else. Indeed, with the GRC approach, we would require the validity of the semidefinite constraints

$$\left[\begin{array}{c|c} \tau & f^T + \rho g^T \\ \hline f + \rho g & A(t) \end{array}\right] \geq 0, \ f \in \mathcal{F}, g \in E$$

for all $f \in \mathcal{F}$ *in the case of* $\rho = 0$ and were allowing "controlled deterioration" of these constraints when $\rho > 0$:

$$\text{dist}\left(\left[\begin{array}{c|c} \tau & f^T + \rho g^T \\ \hline f + \rho g & A(t) \end{array}\right], \mathbf{S}^{m+1}_+\right) \leq \alpha\rho \ \forall (f \in \mathcal{F}, g \in E).$$

When τ and α are large enough, this goal clearly is feasible. In the situation described in item 1) of Exercise, our desire is completely different: we want to keep the semidefinite constraints feasible, compensating for perturbations ρg by replacing the compliance τ with $\tau + \alpha\rho$. As it is shown by our analysis, this goal is infeasible – the "compensation in the value of compliance" should be at least quadratic in ρ.

2) With $\phi(\rho) = (\sqrt{\tau} + \sqrt{\alpha}\rho)^2$, problem (!) makes perfect sense; moreover, given $\tau \geq 0$, $\alpha \geq 0$, $t \in \mathcal{T}$ is feasible for (!) if and only if the system of relations

$$\begin{array}{ll} (a) & \left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0 \ \forall f \in \mathcal{F} \\ (b) & \left[\begin{array}{c|c} 2\alpha & g^T \\ \hline g & A(t) \end{array}\right] \succeq 0 \ \forall g \in E \end{array}$$

Since $\mathcal{F}$ is finite, (a) is just a finite collection of LMIs in t, τ; and since E is a centered at the origin ellipsoid, the results of section 3.4.2 allow to convert the semi-infinite LMI (b) into an equivalent tractable system of LMIs, so that $(a), (b)$ is computationally tractable.

The claim that t is feasible for the semi-infinite constraint

$$\left[\begin{array}{c|c} 2(\sqrt{\tau} + \sqrt{\alpha}\rho)^2 & f^T + \rho g^T \\ \hline f + \rho g & A(t) \end{array}\right] \geq 0 \, \forall (f \in \mathcal{F}, g \in E, \rho \geq 0) \qquad (*)$$

if and only if t satisfies (a) and (b) is evident. Indeed, if t is feasible for the latter semi-infinite LMI, t indeed satisfies (a) and (b) – look what happens when $\rho = 0$ and when $\rho \to \infty$. Vice versa, assume that t satisfies (a) and $()b)$, and let us prove that t satisfies $(*)$ as well. Indeed, given $\rho > 0$, let us set $\mu = \frac{\sqrt{\tau}}{\sqrt{\tau}+\rho\sqrt{\alpha}}$, $\nu = 1-\mu = \frac{\rho\sqrt{\alpha}}{\sqrt{\tau}+\rho\sqrt{\alpha}}$ and $s = \frac{\mu}{\nu}\rho$. For $f \in \mathcal{F}$ and $g \in E$ from (a), (b) it follows that

$$\left[\begin{array}{c|c} 2\tau & f^T \\ \hline f & A(t) \end{array}\right] \succeq 0, \ \left[\begin{array}{c|c} 2s^2\alpha & sg^T \\ \hline sg & A(t) \end{array}\right] \succeq 0;$$

or, which is the same by the Schur Complement Lemma, for every $\epsilon > 0$ one has

$$\|[A(t)+\epsilon I]^{-1/2} f\|_2 \leq \sqrt{2\tau}, \ \|[A(t)+\epsilon I]^{-1/2} g\|_2 \leq \sqrt{2\alpha},$$

whence, by the triangle inequality,

$$\|[A(t)+\epsilon I]^{-1/2}[f+\rho g]\|_2 \leq \sqrt{2\tau} + \sqrt{2\alpha}\rho,$$

meaning that

$$\left[\begin{array}{c|c} [\sqrt{2\tau}+\sqrt{2\alpha}\rho]^2 & f^T+\rho g^T \\ \hline f+\rho g & A(t)+\epsilon I \end{array}\right] \succeq 0.$$

The latter relation holds true for all $(\epsilon > 0, f \in \mathcal{F}, g \in E)$, and thus t is feasible for $(*)$.

Exercise 4.3: 1) The worst-case error of a candidate linear estimate $g^T y$ is

$$\max_{z,\xi:\|z\|_2\leq 1, \|\xi_2\|\leq 1} \|(Az+\xi)^T g - f^T z\|_2,$$

so that the problem of building the best, in the minimax sense, estimate reads

$$\min_{\tau,G} \left\{\tau : |(Az+\xi)^T g - f^T z| \leq \tau \ \forall([z;\xi] : \|z\|_2 \leq 1, \|\xi\|_2 \leq 1)\right\},$$

which is nothing but the RC of the uncertain Least Squares problem

$$\left\{\min_{\tau,g} \left\{\tau : (Az+\xi)^T g - f^T z \leq \tau, f^T z - (Az+\xi)^T g \leq \tau\right\} : \zeta := [z;\xi] \in \mathcal{Z} = B \times \Xi\right\} \qquad (*)$$

in variables g, τ with certain objective and two constraints affinely perturbed by $\zeta = [z;\xi] \in B \times \Xi$. The equivalent tractable reformulation of this RC clearly is

$$\min_{\tau,g} \left\{\tau : \|A^T g - f\|_2 + \|g\| \leq \tau\right\}.$$

2) Now we want of our estimate to satisfy the relations

$$\forall(\rho_z \geq 0, \rho_\xi \geq 0) : |(Az+\xi)^T g - f^T z| \leq \tau + \alpha_z\rho_z + \alpha_\xi\rho_\xi, \ \forall(z : \|z\|_2 \leq 1+\rho_z, \xi : \|\xi\|_2 \leq 1+\rho_\xi),$$

or, which is the same,

$$\forall [z;\xi] : |(Az+\xi)^T g - f^T z| \leq \tau + \alpha_z \text{dist}_{\|\cdot\|_2}(z, B) + \alpha_\xi \text{dist}_{\|\cdot|_2}(\xi, \Xi).$$

This is exactly the same as to say that g should be feasible for the GRC of the uncertainty-affected inclusion

$$(Az+\xi)^T g - f^T z \in \mathbf{Q} = [-\tau, \tau].$$

in the case where the uncertain perturbations are $[z;\xi]$, the perturbation structure for z is given by $\mathcal{Z}_z = B, \mathcal{L}_z = \mathbb{R}^n$ and the norm on $\mathbb{R}^n$ is $\|\cdot\|_2$, and the perturbation structure for ξ is given by $\mathcal{Z}_\xi = \Xi$, $\mathcal{L}_\xi = \mathbb{R}^m$ and the norm on $\mathbb{R}^m$ is $\|\cdot\|_2$. Invoking Proposition 4.2, g is feasible for our GRC if and only if

$$\begin{array}{ll} (a) & (Az+\xi)^T g - f^T z \in \mathbf{Q}\ \forall (z\in B, \xi\in\Xi) \\ (b.1) & |(A^T g - f)^T z| \le \alpha_z\ \forall (z: \|z\|_2 \le 1) \\ (b.2) & |\xi^T g| \le \alpha_\xi\ \forall (\xi : \|\xi\|_2 \le 1) \end{array}$$

or, which is the same, g meets the requirements if and only if

$$\|A^T g - f\|_2 + \|g\|_2 \le \tau,\ \|A^T g - f\|_2 \le \alpha_z,\ \|g\|_2 \le \alpha_\xi.$$

Exercise 4.4: 1) The worst-case error of a candidate linear estimate Gy is

$$\max_{z,\xi:\|z\|_2\le 1, \|\xi_2\|\le 1} \|G(Az+\xi) - Cz\|_2,$$

so that the problem of building the best, in the minimax sense, estimate reads

$$\min_{\tau,G} \left\{\tau : \|G(Az+\xi) - Cz\|_2 \le \tau\ \forall ([z;\xi] : \|z\|_2 \le 1, \|\xi\|_2 \le 1)\right\},$$

which is nothing but the RC of the uncertain Least Squares problem

$$\left\{\min_{\tau,g} \{\tau : \|G(Az+\xi) - Cz\|_2 \le \tau\} : \zeta := [z;\xi] \in \mathcal{Z} = B\times\Xi\right\} \tag{$*$}$$

in variables G, τ. The body of the left hand side of the uncertain constraint is

$$G(Az+\xi) - Cz = L_1^T(G) z R_1 + L_2^T(G)\xi R_2,\ L_1(G) = A^T G^T - C^T, R_1 = 1, L_2(G) = G^T, R_2 = 1$$

that is, we deal with structured norm-bounded uncertainty with two full uncertain blocks: $n\times 1$ block z and $m\times 1$ block ξ, the uncertainty level ρ being 1 (see section 3.3.1). Invoking Theorem 3.4, the system of LMIs

$$\begin{array}{l} \left[\begin{array}{c|c|c} u_0-\lambda & u^T & \\ \hline u & U & GA-C \\ \hline & [GA-C]^T & \lambda I \end{array}\right] \succeq 0, \\ \left[\begin{array}{c|c|c} v_0-\mu & v^T & \\ \hline v & V & G \\ \hline & G^T & \mu I \end{array}\right] \succeq 0 \\ \left[\begin{array}{c|c} \tau - u_0 - v_0 & -u^T - v^T \\ \hline -u-v & \tau I - U - V \end{array}\right] \succeq 0 \end{array} \tag{S}$$

in variables $G, \tau, \lambda, \mu, u_0, u, U, v_0, v, V$ is a tight within the factor $\pi/2$ safe tractable approximation of the RC.

2) Now we want of our estimate to satisfy the relations

$$\forall(\rho_z \ge 0, \rho_\xi \ge 0) : \|G(Az+\xi) - Cz\|_2 \le \tau + \alpha_z\rho_z + \alpha_\xi\rho_\xi,\ \forall(z: \|z\|_2 \le 1+\rho_z, \xi : \|\xi\|_2 \le 1+\rho_\xi),$$

or, which is the same,

$$\forall [z;\xi] : \|G(Az+\xi) - Cz\|_2 \le \tau + \alpha_z \mathrm{dist}_{\|\cdot\|_2}(z,B) + \alpha_\xi \mathrm{dist}_{\|\cdot\|_2}(\xi,\Xi).$$

This is exactly the same as to say that G should be feasible for the GRC of the uncertainty-affected inclusion

$$G(Az+\xi)-Cz\in\mathbf{Q}=\{w:\|w\|_2\leq\tau\}$$

in the case where the uncertain perturbations are $[z;\xi]$, the perturbation structure for z is given by $\mathcal{Z}_z=B,\mathcal{L}_z=\mathbb{R}^n$, the perturbation structure for ξ is given by $\mathcal{Z}_\xi=\Xi,\mathcal{L}_\xi=\mathbb{R}^m$, the global sensitivities w.r.t. z, ξ are, respectively, α_z, α_ξ, and all norms in the GRC setup are the standard Euclidean norms on the corresponding spaces. Invoking Proposition 4.2, G is feasible for our GRC if and only if

$$\begin{array}{ll}(a) & \|G(Az+\xi)-Cz\|_2\in\mathbf{Q}\ \forall(z\in B,\xi\in\Xi)\\ (b) & \|(GA-C)z\|_2\leq\alpha_z\ \forall(z:\|z\|_2\leq 1)\\ (c) & \|G\xi\|_2\leq\alpha_\xi\ \forall(\xi:\|\xi\|_2\leq 1).\end{array}$$

(b), (c) merely say that

$$\|GA-I\|_{2,2}\leq\alpha_z,\|G\|_{2,2}\leq\alpha_\xi,$$

while (a) admits the safe tractable approximation (S).

B.5 Exercises from Lecture 5

Exercise 5.1: From state equations (5.5.1) coupled with control law (5.5.3) it follows that

$$w^N=W_N[\Xi]\zeta+w_N[\Xi],$$

where $\Xi=\{U_t^z,U_t^d,u_t^0\}_{t=0}^N$ is the "parameter" of the control law (5.5.3), and $W_N[\Xi]$, $w_N[\Xi]$ are matrix and vector affinely depending on Ξ. Rewriting (5.5.2) as the system of linear constraints

$$e_j^Tw^N-f_j\leq 0,\ j=1,...,J,$$

and invoking Proposition 4.1, the GRC in question is the semi-infinite optimization problem

$$\begin{array}{lll}\min_{\Xi,\alpha} & \alpha & \\ \text{subject to} & & \\ & e_j^T[W_N[\Xi]\zeta+w_N[\Xi]]-f_j\leq 0\ \forall(\zeta:\|\zeta-\bar{\zeta}\|_s\leq R) & (a_j)\\ & e_j^TW_N[\Xi]\zeta\leq\alpha\ \forall(\zeta:\|\zeta\|_r\leq 1) & (b_j)\\ & \qquad\qquad 1\leq j\leq J. & \end{array}$$

This problem clearly can be rewritten as

$$\begin{array}{ll}\min_{\Xi,\alpha} & \alpha\\ \text{subject to} & \\ & R\|W_N^T[\Xi]e_j\|_{s_*}+e_j^T[W_N[\Xi]\bar{\zeta}+w_N[\Xi]]-f_j\leq 0,\ 1\leq j\leq J\\ & \|W_N^T[\Xi]e_j\|_{r_*}\leq\alpha,\ 1\leq j\leq J\end{array}$$

where

$$s_*=\frac{s}{s-1},\ r_*=\frac{r}{r-1}.$$

Exercise 5.2: The AAGRC is equivalent to the convex program

$$\begin{array}{ll} \min_{\Xi,\alpha} & \alpha \\ \text{subject to} & \\ & R\|W_N^T[\Xi]e_j\|_{s_*} + e_j^T[W_N[\Xi]\bar{\zeta} + w_N[\Xi]] - f_j \leq 0,\ 1 \leq j \leq J \\ & \|[W_N^T[\Xi]e_j]_{d,+}\|_{r_*} \leq \alpha,\ 1 \leq j \leq J \end{array}$$

where

$$s_* = \frac{s}{s-1},\ r_* = \frac{r}{r-1}$$

and for a vector $\zeta = [z; d_0; ...; d_N] \in \mathbb{R}^K$, $[\zeta]_{d,+}$ is the vector obtained from ζ by replacing the z-component with 0, and replacing every one of the d-components with the vector of positive parts of its coordinates, the positive part of a real a being defined as $\max[a, 0]$.

Exercise 5.3: 1) For $\zeta = [z; d_0; ...; d_{15}] \in \mathcal{Z} + \mathcal{L}$, a control law of the form (5.5.3) can be written down as

$$u_t = u_t^0 + \sum_{\tau=0}^{t} u_{t\tau} d_\tau,$$

and we have

$$x_{t+1} = \sum_{\tau=0}^{t} \left[u_\tau^0 - d_\tau + \sum_{s=0}^{\tau} u_{\tau s} d_s\right] = \sum_{\tau=0}^{t} u_\tau^0 + \sum_{s=0}^{t} \left[\sum_{\tau=s}^{t} u_{\tau s} - 1\right] d_s.$$

Invoking Proposition 4.1, the AAGRC in question is the semi-infinite problem

$$\begin{array}{ll} \min_{\{u_t^0, u_{t\tau}\},\alpha} & \alpha \\ \text{subject to} & \\ (a_x) & |\theta\left[\sum_{\tau=0}^{t} u_\tau^0\right]| \leq 0,\ 0 \leq t \leq 15 \\ (a_u) & |u_t^0| \leq 0,\ 0 \leq t \leq 15 \\ (b_x) & |\theta \sum_{s=0}^{t}\left[\sum_{\tau=s}^{t} u_{\tau s} - 1\right] d_s| \leq \alpha \\ & \quad \forall (0 \leq t \leq 15, [d_0; ...; d_{15}] : \|[d_0; ...; d_{15}]\|_2 \leq 1) \\ (b_u) & |\sum_{\tau=0}^{t} u_{t\tau} d_\tau| \leq \alpha \\ & \quad \forall (0 \leq t \leq 15, [d_0; ...; d_{15}] : \|[d_0; ...; d_{15}]\|_2 \leq 1) \end{array}$$

We see that the desired control law is linear ($u_t^0 = 0$ for all t), and the AAGRC is equivalent to the conic quadratic problem

$$\min_{\{u_{t\tau}\},\alpha} \left\{ \alpha : \begin{array}{l} \sqrt{\sum_{s=0}^{t}\left[\sum_{\tau=s}^{t} u_{\tau s} - 1\right]^2} \leq \theta^{-1}\alpha,\ 0 \leq t \leq 15 \\ \sqrt{\sum_{\tau=0}^{t} u_{\tau t}^2} \leq \alpha,\ 0 \leq t \leq 15 \end{array} \right\}.$$

2) In control terms, we want to "close" our toy linear dynamical system, where the initial state is once and for ever set to 0, by a linear state-based non-anticipative control law in such a way that the states $x_1, ..., x_{16}$ and the controls $u_1, ..., u_{15}$ in the closed loop system are "as insensitive to the perturbations $d_0, ..., d_{15}$ as possible," while measuring the changes in the state-control trajectory

$$w^{15} = [0; x_1; ...; x_{16}; u_1, ..., u_{15}]$$

in the weighted uniform norm $\|w^{15}\|_{\infty,\theta} = \max[\theta\|x\|_\infty, \|u\|_\infty]$, and measuring the changes in the sequence of disturbances $[d_0; ...; d_{15}]$ in the "energy" norm $\|[d_0; ...; d_{15}]\|_2$. Specifically, we are interested to find a linear non-anticipating state-based control law that results in the smallest possible constant α satisfying the relation

$$\forall \Delta d^{15} : \|\Delta w^{15}\|_{\infty,\theta} \le \alpha \|\Delta d^{15}\|_2,$$

where Δd^{15} is a shift of the sequence of disturbances, and Δw^{15} is the induced shift in the state-control trajectory.
3) The numerical results are as follows:

θ	α
1.e6	4.0000
10	3.6515
2	2.8284
1	2.3094

Exercise 5.4: 1) Denoting by x_γ^{ij} the amount of information in the traffic from i to j travelling through γ, by q_γ the increase in the capacity of arc γ, and by $O(k)$, $I(k)$ — the sets of outgoing, resp., incoming, arcs for node k, the problem in question becomes

$$\min_{\substack{\{x_\gamma^{ij}\},\\ \{q_\gamma\}}} \left\{ \sum_{\gamma\in\Gamma} c_\gamma q_\gamma : \begin{array}{c} \sum_{(i,j)\in\mathcal{J}} x_\gamma^{ij} \le p_\gamma + q_\gamma \;\forall\gamma \\ \sum_{\gamma\in O(k)} x_\gamma^{ij} - \sum_{\gamma\in I(k)} x_\gamma^{ij} = \left\{\begin{array}{ll} d_{ij}, & k=i \\ -d_{ij}, & k=j \\ 0, & k\notin\{i,j\}\end{array}\right. \\ \forall((i,j)\in\mathcal{J}, k\in V) \\ q_\gamma \ge 0, x_\gamma^{ij} \ge 0 \;\forall((i,j)\in\mathcal{J}, k\in V) \end{array} \right\}. \qquad (*)$$

2) To build the AARC of $(*)$ in the case of uncertain traffics d_{ij}, it suffices to plug into $(*)$, instead of decision variables x_γ^{ij}, affine functions $X_\gamma^{ij}(d) = \xi_\gamma^{ij,0} + \sum_{(\mu,\nu)\in\mathcal{J}} \xi_\gamma^{ij\mu\nu} d_{\mu\nu}$ of $d = \{d_{ij} : (i,j) \in \mathcal{J}\}$ (in the case of (a), the functions should be restricted to be of the form $X_\gamma^{ij}(d) = \xi_\gamma^{ij,0} + \xi_\gamma^{ij} d_{ij}$) and to require the resulting constraints in variables $q_\gamma, \xi_\gamma^{ij\mu\nu}$ to be valid for all realizations of $d \in \mathcal{Z}$. The resulting semi-infinite LO program is computationally tractable (as the AARC of an uncertain LO problem with fixed recourse, see section 5.3.1).

3) Plugging into $(*)$, instead of variables x_γ^{ij}, affine decision rules $X_\gamma^{ij}(d)$ of the just indicated type, the constraints of the resulting problem can be split into 3 groups:

$$\begin{array}{ll} (a) & \sum_{(i,j)\in\mathcal{J}} X_\gamma^{ij}(d) \le p_\gamma + q_\gamma \;\forall\gamma\in\Gamma \\ (b) & \sum_{\substack{(i,j)\in\mathcal{J}\\ \gamma\in\Gamma}} \mathcal{R}_\gamma^{ij} X_\gamma^{ij}(d) = r(d) \\ (c) & q_\gamma \ge 0, X_\gamma^{ij}(d) \ge 0 \,\forall((i,j)\in\mathcal{J}, \gamma\in\Gamma). \end{array}$$

In order to ensure the feasibility of a given candidate solution for this system with probability at least $1-\epsilon$, $\epsilon < 1$, when d is uniformly distributed in a box, the linear equalities (b) *must* be satisfied for all d's, that is, (b) induces a system $A\xi = b$ of linear equality constraints on the vector ξ of coefficients of the affine decision rules $X_\gamma^{ij}(\cdot)$. We can use this system of linear equations, if it is feasible, in order to express ξ as an affine function of a shorter vector η of "free" decision variables, that is, we can easily find H and h in such a way that $A\xi = b$ is equivalent to the existence of η such that $\xi = H\eta + h$. We can now plug $\xi = H\eta + h$ into $(a), (c)$ and forget about (b), thus ending up with a system of constraints of the form

$$\begin{array}{ll} (a') & a_\ell(\eta, q) + \alpha_\ell^T(\eta, q) d \le 0, \; 1 \le \ell \le L = \mathrm{Card}(\Gamma)(\mathrm{Card}(\mathcal{J})+1), \\ (b') & q \ge 0 \end{array}$$

with a_ℓ, α_ℓ affine in $[\eta; q]$ (the constraints in (a') come from the $\mathrm{Card}(\Gamma)$ constraints in (a) and the $\mathrm{Card}(\Gamma)\mathrm{Card}(\mathcal{J})$ constraints $X^{ij}_\gamma(d) \geq 0$ in (c)).

In order to ensure the validity of the uncertainty-affected constraints (a'), as evaluated at a candidate solution $[\eta; q]$, with probability at least $1 - \epsilon$, we can use either the techniques from Lecture 2, or the techniques from section 3.6.4.