SIAM J. OPTIM.
Vol. 13, No. 2, pp. 535–560

ROBUST SOLUTIONS OF UNCERTAIN QUADRATIC AND CONIC-QUADRATIC PROBLEMS*

A. BEN-TAL†, A. NEMIROVSKI†, AND C. ROOS‡

Dedicated to Jochem Zowe on the occasion of his 60th birthday.

Abstract. We consider a conic-quadratic (and in particular a quadratically constrained) optimization problem with uncertain data, known only to reside in some uncertainty set $\mathcal{U}$. The robust counterpart of such a problem leads usually to an NP-hard semidefinite problem; this is the case, for example, when $\mathcal{U}$ is given as the intersection of ellipsoids or as an n-dimensional box. For these cases we build a single, explicit semidefinite program, which approximates the NP-hard robust counterpart, and we derive an estimate on the quality of the approximation, which is essentially independent of the dimensions of the underlying conic-quadratic problem.

Key words. semidefinite relaxation of NP-hard problems, (conic) quadratic programming, robust optimization

AMS subject classification. 90C05

PII. S1052623401392354

1. Introduction.

A conic problem is an optimization problem of the form

$$\text{(CP)} \qquad \min_{x\in\mathbb{R}^n} \left\{c^T x \;:\; Ax - b \in \mathcal{K}\right\},$$

where $\mathcal{K} \subseteq \mathbb{R}^N$ is a closed pointed convex cone with nonempty interior. The *data* associated with (CP) is the triple (A, b, c), with $A \in \mathbb{R}^{N\times n}$, $b \in \mathbb{R}^N$, and $c \in \mathbb{R}^n$.

1.1. Uncertainty in conic problems.

When the data (A, b) associated with the constraint is *uncertain*[1] and is only known to belong to some *uncertainty set* $\mathcal{U}$, we speak about an *uncertain conic problem*, which is in fact a *family* of conic problems:

$$\text{(UCP)} \qquad \left\{\min_{x\in\mathbb{R}^n} \left\{c^T x \;:\; Ax - b \in \mathcal{K}\right\} \;:\; (A, b) \in \mathcal{U}\right\}.$$

The *robust optimization* (RO-) methodology, developed in [1, 2, 3, 4], associates with (UCP) a single deterministic convex problem, the so-called *robust counterpart* (RC):

$$\text{(RC)} \qquad \min_{x\in\mathbb{R}^n} \left\{c^T x \;:\; Ax - b \in \mathcal{K} \quad \forall\,(A, b) \in \mathcal{U}\right\}.$$

*Received by the editors July 16, 2001; accepted for publication (in revised form) February 12, 2002; published electronically October 1, 2002.
http://www.siam.org/journals/siopt/13-2/39235.html.

†Faculty of Industrial Engineering and Management, Technion-Israel Institute of Technology, 32000 Haifa, Israel (morbt@ie.technion.ac.il, nemirovs@ie.technion.ac.il). The research for this paper was done while the first author spent a sabbatical as visiting professor at TU Delft, with the support of TU Delft and the Dutch Organization of Scientific Research (NWO).

‡Faculty of Information Technology and Systems, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands (C.Roos@its.tudelft.nl).

[1]Without loss of generality we assume that the objective function is *certain*. Indeed, if c is uncertain, we can use the following equivalent formulation:

$$\min_{x\in\mathbb{R}^n} \left\{t \;:\; Ax - b \in \mathcal{K}, \quad c^T x - t \le 0\right\},$$

which is a conic problem with a certain objective function.

A feasible/optimal solution of (RC) is called a *robust feasible/optimal* solution of (UCP). The importance of these solutions is motivated and illustrated in [1, 2, 3, 4]. Of course, a crucial issue regarding the usefulness and applicability of the RO-methodology is the extent of the computational effort needed to solve problems such as (RC). At first glance, this looks hopeless, as (RC) is a *semi-infinite* conic problem. Nevertheless, for $\mathcal{K} = \mathbb{R}^N_+$ (the nonnegative orthant), i.e., when (CP) is a linear programming problem, the first two authors have shown [3] that for a very wide class of uncertainty sets $\mathcal{U}$ the resulting (RC)-problem is tractable (i.e., can be solved in time polynomial in the dimensions n, N of (CP)). This is also the case for conic-quadratic problems, i.e., when $\mathcal{K}$ is the Lorentz cone L^N:

$$L^N = \left\{ x \in \mathbb{R}^N \;:\; x_N \geq \sqrt{x_1^2 + \cdots + x_{N-1}^2} \right\},$$

provided that the uncertainty set is an ellipsoid (see [2]); the corresponding results are restated below in Theorems 2.1 and 3.2.

In this paper we deal with (conic) quadratic problems for which the uncertainty sets $\mathcal{U}$ are more general. In particular, we are interested in the case in which $\mathcal{U}$ is given as the intersection of *several* ellipsoids (we call this case the "∩-ellipsoid" case). The situation is then severely aggravated: problem (RC) becomes NP-hard (see [2] and also section 2.2 below).

The goal of this paper is to build *approximate robust counterparts* for the above NP-hard problems, which are computationally tractable and for which a concise statement can be given on the quality of the approximation.

1.2. Approximate robust counterparts. The approximation scheme we use is of the *lift-and-project* type. Specifically, let the uncertainty set $\mathcal{U}$ be given as

$$\mathcal{U} = \left(A^0, b^0\right) + W,$$

where (A^0, b^0) is a *nominal* data vector and W is a compact convex set, symmetric with respect to the origin. (W is interpreted as the *perturbation set.*) Our aim is to approximate the set $\mathcal{X}$ of robust feasible solutions:

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n \;:\; Ax - b \in K \quad \forall\, (A, b) \in \left(A^0, b^0\right) + W \right\}.$$

Towards this aim, we augment the vector x by an additional vector u and look at the following set $\mathcal{R}$, which is given by conic constraints:

$$\mathcal{R} := \{(x, u) \;:\; Px + Qu + r \in \hat{K}\}$$

in terms of some matrices P and Q, a vector r, and a closed convex pointed nonempty cone $\hat{K}$ with nonempty interior.

Definition 1.1. *We say that $\mathcal{R}$ is an* approximate robust counterpart *of $\mathcal{X}$ if the projection of $\mathcal{R}$ onto the plane of x-variables, i.e., the set $\hat{\mathcal{R}} \subseteq \mathbb{R}^n$ given by*

$$\hat{\mathcal{R}} := \{x \;:\; Px + Qu + r \in \hat{K} \text{ for some } u\},$$

is contained in $\mathcal{X}$:

$$\hat{\mathcal{R}} \subseteq \mathcal{X}. \tag{1.1}$$

1.3. Level of conservativeness. Next we introduce a measure, called the *level of conservativeness*, for the proximity of $\hat{\mathcal{R}}$ to $\mathcal{X}$. To this end, let us look at an uncertainty set

$$\mathcal{U}_\rho = \left\{ \left(A^0, b^0\right) + \rho W \right\}, \quad \rho \geq 1.$$

Compared to the original uncertainty set $\mathcal{U} = \mathcal{U}_1$, the perturbations in $\mathcal{U}_\rho$ are increased by a factor ρ. The set of robust feasible solutions corresponding to $\mathcal{U}_\rho$ is

$$\mathcal{X}_\rho := \{x \in \mathbb{R}^n \; : \; Ax - b \in K \quad \forall \, (A, b) \in \mathcal{U}_\rho\}.$$

Clearly, $\mathcal{X}_1 = \mathcal{X}$. As ρ increases from 1, the set $\mathcal{X}_\rho$ shrinks, and eventually we will have

$$\mathcal{X}_\rho \subseteq \hat{\mathcal{R}}. \tag{1.2}$$

The smallest ρ for which this occurs,

$$\rho^* = \inf_{\rho \geq 1} \{\rho \; : \; \mathcal{X}_\rho \subseteq \hat{\mathcal{R}}\}, \tag{1.3}$$

is called the *level of conservativeness* of the approximate counterpart $\mathcal{R}$. Thus we have

$$\mathcal{X}_{\rho^*} \subseteq \hat{\mathcal{R}} \subseteq \mathcal{X}.$$

The implications of this concept are twofold:

(i) If $x \in \hat{\mathcal{R}}$, i.e., if x *can* be augmented to a solution $(x, u) \in \mathcal{R}$, then x *is* a robust feasible solution of problem (CP). This follows from relation (1.1).

(ii) If $x \notin \hat{\mathcal{R}}$, i.e., if x *cannot* be augmented to a solution $(x, u) \in \mathcal{R}$, then x is *not* a robust feasible solution of problem (CP) if its uncertainty set $\mathcal{U}$ is increased to $\mathcal{U}_\rho$, with $\rho \geq \rho^*$.

In real-world applications, the level of uncertainty (the size of vectors in the perturbation set W) is not something that can be specified precisely by the decision maker; it is more likely that it will be specified *up to a factor of order* 1. Thus, for problems for which the level of conservativeness itself is of order 1, the approximate robust counterpart can be as meaningful as the *true* robust counterpart. The main results of the paper show that for *conic-quadratic problems* under "∩-ellipsoid" uncertainty, this is indeed the case: we derive an explicit semidefinite program which is an approximate robust counterpart of the uncertain conic-quadratic problem and whose level of conservativeness is a constant, essentially independent of the dimensions n, N of (CP). The profound implication of these results is that the NP-hardness, associated with uncertain conic-quadratic problems, can be circumvented, and a computationally tractable tool is at hand, capable of producing robust solutions to these difficult problems.

1.4. Organization of the paper. We start in the next section by considering the case of an uncertain convex quadratically constrained problem. The more general case of conic-quadratic problems is considered in section 3. In both cases we first recall the results already known for simple-ellipsoid uncertainty from [2]. The main results concern the cases of ∩-ellipsoid uncertainty, and, as a special case of it, box uncertainty. In the box uncertainty case, we present robust counterparts for which the level of conservativeness is bounded above by a constant, namely, $\pi/2$. The robust counterparts presented in the $\cap$-ellipsoid cases have level of conservativeness at most

$$\left(2\log\left(6\sum_{k=1}^{K}\operatorname{rank} Q_k\right)\right)^{\frac{1}{2}}. \tag{1.4}$$

The matrices Q_k in this expression are symmetric positive semidefinite matrices, of the same order L, and it will always be assumed that their sum is positive definite. Note that the expression under the logarithm in (1.4) may be as large as $6KL$. In many applications, however, it is likely to be much smaller.

2. Approximate robust counterparts of uncertain quadratically constrained problems. A generic convex quadratically constrained problem has the form

$$\min_{x\in\mathbb{R}^n}\left\{c_0^T x \;:\; x^T A_i^T A_i x \le 2b_i^T x + c_i, \quad i=1,\dots,m\right\}, \tag{QC}$$

where the matrices A_i have size $m_i \times n$. Note that (QC) can be written as a conic-quadratic problem:[2]

$$\min_{x\in\mathbb{R}^n}\left\{c_0^T x \;:\; \left\|\begin{pmatrix} A_i x \\ \frac{1}{2}\left(1-2b_i^T x - c_i\right)\end{pmatrix}\right\| \le \frac{1}{2}\left(1+2b_i^T x + c_i\right), \quad i=1,\dots,m\right\}.$$

However, we shall treat the direct formulation (QC). An uncertain (QC)-problem corresponds to the case in which the data $\{(A_i,b_i,c_i) \;:\; i=1,\dots,m\}$ of the problem is uncertain. To model the uncertainty, we use *uncertainty sets* $\mathcal{U}_i$, and we assume

$$(A_i,b_i,c_i)\in\mathcal{U}_i, \quad i=1,\dots,m, \tag{2.1}$$

where the uncertainty set $\mathcal{U}_i$ associated with the ith constraint is given as the intersection of ellipsoids.

In order to construct the robust counterpart (RC) of problem (QC), we should be able to construct the robust counterpart of a single uncertain quadratic constraint

$$x^T A^T A x \le 2b^T x + c \quad \forall\,(A,b,c)\in\mathcal{U}_\rho, \tag{UQC}$$

where $\mathcal{U}_\rho$ is the intersection of K ellipsoids; i.e., it is described as

$$\mathcal{U}_\rho = \left\{(A,b,c) = \left(A^0,b^0,c^0\right) + \sum_{\ell=1}^{L} y_\ell\left(A^\ell,b^\ell,c^\ell\right) \;:\; y\in\rho V\right\},$$

where V is the intersection of K ellipsoids,

$$V = \left\{y\in\mathbb{R}^L \;:\; y^T Q_k y \le 1, \quad k=1,\dots,K\right\},$$

and where each $Q_k \succeq 0$. As stated above, we make the generic assumption that $\sum_{k=1}^{K} Q_k \succ 0$.

[2]When not further specified, $\|.\|$ always denotes the 2-norm $\|.\|_2$.

2.1. Simple ellipsoidal uncertainty. In this case we look for the robust counterpart of the convex quadratic constraint

$$x^T A^T A x \le 2b^T x + c \quad \forall\, (A, b, c) \in \mathcal{U}_{\text{simple}}, \tag{2.2}$$

where

$$\mathcal{U}_{\text{simple}} = \left\{ (A, b, c) = \left(A^0, b^0, c^0\right) + \sum_{\ell=1}^{L} y_\ell \left(A^\ell, b^\ell, c^\ell\right) \ : \ \|y\|^2 \le 1 \right\},$$

with

$$A^\ell \in \mathbb{R}^{M\times n}, \quad b^\ell \in \mathbb{R}^n, \quad c^\ell \in \mathbb{R}, \qquad \ell = 0, \dots, L.$$

This is a special case of (UQC), where $K = 1$ and Q_1 is the identity matrix. This case has been considered already in [2, 6], where the following result is proved.

THEOREM 2.1. *A vector $x \in \mathbb{R}^n$ is a solution of (2.2) if and only if for some $\lambda \in \mathbb{R}$ the pair (x, λ) is a solution of the following linear matrix inequality (LMI):*

$$\left[\begin{array}{c|cccc|c}
c^0 + 2x^T b^0 - \lambda & \frac{1}{2}c^1 + x^T b^1 & \cdots & \frac{1}{2}c^L + x^T b^L & (A^0 x)^T \\
\hline
\frac{1}{2}c^1 + x^T b^1 & \lambda & & & (A^1 x)^T \\
\vdots & & \ddots & & \vdots \\
\frac{1}{2}c^L + x^T b^L & & & \lambda & (A^L x)^T \\
\hline
A^0 x & A^1 x & \cdots & A^L x & I_M
\end{array}\right] \succeq 0.$$

Fundamental in the proof of this result is the so-called S-lemma (see, e.g., [5]).

LEMMA 2.2 (S-lemma). *Let P and Q be symmetric matrices of the same order, and assume that $y^T P y > 0$ for some vector y. Then the implication*

$$z^T P z \ge 0 \quad \Rightarrow \quad z^T Q z \ge 0$$

is valid if and only if $Q \succeq \lambda P$ for some $\lambda \ge 0$.

2.2. Intersection-of-ellipsoids uncertainty. In this case we consider the robust feasible set for (UQC):

$$\mathcal{X}_\rho = \left\{ x \ : \ x^T A^T A x \le 2b^T x + c \quad \forall\, (A, b, c) \in \mathcal{U}_\rho \right\},$$

where

$$\mathcal{U}_\rho = \left\{ (A, b, c) = \left(A^0, b^0, c^0\right) + \rho \sum_{\ell=1}^{L} y_\ell \left(A^\ell, b^\ell, c^\ell\right) \ : \ y^T Q_k y \le 1, \quad k = 1, \dots, K \right\}.$$

Note that the robust counterpart of (UQC) with the $\cap$-ellipsoid uncertainty $\mathcal{U}_\rho$ is, in general, NP-hard. In fact, the associated *analysis* problem "given x, check whether it is robust feasible" is already NP-hard. To support our claim, we observe that the analysis problem in question is at least as difficult as the problem of maximizing a positive definite quadratic form over the unit cube:

$$\text{(MAXQ)} \qquad \text{given } Q \succ 0 \text{ and } q \in \mathbb{R}, \text{ check whether } \max_{y:|y_\ell|\le 1} y^T Q y \le q;$$

the latter problem is known to be NP-hard.[3] Indeed, given data Q, q of (MAXQ) with a $K \times K$ matrix Q, let us find a $K \times K$ matrix D such that $D^T D = Q$ and associate with Q, q the following uncertainty set for (UQC):

$$\mathcal{U}_1 = \left\{ (A, b, c) = (0_{K\times K}, 0_{K\times 1}, q) + \sum_{\ell=1}^{K} y_\ell \left(A^\ell, 0_{K\times 1}, 0\right) \ : \ y_\ell^2 \le 1, \quad \ell = 1, \ldots, K \right\},$$

where the first column of A_ℓ equals the ℓth column of D, and the remaining columns in A_ℓ are zero, $\ell = 1, \ldots, K$. With this setup, one has $(A, b, c) = (Dy, 0_{(K-1)\times K}, 0_{K\times 1}, q)$. If $x = (1, 0, 0, \ldots, 0)^T \in \mathbb{R}^K$, then $Ax = Dy$, and hence checking whether x is robust feasible for (UQC) is exactly the same as checking whether $y^T Q y = \|Dy\|^2$ is $\le q$ for all y with $|y_\ell| \le 1$; thus, the NP-hard problem (MAXQ) is reducible to the analysis problem for (UQC) with a pretty simple ∩-ellipsoid uncertainty ("box uncertainty": $L = K$ and Q_k is the diagonal matrix with the only nonzero diagonal entry, equal to 1, in the cell (k, k)).

The NP-hardness of the robust counterpart of (UQC) in the presence of ∩-ellipsoid uncertainty motivates our current goal—to build a tractable approximate robust counterpart. We introduce some more convenient notations:

$$a[x] = A^0 x, \quad c[x] = 2x^T b^0 + c^0, \quad A_\rho[x] = \rho\left(A^1 x, \ldots, A^L x\right),$$

$$b_\rho[x] = \rho \begin{bmatrix} x^T b^1 \\ \vdots \\ x^T b^L \end{bmatrix}, \qquad d_\rho = \tfrac{1}{2}\rho \begin{bmatrix} c^1 \\ \vdots \\ c^L \end{bmatrix}.$$

Then one may easily verify that $x \in \mathcal{X}_\rho$ holds if and only if

$$y^T Q_k y \le 1,\ k = 1, \ldots, K \Rightarrow (a[x] + A_\rho[x]y)^T (a[x] + A_\rho[x]y) \le 2\left(b_\rho[x] + d_\rho\right)^T y + c[x].$$

The last inequality can be rewritten as

$$y^T A_\rho[x]^T A_\rho[x] y + 2y^T \left(A_\rho[x]^T a[x] - b_\rho[x] - d_\rho\right) \le c[x] - a[x]^T a[x].$$

Hence we obtain that $x \in \mathcal{X}_\rho$ holds if and only if

$$\begin{gathered} y^T Q_k y \le 1,\ \ k = 1, \ldots, K \quad \Rightarrow \\ y^T A_\rho[x]^T A_\rho[x] y + 2y^T \left(A_\rho[x]^T a[x] - b_\rho[x] - d_\rho\right) \le c[x] - a[x]^T a[x]. \end{gathered} \tag{2.3}$$

Observe that if y satisfies $y^T Q_k y \le 1$, then so does $-y$. Hence, $x \in \mathcal{X}_\rho$ holds if and only if

$$\begin{gathered} y^T Q_k y \le 1,\ \ k = 1, \ldots, K \quad \Rightarrow \\ y^T A_\rho[x]^T A_\rho[x] y \pm 2y^T \left(A_\rho[x]^T a[x] - b_\rho[x] - d_\rho\right) \le c[x] - a[x]^T a[x]. \end{gathered}$$

[3] From MAXCUT-related studies it is known that it is NP-hard even to approximate the maximum of a positive definite quadratic form over the unit cube within relative accuracy like 5% [7].

Therefore, we may replace the implication by

$$t^2 \le 1,\ y^T Q_k y \le 1,\ \ k = 1, \dots, K \quad \Rightarrow$$
$$y^T A_\rho[x]^T A_\rho[x] y + 2ty^T \left(A_\rho[x]^T a[x] - b_\rho[x] - d_\rho\right) \le c[x] - a[x]^T a[x].$$

This implication certainly holds if there exist $\lambda_k \ge 0$, $k = 1, \dots, K$, such that for all t and for all y

$$\sum_{k=1}^K \lambda_k y^T Q_k y + \left(c[x] - a[x]^T a[x] - \sum_{k=1}^K \lambda_k\right) t^2$$
$$\ge y^T A_\rho[x]^T A_\rho[x] y + 2ty^T \left(A_\rho[x]^T a[x] - b_\rho[x] - d_\rho\right).$$

We can equivalently express the last condition in a more concise form:

$$\begin{bmatrix} t \\ y \end{bmatrix}^T \begin{bmatrix} c[x] - a[x]^T a[x] - \sum_{k=1}^K \lambda_k & \left(A_\rho[x]^T a[x] - b_\rho[x] - d_\rho\right)^T \\ A_\rho[x]^T a[x] - b_\rho[x] - d_\rho & \sum_{k=1}^K \lambda_k Q_k - A_\rho[x]^T A_\rho[x] \end{bmatrix} \begin{bmatrix} t \\ y \end{bmatrix} \ge 0.$$

In other words, $x \in \mathcal{X}_\rho$ certainly holds if

$$\exists \lambda \ge 0 \ \text{ s.t. } \begin{bmatrix} c[x] - a[x]^T a[x] - \sum_{k=1}^K \lambda_k & \left(A_\rho[x]^T a[x] - b_\rho[x] - d_\rho\right)^T \\ A_\rho[x]^T a[x] - b_\rho[x] - d_\rho & \sum_{k=1}^K \lambda_k Q_k - A_\rho[x]^T A_\rho[x] \end{bmatrix} \succeq 0,$$

which can be rewritten as

$$\exists \lambda \ge 0 \ \text{ s.t. } \begin{bmatrix} c[x] - \sum_{k=1}^K \lambda_k & \left(-b_\rho[x] - d_\rho\right)^T \\ -b_\rho[x] - d_\rho & \sum_{k=1}^K \lambda_k Q_k \end{bmatrix} \succeq \begin{bmatrix} a[x]^T \\ -A_\rho[x]^T \end{bmatrix} \left[a[x] \quad -A_\rho[x]\right].$$

By the Schur complement lemma the latter is equivalent to

$$\exists \lambda \ge 0 \ \text{ s.t. } \begin{bmatrix} c[x] - \sum_{k=1}^K \lambda_k & \left(-b_\rho[x] - d_\rho\right)^T & a[x]^T \\ -b_\rho[x] - d_\rho & \sum_{k=1}^K \lambda_k Q_k & -A_\rho[x]^T \\ a[x] & -A_\rho[x] & I_M \end{bmatrix} \succeq 0.$$

Thus we have proved the following theorem.

THEOREM 2.3. *The set $\mathcal{R}_\rho$ of (x, λ) satisfying $\lambda \ge 0$ and*

$$\begin{bmatrix} c[x] - \sum_{k=1}^K \lambda_k & \left(-b_\rho[x] - d_\rho\right)^T & a[x]^T \\ -b_\rho[x] - d_\rho & \sum_{k=1}^K \lambda_k Q_k & -A_\rho[x]^T \\ a[x] & -A_\rho[x] & I_M \end{bmatrix} \succeq 0 \tag{2.4}$$

is an approximate robust counterpart of the set $\mathcal{X}_\rho$ of robust feasible solutions of (UQC).

Unlike the case in which $\mathcal{U}$ is a single ellipsoid, in the general case of $\cap$-ellipsoids we can no longer use the S-lemma (Lemma 2.2) to get an equivalence between the LMI (2.4) and the uncertain quadratic inequality (UQC). Thus another fundamental tool is needed, and this is offered by our so-called approximate S-lemma (cf. Lemma

A.6 in the appendix). With this tool we are able to derive the main results of this paper: Theorem 2.4, which follows, and Theorem 3.5 in the next subsection.

THEOREM 2.4. *The level of conservativeness of the approximate robust counterpart $\mathcal{R}$ (as given by Theorem 2.3) of the set $\mathcal{X}$ is at most*

$$\tilde{\rho} := \left(2\log\left(6\sum_{k=1}^{K}\operatorname{rank} Q_k\right)\right)^{\frac{1}{2}}. \tag{2.5}$$

Proof. We have to show that when x cannot be extended to a solution (x,λ) of (2.4), then there exists $\zeta_* \in \mathbb{R}^n$ such that

$$\zeta_*^T Q_k \zeta_* \le 1, \quad k = 1, \dots, K, \tag{2.6}$$

and

$$\tilde{\rho}^2 \zeta_*^T A_\rho[x]^T A_\rho[x]\zeta_* + 2\tilde{\rho}\zeta_*^T \left(A_\rho[x]^T a[x] - b_\rho[x] - d\right) \ge c[x] - a[x]^T a[x]. \tag{2.7}$$

The proof is based on Lemma A.6, which can be seen as an "approximate S-lemma." Using the notation of this lemma, let

$$R = \left[\begin{array}{c|c} 0 & \left(A_\rho[x]^T a[x] - b_\rho[x] - d\right)^T \\ \hline A_\rho[x]^T a[x] - b_\rho[x] - d & A_\rho[x]^T A_\rho[x] \end{array}\right],$$

$$R_0 = \left[\begin{array}{c|c} 1 & 0^T \\ \hline 0 & 0 \end{array}\right], \quad R_k = \left[\begin{array}{c|c} 0 & 0^T \\ \hline 0 & Q_k \end{array}\right],$$

and $r_0 = 1$. Note that $R_1, \dots, R_K$ are positive semidefinite, and, due to our generic assumption on the Q_k's,

$$R_0 + \sum_{k=1}^{K} R_k = \left[\begin{array}{c|c} 1 & 0^T \\ \hline 0 & \sum_{k=1}^{K} Q_k \end{array}\right] \succ 0.$$

Moreover, R_0 is dyadic and $r_0 = 1 > 0$. We are therefore in the situation in Lemma A.6 where R_0 is dyadic and $r_0 > 0$. Hence the estimate (2.5) is valid. We proceed by distinguishing two cases.

Case I. We assume in this case that there exist $\lambda_0, \dots, \lambda_K \ge 0$ such that

$$R \preceq \sum_{k=0}^{K} \lambda_k R_k, \tag{2.8}$$

$$\sum_{k=0}^{K} \lambda_k \le c[x] - a[x]^T a[x]. \tag{2.9}$$

Since the LMI (2.4) was shown to imply (2.3), our assumption that x cannot be extended to a solution of (2.4) implies that x cannot be extended to a solution of (2.3). On the other hand, by (2.8),

$$(t, y^T) R \begin{pmatrix} t \\ y \end{pmatrix} \le \sum_{k=0}^{K} \lambda_k (t, y^T) R_k \begin{pmatrix} t \\ y \end{pmatrix} \quad \forall t, y.$$

Hence, using the definition of R and R_k, with $t = 1$,

$$y^T A_\rho[x]^T A_\rho[x]y + 2y^T \left(A_\rho[x]^T a[x] - b_\rho[x] - d\right) \le \lambda_0 + \sum_{k=1}^{K} \lambda_k y^T Q_k y \le \sum_{k=0}^{K} \lambda_k$$

whenever $y^T Q_k y \le 1$, $k = 1, \dots, K$. Therefore, by (2.9),

$$y^T A_\rho[x]^T A_\rho[x]y + 2y^T \left(A_\rho[x]^T a[x] - b_\rho[x] - d\right) \le c[x] - a[x]^T a[x],$$

showing that x *is* a solution of (2.3). Due to this contradiction, Case I cannot occur.

Case II. In this case there do not exist $\lambda_0, \dots, \lambda_K \ge 0$ such that (2.8) and (2.9) hold. Hence, every feasible solution of problem (SDP) (in Lemma A.6) has objective value greater than $c[x] - a[x]^T a[x]$. Thus we have

$$\text{SDP} > c[x] - a[x]^T a[x]. \tag{2.10}$$

By Lemma A.6, there exists $y_* = (t_*, \eta_*)$ such that

$$y_*{}^T R_0 y_* = t_*^2 \le r_0 = 1, \tag{2.11}$$

$$y_*{}^T R_k y_* = \eta_*{}^T Q_k \eta_* \le \tilde{\rho}^2, \quad k = 1, \dots, K, \tag{2.12}$$

$$\begin{aligned} y_*{}^T R y_* &= \eta_*^T A_\rho[x]^T A_\rho[x]\eta_* + 2t_*\eta_*^T \left(A_\rho[x]^T a[x] - b_\rho[x] - d\right) = \text{SDP} \\ &> c[x] - a[x]^T a[x], \end{aligned} \tag{2.13}$$

by (2.10). Setting $\bar{\eta} = \tilde{\rho}^{-1}\eta_*$, the last three relations become

$$\begin{cases} |t_*| \le 1, \\ \bar{\eta}^T Q_k \bar{\eta} \le 1, \quad k = 1, \dots, K, \\ \tilde{\rho}^2 \bar{\eta}^T A_\rho[x]^T A_\rho[x]\bar{\eta} + 2\tilde{\rho}\bar{\eta}^T t_* \left(A_\rho[x]^T a[x] - b_\rho[x] - d\right) > c[x] - a[x]^T a[x]. \end{cases} \tag{2.14}$$

It is easily seen that if $(t_*, \bar{\eta})$ is a solution of (2.14), then either $\zeta_* = \bar{\eta}$ or $\zeta_* = -\bar{\eta}$ is a solution of (2.6)–(2.7).

This completes the proof of Theorem 2.4. □

2.3. Box uncertainty.

THEOREM 2.5. *Consider the uncertain quadratic constraint* (UQC)*, where the uncertainty set is the "box"*

$$\mathcal{U}_\rho = \left\{ (A, b, c) = \left(A^0, b^0, c^0\right) + \rho \sum_{\ell=1}^{L} y_\ell \left(A^\ell, b^\ell, c^\ell\right) \;:\; |y_\ell| \le 1, \quad \ell = 1, \dots, L \right\}. \tag{2.15}$$

Then

(i) *the set $\mathcal{R}_\rho$ of (x, λ) satisfying $\lambda \ge 0$ and*

$$\begin{bmatrix} c[x] - \sum_{\ell=1}^{L} \lambda_\ell & (-b_\rho[x] - d)^T & a[x]^T \\ -b_\rho[x] - d & \text{diag}(\lambda) & -A_\rho[x]^T \\ a[x] & -A_\rho[x] & I_M \end{bmatrix} \succeq 0 \tag{2.16}$$

is an approximate robust counterpart of the set $\mathcal{X}_\rho$ of robust feasible solutions of (UQC)*, and*

(ii) *the level of conservativeness Ω of $\mathcal{R}$ is at most*

$$\Omega \le \frac{\pi}{2}. \tag{2.17}$$

Proof. Part (i) of the theorem is a special case of Theorem 2.3, with $K = L$ and where each Q_k is equal to a diagonal matrix whose only nonzero element is a 1 in the kth position of the diagonal. Thus it remains to prove part (ii). This proof will proceed in two steps. In Step 1 we build an approximate robust counterpart $\hat{\mathcal{R}}$ of (UQC), which is seemingly different from the $\mathcal{R}$ given in part (i) of the theorem, and we prove that the level of conservativeness of $\hat{\mathcal{R}}$ is $\pi/2$. In step 2 we demonstrate that $\hat{\mathcal{R}}$ is in fact equivalent to $\mathcal{R}$.

Step I. (Construction of $\hat{\mathcal{R}}$). The quadratic constraint

$$x^T A^T A x \le 2b^T x + c$$

is equivalent, by the Schur complement lemma, to the LMI

$$\begin{bmatrix} 2b^T x + c & (Ax)^T \\ Ax & I \end{bmatrix} \succeq 0.$$

Thus the robust feasible set of (UQC), $\mathcal{X}_\rho$, corresponding to the uncertainty set $\mathcal{U}_\rho$ in (2.15), is given by

$$\mathcal{X}_\rho = \left\{ x : \begin{bmatrix} 2x^T b^0 + c^0 & \left(A^0 x\right)^T \\ A^0 x & I \end{bmatrix} + \rho \sum_{\ell=1}^{L} y_\ell \begin{bmatrix} 2x^T b^\ell + c^\ell & \left(A^\ell x\right)^T \\ A^\ell x & 0 \end{bmatrix} \succeq 0,\ \|y\|_\infty \le 1 \right\}.$$

An evident sufficient condition for a vector x to belong to $\mathcal{X}_\rho$ is the possibility of extending x by L matrix variables $X^1, \ldots, X^L$, which together satisfy the following LMIs:

$$\begin{aligned} X^\ell &\succeq \pm\rho \begin{bmatrix} 2x^T b^\ell + c^\ell & \left(A^\ell x\right)^T \\ A^\ell x & 0 \end{bmatrix} \equiv \tilde{A}_\ell[x], \quad \ell = 1, \ldots, L, \\ &\begin{bmatrix} 2x^T b^0 + c^0 & \left(A^0 x\right)^T \\ A^0 x & I \end{bmatrix} \succeq \textstyle\sum_{\ell=1}^{L} X^\ell. \end{aligned} \tag{2.18}$$

The system (2.18) is the aforementioned approximate robust counterpart of $\hat{\mathcal{R}}$. The fact that the level of conservativeness of $\mathcal{R}$ is at most $\pi/2$ is then a direct consequence of [4, Theorem 4.4.1, p. 190]. In using the latter, note that rank $\tilde{A}_\ell[x] = 2$.

Step II. ($\hat{\mathcal{R}}$ is equivalent to $\mathcal{R}$). The equivalence is shown in two parts:

II.1. If $(x, \lambda_1, \ldots, \lambda_K)$ is a solution of (2.16), then x can be extended to a solution $\left(x, X^1, \ldots, X^L\right)$ of (2.18).

II.2. If $\left(x, X^1, \ldots, X^L\right)$ a solution of (2.18), then for some $\lambda \ge 0$, $(x, \lambda_1, \ldots, \lambda_K)$ is a solution of (2.16).

The proofs of both parts rely on the following lemma, whose proof depends on Lemma A.8 in the appendix.

LEMMA 2.6. *Let* $c \in \mathbb{R}$, $d \in \mathbb{R}^m$, *and*

$$P = \begin{bmatrix} 2c & d^T \\ d & 0 \end{bmatrix}.$$

Then

(i) *for every* $\lambda > 0$ *the matrix*

$$Y[\lambda, P] := \begin{bmatrix} \lambda + \frac{c^2}{\lambda} & \frac{cd^T}{\lambda} \\ \frac{cd}{\lambda} & \frac{dd^T}{\lambda} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\lambda} \begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}^T \tag{2.19}$$

belongs to the set

$$\mathcal{L}[P, -P] = \{X \;:\; X \succeq P, X \succeq -P\}; \tag{2.20}$$

(ii) *if* $P \neq 0$*, then for every* $X \in \mathcal{L}[P, -P]$ *there exists* $\lambda > 0$ *such that* $X \succeq Y[\lambda, P]$.

Proof. Setting

$$a = (1, 0, \ldots, 0)^T, \qquad b = (c, d_1, \ldots, d_m)^T,$$

one has

$$ab^T + ba^T = \begin{pmatrix} 2c & d^T \\ d & 0 \end{pmatrix}, \quad \lambda aa^T + \frac{1}{\lambda} bb^T = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\lambda} \begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}^T = Y[\lambda, P].$$

Hence, Lemma 2.6 immediately follows from Lemma A.8. □

Proof of II.1. For our case of box uncertainty we have $(Q_k)_{kk} = 1$ and $(Q_k)_{ij} = 0$ ($i \neq k$ or $j \neq k$), and so the system (2.16) reduces (by the Schur complement lemma) to

$$\begin{bmatrix} c[x] - \sum_{\ell=1}^L \lambda_\ell & a[x]^T \\ a[x] & I_M \end{bmatrix} \succeq \sum_{\ell,\, \lambda_\ell > 0} \frac{1}{\lambda_\ell} \left(f_\ell[x] f_\ell[x]^T\right), \tag{2.21}$$

where $\lambda_\ell \geq 0$, $\ell = 1, \ldots, L$, with $\lambda_\ell = 0 \Rightarrow f_\ell[x] = 0$, and $f_1[x], \ldots, f_L[x]$ are the columns of the matrix

$$\begin{bmatrix} (b_\rho[x] + d)^T \\ A_\rho[x]^T \end{bmatrix},$$

i.e.,

$$f_\ell[x] = \begin{bmatrix} \rho x^T b^\ell + \frac{\rho}{2} c^\ell \\ \rho A^\ell x \end{bmatrix} = \rho \begin{bmatrix} x^T b^\ell + \frac{1}{2} c^\ell \\ A^\ell x \end{bmatrix}. \tag{2.22}$$

We rewrite (2.21) as

$$\begin{bmatrix} c[x] & a[x]^T \\ a[x] & I_M \end{bmatrix} \succeq \sum_{\ell,\, \lambda_\ell > 0} \left(\begin{bmatrix} \lambda_\ell & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\lambda_\ell} \left(f_\ell[x] f_\ell[x]^T\right) \right),$$

or, more explicitly,

$$\begin{bmatrix} 2x^T b^0 + c^0 & \left(A^0 x\right)^T \\ A^0 x & I_M \end{bmatrix} \succeq \sum_{\ell,\, \lambda_\ell > 0} \left(\begin{bmatrix} \lambda_\ell & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\lambda_\ell} \left(f_\ell[x] f_\ell[x]^T\right) \right). \tag{2.23}$$

Note that the matrix under the sum has the form of the matrix $Y[\lambda, P]$ in (2.19). Hence, denoting this matrix as X_ℓ whenever $\lambda_\ell > 0$, we may conclude from Lemma 2.6 that

$$\lambda_\ell > 0 \Rightarrow X_\ell \succeq \pm\rho \begin{bmatrix} 2x^T b^\ell + c^\ell & (A^\ell x)^T \\ A^\ell x & 0 \end{bmatrix}. \tag{2.24}$$

Setting $X_\ell = 0$ whenever $\lambda_\ell = 0$, and using (2.23), we ensure that $x, X^1, \ldots, X^L$ is a solution of (2.18). This proves II.1.

Proof of II.2. Assume that x can be extended to a solution $x, X^1, \ldots, X^L$ of (2.18). By Lemma 2.6, for those ℓ's for which

$$\rho \begin{bmatrix} 2x^T b^\ell + c^\ell & (A^\ell x)^T \\ A^\ell x & 0 \end{bmatrix} \neq 0 \tag{2.25}$$

there exist $\lambda_\ell > 0$ such that

$$X^\ell \succeq Y^\ell := \begin{bmatrix} \lambda_\ell & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\lambda_\ell}\left(f_\ell[x] f_\ell[x]^T\right) \succeq \rho \begin{bmatrix} 2x^T b^\ell + c^\ell & (A^\ell x)^T \\ A^\ell x & 0 \end{bmatrix}, \tag{2.26}$$

where the vectors $f_\ell[x]$ are as defined by (2.22). Setting $Y^\ell = 0$ and $\lambda_\ell = 0$ whenever the left-hand side in (2.25) vanishes, it follows that $x, Y^1, \ldots, Y^L$ is a feasible solution of (2.18), which in turn implies

$$\begin{bmatrix} 2x^T b^0 + c^0 & (A^0 x)^T \\ A^0 x & I \end{bmatrix} \succeq \sum_{\ell=1}^{L} Y^\ell = \sum_{\ell,\, \lambda_\ell > 0} \left(\begin{bmatrix} \lambda_\ell & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\lambda_\ell}\left(f_\ell[x] f_\ell[x]^T\right) \right).$$

Via the Schur complement lemma (note that $\lambda_\ell = 0 \Rightarrow f_\ell[x] = 0$), the latter LMI shows that $(x, \lambda_1, \ldots, \lambda_L)$ is a feasible solution of (2.4). This completes the proof of II.2, and thus of Theorem 2.5. □

3. Robust solutions of uncertain conic-quadratic problems. An uncertain conic-quadratic problem (CQP) has the form

$$\text{(CQP)} \qquad \min_{x \in \mathbb{R}^n} \left\{ c^T x \;:\; \left\| A^i x + b_i \right\| \leq a_i^T x + \beta_i, \quad i = 1, \ldots, m \right\},$$

where the data (A^i, b_i, a_i, β_i) is uncertain and is only known to belong to some uncertainty sets $\mathcal{U}_i$,

$$\left(A^i, b_i, a_i, \beta_i\right) \in \mathcal{U}_i, \quad i = 1, \ldots, m.$$

The crucial step in building a robust counterpart for (CQP) is the ability to build a robust counterpart for a single constraint, i.e., the set of solutions $x \in \mathbb{R}^n$ of the semi-infinite inequality system

$$\|Ax + b\| \leq a^T x + \beta, \quad (A, b, a, \beta) \in \mathcal{U}_\rho. \tag{3.1}$$

Here, we deal with the situation in which the uncertainty affecting (3.1) is *sidewise*, i.e., the uncertainty affecting the right-hand side in (3.1) is independent of that affecting the left-hand side. More specifically,

$$\mathcal{U}_\rho = \mathcal{U}_\rho^L \times \mathcal{U}_\rho^R, \tag{3.2}$$

where

$$\mathcal{U}_\rho^L = \left\{ (A, b) = \left(A^0, b^0\right) + \sum_{\ell=1}^{L} y_\ell \left(A^\ell, b^\ell\right) \ : \ y \in \rho \mathcal{V}^L \right\}, \tag{3.3}$$

$$\mathcal{U}_\rho^R = \left\{ (a, \beta) = \left(a^0, \beta^0\right) + \sum_{r=1}^{R} \xi_r \left(a^r, \beta^r\right) \ : \ \xi \in \rho \mathcal{V}^R \right\}. \tag{3.4}$$

The sets $\mathcal{V}^L$ and $\mathcal{V}^R$ are convex *perturbation sets*, and $\rho > 0$ is a parameter expressing the magnitude of the perturbation.

As before, the specific form of $\mathcal{V}^L$ is an *intersection of ellipsoids*, i.e., $\mathcal{V}^L = \mathcal{V}_K^L$, where

$$\mathcal{V}_K^L = \left\{ y \in \mathbb{R}^L \ : \ y^T Q_k y \le 1,\ k = 1, \ldots, K \right\}, \tag{3.5}$$

with

$$Q_k \succeq 0 \quad \text{and} \quad \sum_{k=1}^{K} Q_k \succ 0. \tag{3.6}$$

The form (3.5) includes two important special cases, namely,

- *simple ellipsoidal uncertainty* ($K = 1$),
- *box uncertainty* ($K = L$, $(Q_k)_{kk} = 1$, and $(Q_k)_{ij} = 0$ ($i \neq k$ or $j \neq k$) for $k = 1, \ldots, K$).

For the right-hand side perturbation set $\mathcal{V}^R$ we allow a much more general geometry: $\mathcal{V}^R$ is assumed to be bounded, containing zero, and *semidefinite representable* (sdr), i.e., it can be represented as the projection of a set described by LMIs:

$$\mathcal{V}^R = \left\{ \zeta \in \mathbb{R}^R \ : \ \exists u \in \mathbb{R}^S \ : \ P(\zeta) + Q(u) - T \succeq 0 \right\} \tag{3.7}$$

for some symmetric matrix T, and symmetric matrices $P(\zeta), Q(u)$, which depend linearly on their respective arguments. Specifically,

$$P(\zeta) = \sum_{r=1}^{R} \zeta_r P_r, \qquad Q(u) = \sum_{s=1}^{S} u_s Q_s, \tag{3.8}$$

where $P_r (r = 1, \ldots, R)$ and $Q_s (s = 1, \ldots, S)$ are symmetric matrices, and it is assumed that

$$\exists \bar{\zeta}, \bar{u} \ : \ P(\bar{\zeta}) + Q(\bar{u}) \succ T. \tag{3.9}$$

It is well known that sdr-sets include $\cap$-ellipsoids and many more [4, Lecture 4].

The sidewise uncertainty assumption implies the following fact: x is robust feasible for (3.1) if and only if there exists a τ such that

$$\|Ax + b\| \le \tau, \tag{3.10}$$

$$\tau \le a^T x + \beta. \tag{3.11}$$

This fact allows us to handle (3.1) by treating (3.10) and (3.11) separately. We start with (3.11).

THEOREM 3.1. *A pair (x, τ) satisfies* (3.11), *where $\mathcal{U}^R_\rho$ is given by* (3.4), (3.7), *and* (3.8), *if and only if, for some symmetric matrix V, the triple (x, τ, V) is a solution of the following system of LMIs:*

$$x^Ta^0 + \beta^0 + \mathrm{Tr}\,(TV) \geq \tau, \tag{3.12}$$

$$\mathrm{Tr}\,(VP_r) = \rho\left(x^Ta^r + \beta^r\right), \quad r = 1, \dots, R, \tag{3.13}$$

$$\mathrm{Tr}\,(VQ_s) = 0, \quad s = 1, \dots, S, \tag{3.14}$$

$$V \succeq 0. \tag{3.15}$$

Proof. The pair (x, τ) satisfies (3.11) if and only if

$$\tau \leq x^T\left(a^0 + \rho\sum_{r=1}^R \xi_r a^r\right) + \beta^0 + \rho\sum_{r=1}^R \xi_r\beta^r \quad \forall \xi \in \mathcal{V}^R,$$

which is equivalent to

$$\tau - x^Ta^0 - \beta^0 \leq \inf_{\xi,u}\left\{\sum_{r=1}^R \xi_r\rho\left(x^Ta^r + \beta^r\right) \;:\; P(\xi) + Q(u) \succeq T\right\}. \tag{3.16}$$

The problem on the right-hand side (rhs) of (3.16) is a semidefinite problem:

$$\inf_{\xi,u}\left\{\xi^T\gamma_r[x] \;:\; P(\xi) + Q(u) \succeq T\right\}, \tag{P}$$

where

$$\gamma_r[x] = \rho\left(x^Ta^r + \beta^r\right).$$

The dual problem of (P) is the semidefinite problem

$$\sup_V\{\mathrm{Tr}\,(TV) \;:\; P^*(V) = \gamma_r[x],\, Q^*(V) = 0,\, V \succeq 0\}, \tag{D}$$

where P^* and Q^* are the respective adjoints of P and Q, as given in (3.8). Thus

$$P^*(V) \in \mathbb{R}^R, \qquad P^*(V)_r = \mathrm{Tr}\,(VP_r), \quad r = 1, \dots, R,$$
$$Q^*(V) \in \mathbb{R}^S, \qquad Q^*(V)_s = \mathrm{Tr}\,(VQ_s), \quad s = 1, \dots, S.$$

By assumption (3.9), problem (P) is strictly feasible, and due to the assumption that the set $\mathcal{V}^R$ is bounded, the objective value of (P) is bounded from below. Hence, by SDP duality theory (see, e.g., [4]), problem (D) has an optimal solution and inf (P) = max (D), i.e., there exists a V such that

$$\begin{array}{rcl} \text{rhs of (3.16)} & = & \inf\text{(P)} = \mathrm{Tr}\,(TV), \\ \mathrm{Tr}\,(VP_r) & = & \gamma_r[x] = \rho\left(x^Ta^r + \beta^r\right), \quad r = 1, \dots, R, \\ \mathrm{Tr}\,(VQ_s) & = & 0, \quad s = 1, \dots, S, \\ V & \succeq & 0. \end{array} \tag{3.17}$$

Now (3.16) and (3.17) show that (x, τ, V) indeed satisfies (3.12)–(3.15). □

We now turn to the condition (3.10) with the uncertainty set $\mathcal{U}^L_\rho$ given by (3.3), (3.5), and (3.6). For a general perturbation set as given in (3.5), with $K > 1$, the verification of (3.10) is an NP-hard problem (see [2]). Therefore we shall derive an approximate robust counterpart (Theorem 3.3 below). For the simple ellipsoidal case ($K = 1$) an exact robust counterpart is given by the following result of [2].

THEOREM 3.2. *Consider the condition* (3.10), *where* $\mathcal{U}_\rho^L$ *is given by* (3.3) *and* $\mathcal{V}^L$ *is the ellipsoid* $\mathcal{V}_1^L$ *(see* (3.5)*). Then a pair* (x, τ) *satisfies* (3.10) *if and only if there exists some* $\lambda_1 \geq 0$ *such that the triple* (x, τ, λ_1) *satisfies the following LMI:*

$$\begin{bmatrix} \tau - \lambda_1 & 0 & a[x]^T \\ 0 & \lambda_1 Q_1 & \rho A[x]^T \\ a[x] & \rho A[x] & \tau I_M \end{bmatrix} \succeq 0, \tag{3.18}$$

where

$$a[x] = A^0 x + b^0 \tag{3.19}$$

$$A[x] = \left(A^1 x + b^1, \ldots, A^L x + b^L\right). \qquad \square \tag{3.20}$$

For the general $\cap$-ellipsoids case ($K > 1$), the following theorem gives an approximate robust counterpart of (3.10).

THEOREM 3.3. *The set* $\mathcal{S}_L$ *of triples* $(x, \tau, \lambda) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^K$ *satisfying the LMI*

$$\begin{bmatrix} \tau - \sum_{k=1}^K \lambda_k & 0 & a[x]^T \\ 0 & \sum_{k=1}^K \lambda_k Q_k & \rho A[x]^T \\ a[x] & \rho A[x] & \tau I_M \end{bmatrix} \succeq 0, \quad \lambda \geq 0, \tag{3.21}$$

with $a[x]$ *and* $A[x]$ *as given by* (3.19)–(3.20), *is an approximate robust counterpart of the set of pairs* (x, τ) *satisfying* (3.10), *under the uncertainty set* $\mathcal{U}_\rho^L$ *given by* (3.3) *and* (3.5).

Proof. We have to show that if (x, τ, λ) solves (3.21), then (x, τ) solves (3.10). Now (3.21) is equivalent to the following three conditions:

(i)

$$Y := \begin{bmatrix} \mu & 0 & a[x]^T \\ 0 & \sum_{k=1}^K \lambda_k Q_k & \rho A[x]^T \\ a[x] & \rho A[x] & \tau I_M \end{bmatrix} \succeq 0,$$

(ii) $\mu \geq 0$, $\lambda \geq 0$,

(iii) $\mu + \sum_{k=1}^K \lambda_k \leq \tau$.

Condition (i) implies that for every $y \in \mathbb{R}^L$ and $t \in \mathbb{R}$

$$\begin{bmatrix} t & y^T & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mu & 0 & a[x]^T \\ 0 & \sum_{k=1}^K \lambda_k Q_k & \rho A[x]^T \\ a[x] & \rho A[x] & \tau I_M \end{bmatrix} \begin{bmatrix} t & y^T & 0 \\ 0 & 0 & I \end{bmatrix}^T \succeq 0,$$

which is equivalent to

$$\left[\begin{array}{c|c} \begin{bmatrix} t & y^T \end{bmatrix} \begin{bmatrix} \mu & 0 \\ 0 & \sum_{k=1}^K \lambda_k Q_k \end{bmatrix} \begin{bmatrix} t \\ y \end{bmatrix} & \begin{bmatrix} t & y^T \end{bmatrix} \begin{bmatrix} a[x]^T \\ \rho A[x]^T \end{bmatrix} \\ \hline \begin{bmatrix} a[x] & \rho A[x] \end{bmatrix} \begin{bmatrix} t \\ y \end{bmatrix} & \tau I \end{array}\right] \succeq 0.$$

By the Schur complement lemma, the latter is equivalent to

(i′) $\tau(\mu t^2 + \sum_{k=1}^K \lambda_k y^T Q_k y) \geq \|ta[x] + \rho A[x]y\|^2$.

Therefore, conditions (i)–(iii) reduce to (i′), (ii), and (iii). From these conditions it follows that if (y,t) are chosen such that

$$t^2 \leq 1, \quad y^T Q_k y \leq 1, \qquad k = 1, \ldots, K, \tag{3.22}$$

then

$$\mu t^2 + \sum_{k=1}^K \lambda_k y^T Q_k y \leq \mu + \sum_{k=1}^K \lambda_k \leq \tau, \tag{3.23}$$

and, since $\tau \geq 0$, from (3.23) and (i′),

$$\tau^2 \geq \|ta[x] + \rho A[x]y\|^2 \quad \forall (y,t) \text{ satisfying (3.22)}. \tag{3.24}$$

In particular, for $t = 1$ we get

$$\tau \geq \|a[x] + \rho A[x]y\| \quad \forall y \text{ satisfying (3.22)}. \tag{3.25}$$

Substituting into (3.25) the expression for $a[x]$ and $A[x]$ (see (3.19)–(3.20)), (3.25) becomes explicitly

$$\tau \geq \left\| A^0 x + b^0 + \rho \sum_{\ell=1}^L \lambda_\ell \left(A^\ell x + b^\ell\right) \right\| \quad \forall y \text{ s.t. } y^T Q_k y \leq 1, \ k = 1, \ldots, K. \tag{3.26}$$

Finally, (3.26) is precisely condition (3.10) for $\mathcal{U}_\rho^L$ given by (3.3) and $\mathcal{V}^L = \mathcal{V}_K^L$ given by (3.5). □

Combining the results of Theorems 3.1 and 3.3, we obtain the following result.

COROLLARY 3.4. *The set $\mathcal{S}$ of tuples*

$$(x, \tau, \lambda, V) \ \textit{satisfying}\ (3.12)\text{–}(3.15)\ \textit{and}\ (3.21)$$

is an approximate robust counterpart of the uncertain conic-quadratic constraint (3.1), *where the uncertainty set $\mathcal{U}_\rho$ is given by* (3.2)–(3.9).

The level of conservativeness of $\mathcal{S}$ can be estimated in a way very similar to that used in the case of uncertain quadratic constraints (see the proofs of Theorem 2.4 and Theorem 2.5), and the result is in fact similar.

THEOREM 3.5. (i) *For the case of $\cap$-ellipsoidal uncertainty, with $K > 1$, the level of conservativeness Ω of the approximate robust counterpart $\mathcal{S}$ in Corollary* 3.4 *is at most*

$$\tilde{\Omega} := \left(2 \log \left(6 \sum_{k=1}^K \operatorname{rank} Q_k\right)\right)^{\frac{1}{2}}.$$

(ii) *For the special case of box uncertainty, one has*

$$\Omega \leq \frac{\pi}{2}. \qquad \square$$

Appendix. Some technical lemmas.

LEMMA A.1. *Let* $x = (x_1, \ldots, x_n)$ *and* $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. *If* $\|x\|_2 = 1$ *and the coordinates* ξ_i *of* ξ *are independently identically distributed random variables with*

$$\Pr(\xi_i = 1) = \Pr(\xi_i = -1) = \frac{1}{2},$$

then one has

$$\Pr\left(\left|\xi^T x\right| \le 1\right) \ge \frac{1}{3}. \tag{A.1}$$

Proof.[4] Without loss of generality we may assume that $x \ge 0$ and

$$x_1 \ge x_2 \ge \cdots \ge x_n \ge 0.$$

We define $\theta = x_1$ and

$$s_0 = 0, \quad s_k = s_k(\xi) = \sum_{i=1}^{k} x_i \xi_i, \qquad k = 1, 2, \ldots, n.$$

Then (A.1) is equivalent to $\Pr(|s_n| \le 1) \ge \frac{1}{3}$. To prove this, we define the following events:

$$A_0 = \{\xi \ : \ |s_j| \le 1 - \theta, \ j = 1, \ldots, n\},$$

$$A_k = \{\xi \ : \ |s_j| \le 1 - \theta, \ j = 1, \ldots, k-1, \text{ and } |s_k| > 1 - \theta\}.$$

Note that the events $A_0, A_1, \ldots, A_n$ form a partition of the probability space.

Assuming $A_k \ne \emptyset$, we proceed by deriving a lower bound on the probability that $|s_n| \le 1$ occurs, namely:

$$\Pr(|s_n| \le 1 \mid A_k) \ge p(\theta) := \frac{1}{2}\left(1 - \frac{1 - \theta^2}{(2 - \theta)^2}\right). \tag{A.2}$$

For $k = 0$ this is evident, since the left-hand side is then equal to 1. So let $k \ge 1$ and let us fix a realization $\xi \in A_k$. Then we have

$$1 - \theta < |s_k| \le 1. \tag{A.3}$$

Indeed, the left-hand side of (A.3) follows from the definition of A_k, and the right-hand side from

$$|s_k| = |s_{k-1} + s_k - s_{k-1}| \le |s_{k-1}| + |s_k - s_{k-1}| \le 1 - \theta + x_k \le 1 - \theta + \theta = 1.$$

Because of (A.3) we have the following implication:

$$0 \ge (s_n - s_k) \ \operatorname{sign}(s_k) \ge -2 + \theta \quad \Rightarrow \quad |s_n| \le 1. \tag{A.4}$$

Indeed, if $s_k \ge 0$, then $1 - \theta < s_k \le 1$ and $0 \ge s_n - s_k \ge -2 + \theta$ imply that $s_n \le s_k \le 1$ and $s_n \ge s_k + \theta - 2 > 1 - \theta + \theta - 2 = -1$, and if $s_k \le 0$, then $1 - \theta < -s_k \le 1$ and

[4]This proof is mainly due to P. van der Wal, Delft University of Technology (private communication).

$0 \geq -s_n + s_k \geq -2 + \theta$ imply $s_n \geq s_k \geq -1$ and also $s_n \leq s_k - \theta + 2 < \theta - 1 - \theta + 2 = 1$. So in both cases one has $|s_n| \leq 1$, proving (A.4). Hence we may write

$$\begin{aligned}\Pr\left(|s_n| \leq 1 \mid A_k\right) &\geq \Pr\left(0 \geq (s_n - s_k)\ \operatorname{sign}(s_k) \geq -2 + \theta \mid A_k\right)\\ &\geq \frac{1}{2}\Pr\left(|s_n - s_k| \leq 2 - \theta\right), \quad \text{by symmetry,}\\ &\geq \frac{1}{2}\left(1 - \frac{\operatorname{Var}(s_n - s_k)}{(2-\theta)^2}\right), \quad \text{by the Chebyshev inequality,}\\ &= \frac{1}{2}\left(1 - \frac{\sum_{j=k+1}^{n} x_k^2}{(2-\theta)^2}\right) \geq \frac{1}{2}\left(1 - \frac{1-\theta^2}{(2-\theta)^2}\right) = p(\theta).\end{aligned}$$

Thus we have proved (A.2). Since $(A_0, A_1, \ldots, A_n)$ is a partition of the probability space, it follows that

$$\Pr\left(|s_n| \leq 1\right) \geq p(\theta) \geq \min_{0 \leq \theta \leq 1} \frac{1}{2}\left(1 - \frac{1-\theta^2}{(2-\theta)^2}\right) = p\left(\frac{1}{2}\right) = \frac{1}{3}.$$

This proves the lemma. □

Based on numerical experiments, we believe that Lemma A.1 can be improved as stated in the conjecture below.

CONJECTURE A.2. *Let x and ξ be as defined in Lemma* A.1. *Then*

$$\Pr\left(\left|\xi^T x\right| \leq 1\right) \geq \frac{1}{2}.$$

LEMMA A.3. *Let* $\operatorname{rank} B = k$, $B \succeq 0$, *and let ξ be as defined in Lemma* A.1. *Then*

$$\Pr\left(\xi^T B \xi \geq \alpha \operatorname{Tr} B\right) \leq 2k e^{-\frac{\alpha}{2}} \quad \forall \alpha > 0.$$

Proof. Writing

$$B = \sum_{i=1}^{k} b_i b_i^T,$$

the statement in the lemma can be rewritten as

$$\Pr\left(\sum_{i=1}^{k}\left(b_i^T \xi\right)^2 \geq \alpha \sum_{i=1}^{k}\|b_i\|^2\right) \leq 2k e^{-\frac{\alpha}{2}} \quad \forall \alpha > 0.$$

We have

$$\begin{aligned}p_k &:= \Pr\left(\sum_{i=1}^{k}\left(b_i^T \xi\right)^2 \geq \alpha \sum_{i=1}^{k}\|b_i\|^2\right) = \Pr\left(\sum_{i=1}^{k}\left(\left(b_i^T \xi\right)^2 - \alpha\|b_i\|^2\right) \geq 0\right)\\ &\leq \Pr\left(\max_i\left(\left(b_i^T \xi\right)^2 - \alpha\|b_i\|^2\right) \geq 0\right) \leq \sum_{i=1}^{k}\Pr\left(\left(b_i^T \xi\right)^2 \geq \alpha\|b_i\|^2\right)\\ &= \sum_{i=1}^{k}\Pr\left(\left|b_i^T \xi\right| \geq \sqrt{\alpha}\,\|b_i\|\right) = 2\sum_{i=1}^{k}\Pr\left(b_i^T \xi \geq \sqrt{\alpha}\,\|b_i\|\right).\end{aligned}$$

For any random variable y with distribution F, we have for any $\rho \geq 0$,

$$\mathcal{E}\left(e^{\rho y}\right) = \int_{-\infty}^{0} e^{\rho y} dF(y) + \int_{0}^{\infty} e^{\rho y} dF(y) \geq 0 + \int_{0}^{\infty} dF(y) = \operatorname{Pr}(y \geq 0).$$

Hence,

$$\operatorname{Pr}\left(b_i^T \xi \geq \sqrt{\alpha}\, \|b_i\|\right) \leq \mathcal{E}\left(e^{\rho(b_i^T\xi - \sqrt{\alpha}\|b_i\|)}\right) = \mathcal{E}\left(e^{\rho\,(b_i^T \xi)}\right) e^{-\rho\sqrt{\alpha}\,\|b_i\|}.$$

Furthermore, using the inequality $\cosh t \leq e^{\frac{1}{2}t^2}$, we have

$$\mathcal{E}\left(e^{\rho(b_i^T\xi)}\right) = \prod_{j=1}^{n} \mathcal{E}(e^{\rho b_{ij}\xi_j}) = \prod_{j=1}^{n} \cosh(\rho b_{ij}) \leq \prod_{j=1}^{n} e^{\frac{1}{2}\rho^2 b_{ij}^2} = e^{\frac{1}{2}\rho^2\|b_i\|^2}.$$

Substitution gives

$$\operatorname{Pr}\left(b_i^T \xi \geq \sqrt{\alpha}\, \|b_i\|\right) \leq e^{\frac{1}{2}\rho^2\|b_i\|^2 - \rho\sqrt{\alpha}\|b_i\|}, \quad \rho \geq 0.$$

The right-hand side is minimal if $\rho = \sqrt{\alpha}/\|b_i\|$. Thus we obtain

$$\operatorname{Pr}\left(b_i^T \xi \geq \sqrt{\alpha}\, \|b_i\|\right) \leq e^{-\frac{\alpha}{2}}.$$

From this we derive the inequality

$$p_k \leq 2\sum_{i=1}^{k} \operatorname{Pr}\left(b_i^T \xi \geq \sqrt{\alpha}\, \|b_i\|\right) \leq 2\sum_{i=1}^{k} e^{\frac{-\alpha}{2}} = 2ke^{-\frac{\alpha}{2}},$$

which completes the proof of the lemma. $\square$

LEMMA A.4. *Let B denote a symmetric $n \times n$ matrix and ξ be as defined in Lemma* A.1. *Then*

$$\operatorname{Pr}\left(\xi^T B \xi \leq \operatorname{Tr} B\right) > \frac{1}{8n^2}. \tag{A.5}$$

Proof. Consider the random variable

$$\gamma := \sum_{i<j} \xi_i \xi_j A_{ij} = \frac{1}{2}(\xi^T B \xi - \operatorname{Tr} B).$$

Then (A.5) is equivalent to

$$\omega := \operatorname{Pr}(\gamma \leq 0) > \frac{1}{8n^2}. \tag{A.6}$$

Let $\mu(dt)$ be the distribution of γ, and let

$$I_\ell = \int_{-\infty}^{\infty} |t|^\ell \mu(dt), \quad \ell = 1, 2, \ldots.$$

Since $\mathcal{E}(\gamma) = 0$, we have

$$\int_{t\leq 0} |t|\, \mu(dt) = \int_{t \geq 0} |t|\, \mu(dt).$$

Hence

$$I_1 = 2\int_{t\le 0} |t|\,\mu(dt) = 2\int_{t\le 0} |t|\,\frac{\mu(dt)}{\Pr(\gamma\le 0)} \times \Pr(\gamma \le 0)$$
$$\le 2\left(\int_{t\le 0} t^2\mu(dt)\right)^{\frac{1}{2}} \sqrt{\Pr(\gamma\le 0)} \le 2\omega^{\frac{1}{2}} I_2^{\frac{1}{2}}.$$

Further,

$$I_2 = \int_{-\infty}^{\infty} t^2\mu(dt) = \int_{-\infty}^{\infty} |t|^{\frac{1}{2}}\,|t|^{\frac{3}{2}}\,\mu(dt) \le I_1^{\frac{1}{2}} I_2^{\frac{1}{2}} \le \sqrt{2}\,\omega^{\frac{1}{4}} I_2^{\frac{1}{4}} I_3^{\frac{1}{2}}.$$

Thus it follows that

$$\omega \ge \frac{I_2^3}{16 I_3^2}. \tag{A.7}$$

Also

$$I_2 = \mathcal{E}\left(\gamma^2\right) = \mathcal{E}\left(\sum_{i<j,\,k<\ell} \xi_i\xi_j\xi_k\xi_\ell A_{ij}A_{k\ell}\right) = \sum_{i<j} A_{ij}^2$$

and

$$I_3 \le \mathcal{E}\left(\left|\sum_{i<j}\xi_i\xi_j A_{ij}\right|^3\right) \le \mathcal{E}\left(\left(\sum_{i<j}\xi_i\xi_j A_{ij}\right)^2 \left|\sum_{i<j}\xi_i\xi_j A_{ij}\right|\right)$$
$$\le \left(\sum_{i<j} A_{ij}^2\right)\sum_{i<j}|A_{ij}| \le \left(\sum_{i<j} A_{ij}^2\right)\sqrt{\frac{n(n-1)}{2}}\sqrt{\sum_{i<j} A_{ij}^2}.$$

The last inequality uses that $\sum_{i=1}^k |\alpha_i| \le \sqrt{k}\sqrt{\sum_{i=1}^k \alpha_i^2}$. Putting the above estimates for I_2 and I_3 into (A.7), we get

$$\omega \ge \frac{1}{16}\,\frac{2}{n(n-1)} > \frac{1}{8n^2},$$

and hence the lemma is proved. □

CONJECTURE A.5. *Let B and ξ be as defined in Lemma* A.4. *Then*

$$\Pr\left(\xi^T B\xi \le \mathrm{Tr}\,B\right) \ge \frac{1}{4}.$$

LEMMA A.6 (approximate S-lemma). *Let $R, R_0, R_1, \ldots, R_K$ be symmetric $n\times n$ matrices such that*

$$R_1,\ldots,R_K \succeq 0, \tag{A.8}$$

and assume that

$$\exists \lambda_0,\lambda_1,\ldots,\lambda_K \ge 0 \quad s.t. \quad \sum_{k=0}^{K}\lambda_k R_k \succ 0. \tag{A.9}$$

Consider the following quadratically constrained quadratic program,

$$QCQ = \max_{y\in\mathbb{R}^n} \left\{ y^T R y \ : \ y^T R_0 y \le r_0,\ y^T R_k y \le 1,\ k=1,\dots,K \right\} \tag{A.10}$$

and the semidefinite optimization problem

$$SDP = \min_{\mu_0,\mu_1,\dots,\mu_K} \left\{ r_0\mu_0 + \sum_{k=1}^K \mu_k \ : \ \sum_{k=0}^K \mu_k R_k \succeq R,\ \mu \ge 0 \right\}. \tag{A.11}$$

Then

(i) *If problem* (A.10) *is feasible, then problem* (A.11) *is bounded below and*

$$SDP \ge QCQ. \tag{A.12}$$

Moreover, there exist $y_* \in \mathbb{R}^n$ *such that*

$$y_*^T R y_* = SDP, \tag{A.13}$$

$$y_*^T R_0 y_* \le r_0, \tag{A.14}$$

$$y_*^T R_k y_* \le \tilde{\rho}^2, \quad k=1,\dots,K, \tag{A.15}$$

where (cf. (2.5)*)*

$$\tilde{\rho} := \left(2\log\left(6\sum_{k=1}^K \operatorname{rank} R_k \right)\right)^{\frac{1}{2}}$$

if R_0 *is a dyadic matrix, and*

$$\tilde{\rho} = \left(2\log\left(16n^2\sum_{k=1}^K \operatorname{rank} R_k \right)\right)^{\frac{1}{2}} \tag{A.16}$$

otherwise.

(ii) *If*

$$r_0 > 0, \tag{A.17}$$

then (A.10) *is feasible, problem* (A.11) *is solvable, and*

$$0 \le QCQ \le SDP \le \tilde{\rho}^2 QCQ. \tag{A.18}$$

Remark A.7. We claim that the usual S-lemma (cf. Lemma 2.2) can be obtained as a corollary of Lemma A.6. The "if" part of the S-lemma being evident, we focus below on the "only if" part.

1^0. Observe that it suffices to prove the following statement:

(!) *Assume that the set* $\{z : z^T P z > 0\}$ *is nonempty and that*

$$z \ne 0, \quad z^T P z \ge 0 \Rightarrow z^T Q z > 0. \tag{A.19}$$

Then

$$\exists \lambda \ge 0 : \quad Q \succeq \lambda P. \tag{A.20}$$

Indeed, let P, Q be such that

$$\{z : z^T P z > 0\} \neq \emptyset \quad \text{and} \quad z^T P z \geq 0 \Rightarrow z^T Q z \geq 0.$$

Then the pair $(P, Q + \epsilon I)$ for $\epsilon > 0$ clearly satisfies the premise in (!). Believing in (!), we therefore conclude that for every $\epsilon > 0$ there exists $\lambda(\epsilon) \geq 0$ such that $Q + \epsilon I \succeq \lambda(\epsilon) P$. As $\epsilon \to +0$, $\lambda(\epsilon)$ remains bounded due to $\lambda(\epsilon)\bar{z}^T P \bar{z} \leq \bar{z}^T (Q + \epsilon I)\bar{z}$, where $\bar{z}$ is such that $\bar{z}^T P \bar{z} > 0$. Since $\lambda(\epsilon) \geq 0$ remains bounded as $\epsilon \to +0$, there exists an accumulation point $\lambda \geq 0$ of $\lambda(\epsilon)$ as $\epsilon \to +0$; since $Q + \epsilon I \succeq \lambda(\epsilon) P$, one clearly has $Q \succeq \lambda P$, as required.

2^0. To prove (!), assume that the premise in (!) holds true, and observe that then the optimal value $QCQ(\epsilon)$ in the optimization problem

$$\max_x \left\{ -x^T Q x : -x^T P x \leq 1, \, \epsilon x^T x \leq 1 \right\} \tag{A.21}$$

remains bounded as $\epsilon \to +0$. Indeed, otherwise there clearly would exist a sequence of vectors x_i, $\|x_i\| \to \infty$ as $i \to \infty$, such that $x_i^T P x_i \geq -1$ and $x_i^T Q x_i \to -\infty$ as $i \to \infty$. By evident reasons, this would imply the existence of a unit vector $\bar{x}$ such that $\bar{x}^T P \bar{x} \geq 0$ and $\bar{x}^T Q \bar{x} \leq 0$, which would contradict (A.19). Now, the data

$$R = -Q, \quad R_0 = -P, \quad R_1 = \epsilon I, \quad r_0 = 1, \quad K = 1$$

clearly satisfy the premises (A.8) and (A.9) of Lemma A.6, and with these data, (A.10) coincides with (A.21). Since $r_0 = 1 > 0$, part (ii) of Lemma A.6 applies. Thus problem (A.11) is solvable and (A.18) holds. Hence, for every $\epsilon > 0$ there exist $\mu_0(\epsilon) \geq 0$ and $\mu_1(\epsilon) \geq 0$ such that

$$-\mu_0(\epsilon) P + \mu_1(\epsilon)\epsilon I \succeq -Q, \qquad \mu_0(\epsilon) + \mu_1(\epsilon) \leq \bar{\rho}^2 QCQ(\epsilon).$$

Since $QCQ(\epsilon)$ remains bounded as $\epsilon \to 0$, so are $\mu_0(\epsilon)$, $\mu_1(\epsilon)$; therefore there exists an accumulation point $(\mu_1 \geq 0, \mu_2 \geq 0)$ of $(\mu_0(\epsilon), \mu_1(\epsilon))$ as $\epsilon \to +0$, and $\lambda = \mu_1$ clearly satisfies the conclusion in (A.20). ☐

Proof. Notice that problem (A.11) is the semidefinite dual of

$$RQCQ = \max_{X \succeq 0} \{ \operatorname{Tr} RX \ : \ \operatorname{Tr} R_0 X \leq r_o, \, \operatorname{Tr} R_k X \leq 1, \, k = 1, \dots, K \}. \tag{A.22}$$

The latter problem is the standard semidefinite relaxation of the quadratically constrained quadratic problem (A.10), so we have

$$RQCQ \geq QCQ. \tag{A.23}$$

In part (i) of the lemma, (A.10) is assumed to be feasible; hence (A.22) is feasible as well, and hence, by weak duality, between problem (A.11) and its dual (A.22), problem (A.11) is bounded below. Now assumption (A.9) ensures that (A.11) is strictly feasible; thus from semidefinite duality theory, problem (A.22) is solvable and

$$SDP = RQCQ. \tag{A.24}$$

By (A.23) and (A.24), $SDP \geq QCQ$, which completes the proof of the first part of claim (i) in the lemma. To prove the second part, we first simplify the system (A.13)–(A.15). Letting

X_* denote an optimal solution of problem (A.22), we introduce

$$\hat{R} = X_*^{\frac{1}{2}} R X_*^{\frac{1}{2}}. \tag{A.25}$$

Let

$$\hat{R} = U\tilde{R}U^T \qquad (U^TU = I,\ \tilde{R} = \operatorname{diag}(r_1, \ldots, r_n)) \tag{A.26}$$

be the eigenvalue decomposition of $\hat{R}$. Choosing

$$y_* = X_*^{\frac{1}{2}} Uu, \quad u \in \mathbb{R}^n, \tag{A.27}$$

we have

$$y_*^T R y_* = u^T U^T X_*^{\frac{1}{2}} R X_*^{\frac{1}{2}} Uu = u^T U^T \hat{R} Uu = u^T \tilde{R} u = \sum_{i=1}^n r_i u_i^2.$$

Also

$$SDP = RQCQ = \operatorname{Tr} RX_* = \operatorname{Tr} \hat{R} = \operatorname{Tr} \tilde{R} = \sum_{i=1}^n r_i,$$

and thus (A.13) is equivalent to

$$\text{(a)} \qquad \sum_{i=1}^n r_i u_i^2 = \sum_{i=1}^n r_i.$$

Now, defining

$$\hat{R}_k = X_*^{\frac{1}{2}} R_k X_*^{\frac{1}{2}}, \quad \tilde{R}_k = U^T \hat{R}_k U, \qquad k = 0, 1, \ldots, K,$$

and using (A.27), we obtain

$$y_*^T R_k y_* = u^T U^T X_*^{\frac{1}{2}} R_k X_*^{\frac{1}{2}} Uu = u^T U^T \hat{R}_k Uu = u^T \tilde{R}_k u. \tag{A.28}$$

Since X_* solves RQCQ,

$$r_0 \geq \operatorname{Tr} R_0 X_* = \operatorname{Tr} \hat{R}_0 = \operatorname{Tr} \tilde{R}_0 \tag{A.29}$$

and

$$1 \geq \operatorname{Tr} R_k X_* = \operatorname{Tr} \hat{R}_k = \operatorname{Tr} \tilde{R}_k, \quad k = 1, \ldots, K. \tag{A.30}$$

From (A.28) and (A.29) we see that (A.14) holds if

$$\text{(b)} \qquad u^T \tilde{R}_0 u \leq \operatorname{Tr} \tilde{R}_0,$$

and from (A.28) and (A.30), relation (A.15) holds if

$$\text{(c)} \qquad u^T \tilde{R}_k u \leq \tilde{\rho}^2 \operatorname{Tr} \tilde{R}_k, \quad k = 1, \ldots, K.$$

We conclude that if there exists a $\bar{u}$ satisfying

$$\sum_{i=1}^{n} r_i \bar{u}_i^2 = \sum_{i=1}^{n} r_i, \tag{A.31}$$

$$\bar{u}^T \tilde{R}_0 \bar{u} \leq \operatorname{Tr} \tilde{R}_0, \tag{A.32}$$

$$\bar{u}^T \tilde{R}_k \bar{u} \leq \tilde{\rho}^2 \operatorname{Tr} \tilde{R}_k, \quad k = 1, \ldots, K, \tag{A.33}$$

then $y_* = X_*^{\frac{1}{2}} U \bar{u}$ satisfies (A.13)–(A.15). Note that (A.31) is automatically satisfied if $\bar{u}$ is a ± 1-vector. Thus it suffices to show that (A.32) and (A.33) can be satisfied by a ± 1-vector $\bar{u}$.

Let us pretend for a moment that the vector $\bar{u}$ is a *random* ± 1-vector such that $\Pr(\bar{u}_i = 1) = \Pr(\bar{u}_i = -1) = \frac{1}{2}$ for each i. Let B denote the event that $\bar{u}$ satisfies (A.32), and C_k the event that $\bar{u}^T \tilde{R}_k \bar{u} \leq \tilde{\rho}^2 \operatorname{Tr} \tilde{R}_k$, and $C = \cap_k C_k$, i.e., C denotes the event that $\bar{u}$ satisfies (A.33). Then we only need to show that

$$\Pr(B \cap C) > 0. \tag{A.34}$$

Since

$$B \subseteq (B \cup C) \cap C^c,$$

where ^{c} refers to the complement of the event, we may write

$$\begin{aligned} \Pr(B \cup C) &\geq \Pr(B) - \Pr(C^c) = \Pr(B) - \Pr((\cap_k C_k)^c) \\ &= \Pr(B) - \Pr(\cup_k C_k^c) \geq \Pr(B) - \sum_{k=1}^{K} \Pr(C_k^c). \end{aligned}$$

Hence, (A.34) will certainly hold if for some $p_0 > 0$,

$$\Pr(B) > p_0, \tag{A.35}$$

$$\sum_{k=1}^{K} \Pr(C_k^c) \leq p_0. \tag{A.36}$$

We first consider the case in which R_0 is dyadic. Then $\hat{R}_0$ and $\tilde{R}_0$ are also dyadic, and hence we may write, for a suitable vector b,

$$\tilde{R}_0 = bb^T.$$

Then condition (A.32) is equivalent to

$$\bar{u}^T \frac{b}{\|b\|} \leq 1.$$

Hence, in the dyadic case, (A.35) is equivalent to

$$\Pr\left(\left|u^T x\right| \leq 1\right) > p_0, \tag{A.37}$$

where x is the unit vector $b/\|b\|$. By Lemma A.1 this inequality certainly holds if $p_0 = \frac{1}{3}$. On the other hand, by Lemma A.3, for each k,

$$\Pr(C_k^c) = \Pr\left(\bar{u}^T \tilde{R}_k \bar{u} > \tilde{\rho}^2 \operatorname{Tr} \tilde{R}_k\right) \leq 2\,(\operatorname{rank} \tilde{R}_k) e^{-\frac{\tilde{\rho}^2}{2}}.$$

Since $\operatorname{rank} \tilde{R}_k \leq \operatorname{rank} R_k$, we obtain

$$\sum_{k=1}^{K} \Pr(C_k^c) \leq 2\, e^{-\frac{\tilde{\rho}^2}{2}} \sum_{k=1}^{K} \operatorname{rank} R_k,$$

and so inequalities (A.35) and (A.36) will hold if $p_0 = \frac{1}{3}$ and $\tilde{\rho}$ is such that

$$2\, e^{-\frac{\tilde{\rho}^2}{2}} \sum_{k=1}^{K} \operatorname{rank} R_k = \frac{1}{3} = p_0. \tag{A.38}$$

One may easily verify that the value of $\tilde{\rho}$ as given by (2.5) is indeed the solution of (A.38). Thus the proof is complete for the case in which R_0 is dyadic.

We finally consider the general case, where R_0 is an arbitrary symmetric matrix. Then we apply Lemma A.4, which gives that (A.35) holds for $p_0 = 1/(8n^2)$. Then solving $\tilde{\rho}$ from (A.38) with this value of p_0, we get the value given in (A.16).

To complete the proof of the lemma we need only to prove $SDP \leq \tilde{\rho}^2 QCQ$, the last inequality in (A.18). For this, let y_* satisfy (A.13)–(A.15). Then, since $\tilde{\rho} > 1$, the vector

$$\bar{y} = \frac{y_*}{\tilde{\rho}}$$

is feasible for problem (A.10). Therefore, using (A.13),

$$QCQ \geq \bar{y}^T R \bar{y} = \frac{1}{\tilde{\rho}^2} y_*^T R y_* = \frac{SDP}{\tilde{\rho}^2},$$

and hence the proof is complete. ◻

LEMMA A.8. *Let $a, b \in \mathbb{R}^n$ be two nonzero vectors and X a symmetric $n \times n$ matrix. Then*

$$X \succeq \pm\left(ab^T + ba^T\right) \tag{A.39}$$

holds if and only if

$$\exists \rho > 0 \quad s.t. \quad X \succeq \rho aa^T + \frac{1}{\rho} bb^T. \tag{A.40}$$

Proof. Suppose (A.40) holds. Then, for arbitrary $y \in \mathbb{R}^n$ one has

$$\begin{aligned} y^T X y &\geq y^T \left(\rho aa^T + \frac{1}{\rho} bb^T\right) y = \rho \left(a^T y\right)^2 + \frac{1}{\rho}\left(b^T y\right)^2 \\ &\geq 2\left|a^T y\right|\left|b^T y\right| \geq \left|y^T\left(ab^T + ba^T\right) y\right|; \end{aligned}$$

hence (A.39) follows. On the other hand, if (A.40) does not hold, then the system

$$X \succeq \rho aa^T + \mu bb^T, \qquad \begin{pmatrix} \rho & 1 \\ 1 & \mu \end{pmatrix} \succeq 0 \tag{A.41}$$

does not have a solution (ρ, μ). This implies that the optimal value p^* of the semidefinite optimization problem

$$p^* = \min_{t,\rho,\mu} \left\{ t : tI + X \succeq \rho aa^T + \mu bb^T, \quad \begin{pmatrix} \rho & 1 \\ 1 & \mu \end{pmatrix} \succeq 0 \right\} \tag{SDP}$$

is positive. Clearly (SDP) is strictly feasible and bounded below. Hence its dual problem (SDD),

(SDD)

$$\max_{\substack{v,\, v_1,\, v_2,\\ U \succeq 0}} \left\{ -\mathrm{Tr}\,(UX) - 2v : \mathrm{Tr}\,(U) = 1, \begin{pmatrix} v_1 & v \\ v & v_2 \end{pmatrix} \succeq 0,\ v_1 = b^T U b,\ v_2 = a^T U a \right\},$$

is solvable and has the same optimal value $p^* > 0$. A feasible solution (U, v, v_1, v_2) to (SDD) satisfies

$$|v| \leq \sqrt{v_1 v_2} = \sqrt{(a^T U a)\,(b^T U b)}.$$

Hence, $p^* > 0$ implies the existence of $U \succeq 0$ such that

$$2\sqrt{(a^T U a)\,(b^T U b)} > \mathrm{Tr}\,(UX). \tag{A.42}$$

Let

$$\bar{a} = U^{\frac{1}{2}} a, \quad \bar{b} = U^{\frac{1}{2}} b, \quad \bar{X} = U^{\frac{1}{2}} X U^{\frac{1}{2}}.$$

Then (A.42) can be rewritten as

$$\mathrm{Tr}\,\bar{X} < 2\sqrt{(\bar{a}^T \bar{a})\,(\bar{b}^T \bar{b})} = 2\,\|\bar{a}\|\,\|\bar{b}\|. \tag{A.43}$$

Now suppose that X satisfies (A.39). Then it follows that

$$\bar{X} \succeq \pm\left(\bar{a}\bar{b}^T + \bar{b}\bar{a}^T\right).$$

Define $Q = \bar{a}\bar{b}^T + \bar{b}\bar{a}^T$, and let $\lambda(Q)$ be the vector of eigenvalues of Q. It then follows that

$$\mathrm{Tr}\,\bar{X} \geq \|\lambda(Q)\|_1 = \left\|\lambda(\bar{a}\bar{b}^T + \bar{b}\bar{a}^T)\right\|_1 = 2\,\|\bar{a}\|\,\|\bar{b}\|.$$

This contradicts (A.43). Hence the proof is complete. □

REFERENCES

[1] A. Ben-Tal and A. Nemirovski, *Robust truss topology design via semidefinite programming,* SIAM J. Optim., 7 (1997), pp. 991–1016.

[2] A. Ben-Tal and A. Nemirovski, *Robust convex optimization,* Math. Oper. Res., 23 (1998), pp. 769–805.

[3] A. Ben-Tal and A. Nemirovski, *Robust solutions of uncertain linear programs,* Oper. Res. Lett., 25 (1999), pp. 1–13.

[4] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization. Analysis, Algorithms, and Engineering Applications*, MPS/SIAM Ser. Optim. MP02, SIAM, Philadelphia, 2001.

[5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory,* SIAM Stud. Appl. Math. 15, SIAM, Philadelphia, 1994.

[6] L. El Ghaoui and H. Lebret, *Robust solutions to least-squares problems with uncertain data,* SIAM J. Matrix Anal. Appl., 18 (1997), pp 1035–1064.

[7] J. Håstad, *Some optimal in-approximability results,* in Proceedings of the 29th ACM Symposium on Theory of Computing, El Paso, TX, 1997, SIGACT, ACM, New York, 1997, pp. 1–10.