

ON TRACTABLE APPROXIMATIONS OF UNCERTAIN LINEAR MATRIX INEQUALITIES AFFECTED BY INTERVAL UNCERTAINTY*

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Abstract. We present efficiently verifiable sufficient conditions for the validity of specific NP-hard semi-infinite systems of linear matrix inequalities (LMIs) arising from LMIs with uncertain data and demonstrate that these conditions are “tight” up to an absolute constant factor. In particular, we prove that given an $n \times n$ interval matrix $\mathcal{U}_\rho = \{A \mid |A_{ij} - A_{ij}^*| \leq \rho C_{ij}\}$, one can build a computable lower bound, accurate within the factor $\frac{\pi}{2}$, on the supremum of those ρ for which all instances of \mathcal{U}_ρ share a common quadratic Lyapunov function. We then obtain a similar result for the problem of quadratic Lyapunov stability synthesis. Finally, we apply our techniques to the problem of maximizing a homogeneous polynomial of degree 3 over the unit cube.

Key words. robust semidefinite optimization, data uncertainty, Lyapunov stability synthesis, relaxations of combinatorial problems

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1. Introduction. In this paper, we focus on the following “matrix cube” problem:

MatrCube: Given an affine mapping $u \rightarrow \mathcal{B}(u) = B^0 + \sum_{\ell=1}^L u_\ell B^\ell$ from \mathbf{R}^L to the space \mathbf{S}^m of $m \times m$ real symmetric matrices and $\rho > 0$, check whether the image

$$\mathcal{C}[\rho] = \{A \mid \exists(u, \|u\|_\infty \leq \rho) : A = \mathcal{B}(u)\}$$

of the box $\{\|u\|_\infty \leq \rho\}$ under this mapping is contained in the cone \mathbf{S}_+^m of positive semidefinite matrices.

Problem **MatrCube** is closely related to what is called *uncertain semidefinite programming with interval uncertainty*. Specifically, consider a linear matrix inequality (LMI)

$$(1) \quad A_0 + \sum_{j=1}^n x_j A_j \succeq 0;$$

here $x \in \mathbf{R}^n$ is the vector of variables, $A_0, \dots, A_n \in \mathbf{S}^m$, and $A \succeq B$ means that $A - B \in \mathbf{S}_+^m$. Assume that the *data* $[A_0, \dots, A_n]$ of the LMI “are uncertain”—we only know that the data belong to a given *uncertainty set* \mathcal{U} . Our aim is to find *robust* solutions of the resulting “uncertain LMI,” i.e., solutions x of the semi-infinite system of LMIs

$$(2) \quad A_0 + \sum_{j=1}^n x_j A_j \succeq 0 \quad \forall [A_0, \dots, A_n] \in \mathcal{U}.$$

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We say that the uncertainty is *interval* if \mathcal{U} is the image of a box under an affine mapping:

$$(3) \quad \mathcal{U} = \mathcal{U}_\rho = \left\{ [A_0, \dots, A_n] = [A_0^0, \dots, A_n^0] + \sum_{\ell=1}^L u_\ell [A_0^\ell, \dots, A_n^\ell] \mid \|u\|_\infty \leq \rho \right\}.$$

As an example, consider the following *Lyapunov stability analysis* problem.

(LSA): Given an uncertain linear time-varying system

$$(4) \quad \frac{d}{dt}x(t) = A(t)x(t), \quad A(t) \in \mathcal{A} \quad \forall t,$$

where \mathcal{A} is a given compact set of matrices, check whether the system admits a quadratic Lyapunov function, i.e., whether there exists a positive definite matrix X such that

$$\frac{d}{dt}(x^T(t)X^{-1}x(t)) < 0$$

for all nonzero trajectories $x(t)$ of (4).

Recall that the existence of a quadratic Lyapunov function is a standard *sufficient* condition for the stability of the system (i.e., for the fact that $x(t) \rightarrow 0, t \rightarrow \infty$, for every trajectory of the system, whatever is a (measurable) choice of $A(\cdot)$ taking values in \mathcal{A}). It is easily seen that the existence of a quadratic Lyapunov function for (4) is equivalent to the solvability of the semi-infinite system of LMIs

$$(5) \quad \begin{array}{l} \text{(a)} \quad X \succeq I, \\ \text{(b)} \quad AX + XA^T \preceq -I \quad \forall A \in \mathcal{A}, \end{array}$$

and every solution X of the latter system defines a quadratic Lyapunov function for (4). Note that (5) is of the form of (2), so that finding a quadratic Lyapunov stability certificate for uncertain linear dynamic system (4) is exactly the same as solving a semi-infinite system of LMIs (2) associated with an appropriately chosen uncertainty set \mathcal{U} . Note also that the latter set is an interval uncertainty, provided that \mathcal{A} is also; e.g., provided that \mathcal{A} is an interval matrix:

$$(6) \quad \mathcal{A} = \mathcal{A}_\rho = \{A : |A_{ij} - A_{ij}^*| \leq \rho D_{ij} \quad \forall i, j\}.$$

(A^* is the “nominal” matrix, $D = [D_{ij} \geq 0]_{i,j}$ is a “perturbation scale,” and $\rho > 0$ is a “perturbation level.”)

The Lyapunov analysis example, as well as other examples which can be found in [1, 2, 4, 6], demonstrates the importance of robust solutions to semidefinite problems affected by data uncertainty, in particular by interval uncertainty. Theoretically speaking, the major difficulty with this concept is that (2) is a semi-infinite system of LMIs, and as such it can be computationally intractable. However, the set \mathcal{X} of solutions to (2) is clearly closed and convex; it follows that, essentially (for details, see [8]), the “computational tractability” of (2) (i.e., the ability to find efficiently a point in \mathcal{X} or to maximize efficiently a linear function over \mathcal{X}) is equivalent to the possibility of solving efficiently the following associated *analysis* problem.

Anal[x]: Given a candidate solution x , check whether it satisfies (2).

The role of the matrix cube problem in the context of uncertain semidefinite programming comes from the evident fact that *in the case of interval uncertainty* (3), *problem*

$\text{Anal}[x]$ is equivalent to problem MatrCube with the data

$$B^\ell = A_0^\ell + \sum_{j=1}^n x_j A_j^\ell, \quad \ell = 0, 1, \dots, L.$$

Unfortunately, the matrix cube problem in general is NP-hard. This is so even in the case in which all “edges” B^1, \dots, B^L of the “matrix box” $\mathcal{C}[\rho]$ are of rank ≤ 2 (see [9] or section 4 below). Consequently, one is forced to look for *verifiable sufficient conditions* for the inclusion $\mathcal{C}[\rho] \subset \mathbf{S}_+^m$. The simplest condition of this type is evident:

(S) Assume that there exist matrices X_1, \dots, X_L satisfying the system of LMIs

$$(7) \quad \begin{aligned} (a) \quad & X_\ell \succeq \pm \rho B^\ell, \quad \ell = 1, \dots, L, \\ (b) \quad & \sum_{\ell=1}^L X_\ell \preceq B^0. \end{aligned}$$

Then $\mathcal{C}[\rho] \subset \mathbf{S}_+^m$.

In the context of semi-infinite system of LMIs (2) with interval uncertainty (3), condition (S) results in the following system of LMIs in variables $x, \{X_\ell\}$:

$$(8) \quad \begin{aligned} X_\ell & \succeq \pm \rho \left[A_0^\ell + \sum_{j=1}^n x_j A_j^\ell \right], \quad \ell = 1, \dots, L, \\ \sum_{\ell=1}^L X_\ell & \preceq A_0^0 + \sum_{j=1}^n x_j A_j^0. \end{aligned}$$

This system is a “computationally tractable conservative approximation” of (2) in the sense that whenever x can be extended to a feasible solution of (8), x is feasible for (2) (by (S)).

The main result of this paper is as follows:

(N) The simple sufficient condition (S) for the inclusion $\mathcal{C}[\rho] \subset \mathbf{S}_+^m$ is not too conservative, provided that the edges B^1, \dots, B^L of the matrix box $\mathcal{C}[\rho]$ are of small ranks. Specifically, if (S) is not satisfied, then a $\vartheta(\mu)$ -enlargement $\mathcal{C}[\vartheta(\mu)\rho]$ of the box $\mathcal{C}[\rho]$ is not contained in \mathbf{S}_+^m . Here

$$\mu = \max_{\ell=1, \dots, L} \text{rank}(B^\ell),$$

and $\vartheta(\mu)$ is a certain universal function such that

$$\begin{aligned} \vartheta(1) &= 1; \quad \vartheta(2) = \frac{\pi}{2} = 1.57 \dots; \quad \vartheta(3) = 1.73 \dots; \quad \vartheta(4) = 2; \\ \vartheta(\mu) &\leq \frac{\pi\sqrt{\mu}}{2} \quad \forall \mu. \end{aligned}$$

Note that in typical semi-infinite systems of LMIs arising in control, perturbation of a single data entry results in small rank perturbations of the LMIs; whenever this is the case, (S) and (N) allow us to build a “tight” (up to a moderate absolute constant factor), computationally tractable conservative approximation of the semi-infinite system in question, provided that the uncertainty is interval. For example, in the Lyapunov stability analysis system (5), by perturbing a single entry in A , we

perturb the left-hand side of the semi-infinite LMI (5.b) by a matrix of rank ≤ 2 ; as we shall see, this observation combined with (N) allows us to build efficiently a lower bound, tight up to the factor $\frac{\pi}{2}$, for the “Lyapunov stability radius” of an interval matrix (i.e., for the supremum of those $\rho > 0$ for which all instances of the interval matrix (6) share a common quadratic Lyapunov function).

The rest of this paper is organized as follows. In section 2, we prove our main result (N). Section 3 is devoted to control applications of this result, specifically, those in Lyapunov stability analysis and synthesis. In section 4, we establish links between the matrix cube problem and the problem of maximizing a positive definite quadratic form over the unit cube; in particular, we demonstrate that (N) allows us to rederive the “ $\frac{\pi}{2}$ theorem” of Nesterov [12], which states that the standard semidefinite bound on the maximum of a positive definite quadratic form over the unit cube is tight within the factor $\frac{\pi}{2}$. In the concluding section 5, we apply our techniques to the problem of maximizing a homogeneous polynomial of degree 3 over the unit cube.

In what follows, we frequently use the *semidefinite duality*; for the reader’s convenience, we list here the relevant results (for proofs, see, e.g., [11]). Consider a semidefinite problem

$$(Pr) \quad \min_x \left\{ c^T x : \sum_{j=1}^n x_j A_j - A_0 \succeq 0 \right\};$$

here $x \in \mathbf{R}^n$, $A_0, \dots, A_n \in \mathbf{S}^m$. It is assumed that no nontrivial linear combination of the matrices A_1, \dots, A_n is zero.

The *semidefinite dual* of (Pr) is the problem

$$(Dl) \quad \max_X \{ \text{Tr}(A_0 X) : \text{Tr}(A_j X) = c_j, j = 1, \dots, n, X \succeq 0 \}.$$

The duality is symmetric: (Dl) can be straightforwardly rewritten in the form of (Pr), and the semidefinite dual of this reformulation is (equivalent to) (Pr). The *semidefinite duality theorem* says that if (Pr) is bounded below and strictly feasible (i.e., $\sum_j \bar{x}_j A_j - A_0 \succ 0$, for certain \bar{x} , where $A \succ B$ means that $A - B$ is positive definite), then (Dl) is solvable and has the same optimal value as (Pr).

2. The matrix cube problem. The formal statement of our main result (N) is given by the following.

THEOREM 2.1. *Consider problem MatrCube along with system of LMIs (7) in matrix variables X_1, \dots, X_L , and let*

$$\mu = \max_{1 \leq \ell \leq L} \text{rank}(B^\ell)$$

(note $1 \leq \ell$ in the max!). Then

(i) if system (7) is solvable, the matrix box $\mathcal{C}[\rho]$ is contained in the positive semidefinite cone \mathbf{S}_+^m ;

(ii.a) if system (7) is unsolvable, the $\vartheta(\mu)$ -enlargement $\mathcal{C}[\vartheta(\mu)\rho]$ of the matrix box $\mathcal{C}[\rho]$ is not contained in the positive semidefinite cone, where the function $\vartheta(\cdot)$ is given by

$$(9) \quad \frac{1}{\vartheta(k)} = \min_\alpha \left\{ \int_{\mathbf{R}^k} \left| \sum_{i=1}^k \alpha_i u_i^2 \right| (2\pi)^{-k/2} \exp \left\{ -\frac{u^T u}{2} \right\} du \mid \sum_{i=1}^k |\alpha_i| = 1 \right\};$$

(ii.b) the function $\vartheta(\cdot)$ satisfies the relations

$$(10) \quad \vartheta(k) \leq \frac{\pi\sqrt{k}}{2} \quad \forall k; \quad \vartheta(2) = \frac{\pi}{2}.$$

Proof. (i) is evident: If $\{X_\ell\}_{\ell=1}^L$ solves (7), then $u_\ell B^\ell \succeq -X_\ell$ for all ℓ and all $u_\ell \in [-\rho, \rho]$ by (7.a), so that

$$\|u\|_\infty \leq \rho \Rightarrow B^0 + \sum_{\ell=1}^{\ell} u_\ell B^\ell \succeq B^0 - \sum_{\ell=1}^L X_\ell \succeq 0$$

(we have used (7.b)), and thus $\mathcal{C}[\rho] \subset \mathbf{S}_+^m$.

(ii.a): Assume that (7) is unsolvable, and let us prove that in this case $\mathcal{C}[\vartheta(\mu)\rho] \not\subset \mathbf{S}_+^m$.

¹0. Since (7) is unsolvable, the optimal value in the semidefinite program

$$(P) \quad \min_{t, \{X_\ell\}} \left\{ t \mid tI + B^0 \succeq \sum_{\ell=1}^L X_\ell, X_\ell \succeq \pm \rho B^\ell, \ell = 1, \dots, L \right\}$$

is positive. Since (P) is strictly feasible, it follows from the semidefinite duality theorem that the semidefinite dual of (P), i.e., the program

$$(D) \quad \max_{U, \{Y_\ell, Z_\ell\}} \left\{ \rho \sum_{\ell=1}^L \text{Tr}([Y_\ell - Z_\ell]B^\ell) - \text{Tr}(UB^0) \mid \begin{array}{l} \text{Tr}(U) = 1, U \succeq 0, \\ Y_\ell + Z_\ell = U, \ell = 1, \dots, L, \\ Y_\ell, Z_\ell \succeq 0, \ell = 1, \dots, L, \end{array} \right\}$$

is solvable with a positive optimal value.

²0. To proceed, we need the following simple result.

LEMMA 2.2. *Let $U \succeq 0$ and B be a symmetric matrix of the same size as U . Then*

$$(11) \quad \max_{Y, Z \succeq 0: Y+Z=U} \text{Tr}([Y - Z]B) = \max_{V=V^T, \|V\| \leq 1} \text{Tr}(VU^{1/2}BU^{1/2}) = \|\lambda(U^{1/2}BU^{1/2})\|_1,$$

where $\lambda(Z)$ is the vector of eigenvalues of a symmetric matrix Z (counted with their multiplicities) and $\|Z\| = \|\lambda(Z)\|_\infty$ is the operator norm of Z .

Proof. We clearly have

$$\begin{aligned} \max_{Y, Z \succeq 0: Y+Z=U} \text{Tr}([Y - Z]B) &= \max_{P, Q \succeq 0: P+Q=I} \text{Tr}([U^{1/2}PU^{1/2} - U^{1/2}QU^{1/2}]B) \\ &= \max_{P, Q \succeq 0: P+Q=I} \text{Tr}([P - Q][U^{1/2}BU^{1/2}]) \\ &= \max_{V=V^T: \|V\| \leq 1} \text{Tr}(V[U^{1/2}BU^{1/2}]), \end{aligned}$$

as stated in the first equality in (11). To get the second equality, it suffices to consider the case in which the matrix $U^{1/2}BU^{1/2}$ is diagonal; in that case the equality becomes evident. \square

In view of Lemma 2.2, the fact that (D) is solvable with positive optimal value means that there exists $U \succeq 0$ such that

$$(12) \quad \rho \sum_{\ell=1}^L \|\lambda(U^{1/2}B^\ell U^{1/2})\|_1 > \text{Tr}(U^{1/2}B^0 U^{1/2}).$$

We are about to provide a probabilistic interpretation of (12), and this interpretation will lead us to (ii.a).

3⁰. Let us write $\xi \sim \mathcal{N}(0, I_k)$ to express that ξ is a random Gaussian k -dimensional vector with zero mean and unit covariance matrix, and let

$$p_k(u) = (2\pi)^{-k/2} \exp\{-u^T u/2\}$$

be the corresponding Gaussian density. We need the following fact.

LEMMA 2.3. *Whenever k is an integer and B is a symmetric $m \times m$ matrix with $\text{rank}(B) \leq k$ and $\xi \sim \mathcal{N}(0, I_m)$, one has*

$$\mathbf{E} \{|\xi^T B \xi|\} \geq \frac{\|\lambda(B)\|_1}{\vartheta(k)}.$$

Proof. It suffices to consider the case in which B is diagonal; in this case the relation in question immediately follows from the definition of $\vartheta(\cdot)$. \square

4⁰. Let $\xi \sim \mathcal{N}(0, I_m)$. We have

$$\begin{aligned} \mathbf{E} \left\{ \vartheta(\mu) \rho \sum_{\ell=1}^L |\xi^T U^{1/2} B^\ell U^{1/2} \xi| \right\} &= \rho \sum_{\ell=1}^L \vartheta(\mu) \mathbf{E} \left\{ |\xi^T U^{1/2} B^\ell U^{1/2} \xi| \right\} \\ &\geq \rho \sum_{\ell=1}^L \|\lambda(U^{1/2} B^\ell U^{1/2})\|_1 \\ &\text{[by Lemma 2.3 and in view of } \text{rank}(U^{1/2} B^\ell U^{1/2}) \leq \text{rank}(B^\ell) \leq \mu] \\ &> \text{Tr}(U^{1/2} B^0 U^{1/2}) \quad \text{[by (12)]} \\ &= \mathbf{E} \{ \xi^T U^{1/2} B^0 U^{1/2} \xi \} \quad \text{[evident],} \end{aligned}$$

so that there exists $r \in \mathbf{R}^m$ such that

$$\sum_{\ell=1}^L \vartheta(\mu) \rho |r^T U^{1/2} B^\ell U^{1/2} r| > r^T U^{1/2} B^0 U^{1/2} r.$$

Consequently, there exists a collection $\{\epsilon_\ell = \pm 1, \ell = 1, \dots, L\}$ such that

$$r^T \left[\sum_{\ell=1}^L \vartheta(\mu) \rho \epsilon_\ell U^{1/2} B^\ell U^{1/2} \right] r > r^T U^{1/2} B^0 U^{1/2} r,$$

i.e., the matrix $B^0 - \sum_{\ell=1}^L \vartheta(\mu) \rho \epsilon_\ell B^\ell$ is not positive semidefinite. Thus, $\mathcal{C}[\vartheta(\mu) \rho] \notin \mathbf{S}_+^m$, as claimed in (ii.a).

(ii.b): Let $\alpha \in \mathbf{R}^k$, $\|\alpha\|_1 = 1$, $\beta = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} \in \mathbf{R}^{2k}$, and $\xi \sim \mathcal{N}(0, I_{2k})$. Setting

$$J = \int \left| \sum_{i=1}^k u_i^2 \alpha_i \right| p_k(u) du,$$

we have

$$(13) \quad \mathbf{E} \left\{ \left| \sum_{i=1}^{2k} \xi_i^2 \beta_i \right| \right\} \leq \mathbf{E} \left\{ \left| \sum_{i=1}^k \xi_i^2 \alpha_i \right| + \left| \sum_{i=k+1}^{2k} \xi_i^2 \alpha_{i-k} \right| \right\} = 2J.$$

On the other hand, setting $\eta_i = (\xi_i - \xi_{i+k})/\sqrt{2}$, $\zeta_i = (\xi_i + \xi_{i+k})/\sqrt{2}$, we get

$$(14) \quad \left| \sum_{i=1}^{2k} \xi_i^2 \beta_i \right| = \left| \sum_{i=1}^k 2\alpha_i \eta_i \zeta_i \right| = 2 |\hat{\eta}^T \zeta|, \quad \hat{\eta} = \begin{bmatrix} \alpha_1 \eta_1 \\ \vdots \\ \alpha_k \eta_k \end{bmatrix}, \quad \zeta = \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_k \end{bmatrix}.$$

Note that $\zeta \sim \mathcal{N}(0, I_k)$ and $\hat{\eta}, \zeta$ are independent. Setting

$$\tilde{\eta} = \begin{bmatrix} |\alpha_1 \eta_1| \\ \vdots \\ |\alpha_k \eta_k| \end{bmatrix},$$

we have

$$(15) \quad \begin{aligned} \mathbf{E} \{ |\hat{\eta}^T \zeta| \} &= \mathbf{E} \{ \|\hat{\eta}\|_2 \} \int |t| p_1(t) dt \\ &\quad [\text{since } \hat{\eta}, \zeta \text{ are independent and } \zeta \sim \mathcal{N}(0, I_k)] \\ &= \mathbf{E} \{ \|\hat{\eta}\|_2 \} \frac{2}{\sqrt{2\pi}} = \frac{2}{\sqrt{2\pi}} \mathbf{E} \{ \|\tilde{\eta}\|_2 \} \\ &\geq \frac{2}{\sqrt{2\pi}} \|\mathbf{E} \{ \tilde{\eta} \} \|_2 = \frac{2}{\sqrt{2\pi}} \sqrt{\sum_{i=1}^k \alpha_i^2 \left(\frac{2}{\sqrt{2\pi}} \right)^2} \geq \frac{2}{\pi \sqrt{k}}. \end{aligned}$$

Combining (13), (14), and (15), we get $2J \geq \frac{4}{\pi \sqrt{k}}$, i.e., $\frac{1}{J} \leq \frac{\pi \sqrt{k}}{2}$, which yields the first relation in (10).

The second relation in (10) is given by the following computation:

$$\begin{aligned} \frac{1}{\vartheta(2)} &= \min_{\substack{\alpha \in \mathbf{R}^2, \\ \|\alpha\|_1=1}} \left\{ \int |\alpha_1 u_1^2 + \alpha_2 u_2^2| p_2(u) du \right\} = \min_{\theta \in [0,1]} \int |\theta u_1^2 - (1-\theta) u_2^2| p_2(u) du \\ &= \frac{1}{2} \int |u_1^2 - u_2^2| p_2(u) du \end{aligned}$$

[since the function to be minimized is convex in θ and symmetric w.r.t. $\theta = 1/2$]

$$= \left[\int |t| p_1(t) dt \right]^2 = \frac{2}{\pi}. \quad \square$$

Let us reformulate Theorem 2.1 in a more convenient form as follows.

COROLLARY 2.4. *Consider a semi-infinite system of LMIs (2) with interval data (see (3))*

$$(Sys[\rho]) \quad A_0 + \sum_{j=1}^n x_j A_j \succeq 0 \quad \forall [A_0, A_1, \dots, A_n] \in \mathcal{U}_\rho,$$

$$\mathcal{U}_\rho = \left\{ [A_0, A_1, \dots, A_n] = [A_0^0, A_1^0, \dots, A_n^0] + \sum_{\ell=1}^L u_\ell [A_0^\ell, A_1^\ell, \dots, A_n^\ell] \mid \|u\|_\infty \leq \rho \right\},$$

and let

$$B^\ell[x] = A_0^\ell + \sum_{j=1}^n x_j A_j^\ell, \quad \ell = 0, 1, \dots, L,$$

$$\mu = \max_{\substack{x, \\ 1 \leq \ell \leq L}} \text{rank}(B^\ell[x])$$

(note $1 \leq \ell$ in the max!).

The system of LMIs in variables $x, \{X_\ell\}$

$$X_\ell \succeq \pm \rho B^\ell[x], \quad \ell = 1, \dots, L,$$

(Appr[ρ])

$$\sum_{\ell=1}^L X_\ell \preceq B^0[x]$$

is a $\vartheta(\mu)$ -tight approximation of (Sys[ρ]), i.e.,

(i) if x can be extended to a feasible solution of (Appr[ρ]), then x is feasible for (Sys[ρ]);

(ii) if x cannot be extended to a feasible solution of (Appr[ρ]), then x is not feasible for (Sys[$\vartheta(\mu)\rho$]).

2.1. Simplification of (7). From the computational viewpoint, a shortcoming of the sufficient condition (7) for the inclusion $\mathcal{C}[\rho] \subset \mathbf{S}_+^m$ is that the sizes of the LMI system (7), although polynomial in the sizes of **MatrCube**, are “large”: The system has $2L + 1$ “big” ($m \times m$) LMIs and has $\frac{Lm(m+1)}{2}$ scalar decision variables. Our local goal is to demonstrate that in the case in which $\mu \equiv \max_{1 \leq \ell \leq L} \text{rank}(B^\ell)$ is small, (7) can be reduced to a much smaller system of LMIs.

PROPOSITION 2.5. (i) Let $S \in \mathbf{S}^m$ be a matrix of rank $k > 0$, so that

$$S = P^T R P$$

with invertible $k \times k$ symmetric matrix R and $k \times m$ matrix P of rank k .

(i.1) A matrix $X \in \mathbf{S}^m$ satisfies the relation $X \succeq \pm S$ if and only if there exist $k \times k$ symmetric matrices Y, Z satisfying the relations

$$(a) \quad X \succeq \frac{1}{2} P^T (Y + Z) P,$$

$$(16) \quad (b) \quad \begin{bmatrix} Y & R \\ R & Z \end{bmatrix} \succeq 0.$$

(i.2) In particular, $X \succeq \pm S$ if and only if there exists $U \succeq \pm R$ such that $X \succeq P^T U P$.

(ii) Consequently, the solvability of (7) is equivalent to the solvability of the system of LMIs

$$(a) \quad \begin{bmatrix} Y_\ell & \rho R_\ell \\ \rho R_\ell & Z_\ell \end{bmatrix} \succeq 0, \quad \ell = 1, \dots, L,$$

$$(17) \quad (b) \quad \sum_{\ell=1}^L P_\ell^T (Y_\ell + Z_\ell) P_\ell \preceq 2B^0$$

in matrix variables $Y_\ell, Z_\ell \in \mathbf{S}^{k_\ell}$, $\ell = 1, \dots, L$. Here $k_\ell = \text{rank}(B^\ell)$ (without loss of generality, we can assume that $k_\ell > 0$), and $P_\ell, R_\ell = R_\ell^T$ are $k_\ell \times m$ and $k_\ell \times k_\ell$ matrices of rank k_ℓ such that $B^\ell = P_\ell^T R_\ell P_\ell$, $\ell = 1, \dots, L$.

Proof. (i.1), “if” part: Assume that X, Y, Z satisfy (16); we should prove that then $X \succeq \pm S$. To this end it suffices to verify that if Y, Z satisfy (16.b), then $\frac{1}{2}(Y + Z) \succeq \pm R$, which is immediate:

$$(16.b) \Rightarrow \left\{ 0 \leq \begin{bmatrix} \xi \\ \epsilon \xi \end{bmatrix}^T \begin{bmatrix} Y & R \\ R & Z \end{bmatrix} \begin{bmatrix} \xi \\ \epsilon \xi \end{bmatrix} \quad \forall (\xi \in \mathbf{R}^k, \epsilon = \pm 1) \right\}$$

$$\Leftrightarrow \left\{ 0 \leq \xi^T(Y + Z)\xi + 2\epsilon \xi^T R \xi \quad \forall (\xi \in \mathbf{R}^k, \epsilon = \pm 1) \right\} \Rightarrow \frac{1}{2}(Y + Z) \succeq \pm R.$$

(i.1), “only if” part: Let $X \succeq \pm S$. We should prove that there exist Y, Z satisfying (16). Assume, on the contrary, that the system of LMIs (16) in variables Y, Z is unsolvable, and consider the semidefinite program

$$(18) \quad t^* = \min_{t, Y, Z} \left\{ t : \begin{array}{l} tI + 2X - P^T(Y + Z)P \succeq 0, \\ \begin{bmatrix} Y & R \\ R & Z \end{bmatrix} \succeq 0 \end{array} \right\}.$$

Since P is of rank k , the intersections of the levels sets of the objective with the (nonempty!) feasible set of the problem are bounded, whence the problem is solvable; unsolvability of (16) implies that the optimal value t^* in the problem is positive. Since (18) clearly is strictly feasible, it follows that the semidefinite dual of (18), which is the semidefinite program

$$(19) \quad \min_{U, V, W, Q} \left\{ -2 \text{Tr}(UX) - 2 \text{Tr}(RQ^T) : \begin{array}{l} V = PUP^T \\ W = PUP^T \\ \begin{bmatrix} V & Q \\ Q^T & W \end{bmatrix} \succeq 0 \\ \text{Tr}(U) = 1 \\ U, V, W \succeq 0 \end{array} \right\},$$

is solvable with the same positive optimal value t^* . In other words, there exist $U \succeq 0$ and Q such that

$$(20) \quad \begin{array}{l} \text{(a)} \quad \text{Tr}(UX) < \text{Tr}(RQ^T), \\ \text{(b)} \quad \begin{bmatrix} PUP^T & Q \\ Q^T & PUP^T \end{bmatrix} \succeq 0. \end{array}$$

From (20.b), by standard arguments, it follows that $Q = PU^{1/2}MU^{1/2}P^T$ for appropriately chosen M such that $M^T M \leq I$. Consequently, (20.a) reads

$$\text{Tr}(\underbrace{U^{1/2}XU^{1/2}}_{\bar{X}}) < \text{Tr}(RP^T U^{1/2}M^T U^{1/2}P) = \text{Tr}(\underbrace{(U^{1/2}SU^{1/2})}_{\bar{S}}M^T).$$

Since $M^T M \leq I$, the quantity $\text{Tr}(\bar{S}M^T)$ is $\leq \|\lambda(\bar{S})\|_1$, and we come to the relation $\text{Tr}(\bar{X}) < \|\lambda(\bar{S})\|_1$. This is the desired contradiction, since from $X \succeq \pm S$ it follows that $\bar{X} \succeq \pm \bar{S}$, whence $\text{Tr}(\bar{X}) \geq \|\lambda(\bar{S})\|_1$. (Notice what happens in the orthonormal basis, where \bar{S} becomes diagonal.) Thus (i.1) is proved.

(i.2): If $X \succeq P^T U P$ with $U \succeq \pm R$, then of course $X \succeq \pm P^T R P = \pm S$. Conversely, if $X \succeq \pm S$, then by (i.2) there exist Y, Z satisfying (16). Setting $U = \frac{1}{2}(Y+Z)$, we have $X \succeq P^T U P$ by (16.a), and applying (i.1) to R rather than to S , we have $U \succeq \pm R$.

(ii) is an immediate consequence of (i). □

Note that when the ranks k_ℓ of the matrices B^ℓ , $\ell = 1, \dots, L$, are much less than the size m of these matrices, system (17) is much better suited for numerical processing than (7). Indeed, the latter system has $2L + 1$ “big” ($m \times m$) LMIs and totally $\frac{Lm(m+1)}{2}$ scalar decision variables, while the former system has a single “big” LMI, L “small” ones (of the sizes at most $2\mu \times 2\mu$, $\mu = \max_{1 \leq \ell \leq L} k_\ell$), and no more than $L\mu(\mu + 1)$ scalar decision variables. A shortcoming of the reformulated system, when compared with the original one, is that when the matrices B^ℓ depend affinely on certain vectors of parameters x (as is the case in the semi-infinite LMI (2) with interval uncertainty (3)), system (7) always is a system of LMIs in variables x , $\{X_\ell\}$ (cf. $(A[\rho])$), while (17) is a system of LMIs in x, Y_ℓ, Z_ℓ only under the additional (and restrictive) assumption that the matrices P_ℓ are independent of x . In section 3.2 we shall see that in certain important applications this shortcoming can be avoided.

3. Application I: Quadratic Lyapunov stability analysis and synthesis.

3.1. Lyapunov stability analysis/synthesis. Consider a controlled time-varying linear dynamic system

$$\begin{aligned} \text{(a)} \quad \frac{d}{dt}x(t) &= A(t)x(t) + B(t)u(t) && \text{[open-loop system]}, \\ \text{(b)} \quad u(t) &= Kx(t) && \text{[feedback]}, \\ \text{(21)} \quad & \downarrow \\ \text{(c)} \quad \frac{d}{dt}x(t) &= [A(t) + B(t)K]x(t) && \text{[closed-loop system]} \end{aligned}$$

(x is n -dimensional, u is m -dimensional), which is uncertain in the sense that the dependency $t \mapsto [A(t), B(t)]$ is not known in advance; all we know is that

$$\forall t: [A(t), B(t)] \in \mathcal{U}_\rho = \{[A, B] \mid |A_{ij} - B_{ij}^*| \leq \rho C_{ij}, |B_{i\ell} - B_{i\ell}^*| \leq \rho D_{i\ell} \quad \forall i, j, \ell\}. \tag{22}$$

Here A^*, B^* are given “nominal” data; C, D are given “scale matrices” with nonnegative entries; and $\rho \geq 0$ is the “perturbation level.”

Consider the following pair of problems.

Lyapunov stability analysis: Given A^*, B^*, C, D , and a feedback K , find the supremum R_*^a of those $\rho \geq 0$ for which all instances $A + BK, [A, B] \in \mathcal{U}_\rho$, of the closed-loop system matrix (21.c) share a common quadratic Lyapunov function:

$$\begin{aligned}
 & \text{(LA)} \\
 & R_*^a = \sup_{\rho, X} \left\{ \rho : X \succeq I, [A + BK]X + X[A + BK]^T \preceq -I \forall [A, B] \in \mathcal{U}_\rho \right\} \\
 & = \sup_{\rho, X} \left\{ \rho : \begin{array}{l} X \succeq I \\ \forall (u_{ij}, |u_{ij}| \leq \rho, u^{i\ell}, |u^{i\ell}| \leq \rho) : \\ \sum_{i,j} u_{ij} \underbrace{C_{ij}[E^{ij}X + XE^{ij}]}_{A_{ij}[X]} + \sum_{i,\ell} u^{i\ell} \underbrace{D_{i\ell}[F^{i\ell}KX + XK^T(F^{i\ell})^T]}_{A^{i\ell}[X]} \\ \preceq \underbrace{[-I - (A^* + B^*K)X - X(A^* + B^*K)^T]}_{A[X]} \end{array} \right\},
 \end{aligned}$$

where E^{ij} are the basic $n \times n$ matrices (1 in cell ij , zeros in other cells), and $F^{i\ell}$ are the basic $n \times m$ matrices.

Lyapunov stability synthesis: Given A^*, B^*, C, D , find the supremum R_*^s of those $\rho \geq 0$ for which there exists a feedback K such that all instances $A + BK$, $[A, B] \in \mathcal{U}_\rho$, of the closed-loop system matrix (21.c) share a common quadratic Lyapunov function:

$$\begin{aligned}
 & \text{(LS)} \\
 & R_*^s = \sup_{\rho, X, K} \left\{ \rho : X \succeq I, [A + BK]X + X[A + BK]^T \preceq -I \forall [A, B] \in \mathcal{U}_\rho \right\} \\
 & = \sup_{\rho, X, Z} \left\{ \rho : X \succeq I, AX + XA^T + BZ + Z^T B^T \preceq -I \forall [A, B] \in \mathcal{U}_\rho, [Z = KX] \right\} \\
 & = \sup_{\rho, X, Z} \left\{ \rho : \begin{array}{l} X \succeq I \\ \forall (u_{ij}, |u_{ij}| \leq \rho, u^{i\ell}, |u^{i\ell}| \leq \rho) : \\ \sum_{i,j} u_{ij} \underbrace{C_{ij}[E^{ij}X + XE^{ij}]}_{B_{ij}[X]} + \sum_{i,\ell} u^{i\ell} \underbrace{D_{i\ell}[F^{i\ell}Z + Z^T(F^{i\ell})^T]}_{B^{i\ell}[Z]} \\ \preceq \underbrace{[-I - A^*X - X(A^*)^T - B^*Z - Z(B^*)^T]}_{B[X, Z]} \end{array} \right\},
 \end{aligned}$$

where E^{ij} are the basic $n \times n$, and $F^{i\ell}$ are the basic $n \times m$ matrices.

As we can see, both problems (LA) and (LS) deal with *solvability of semi-infinite systems of LMIs*. Consider the approximations of these systems as follow:

$$\text{(ALA)} \quad \rho_*^a = \max_{\rho, X, \{X^{ij}, Y^{i\ell}\}} \left\{ \rho : \begin{array}{l} A_{ij}[X] \leq X^{ij}, -A_{ij}[X] \leq X^{ij} \forall i, j, \\ A^{i\ell}[X] \leq Y^{i\ell}, -A^{i\ell}[X] \leq Y^{i\ell} \forall i, \ell, \\ \rho \left[\sum_{i,j} X^{ij} + \sum_{i,\ell} Y^{i\ell} \right] \preceq A[X], \end{array} \right\}$$

$$\text{(ALS)} \quad \rho_*^s = \max_{\rho, X, Z, \{X^{ij}, Y^{i\ell}\}} \left\{ \rho : \begin{array}{l} B_{ij}[X] \leq X^{ij}, -B_{ij}[X] \leq X^{ij} \forall i, j, \\ B^{i\ell}[Z] \leq Y^{i\ell}, -B^{i\ell}[Z] \leq Y^{i\ell} \forall i, \ell, \\ \rho \left[\sum_{i,j} X^{ij} + \sum_{i,\ell} Y^{i\ell} \right] \preceq B[X, Z]. \end{array} \right\}$$

Note that both (ALA) and (ALS) are generalized eigenvalue problems (see [3, 10]) and as such are “computationally tractable.”

Taking into account that the ranks of the matrices $A_{ij}[X]$, $A^{i\ell}[X]$, $B_{ij}[X]$, $B^{i\ell}[Z]$ never exceed 2 and applying Corollary 2.4, we come to the following result.

THEOREM 3.1. (i) *Consider the Lyapunov stability analysis problem and assume that the matrix $A^* + B^*K$ of the nominal closed-loop system is stable (i.e., all its eigenvalues are in the open left half-plane). Then problem (ALA) is an approximation of (LA) (i.e., the ρ, X -component of a feasible solution of (ALA) is a feasible solution of (LA)), and the optimal value of (ALA) coincides with the one of (LA) within the factor $\frac{\pi}{2}$:*

$$\rho_*^a \leq R_*^a \leq \frac{\pi}{2} \rho_*^a.$$

(ii) *Consider the Lyapunov stability synthesis problem and assume that the nominal system is stabilizable (i.e., there exists a feedback K^* such that the matrix $A^* + B^*K^*$ is stable). Then problem (ALS) is an approximation of (LS) (i.e., the ρ, X, Z -component of a feasible solution of (ALS) is a feasible solution of (LS)), and the optimal value in (ALS) coincides with the one of (LS) within the factor $\frac{\pi}{2}$:*

$$\rho_*^s \leq R_*^s \leq \frac{\pi}{2} \rho_*^s.$$

3.2. Simplifications of (ALA) and (ALS). Although the dimensions of the approximating semidefinite problems (ALA) and (ALS) are polynomial in the dimensions of the original system (21), they are nevertheless of huge design dimension. (They have a matrix variable per every uncertain entry in the data of (21).) This fact may render the approximating problems too difficult for practical use. We are about to demonstrate that the design dimensions of (ALA) and (ALS) can be reduced dramatically.

Consider a “generic problem” of the same structure as (ALA), (ALS):

We are given $\rho > 0$ and $L + 1$ symmetric $m \times m$ matrices $B^0[x]$, $B^1[x], \dots, B^L[x]$ affinely depending on vector x of design variables, with $B^\ell[x]$, $\ell \geq 1$, of the form

$$(23) \quad B^\ell[x] = a_\ell b_\ell^T[x] + b_\ell[x] a_\ell^T,$$

where $a_\ell \neq 0$ and the vectors $b_\ell[x] \neq 0$ are affine in x . We associate with these data the semi-infinite system of LMIs in variables x, u

$$(24) \quad B^0[x] + \sum_{\ell=1}^L u_\ell B^\ell[x] \succeq 0 \quad \forall (u : \|u\|_\infty \leq \rho),$$

along with its “tractable conservative approximation”—the system of LMIs in variables x and additional matrix variables X_1, \dots, X_L as follows:

$$(a) \quad X_\ell \succeq \pm \rho B^\ell[x], \quad \ell = 1, \dots, L,$$

$$(\mathcal{P}[\rho]) \quad (b) \quad \sum_{\ell=1}^L X_\ell \preceq B^0[x].$$

The problem is to simplify $(\mathcal{P}[\rho])$, i.e., to pass from this system to a system of LMIs in variables x and, perhaps, additional variables λ in

such a way that the new system (let it be called $(\mathcal{S}[\rho])$) is of smaller design dimension than $(\mathcal{P}[\rho])$ and is equivalent to $(\mathcal{P}[\rho])$ in the sense that an x can be extended to a feasible solution of $(\mathcal{S}[\rho])$ if and only if x can be extended to a feasible solution of $(\mathcal{P}[\rho])$.

Note that both (ALA) and (ALS) are of the form of $(\mathcal{P}[\rho])$.

The simplification of $(\mathcal{P}[\rho])$ to follow is similar to the construction presented in Proposition 2.5; it turns out that the specific form (23) of the dependence of B^ℓ on x allows us to end up with an analogy of (17) which is a system of LMIs in x and additional variables. The key to our simplification is the following simple fact (which can be viewed as certain strengthening of Proposition 2.5(i) for the case in which $S = ab^T + ba^T$).

LEMMA 3.2. *Let $a, b \in \mathbf{R}^m$ be two nonzero vectors, and let X be an $m \times m$ symmetric matrix. Then $X \succeq \pm[ab^T + ba^T]$ if and only if there exists a positive real λ such that*

$$(25) \quad X \succeq \lambda aa^T + \frac{1}{\lambda} bb^T.$$

Proof. “If” part: It suffices to prove that if $\lambda > 0$, then $\lambda aa^T + \frac{1}{\lambda} bb^T \succeq \pm[ab^T + ba^T]$, which is immediate:

$$\forall \xi : \xi^T \left[\lambda aa^T + \frac{1}{\lambda} bb^T \right] \xi = \lambda (a^T \xi)^2 + \frac{1}{\lambda} (b^T \xi)^2 \geq 2|a^T \xi| |b^T \xi| \geq |\xi^T [ab^T + ba^T] \xi|.$$

“Only if” part: Assume that $a, b \neq 0$ and $X \succeq \pm[ab^T + ba^T]$; we should prove that there exists $\lambda > 0$ such that $X \succeq [\lambda aa^T + \frac{1}{\lambda} bb^T]$, or, a statement which is clearly equivalent, that the system of LMIs

$$(26) \quad \begin{aligned} X &\succeq \lambda aa^T + \mu bb^T, \\ \begin{bmatrix} \mu & 1 \\ 1 & \lambda \end{bmatrix} &\succeq 0 \end{aligned}$$

is solvable. Assume, on the contrary, that the system is unsolvable. Since $a, b \neq 0$, the semidefinite problem

$$(27) \quad \min_{t, \lambda, \mu} \left\{ t : \begin{aligned} tI + X &\succeq \lambda aa^T + \mu bb^T, \\ \begin{bmatrix} \mu & 1 \\ 1 & \lambda \end{bmatrix} &\succeq 0 \end{aligned} \right\}$$

clearly is solvable; but then the infeasibility of (27) means that the optimal value in problem (27) is positive. Since the problem clearly is strictly feasible, the problem

$$(28) \quad \max_{U, p, q, r} \left\{ -\text{Tr}(UX) - 2r : \begin{aligned} p &= b^T U b \\ q &= a^T U a \\ \text{Tr}(U) &= 1 \\ \begin{bmatrix} p & r \\ r & q \end{bmatrix} &\succeq 0 \\ U &\succeq 0 \end{aligned} \right\},$$

which is the semidefinite dual of (27), is solvable with positive optimal value. Since at a feasible solution to this problem one clearly has $|r| \leq \sqrt{pq} = \sqrt{(a^T U a)(b^T U b)}$,

the latter fact is equivalent to the existence of $U \succeq 0$ such that

$$(29) \quad 2\sqrt{(a^T U a)(b^T U b)} > \text{Tr}(UX).$$

Setting $\bar{a} = U^{1/2}a$, $\bar{b} = U^{1/2}b$, $\bar{X} = U^{1/2}XU^{1/2}$ and taking into account that $X \succeq \pm[ab^T + ba^T]$, we get

$$(30) \quad \begin{aligned} \text{(a)} \quad & \bar{X} \succeq \pm Q, \quad Q = \bar{a}\bar{b}^T + \bar{b}\bar{a}^T, \\ \text{(b)} \quad & \text{Tr}(\bar{X}) < 2\|\bar{a}\|_2\|\bar{b}\|_2. \end{aligned}$$

This is the desired contradiction. Indeed, from (30.a) it follows that $\text{Tr}(\bar{X}) \geq \|\lambda(Q)\|_1$ (pass to the orthonormal basis where Q is diagonal); on the other hand, an immediate computation demonstrates that $\|\lambda(Q)\|_1 = 2\|\bar{a}\|_2\|\bar{b}\|_2$, which is $> \text{Tr}(\bar{X})$ by (30.b). \square

Lemma 3.2 underlies the following proposition.

PROPOSITION 3.3. *The LMI system $(\mathcal{P}[\rho])$ is equivalent to the following system of LMIs in variables x and additional variables $Y \in \mathbf{S}^m$, $\lambda \in \mathbf{R}^L$:*

$$(31) \quad \begin{aligned} \text{(a)} \quad & \begin{bmatrix} Y - \sum_{\ell=1}^L \lambda_\ell a_\ell a_\ell^T & b_1[x] & b_2[x] & \dots & b_\ell[x] \\ b_1^T[x] & \lambda_1 & & & \\ b_2^T[x] & & \lambda_2 & & \\ \vdots & & & \ddots & \\ b_L^T[x] & & & & \lambda_L \end{bmatrix} \succeq 0, \\ \text{(b)} \quad & \rho Y \preceq B^0[x]. \end{aligned}$$

Proof. We should prove that if x can be extended to a feasible solution of $(\mathcal{P}[\rho])$, then x can be extended to a feasible solution of (31), and vice versa.

1⁰. Assume that $x, \{X_\ell\}$ is a feasible solution of $(\mathcal{P}[\rho])$, and let $J(x)$ be the set of those ℓ for which $b_\ell[x] = 0$. Let us extend x to a feasible solution of (31) as follows:

1. For $\ell \in J(x)$, we set $\lambda_\ell = 0$.
2. For $\ell \notin J(x)$, we have $a_\ell \neq 0$, $b_\ell[x] \neq 0$, and $\rho^{-1}X_\ell \succeq \pm[a_\ell b_\ell^T[x] + b_\ell[x]a_\ell^T]$. Applying Lemma 3.2, we can find $\lambda_\ell > 0$ such that $\rho^{-1}X_\ell \succeq \lambda_\ell a_\ell a_\ell^T + \lambda_\ell^{-1} b_\ell[x] b_\ell^T[x]$.

3. After we have defined $\lambda_\ell \geq 0$ for all $\ell = 1, \dots, L$, we set

$$Y = \sum_{\ell \notin J(x)} [\lambda_\ell a_\ell a_\ell^T + \lambda_\ell^{-1} b_\ell[x] b_\ell^T[x]].$$

Let us prove that $x, Y, \{\lambda_\ell\}$ is feasible for (31). Indeed, (31.a) is readily given by the definition of Y and the Schur complement lemma. (Note that a zero λ_ℓ on the diagonal of the left-hand side matrix in (31.a) corresponds to a zero row and a zero column.) Further, from the origin of λ_ℓ , $\ell \notin J(x)$, it follows that $Y \preceq \sum_{\ell \notin J(x)} \rho^{-1}X_\ell \preceq \rho^{-1} \sum_{\ell=1}^L X_\ell$, and since $x, \{X_\ell\}$ is feasible for $(\mathcal{P}[\rho])$, we conclude that $\rho Y \preceq B^0[x]$, i.e., (31.b) is valid. Thus, $x, Y, \{\lambda_\ell\}$ is feasible for (31).

2⁰. Now assume that $x, Y, \{\lambda_\ell\}$ is feasible for (31), and let us prove that x can be extended to a feasible solution of $(\mathcal{P}[\rho])$. Let, as above, $J(x)$ be the set of those ℓ for

which $b_\ell[x] = 0$. Note that from (31.a) it follows that $\lambda_\ell \geq 0$ for all ℓ , and $\lambda_\ell > 0$ for $\ell \notin J(x)$. Let us set

$$X_\ell = \rho \begin{cases} 0, & \ell \in J(x), \\ \lambda_\ell a_\ell a_\ell^T + \lambda_\ell^{-1} b_\ell[x] b_\ell^T[x], & \ell \notin J(x). \end{cases}$$

Applying Lemma 3.2, we see that $(\mathcal{P}[\rho].a)$ holds true. Now, by the Schur complement lemma from (31.a), it follows that

$$\rho^{-1} \sum_\ell X_\ell = \sum_{\ell \notin J(x)} [\lambda_\ell a_\ell a_\ell^T + \lambda_\ell^{-1} b_\ell[x] b_\ell^T[x]] \preceq Y;$$

this observation combined with (31.b) implies the validity of $(\mathcal{P}[\rho].b)$. Thus, $x, \{X_\ell\}$ is feasible for $(\mathcal{P}[\rho])$. \square

We have reduced the system of LMIs $(\mathcal{P}[\rho])$ to (31). In the original system, there are $(\dim x + L \dim X)$ scalar design variables, while in the resulting system there are just $(\dim x + \dim X + L)$ design variables. To realize how large the reduction in the design dimension can be, consider the case in which $(\mathcal{P}[\rho])$ is the problem (ALS). Here $x = X$ is a symmetric $n \times n$ matrix, and L is the total number of uncertain entries in the underlying uncertain interval matrix $[A, B]$. Here the original system $(\mathcal{P}[\rho])$ has $L + 1$ symmetric $n \times n$ matrix variables, i.e., totally $\frac{(L+1)n(n+1)}{2}$ scalar design variables, and $(2L + 1)$ “large” $(n \times n)$ LMIs. The reformulated system (31) has just two symmetric $n \times n$ matrix variables X, Y , and $L \leq n^2 + nm$ scalar variables λ_ℓ , i.e., totally $L + n(n + 1) \leq 2n^2 + n(m + 1)$ scalar design variables. As for LMIs, system (31) has one “large” $(n \times n)$ LMI (b) and one “very large” $((n + L) \times (n + L))$ LMI (a); note, however, that this LMI is of very simple “arrow” structure and is very sparse. Thus, (31) seems to be much better suited for numerical processing than $(\mathcal{P}[\rho])$.

3.3. Extensions. An LMI region is a set \mathcal{H} in the complex plane \mathbf{C} representable as

$$\mathcal{H} = \{z \in \mathbf{C} \mid f_{\mathcal{H}}(z) \equiv P + Qz + Q^T \bar{z} \prec 0\},$$

where $P = P^T$ and Q are real $k \times k$ matrices and \bar{z} is the complex conjugate of z . The simplest examples of LMI regions are

1. *open left half-plane:* $f_{\mathcal{H}}(z) = z + \bar{z}$;
2. *open disk* $\{z \mid |z + q| \leq r\}$, $q \in \mathbf{R}, r > 0$: $f_{\mathcal{H}}(z) = \begin{pmatrix} -r & \bar{z} + q \\ z + q & -r \end{pmatrix}$;
3. *the interior of the sector* $\{z \mid \pi - \theta \leq |\arg(z)| \leq \pi\}$ ($-\pi < \arg(z) \leq \pi$, $0 < \theta < \frac{\pi}{2}$):

$$f_{\mathcal{H}}(z) = \begin{pmatrix} (z + \bar{z}) \sin \theta & -(z - \bar{z}) \cos \theta \\ (z - \bar{z}) \cos \theta & (z + \bar{z}) \sin \theta \end{pmatrix};$$

4. *the stripe* $\{z \mid h_1 < \Re(z) < h_2\}$: $f_{\mathcal{H}}(z) = \begin{pmatrix} 2h_1 - (z + \bar{z}) & 0 \\ 0 & (z + \bar{z}) - 2h_2 \end{pmatrix}$.

It is known (see, e.g., [5]) that the spectrum $\Sigma(A)$ of a real $n \times n$ matrix A belongs to \mathcal{H} if and only if there exists $Y \in \mathbf{S}^m$, $Y \succ 0$, such that the $k \times k$ block matrix $\mathcal{M}[X, A]$ with the $m \times m$ blocks

$$\mathcal{M}_{ij}[X, A] = P_{ij}X + Q_{ij}AX + Q_{ji}XA^T, \quad i, j = 1, \dots, m,$$

is negative definite. We can treat such an X as a certificate of the inclusion $\Sigma(A) \subset \mathcal{H}$, and for homogeneity reasons we can normalize this certificate to satisfy the relations $X \succeq I$, $\mathcal{M}[X, A] \preceq -I$. From now on, we speak about normalized certificates only.

The problem we are interested in now is as follows. Given an LMI region \mathcal{H} and an ‘‘uncertain interval matrix’’

$$\mathcal{U}_\rho = \{[A, B] \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times m} \mid |A_{ij} - A_{ij}^*| \leq \rho C_{ij}, |B_{i\ell} - B_{i\ell}^*| \leq \rho D_{i\ell}, 1 \leq i, j \leq n, 1 \leq \ell \leq m\}$$

of the open-loop system (21), we ask what is the supremum R_* of those $\rho \geq 0$ for which there exists a linear feedback $K \in \mathbf{R}^{m \times n}$ such that all instances $A + BK$, $[A, B] \in \mathcal{U}_\rho$, of the uncertain matrix of the closed-loop system share a common certificate X of the inclusion $\Sigma(A + BK) \subset \mathcal{H}$. This important problem in control is a natural extension of the Lyapunov stability synthesis problem. The problem can be treated in the same fashion as Lyapunov analysis/synthesis. Indeed, $X \succeq I$ certifies the inclusion $\Sigma(A + BK) \subset \mathcal{H}$ for all $[A, B] \in \mathcal{U}_\rho$ if and only if (X, K) solves the semi-infinite system of matrix inequalities

$$\mathcal{M}[X, A + BK] \preceq -I \quad \forall [A, B] \in \mathcal{U}_\rho.$$

Passing from the variables X, K to $X, Z = KX$, we convert this system to the semi-infinite system of LMIs

$$\forall ([A, B] \in \mathcal{U}_\rho) :$$

$$\mathcal{N}(X, Z, A, B) \equiv [P_{ij}X + Q_{ij}AX + Q_{ij}BZ + Q_{ji}XA^T + Q_{ji}Z^T B^T]_{1 \leq i, j \leq k} \preceq -I, \quad (32)$$

where $[M_{ij}]_{1 \leq i, j \leq k}$ denotes block matrix with blocks M_{ij} . We see that (X, Z) solves (32) if and only if (X, Z) solves the semi-infinite system of LMIs

$$(\mathcal{I}[\rho])$$

$$\forall (\{u_{ij}, |u_{ij}| \leq 1\}, \{v_{i\ell}, |v_{i\ell}| \leq 1\}) :$$

$$\rho \left(\sum_{(i,j): C_{ij} > 0} u_{ij} \mathcal{N}^0(X, Z, C_{ij}E^{ij}, 0) + \sum_{(i,\ell): D_{i\ell} > 0} v_{i\ell} \mathcal{N}^0(X, Z, 0, D_{i\ell}F^{i\ell}) \right) - I - \mathcal{N}(X, Z, A^*, B^*) \succeq 0,$$

$$\mathcal{N}^0(X, Z, A, B) = [Q_{ij}AX + Q_{ij}BZ + Q_{ji}XA^T + Q_{ji}Z^T B^T]_{1 \leq i, j \leq k},$$

where the basic matrices E^{ij} , $F^{i\ell}$ are the same as in (LA), (LS). As before, an evident sufficient condition for $X \succeq I$ and Z to solve $(\mathcal{I}[\rho])$ is the existence of matrices X^{ij} , $(i, j) \in \mathcal{C} = \{(i, j) \mid C_{ij} > 0\}$, and $Z^{i\ell}$, $(i, \ell) \in \mathcal{D} = \{(i, \ell) \mid D_{i\ell} > 0\}$, such that $(X, Z, X^{ij}, Z^{i\ell})$ solves the system of LMIs

$$X^{ij} \succeq \mathcal{N}^0(X, Z, C_{ij}E^{ij}, 0), \quad X^{ij} \succeq -\mathcal{N}^0(X, Z, C_{ij}E^{ij}, 0), \quad (i, j) \in \mathcal{C},$$

$$Z^{i\ell} \succeq \mathcal{N}^0(X, Z, 0, D_{i\ell}F^{i\ell}), \quad Z^{i\ell} \succeq -\mathcal{N}^0(X, Z, 0, D_{i\ell}F^{i\ell}), \quad (i, \ell) \in \mathcal{D},$$

$$(\mathcal{II}[\rho])$$

$$\rho \left(\sum_{(i,j) \in \mathcal{C}} X^{ij} + \sum_{(i,\ell) \in \mathcal{D}} Z^{i\ell} \right) \preceq -I - \mathcal{N}(X, Z, A^*, B^*),$$

$$X \succeq I.$$

Invoking Theorem 2.1, we arrive at the following result.

THEOREM 3.4. *Let the system $(\mathcal{I}[0])$ (or, which is the same, $(\mathcal{II}[0])$) be solvable, and let*

$$\mu = \max \left[\max_{X,Z,i,j} \text{rank}(\mathcal{N}^0(X, Z, C_{ij}E^{ij}, 0)), \max_{X,Z,i,\ell} \text{rank}(\mathcal{N}^0(X, Z, 0, D_{i\ell}F^{i\ell})) \right].$$

Then

(i) *if $(\mathcal{II}[\rho])$ is solvable, then so is $(\mathcal{I}[\rho])$, and the (X, Z) -component of a solution of the former system solves the latter system;*

(ii) *if $(\mathcal{II}[\rho])$ is unsolvable, then so is $(\mathcal{I}[\vartheta(\mu)\rho])$, where $\vartheta(\cdot)$ is the function given in (9).*

In particular,

$$(33) \quad \frac{\sup \{ \rho : (\mathcal{I}[\rho]) \text{ is solvable} \}}{\sup \{ \rho : (\mathcal{II}[\rho]) \text{ is solvable} \}} \leq \vartheta(\mu).$$

Note that the denominator in (33) is the optimal value in an explicit generalized eigenvalue problem and thus is efficiently computable. Note also that one always has $\mu \leq 2k$, and that, for our list of the 4 simple LMI regions, $\mu = 2$ in cases 1 and 2 (“half-plane” and “disk”), and $\mu = 4$ in cases 3 and 4 (“sector” and “stripe”).

There are many other applications of Theorem 2.1 to semi-infinite systems of LMIs (2) arising in control, provided that the uncertainty set \mathcal{U} in (2) is an interval uncertainty. In a typical control application, all matrices A_j in (2) share a common block-diagonal structure and are such that when perturbing a single data entry, every diagonal block in the matrix $A_0 + \sum_j x_j A_j$ is perturbed by a small rank matrix, which is exactly the case considered in Theorem 2.1.

4. Application II: Quadratic maximization over the unit cube. Here we demonstrate that the **MatrCube** problem in its simplest form, where all the edge matrices B^ℓ are very specific matrices of ranks ≤ 2 , is equivalent to the problem

$$(34) \quad \omega_*(Q) = \max_x \{ x^T Q x : \|x\|_\infty \leq 1 \} \quad [Q \succ 0]$$

of maximizing a positive definite quadratic form over the unit cube. On one hand, this observation says that **MatrCube** (already in the case of “rank 2 edges”) is NP-hard (since (34) is). On the other hand, our observation allows us to extract from Theorem 2.1 a certain statement about the possibility of building efficiently a tight bound on the optimal value in (34). As it turns out, this bound is exactly the one given by the standard semidefinite relaxation of (34), and the corresponding “tightness” statement coming from Theorem 2.1 is nothing but the “ $\frac{\pi}{2}$ Theorem” of Nesterov [12].

The link between the quadratic maximization over the unit cube and the matrix cube problem is given by the following simple observation.

PROPOSITION 4.1. *Assume that Q in (34) is positive definite. Then*

$$(35) \quad \begin{aligned} (a) \quad & \omega \geq \omega_*(Q) \equiv \max_{x: \|x\|_\infty \leq 1} x^T Q x, \\ & \updownarrow \\ (b) \quad & \omega \xi^T Q^{-1} \xi \geq \|\xi\|_1^2 \quad \forall \xi, \\ & \updownarrow \\ (c) \quad & \omega Q^{-1} + \{A \in \mathbf{S}^m : |A_{ij}| \leq 1, 1 \leq i, j \leq m\} \subset \mathbf{S}_+^m. \end{aligned}$$

Proof. Relation (a) means that the ellipsoid $\{x : x^T Q x \leq \omega\}$ contains the unit cube $\{x : \|x\|_\infty \leq 1\}$. Passing to polars, this is exactly the same as saying that the polar of the ellipsoid, which is the ellipsoid $\{\xi : \xi^T Q^{-1} \xi \leq \omega^{-1}\}$, is contained in the polar of the unit cube, which is the set $\{\xi : \|\xi\|_1 \leq 1\}$. But the latter inclusion is exactly what is stated in (b). Thus we have proved the equivalence (i).

Now, (c) says exactly that

$$(36) \quad \omega \xi^T Q^{-1} \xi + \min_A \{ \xi^T A \xi : A = A^T, |A_{ij}| \leq 1 \} \geq 0 \quad \forall \xi.$$

The minimum in the left-hand side of this relation is equal to $-\|\xi\|_1^2$. (Indeed, $\xi^T A \xi \geq -\|\xi\|_1^2$ whenever $|A_{ij}| \leq 1$ for all i, j , and $\xi^T A \xi = -\|\xi\|_1^2$ for $A_{ij} = -\text{sign}(\xi_i)\text{sign}(\xi_j)$, $i, j = 1, \dots, m$.) Thus, (36) is equivalent to the relation $\omega \xi^T Q^{-1} \xi - \|\xi\|_1^2 \geq 0$ for all ξ , which is nothing but (b). Thus we have proved the equivalence (ii). \square

Now, let S^{ij} be the basic symmetric matrices (so that S^{ii} has a single nonzero entry, equal to 1, in the cell (i, i) , and S^{ij} , $i \neq j$, has exactly two nonzero entries, both equal to 1, in the cells (i, j) and (j, i)). Relation (35.b) says exactly that the matrix box

$$c \left[\frac{1}{\omega} \right] = \left\{ Q^{-1} + \sum_{1 \leq i \leq j \leq m} u_{ij} S^{ij} : \|u\|_\infty \leq \frac{1}{\omega} \right\}$$

is contained in the positive semidefinite cone. According to Theorem 2.1, a *sufficient* condition for this inclusion is the solvability of the system of LMIs as follows:

$$(37) \quad \begin{aligned} X^{ij} &\succeq \pm \rho S^{ij}, \quad 1 \leq i \leq j \leq m, \\ \sum_{1 \leq i \leq j \leq m} X^{ij} &\preceq Q^{-1}, \quad \rho = \frac{1}{\omega}. \end{aligned}$$

Moreover, since the ranks of the edge matrices S^{ij} are ≤ 2 , Theorem 2.1 says that the solvability of (37) is a “tight, within the factor $\frac{\pi}{2}$ ” sufficient condition for the validity of (35.b). Taking into account that the smallest value of ω for which (35.b) is valid is exactly $\omega_*(Q)$ (Proposition 4.1), we arrive at the following.

PROPOSITION 4.2. *Let $Q \succ 0$. Consider the semidefinite program*

$$(38) \quad \rho(Q) = \max_{\rho, X^{ij}} \left\{ \rho : \begin{aligned} X^{ij} &\succeq \pm \rho S^{ij}, \quad 1 \leq i \leq j \leq m \\ \sum_{1 \leq i \leq j \leq m} X^{ij} &\preceq Q^{-1} \end{aligned} \right\}.$$

The reciprocal of the optimal value in this problem is an upper bound on the optimal value $\omega_(Q)$ in the problem of quadratic maximization (34), and this bound is tight within the factor $\frac{\pi}{2}$:*

$$(39) \quad \omega_*(Q) \leq \frac{1}{\rho(Q)} \leq \frac{\pi}{2} \omega_*(Q).$$

Proposition 4.2 says that a certain quantity which is efficiently computable via semidefinite programming (namely, $1/\rho(Q)$) is a tight, within the factor $\pi/2$, upper

bound on the maximum $\omega_*(Q)$ of the positive definite quadratic form $x^T Q x$ over the unit cube. We are about to demonstrate that our bound is nothing but the standard semidefinite upper bound

$$\begin{aligned} \omega^*(Q) &= \max_X \{ \text{Tr}(QX) : X_{ii} \leq 1, i = 1, \dots, m, X \succeq 0 \} \\ (40) \quad &= \min_{\lambda} \left\{ \sum_{i=1}^m \lambda_i : \text{Diag}\{\lambda\} \succeq Q \right\} \end{aligned}$$

on $\omega_*(Q)$.

PROPOSITION 4.3. For $Q \succ 0$, one has $\frac{1}{\rho(Q)} = \omega^*(Q)$.

Proof. Let e_i be the standard basic orthonormal vectors in \mathbf{R}^m , so that $S^{ij} = \frac{1}{1+\delta_{ij}} [e_i e_j^T + e_j e_i^T]$, where δ_{ij} are the Kronecker symbols. Applying Lemma 3.2, we see that

$$\begin{aligned} \rho(Q) &= \max \left\{ \rho : \exists X^{ij} : \begin{array}{l} X^{ij} \succeq \pm \frac{\rho}{1+\delta_{ij}} [e_i e_j^T + e_j e_i^T], 1 \leq i \leq j \leq m \\ \sum_{1 \leq i \leq j \leq m} X^{ij} \preceq Q^{-1} \end{array} \right\} \\ (41) \quad &= \max \left\{ \rho : \exists \{H_{ij} > 0\}_{1 \leq i \leq j \leq m} : \sum_{1 \leq i \leq j \leq m} \frac{1}{1+\delta_{ij}} [H_{ij} e_i e_i^T + H_{ij}^{-1} e_j e_j^T] \preceq \frac{1}{\rho} Q^{-1} \right\}. \end{aligned}$$

Let \mathcal{H} be the set of all $m \times m$ matrices $H = [H_{ij}]$ with positive entries such that $H_{ij} H_{ji} \geq 1$ for all i, j . It is immediately seen that (41) can be rewritten as

$$\begin{aligned} (42) \quad \rho^{-1}(Q) &= \min \{ \omega : \exists (H \in \mathcal{H}) : \Lambda(H) \preceq \omega Q^{-1} \} \\ &= \min \{ \omega : \exists (H \in \mathcal{H}) : Q \preceq \omega \Lambda^{-1}(H) \}, \end{aligned}$$

where $\Lambda(H)$ is the diagonal matrix with the diagonal entries

$$\Lambda_{ii}(H) = \sum_{j=1}^m H_{ij}, \quad i = 1, \dots, m.$$

LEMMA 4.4. The matrices which can be represented as $\Lambda^{-1}(H)$, $H \in \mathcal{H}$, are exactly the positive definite diagonal matrices with trace ≤ 1 .

Proof. A matrix $M = \text{Diag}\{\mu_i\}$ with $\mu_i > 0$ and $s \equiv \sum_i \mu_i \leq 1$ is $\Lambda^{-1}(H)$ for H given by $H_{ij} = \frac{\mu_j}{s \mu_i}$; note that $H \in \mathcal{H}$ due to $s \leq 1$. It remains to prove that if $H \in \mathcal{H}$, then $\text{Tr}(\Lambda^{-1}(H)) \leq 1$. To this end observe that

(*) For positive reals μ_1, \dots, μ_m , one has

$$\sum_i \mu_i \leq 1 \Leftrightarrow \forall \{a_i > 0\} : \sum_i \frac{a_i^2}{\mu_i} \geq \left(\sum_i a_i \right)^2.$$

Indeed, \Rightarrow is given by the evident relation $\min_{\mu_i > 0: \sum_i \mu_i \leq 1} \sum_i a_i^2 / \mu_i = (\sum_i a_i)^2$ for all $a_i > 0$. To verify \Leftarrow , set $a_i = \mu_i$ in the inequality $\sum_i a_i^2 / \mu_i \geq (\sum_i a_i)^2$.

In view of (*), in order to prove that $H \in \mathcal{H}$ implies $\text{Tr}(\Lambda^{-1}(H)) \leq 1$, it suffices

to verify that if $H \in \mathcal{H}$ and $a_i > 0$, then $\sum_i a_i^2 \Lambda_{ii}(H) \geq (\sum_i a_i)^2$, which is immediate:

$$\begin{aligned} \sum_{i=1}^m a_i^2 \Lambda_{ii}(H) &= \sum_{i,j=1}^m a_i^2 H_{ij} = \sum_{i=1}^m a_i^2 H_{ii} + \sum_{i < j} [a_i^2 H_{ij} + a_j^2 H_{ji}] \\ &\geq \sum_i a_i^2 + 2 \sum_{i < j} a_i a_j \quad [\text{since } H_{ij} > 0, H_{ij} H_{ji} \geq 1] \\ &= \left(\sum_i a_i \right)^2. \quad \square \end{aligned}$$

By Lemma 4.4, as H runs through \mathcal{H} , the matrix $\Lambda^{-1}(H)$ runs through the entire set of positive definite diagonal matrices with trace ≤ 1 , so that the matrix $\omega \Lambda^{-1}(H)$ runs through the entire set of positive definite diagonal matrices with trace $\leq \omega$. Consequently, (42) implies that

$$\rho^{-1}(Q) = \min \left\{ \sum_{i=1}^m \lambda_i : Q \preceq \text{Diag}\{\lambda\} \right\},$$

so that $\rho^{-1}(Q) = \rho^*(Q)$ by (40). \square

Note that the fact that the bound (40) on the optimal value $\omega_*(Q)$ of (34) is tight within the factor $\frac{\pi}{2}$ is known; it is the “ $\frac{\pi}{2}$ Theorem” of Nesterov [12], established originally via a construction based on the famous MAXCUT-related “random hyperplane” technique of Goemans and Williamson [7]. Surprisingly, the alternative proof that we have developed, although it exploits randomization, seemingly uses nothing like the random hyperplane technique.

5. Maximizing a homogeneous polynomial of degree 3 over the unit cube. Let $B[x^1, x^2, x^3]$ be a symmetric 3-linear form on \mathbf{R}^m , and let $P[x] = B[x, x, x]$ be the associated homogeneous polynomial; i.e.,

$$P[x] = \sum_{j=1}^m x_j (x^T B_j x), \quad \text{where } B_j \in S^m.$$

Consider the problem of computing

$$\omega(P) = \max_x \{P[x] : \|x\|_\infty \leq 1\}$$

along with the semidefinite program

$$(43) \quad \omega^*(P) = \min_{\lambda, X^1, \dots, X^m} \left\{ \sum_{j=1}^m \lambda_j : \sum_j X^j \preceq \text{Diag}\{\lambda\}, X^j \succeq \pm B_j, j = 1, \dots, m \right\},$$

where B_j are the matrices of the symmetric bilinear forms $B[e_j, \cdot, \cdot]$; here $e_j, j = 1, \dots, m$, are the standard basic orths in \mathbf{R}^m . We intend to demonstrate that $\omega^*(P)$ is an upper bound on $\omega(P)$, and that the quality of this bound basically depends only on the “width”

$$d(P) = \max_{1 \leq j \leq m} \text{rank}(B_j)$$

of P .

THEOREM 5.1. *One has*

$$(44) \quad \omega(P) \leq \omega^*(P) \leq 4.652\vartheta(d(P)) \ln(m+1)\omega(P) \leq 7.31\sqrt{d(P)} \ln(m+1)\omega(P),$$

where $\vartheta(\cdot)$ is given by (9).

Proof. The proof is very much in the spirit of the matrix cube Theorem 2.1; it uses a *probabilistic* argument in order to validate the solvability/unsolvability of a certain deterministic inequality system.

1^o. Let λ, X_1, \dots, X_m be a feasible solution of (43). We have

$$\begin{aligned} \|x\|_\infty \leq 1 \Rightarrow P[x] &= \sum_j x_j(x^T B_j x) \leq \sum_j |x_j|(x^T X_j x) \leq \sum_j x^T X_j x \\ &\leq x^T \text{Diag}\{\lambda\}x \leq \sum_j \lambda_j, \end{aligned}$$

which gives the first inequality in (44).

2^o. Let us prove the second inequality in (44); without loss of generality we may assume that $P \neq 0$, so that $\omega(P) > 0$. Problem (43) is strictly feasible and bounded below, so that its optimal value $\omega^*(P)$ is equal to that of its (solvable) semidefinite dual problem:

$$(45) \quad \omega^*(P) = \max_{U, Y_j, Z_j} \left\{ \begin{array}{l} U, Y_j, Z_j \succeq 0, \\ \sum_j \text{Tr}([Y_j - Z_j]B_j) : Y_j + Z_j = U, \\ U_{jj} = 1, j = 1, \dots, m. \end{array} \right\}$$

Invoking Lemma 2.2, we see that there exists U such that

$$(46) \quad \begin{array}{ll} \text{(a)} & U \succeq 0, \\ \text{(b)} & U_{jj} = 1, \quad j = 1, \dots, m, \\ \text{(c)} & \sum_j \|\lambda(U^{1/2}B_jU^{1/2})\|_1 = \omega^*(P). \end{array}$$

Now let $V = U^{1/2}$ and let $\xi \sim \mathcal{N}(0, I_m)$. By Lemma 2.3 and (46.c) we have

$$(47) \quad \vartheta(d(P)) \mathbf{E} \left\{ \sum_j |\xi^T V B_j V \xi| \right\} \geq \sum_j \|\lambda(V B_j V)\|_1 = \omega^*(P).$$

At the same time, by (46.b) the Euclidean norms of the rows of V are equal to 1, so that

$$\|V\xi\|_\infty = \max_{1 \leq j \leq m} |\zeta_j|, \quad \zeta_j \sim \mathcal{N}(0, 1),$$

whence, as is well known,

$$(48) \quad \mathbf{E} \{ \|Y\xi\|_\infty^2 \} \leq 2 \ln(m+1).$$

To make the paper self-contained, here is a derivation of (48). We have

$$\begin{aligned}
 t > 0 &\Rightarrow \\
 \psi(t) &\equiv \text{Prob} \{|\zeta_j| > t\} = 2 \int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\tau^2}{2}\right\} d\tau \\
 &\leq 2 \int_t^\infty \frac{1}{\sqrt{2\pi}} \frac{\tau}{t} \exp\left\{-\frac{\tau^2}{2}\right\} d\tau = \sqrt{\frac{2}{\pi}} t^{-1} \exp\left\{-\frac{t^2}{2}\right\} \\
 &\Rightarrow \\
 \text{Prob} \left\{ \max_{j \leq m} |\zeta_j| > t \right\} &\leq \min[1, m\psi(t)] \leq \min \left[1, m\sqrt{\frac{2}{\pi}} t^{-1} \exp\left\{-\frac{t^2}{2}\right\} \right] \\
 &\Rightarrow \\
 \mathbf{E} \left\{ \max_{j \leq m} |\zeta_j|^2 \right\} &= 2 \underbrace{\int_{t>0} t \min[1, m\psi(t)] dt}_{J_m} \leq 2 \int_{t>0} \min \left[t, m\sqrt{\frac{2}{\pi}} \exp\left\{-\frac{t^2}{2}\right\} \right] dt \\
 &\leq 2 \int_0^\tau t dt + 2\sqrt{\frac{2}{\pi}} m \int_\tau^\infty \exp\left\{-\frac{t^2}{2}\right\} dt \\
 &\leq \tau^2 + 2\sqrt{\frac{2}{\pi}} m \tau^{-1} \exp\left\{-\frac{\tau^2}{2}\right\}.
 \end{aligned}$$

The resulting bound

$$(49) \quad \mathbf{E} \left\{ \max_{j \leq m} |\zeta_j|^2 \right\} \leq 2J_m \leq \tau^2 + 2\sqrt{\frac{2}{\pi}} m \tau^{-1} \exp\left\{-\frac{\tau^2}{2}\right\}$$

is valid for all m and all $\tau > 0$. Assuming $m \geq 3$ and setting $\tau = \sqrt{2 \ln(m/2)}$, one can easily conclude from (49) that $2J_m \leq 2 \ln(m + 1)$ for all $m \geq 25$. Numerical computation of J_m for $m \leq 25$ demonstrates that the latter inequality holds true for all m .

Combining (47), (48), we get

$$(50) \quad \mathbf{E} \left\{ \sum_j |\xi^T V B_j V \xi| \right\} \geq \frac{\omega^*(P)}{2\vartheta(d(P)) \ln(m + 1)} \mathbf{E} \{ \|V \xi\|_\infty^2 \},$$

and the left-hand side in this inequality is positive. It follows that there exist $\eta \in \mathbf{R}^m$ and a vector $\epsilon \in \mathbf{R}^m$ with entries ± 1 such that

$$\eta^T \left[\sum_j \epsilon_j B_j \right] \eta \geq \frac{\omega^*(P)}{2\vartheta(d(P)) \ln(m + 1)} \quad \text{and} \quad \|\eta\|_\infty = 1,$$

whence

$$(51) \quad \max_{\epsilon, \eta} \{ B[\epsilon, \eta, \eta] : \|\epsilon\|_\infty \leq 1, \|\eta\|_\infty \leq 1 \} \geq \frac{\omega^*(P)}{2\vartheta(d(P)) \ln(m + 1)}.$$

On the other hand,

$$B[x + ty, x + ty, x + ty] = B[x, x, x] + 3tB[x, x, y] + 3t^2B[x, y, y] + t^3B[y, y, y],$$

whence

$$\forall t \neq 0 \quad \forall x, y :$$

$$B[x, y, y] = \frac{B[x + ty, x + ty, x + ty] + B[x - ty, x - ty, x - ty] - 2B[x, x, x]}{6t^2}.$$

It follows that

$$\max_{\epsilon, \eta} \{B[\epsilon, \eta, \eta] : \|\epsilon\|_\infty \leq 1, \|\eta\|_\infty \leq 1\} \leq \frac{(1+t)^3 + 1}{3t^2} \omega(P) \quad \forall t > 0,$$

which combines with (51) to yield the relation

$$\frac{\omega^*(P)}{2\vartheta(d(P)) \ln(m+1)} \leq \min_{t>0} \frac{(1+t)^3 + 1}{3t^2} \omega(P) \leq 2.326\omega(P),$$

and the second inequality in (44) follows. The third inequality follows from $\vartheta(d) \leq \frac{\pi\sqrt{d}}{2}$; see (10). \square

Remark. There are two simple cases in which $d(P)$ is small. The first is when $P[x]$ is “of small rank”: $P[x] = \sum_{\ell=1}^L (p_\ell^T x)^3$ with a small L (since clearly $d(P) \leq L$). The second case is when P is of a “band” structure; i.e., the quantity

$$\kappa = \max_{1 \leq i \leq j \leq k \leq m} \{k - i : B[e_i, e_j, e_k] \neq 0\}$$

is small (since clearly $d(P) \leq 2\kappa + 1$).

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