Statistical Inference via Convex Optimization

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Fact: Many inference procedures in Statistics reduce to optimization

Example: MLE – Maximum Likelihood Estimation

Problem: Given a parametric family \( \{ p_\theta(\cdot) : \theta \in \Theta \} \) of probability densities on \( \mathbb{R}^d \) and a random observation \( \omega \) drawn from some density \( p_{\theta^*}(\cdot) \) from the family, estimate the parameter \( \theta^* \).

Maximum Likelihood Estimate: Given \( \omega \), maximize \( p_\theta(\omega) \) over \( \theta \in \Theta \) and use the maximizer \( \hat{\theta} = \hat{\theta}(\omega) \) as an estimate of \( \theta^* \).

Note: In MLE, optimization is used for number crunching only and has nothing to do with motivation and performance analysis of MLE.

Fact: Most of traditional applications of Optimization in Statistics are of “number crunching” nature. While often vitally important, “number crunching” applications are beyond our scope.
What is in our scope, are inference routines motivated and justified by Optimization Theory – Convex Analysis, Optimality Conditions, Duality...

As a matter of fact, our "working horse" will be Convex Optimization. This choice is motivated by

- nice geometry of convex sets, functions, and optimization problems

- computational tractability of convex optimization implying computational efficiency of statistical inferences stemming from Convex Optimization.
SPARSITY-ORIENTED SIGNAL PROCESSING

- Signal Recovery from Indirect Observations
- Sparse $\ell_1$ Recovery: Motivation
- Validating $\ell_1$ Recovery
  - $s$-Goodness and Nullspace Property
  - Quantifying Nullspace Property
  - Regular and Penalized $\ell_1$ Recoveries
  - Restricted Isometry Property
  - Tractability Issues
Basic Signal Processing problem is to recover unknown signal \( x_\ast \in \mathbb{R}^n \) from its observation

\[
y = A(x_\ast) + \xi
\]

- \( x \mapsto A(x) : \mathbb{R}^n \to \mathbb{R}^m \): known "signal-to-observation" transformation
- \( \xi \): observation noise.
- In many applications, the signal-to-observation transformation is just linear:
  \[
  A(x) = Ax \text{ for some known } m \times n \text{ matrix } A.
  \]
- Assume from now on that \( A(\cdot) \) is linear
  \( \Rightarrow \) the recovery problem is just to solve a system of linear equations
  \[
  Ax = b := Ax_\ast
  \]

given \( m \times n \) matrix \( A \) and a noisy observation \( y \) of the "true" right hand side \( b \).
Problem of interest: to solve a linear system

\[ Ax = b :\equiv Ax^* \]

given \( m \times n \) matrix \( A \) and a noisy observation \( y \) of the “true” right hand side \( b \).

As of now, there are two typical settings of the problem:

- \( m \geq n \) (typically, \( m \gg n \)) — we have (much) more observations than unknowns. This is the classical case studied in numerical Linear Algebra (where noise is non-random) and Statistics (where noise is random).

Unless \( A \) is “pathological,” the only difficulty here is the presence of noise. The challenge is to reproduce well the true signal while suppressing as much as possible the influence of noise.

- \( m < n \) (and even \( m \ll n \)) — we have (much) less observations than unknowns. Till early 2000’s, this case was thought of as completely meaningless. Indeed, as Linear Algebra says, an under-determined (with more unknowns than equations) system of linear equations either has no solutions at all, or has infinitely many solutions which can be arbitrarily far away from each other.

\[ \Rightarrow \text{When } m < n, \text{ the true signal cannot be recovered from observations even in the noiseless case!} \]

Remedy: Add some information on the true signal.
Problem of interest: to solve a linear system
\[ Ax = b : = Ax_\ast \]
given \( m \times n \) matrix \( A \) and a noisy observation \( y \) of the “true” right hand side \( b \) in the case of \( m \ll n \)

Sparsity-oriented remedy: Reduce the problem to the one where the signal is sparse – has \( s \ll n \) nonzero entries, and utilize sparsity in your recovery routine.

Fact: Many real-life signals \( x \) when presented by their coefficients in properly selected basis ("dictionary") \( B \):
\[ x = Bu \]
- columns of \( B \): vectors of basis \( B \)
- \( u \): coefficients of \( x \) in basis \( B \)
become sparse (or nearly so): \( u \) has just \( s \ll n \) nonzero entries (or can be well approximated by vector with \( s \ll n \) nonzero entries).
Illustration: 25 sec fragment of audio signal “Mail must go through” (dimension 1,058,400) and its Discrete Fourier Transform:

How mail goes through in time domain

How mail goes through in frequency domain

<table>
<thead>
<tr>
<th>% of leading Fourier coefficients kept</th>
<th>energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>25%</td>
<td>99.8%</td>
</tr>
<tr>
<td>15%</td>
<td>99.6%</td>
</tr>
<tr>
<td>5%</td>
<td>98.2%</td>
</tr>
<tr>
<td>1%</td>
<td>79.0%</td>
</tr>
</tbody>
</table>
**Illustration:** The $256 \times 256$ image

This can be thought of as $256^2 = 65536$-dimensional vector (write down the intensities of pixels column by column). “As is,” this vector is not sparse and cannot be approximated well by highly sparse vectors.

This is what happens when we keep several leading (i.e., largest in magnitude) entries and zero out all other entries:

1% of leading entries kept

10% of leading entries kept

25% of leading entries kept

50% of leading entries kept
However, the image (same as other “non-pathological” images) is nearly sparse when represented in wavelet basis:

<table>
<thead>
<tr>
<th>Percentage Kept</th>
<th>Wavelet Coeff.</th>
<th>Energy Kept</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>1% of leading wavelet coeff.</td>
<td>(99.70% of energy) kept</td>
</tr>
<tr>
<td>5%</td>
<td>5% of leading wavelet coeff.</td>
<td>(99.93% of energy) kept</td>
</tr>
<tr>
<td>10%</td>
<td>10% of leading wavelet coeff.</td>
<td>(99.96% of energy) kept</td>
</tr>
<tr>
<td>25%</td>
<td>25% of leading wavelet coeff.</td>
<td>(99.99% of energy) kept</td>
</tr>
</tbody>
</table>
Similar, albeit less intense, phenomenon takes place when representing typical images in frequency domain:

<table>
<thead>
<tr>
<th>Images</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td>1% of leading Fourier coeff. (99.70% of energy) kept</td>
</tr>
<tr>
<td><img src="image2.png" alt="Image" /></td>
<td>5% of leading Fourier coeff. (99.93% of energy) kept</td>
</tr>
<tr>
<td><img src="image3.png" alt="Image" /></td>
<td>1% of leading Fourier coeff. (96.41% of energy) kept</td>
</tr>
<tr>
<td><img src="image4.png" alt="Image" /></td>
<td>5% of leading Fourier coeff. (99.46% of energy) kept</td>
</tr>
<tr>
<td><img src="image5.png" alt="Image" /></td>
<td>10% of leading Fourier coeff. (99.96% of energy) kept</td>
</tr>
<tr>
<td><img src="image6.png" alt="Image" /></td>
<td>25% of leading Fourier coeff. (99.99% of energy) kept</td>
</tr>
<tr>
<td><img src="image7.png" alt="Image" /></td>
<td>10% of leading Fourier coeff. (99.76% of energy) kept</td>
</tr>
<tr>
<td><img src="image8.png" alt="Image" /></td>
<td>25% of leading Fourier coeff. (99.95% of energy) kept</td>
</tr>
</tbody>
</table>
When recovering a signal $x_*$ admitting a sparse (or nearly so) representation $Bu_*$ in a *known* basis $B$ from observations

$$y = Ax_* + \xi,$$

the situation reduces to the one when the signal to be recovered is just sparse. Indeed, we can first recover sparse $u_*$ from observations

$$y = Ax_* + \xi = [AB]u_* + \xi.$$

After an estimate $\hat{u}$ of $u_*$ is built, we can estimate $x_*$ by $B\hat{u}$.

$\Rightarrow$ In fact, sparse recovery is about how to recover a *sparse* $n$-dimensional signal $x$ from $m \ll n$ observations

$$y = Ax_* + \xi.$$
How to recover a sparse (or nearly so) $n$-dimensional signal $x$ from $m \ll n$ observations

$$y = Ax_\ast + \xi$$

To get an idea, consider the case when $x_\ast$ is exactly sparse – has $s \ll n$ nonzero entries – and there is no observation noise:

$$y = Ax_\ast$$

- If we knew the positions $i_1, \ldots, i_s$ of the nonzero entries in $x_\ast$, we could recover $x_\ast$ by solving the system with just $s$ unknowns:

$$y = \begin{bmatrix} A_{i_1}, \ldots, A_{i_s} \end{bmatrix} \cdot \begin{bmatrix} x_{i_1}; \ldots; x_{i_s} \end{bmatrix}.$$ (!)

When $s \leq m$ (which, with $s \ll n$, still allows for $m \ll n$), we would get over-determined system of linear equations on the nonzero entries in $x$. Assuming $A$ “non-pathologic,” so that every $s \leq m$ columns of $A$ are linearly independent, (!) has a unique solution which can be easily found.

**But:** We never know in advance where the nonzeros in $x$ are located!
How to recover a sparse (or nearly so) \( n \)-dimensional signal \( x \) from \( m \ll n \) observations

\[
y = Ax_\ast + \xi
\]

♠ A straightforward way to account for the fact that we never know where the nonzeros in \( x_\ast \) stand, is to look for the sparsest solution to the system \( y = Ax \). This amounts to solving the optimization problem

\[
\min_{x} \text{nnz}(x) \text{ s.t. } y = Ax \tag{!}
\]

• \( \text{nnz}(x) \): # of nonzero entries in \( x \).
• It is easily seen that if \( x_\ast \) is \( s \)-sparse and every \( 2s \) columns in \( A \) are linearly independent (which is so when \( 2s \leq m \), unless \( A \) is pathological), then \( x_\ast \) is the unique optimal solution to (!), and thus our procedure recovers \( x_\ast \) exactly.

But: \( \text{nnz}(z) \) is a bad (nonconvex and discontinuous) function, so that (!) is a disastrously complicated combinatorial problem. Seemingly, the only way to solve (!) is to use brute force search where we test one by one all collections of potential locations of nonzero entries in a solution. Brute force is completely unrealistic: to recover \( s \)-sparse signal, it would require looking through \textit{at least}

\[
N = \binom{n}{s-1} = \frac{n!}{(s-1)!(n-s+1)!}
\]
candidate solutions.

• with \( s = 17, n = 128 \), \( N \) is as large as \( 1.49 \cdot 10^{21} \)
• with \( s = 49, n = 1024 \), \( N \) is as large as \( 3.94 \cdot 10^{84} \)
How to recover a sparse (or nearly so) \( n \)-dimensional signal \( x \) from \( m \ll n \) observations

\[ y = Ax_* + \xi \]

• Solving problem

\[ \min_x \text{nnz}(x) \text{ s.t. } y = Ax \] (1)

would yield the desired recovery, but (1) is heavily computationally intractable...

♠ **Partial remedy:** Replace the difficult to minimize objective \( \text{nnz}(\theta) \) with an “easy-to-minimize” objective, specifically, with \( \| \theta \|_1 = \sum_i |\theta_i| \), thus arriving at \( \ell_1 \)-recovery

\[ \hat{x} = \text{argmin}_x \{ \sum_i |x_i| : Ax = y := Ax_* \} \] (2)

♠ **Observation:** (2) is just an LO program!

Indeed,

• the constraints in (2) are linear equalities.

• \( |x_i| = \max[x_i, -x_i] \), so that the terms in the objective can be “linearized.”

♠ **The LO reformulation of (2) is**

\[ \min_{x,z} \left\{ \sum_j z_j : Ax = y, z_j \geq x_j, z_j \geq -x_j \forall j \leq n \right\} . \]
• **In the noiseless case**, $\ell_1$ recovery is given by

$$\hat{x} = \text{argmin}_x \{\sum_i |x_i| : Ax = y := Ax^*\}$$

♠ When the observation $y$ is noisy:

$$y = Ax^* + \xi$$

the constraint $Ax = y$ on a candidate recovery should be relaxed.

• **When we know an upper bound** $\delta$ on some norm $\|\xi\|$ of the noise $\xi$, a natural version of $\ell_1$ recovery is

$$\hat{x} \in \text{Argmin}_x \{\sum_i |x_i| : \|Ax - y\| \leq \delta\} \quad (\ast)$$

**Note:** When $\|\xi\| = \|\xi\|_{\infty} := \max_i |\xi_i|$ (“uniform norm”), (\ast) reduces to the LO program

$$\min_{x,z} \left\{ \sum_j z_j : \begin{array}{l}
-z_j \leq x_j \leq z_j, \ 1 \leq j \leq n \\
y_i - \delta \leq [Ax]_i \leq y_i + \delta, \ 1 \leq i \leq m
\end{array} \right\}$$

• **When the noise** $\xi$ is random with zero mean, there are reasons to define $\ell_1$ recovery by **Dantzig Selector**:

$$\hat{x} \in \text{Argmin}_x \{\sum_i |x_i| : \|Q(Ax - y)\|_{\infty} \leq \delta\}$$

with $M \times m$ **contrast matrix** $Q$ and $\delta > 0$ chosen according to noise’s structure and intensity. This again is reducible to LO program, specifically,

$$\min_{x,z} \left\{ \sum_j z_j : \begin{array}{l}
-z_j \leq x_j \leq z_j, \ 1 \leq j \leq n \\
-\delta \leq [QAx - Qy]_i \leq \delta, \ 1 \leq i \leq M
\end{array} \right\}$$

• **Note:** In Dantzig Selector proper, $Q = A^T$. 

1.12
(?) How to recover a sparse (or nearly so) $n$-dimensional signal $x$ from $m \ll n$ observations

$$y = Ax_* + \xi$$

(!) Use $\ell_1$ minimization

$$\hat{x} \in \text{Argmin}_x \{\sum_i |x_i| : \|Ax - y\| \leq \delta\}$$

♣ Compressed Sensing theory shows that under appropriate assumptions on $A$, in a meaningful range of sizes $m$, $n$ and sparsities $s$, $\ell_1$-minimization recovers the unknown signal $x_*$

— exactly, when $x_*$ is $s$-sparse and there is no observation noise,

— within inaccuracy $\leq C(A)[\delta_n + \delta_s]$ in the general case

• $\delta_n$: magnitude of noise

• $\delta_s$: deviation of $x_*$ from its best $s$-sparse approximation
♠ **Bad news:** “Appropriate assumptions on $A$” are difficult to verify
Partial remedy: there are conservative verifiable sufficient conditions for “appropriate assumptions.”

♠ **Good news:** For $A$ drawn at random from natural distributions, “appropriate assumptions” are satisfied with overwhelming probability.

- E.g., when entries in $m \times n$ matrix $A$ are, independently of each other, sampled from Gaussian distribution, the resulting matrix, with probability approaching 1 as $m, n$ grow, ensures the validity of $\ell_1$ recovery of sparse signals with as many as

$$s = O(1) \frac{m}{\ln(n/m)}$$

nonzero entries.

♠ **More good news:** In many applications (Imaging, Radars, Magnetic Resonance Tomography,...), signal acquisition via randomly generated matrices $A$ makes perfect sense and results in significant acceleration of the acquisition process. In these applications, signals of interest are sparse in properly selected bases

⇒ *With accelerated acquisition, no information is lost!*
Example: Single-Pixel Camera:

Low-cost, fast, sensitive optical detection

Image encoded by DMD and random basis

Compressed, encoded image data sent via RF for reconstruction

DMD

RNG

PD

A/D

Xmtr

Rcvr

DSP
How it works:
Sparse recovery via Dantzig Selector

- 32 nonzero entries
  - No noise
  - Recovery error: $7.8 \cdot 10^{-9}$

- 32 nonzero entries
  - Noise's StD 0.01
  - Recovery error: 0.064

- 32 nonzero entries
  - Noise's StD 0.10
  - Recovery error: 0.66

- 64 nonzero entries
  - No noise
  - Recovery error: 1.47

$: signal \quad +: recovery
256 \times 512 \text{ Gaussian sensing matrix } A
Validity of sparse signal recovery via $\ell_1$ minimization

♠ **Notational convention:** From now on, for a vector $x \in \mathbb{R}^n$

- $I_x = \{ j : x_j \neq 0 \}$ is the support of $x$.
- for a subset $I$ of the index set $\{ 1, ..., n \}$, $x_I$ is the vector obtained from $x$ by zeroing out entries with indexes *not* in $I$, and $I^o$ is the complement of $I$:

  $I^o = \{ i \in \{ 1, ..., n \} : i \notin I \}$.

- for $s \leq n$, $x^s$ is the vector obtained from $x$ by zeroing our all but the $s$ largest in magnitude entries.

$x^s$ is the best $s$-sparse approximation of $x$ in any one of the $\ell_p$ norms, $1 \leq p \leq \infty$.

- for $s \leq n$ and $p \in [1, \infty]$, we set

  $$\| x \|_{s,p} = \| x^s \|_p.$$
Validity of $\ell_1$ minimization in the noiseless case

♣ The minimal requirement on sensing matrix $A$ which makes $\ell_1$-minimization valid is to guarantee the correct recovery of exactly $s$-sparse signals in the noiseless case, and we start with investigating this property.

♠ $s$-Goodness: An $m \times n$ sensing matrix $A$ is called $s$-good, if whenever the true signal $x$ underlying noiseless observations is $s$-sparse, this signal will be recovered exactly by $\ell_1$-minimization.

Equivalently: $A$ is $s$-good, if

\[ \text{nnz}(x_*) \leq s \]
\[ \Rightarrow x_* \text{ is the unique optimal solution to } \min_x \{ \|x\|_1 : Ax = Ax_* \} \]

♠ Necessary and sufficient condition for $s$-goodness is Nullspace Property:

For every $0 \neq z \in \text{Ker} A := \{z : Az = 0\}$ it holds

\[ \|z\|_{s,1} < \frac{1}{2}\|z\|_1. \]

• Nullspace Property can be derived from LO Optimality Conditions, same as can be verified directly.
\textbf{\textit{s-goodness}} \Rightarrow \textbf{Nullspace Property}:

Nullspace Property does \textit{not} take place

\[ \exists 0 \neq z \in \text{Ker} A : \| z^s \|_1 \geq \frac{1}{2} \| z \|_1 \]

\[ A z^s = A[z^s - z], \| z^s \|_1 \geq \| z^s - z \|_1 \]

\textit{s}-sparse signal \( x_* = z^s \) is not the unique optimal solution to \( \min_x \{ \| x \|_1 : Ax = Ax_* \} \) – contradiction

\textbf{Nullspace Property} \Rightarrow \textbf{s-goodness}: Let Nullspace Property take place and \( x_* \) be \( s \)-sparse, and let \( u \) be an optimal solution to \( \min_x \{ \| x \|_1 : Ax = Ax_* \} \).

Denoting by \( I \) the support of \( x_* \), for \( z = u - x_* \) we have \( z \in \text{Ker} A \) and

\[
\begin{align*}
    z_I &= u_I - [x_*]_I = u_I - x_* \quad \& \quad z_{Io} = u_{Io} \\
    \Rightarrow \quad \| z_I \|_1 &\geq \| x_* \|_1 - \| u_I \|_1 \quad \& \quad \| z_{Io} \|_1 = \| u_{Io} \|_1 \\
    \Rightarrow \quad \| z_I \|_1 - \| z_{Io} \|_1 &\geq \| x_* \|_1 - \| u_I \|_1 - \| u_{Io} \|_1 \\
    &= \| x_* \|_1 - \| u \|_1 \geq 0 \\
    \Rightarrow \quad \| z_I \|_1 - \| z_{Io} \|_1 &\geq 0 \\
    \Rightarrow \quad \| z \|_{s,1} &\geq \| z_I \|_1 \geq \frac{1}{2} [\| z_I \|_1 + \| z_{Io} \|_1] = \frac{1}{2} \| z \|_1 \\
    \Rightarrow \quad z &= 0
\end{align*}
\]
Questions to be addressed:

What happens when $A$ is $s$-good, but $\ell_1$ recovery is “imperfect,” e.g.

- $x$ is not exactly $s$-sparse, and/or
- there is observation noise

How to verify, given $A$ and $s$, that $A$ is $s$-good
In order to address the above questions, we need to “quantify” Nullspace Property.

Nullspace Property states that

\[
\{ z \in \text{Ker} A & \|z\|_1 = 1 \} \Rightarrow \|z\|_{s,1} < 1/2 ,
\]

or, which is the same,

\[
\exists \kappa < 1/2 : \|z\|_{s,1} \leq \kappa \|z\|_1 \ \forall z \in \text{Ker} A \quad (!) \]

Equivalent form of necessary and sufficient condition (!) for \( s \)-goodness of \( m \times n \) sensing matrix \( A \) reads:

\( A \) is \( s \)-good if and only if for some constants \( \kappa < 1/2, C < \infty, \) and some norm \( \| \cdot \| \) it holds

\[
\|x\|_{s,1} \leq C \|Ax\| + \kappa \|x\|_1 \ \forall x \in \mathbb{R}^n \quad (!!) 
\]

Indeed, (!!) clearly implies (!). Assume (!), and let \( \bar{x} \) be \( \| \cdot \|_1 \)-closest to \( x \) element of \( \text{Ker} A \), so that \( \|x - \bar{x}\|_1 \leq c \|Ax\| \) with \( c \) independent of \( x \). We have

\[
\|x\|_{s,1} \leq \|\bar{x}\|_{s,1} + \|x - \bar{x}\|_1 \leq \kappa \|\bar{x}\|_1 + \|x - \bar{x}\|_1 \\
\leq \kappa \|x\|_1 + \lfloor 1 + \kappa \rfloor \|x - \bar{x}\|_1 \leq [1 + \kappa] c \|Ax\| + \kappa \|x\|_1
\]

1.21
\[
\exists C : \|x\|_{s,1} \leq C \|Ax\| + \kappa \|x\|_1 \ \forall x \in \mathbb{R}^n
\]

\[\text{!!}\]

\[\text{♠ It makes sense to rewrite the latter condition in a more flexible form linking }\]

- \(m \times n\) sensing matrix \(A\),
- sparsity level \(s\),
- \(m \times N\) contrast matrix \(H\),
- norm \(\| \cdot \|\) on \(\mathbb{R}^N\),
- condition’s parameter \(q \in [1, \infty]\), and
- parameter \(\kappa \in (0, 1/2)\)

**Condition** \(Q_q(s, \kappa)\):

\[
\|x\|_{s,q} := \|x^s\|_q \leq s^\frac{1}{q} \|H^T Ax\| + \kappa s^{\frac{1}{q}-1} \|x\|_1 \ \forall x \in \mathbb{R}^n
\]

\[\text{♠ We treat condition } Q_q(s, \kappa) \text{ as a condition on contrast matrix } H \text{ and norm } \| \cdot \| .\]

\[\text{♠ Note: } A \text{ is } s\text{-good if and only if the Nullspace Property holds, or, which is the same, if and only if the condition } Q_1(s, \kappa) \text{ with some } \kappa < 1/2 \text{ is satisfiable (e.g., with } N = n, H = CA^T \text{ with properly selected } C, \text{ and } \| \cdot \| = \| \cdot \|_\infty).\]

1.22
Condition $Q_q(s, \kappa)$:

$$
\|x\|_{s,q} := \|x^s\|_q \leq s^{\frac{1}{q}} \|H^T A x\| + \kappa s^{\frac{1}{q} - 1} \|x\|_1 \forall x \in \mathbb{R}^n
$$

♠ Immediate observations:

• **The larger is** $q$, **the stronger is** $Q_q(s, \kappa)$: If $H, \| \cdot \|$ satisfy $Q_q(s, \kappa)$ and $p \in [1, q]$, then $H, \| \cdot \|$ satisfy $Q_p(s, \kappa)$.

Indeed, if $H, \| \cdot \|$ satisfy $Q_q(s, \kappa)$, then

$$
\|x\|_{s,p} \leq \frac{1}{s^p} \|x\|_{s,q} s^p \leq s^p \frac{1}{q} \left[ s^{\frac{1}{q}} \|H^T A x\| + \kappa s^{\frac{1}{q} - 1} \|x\|_1 \right] = s^p \|H^T A x\| + \kappa s^{p - 1} \|x\|_1.
$$

• **Satisfiability of the weakest condition** $Q_1(s, \kappa)$ for some $\kappa < 1/2$ is necessary and sufficient for $s$-goodness of $A$.

♠ **Fact:** Conditions $Q_q(s, \kappa)$ underly instructive bounds on recovery error for imperfect $\ell_1$ recovery.

1.23
Example A: Regular $\ell_1$-Recovery

♣ Regular $\ell_1$ recovery of signal $x$ from observations

$$y = Ax + \eta$$

is given by

$$\hat{x}_{\text{reg}}(y) \in \text{Argmin}_{u} \left\{ \|u\|_1 : \|H^T(Au - y)\| \leq \rho \right\}$$

where $H, \| \cdot \|, \rho \geq 0$ are construction’s parameters.

♠ Theorem. Let $s$ be a positive integer, $q \in [1, \infty]$ and $\kappa \in (0, 1/2)$. Assume that $H, \| \cdot \|$ satisfy $Q_q(s, \kappa)$, and let

$$\Xi_\rho = \{ \eta : \|H^T\eta\| \leq \rho \}.$$ 

Then for all $x \in \mathbb{R}^n$ and $\eta \in \Xi_\rho$ one has

$$\|\hat{x}_{\text{reg}}(Ax + \eta) - x\|_p \leq \frac{4(2s)^{\frac{1}{p}}}{1 - 2\kappa} \left[ \rho + \frac{\|x - x^s\|_1}{2s} \right], \quad 1 \leq p \leq q.$$ 

Note: Regular $\ell_1$ recovery requires a priori information on noise needed to select $\rho$ with “meaningful” $\Xi_\rho$ and does not require a priori information on sparsity $s.$
∀η ∈ Ξρ = \{η : ∥HTη∥ ≤ ρ\} ∀x:

\[ \|\hat{x}_{\text{reg}}(Ax + \eta) - x\|_p \leq \frac{4(2s)^{\frac{1}{p}}}{1 - 2\kappa} \left[ \rho + \frac{\|x - x^s\|_1}{2s} \right], \ 1 ≤ p ≤ q. \]

♠ **Comments:**

**A.** ρ stems from observation errors:
- • η ≡ 0 ⇒ we can set ρ = 0, resulting in zero recovering error for exactly s-sparse signals
- • η is “uncertain but bounded” : η ∈ U for some known and bounded U
  ⇒ we can set ρ = maxu∈U ∥HTu∥
- • η ∼ N(0, σ²Im) ⇒ given tolerance β and setting
  \[
  ρ = σ\sqrt{2\ln(N/β)} \max_i ∥\text{Col}_i[H]∥_2
  \]
  we get
  \[
  \text{Prob}\{η : ∥HTη∥_∞ ≤ ρ\} ≥ 1 - β
  \]
  When ∥·∥ = ∥·∥_∞, this allows to build explicitly “confidence domains” for regular ℓ₁ recovery.

**B.** Pay attention to the factor \( s^{-1} \) at the “near-sparsity” term \( ∥x - x^s∥_1 \).

**C.** Adjusting H and ∥·∥, we can, to some extent, account for the nature of observation errors.

1.25
Example B: Penalized $\ell_1$ Recovery

Penalized $\ell_1$ recovery of signal $x$ from observations

$$y = Ax + \eta$$

is given by

$$\hat{x}_{\text{pen}}(y) \in \text{Argmin}_u \{ \|u\|_1 + \lambda \|H^T(Au - y)\| \}$$

where $H$, $\| \cdot \|$, $\lambda > 0$ are construction's parameters.

♠ Theorem. Given $A$, positive integer $s$, and $q \in [1, \infty]$, assume that $H$, $\| \cdot \|$ satisfy $Q_q(s, \kappa)$ with $\kappa < 1/2$, and let $\lambda \geq 2s$. Then for all $\eta \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$, for $1 \leq p \leq q$ it holds

$$\|\hat{x}_{\text{pen}}(Ax + \eta) - x\|_p \leq \frac{4\lambda_p^1 \left( \frac{1}{2} + \frac{\lambda}{4s} \right)}{1 - 2\kappa} \left[ \|H^T\eta\| + \frac{\|x - x^s\|_1}{2s} \right].$$

In particular, with $\lambda = 2s$, for $1 \leq p \leq q$ it holds

$$\|\hat{x}_{\text{pen}}(Ax + \eta) - x\|_p \leq \frac{4(2s)^p_1}{1 - 2\kappa} \left[ \|H^T\eta\| + \frac{\|x - x^s\|_1}{2s} \right].$$

Note: Penalized $\ell_1$ recovery requires a priori knowledge of sparsity level $s$ and does not require any information on noise.

Note: When $\lambda = 2s$, for all $x$ it holds

$$\forall (\rho \geq 0, \eta \in \Xi_\rho) := \{ \eta : \|H^T\eta\| \leq \rho \} :$$

$$\|\hat{x}_{\text{pen}}(Ax + \eta) - x\|_p \leq \frac{4(2s)^p_1}{1 - 2\kappa} \left[ \rho + \frac{\|x - x^s\|_1}{2s} \right], \ 1 \leq p \leq q.$$
\[ H, \| \cdot \| \text{ satisfy } Q_q(s, \kappa) \]
\[ y = Ax + \eta, \eta \sim \mathcal{N}(0, \sigma^2 I_N) \]
\[ x \in \mathbb{R}^n \text{ is } s\text{-sparse} \]

\[ \Downarrow \]

\[ \text{Prob} \left\{ \| \hat{x}_{\text{reg}}(Ax + \eta) - x \|_p \leq C(H, \kappa, \ln(1/\epsilon)) \sigma_{sp}^{1/p} \right\} \geq 1 - \epsilon \]

\[ \text{Prob} \left\{ \| \hat{x}_{\text{pen}}(Ax + \eta) - x \|_p \leq C(H, \kappa, \ln(1/\epsilon)) \sigma_{sp}^{1/p} \right\} \geq 1 - \epsilon \]

\[ 1 \leq p \leq q \]

**Note:** Given direct observations \( y = x + \eta \) of \( s\)-dimensional signal \( x \) with \( \eta \sim \mathcal{N}(0, \sigma^2 I_s) \), the expected \( \| \cdot \|_p \)-norm of recovery error in optimal recovery is \( O(1)\sigma_{sp}^{1/p} \).
How to Verify Validity Conditions for $\ell_1$-Recovery?

♣ **Bad news:** Given $A$ and $s$, the Nullspace Property is difficult to verify. Similarly, when $q < \infty$ and $\kappa < 1/2$, it is difficult to verify whether the condition $Q_q(s, \kappa)$ is satisfied by given $H, \| \cdot \|$, same as it is difficult to verify whether the condition is satisfiable at all.

♠ **Relatively good news:** There are natural ensembles of random sensing matrices for which properly selected $H, \| \cdot \|$ with overwhelming probability satisfy $Q_2(s, \kappa)$ and thus are $s$-good.

♣ **Definition.** An $m \times n$ sensing matrix $A$ satisfies Restricted Isometry Property (RIP($\delta, k$), if

\[
(1 - \delta) \|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta) \|x\|_2^2 \ \forall (x: \text{nnz}(x) \leq k).
\]

♠ **Theorem** Let $m \times n$ sensing matrix $A$ satisfy RIP($\delta, 2s$) for some $\delta < 1/3$ and positive integer $s$. Then

- The pair $\left( H = \frac{s^{-1/2}}{1-\delta} I_m, \| \cdot \|_2 \right)$ satisfies the condition $Q_2(s, \frac{\delta}{1-\delta})$;
- The pair $\left( H = \frac{1}{1-\delta} A, \| \cdot \|_\infty \right)$ satisfies the condition $Q_2(s, \frac{\delta}{1-\delta})$.

1.28
♠ Theorem Given $\delta \in (0, \frac{1}{5}]$, with properly selected positive $c = c(\delta)$, $d = d(\delta)$, $f = f(\delta)$ for all $m \leq n$ and all positive integers $k$ such that

$$k \leq \frac{m}{c \ln(n/m) + d}$$

the probability for a random $m \times n$ matrix $A$ with independent $\mathcal{N}(0, \frac{1}{m})$ entries to satisfy $\text{RIP}(\delta, k)$ is at least

$$1 - \exp\{-fm\}.$$ 

Similar result holds true for Rademacher matrices – those with i.i.d. entries taking values $\pm 1/\sqrt{m}$ with probabilities 0.5.

Note: $k$ can be “nearly” as large as $m$!
Sketch of the proof

Let $A$ be Gaussian random $m \times n$ matrix from Theorem, $I \subset \{1, \ldots, n\}$ be fixed $k$-element index set, and $A_I = [A_{ij} : i \leq m, j \in I]$. Let us fix $\alpha \in (0, 0.1]$. 

Fact: For fixed $u \in \mathbb{R}^k$ with $\|u\|_2 = 1$ one has

$$\text{Prob}\{ A : \|A_I u\|_2^2 \not\in [1 - \alpha, 1 + \alpha] \} \leq 2e^{-m/5} \alpha^2. $$

[observe that $A_I u \sim \mathcal{N}(0, \frac{1}{m} I_m)$ and use standard bounds on the tails of $\chi^2$-distribution]

Let $\Gamma$ be $\alpha$-net on the unit sphere $S_k$ in $\mathbb{R}^k$. Then

$$\text{Prob}\{ A : \exists u \in S_k : \|A_I u\|_2^2 \not\in [1 - 4\alpha, 1 + 4\alpha] \} \leq \pi := 2|\Gamma|e^{-m/5} \alpha^2$$

[By Fact, $\text{Prob}\{ A : \exists u \in S_k \forall u \in \Gamma \} \geq 1 - \pi$. Since the quadratic form $f(u) := u^T A_I^T A_I u$ is Lipschitz continuous on $S_k$ with constant $2M := 2 \max_{u \in S_k} \|A_I u\|_2^2$, we have

$$A \in \mathcal{E} \Rightarrow \left\{ \begin{array}{l}
\min_{u \in S_k} f(u) \geq \min_{u \in \Gamma} -\alpha M \leq 1 - (1 + \alpha)M \\
M = \max_{u \in S_k} f(u) \leq \max_{u \in \Gamma} f(u) + \alpha M \leq (1 + \alpha)M + 1
\end{array} \right.,$$

and the conclusion follows.]

Let $\forall (I, |I| = k)$:

$$\text{Prob}\{ A : (1 - 4\alpha)I_k \preceq A_I^T A_I \preceq (1 + 4\alpha)I_k \} \geq 1 - 2\left[1 + 2/\alpha\right]^k e^{-m/5} \alpha^2$$

[Comparing volumes, the cardinality of a minimal $\alpha$-net on $S_k$ is $\leq \mathcal{F}$]

$$\text{Prob}\{ A \text{ is not RIP}(4\alpha, k) \} \leq \binom{n}{k} \left[1 + 2/\alpha\right]^k e^{-m/5} \alpha^2$$

Theorem.

1.30
♠ **Bad news:** No (series of) explicitly computable (even by a randomized computation) RIP(0.1, k) “low” \((2m \leq n)\) \(m \times n\) matrices with “large” \(k\) (namely, \(k \gg \sqrt{m}\)) are known.

♥ The natural idea – “generate at random a low \(m \times n\) matrix and check whether it satisfies RIP(0.1, k)” with “large” \(k\); if yes, output the matrix” – fails – fails: while typical random matrices do possess RIP(0.1, k) with “large” \(k\), we do not know how to verify this property in a computationally efficient fashion.
Designing/checking RIP matrices is similar to other situations where we do know that a typical randomly selected object possesses some property, but we neither can point out an individual object with this property, nor can check efficiently whether a given object possesses it. Some examples:

- **Complexity of Boolean functions** [Shannon, 1949]: For a Boolean function \( f \) of \( n \) Boolean variables, the minimal number of AND, OR, NOT switches in a circuit computing the function is upper-bounded by \( O(1) \frac{2^n}{n} \), and as \( n \) grows, this bound becomes sharp with overwhelming probability.

  **However:** No individual functions with nonlinear “Boolean complexity” are known...

- **Lindenstrauss-Johnson Theorem** For a Gaussian “low” \( m \times n \) matrix \( A \), the image \( \{ Ax : x \in B_n \} \) of the unit \( n \)-dimensional box \( B_n = \{ x \in \mathbb{R}^n : \| x \|_{\infty} \leq 1 \} \) under the mapping \( x \mapsto Ax \) with overwhelming, as \( n \to \infty \), probability is in-between two similar ellipsoids with the ratio of linear sizes not exceeding \( 1 + O(1)(n/m)^2 \).

  **However:** No individual matrices \( A \) with \( AB_n \) reasonably close to an ellipsoid are known...

**Note:** For every \( \epsilon \in (0, 1) \) and every \( n \), one can explicitly point out a polytope \( P \) given by \( O(1)n \ln(1/\epsilon) \) linear inequalities on \( O(1)n \ln(1/\epsilon) \) variables such that the projection of \( P \) onto the plane of the first \( n \) variables is in-between \( \{ x \in \mathbb{R}^n : \| x \|_2 \leq 1 \} \) and \( \{ x \in \mathbb{R}^n : \| x \|_2 \leq 1 + \epsilon \} \). However, this “fast polyhedral approximation” of Euclidean ball deals with polytopes \( P \) quite different from boxes...
We have seen that RIP-matrices $A$ yield easy-to-satisfy condition $Q_2(s, \kappa)$. Unfortunately, RIP is difficult to verify...

Good news: Condition $Q_\infty(s, \kappa)$ is fully computationally tractable.
Theorem Let $A$ be an $m \times n$ sensing matrix, $s$ be a sparsity level, and $\kappa \geq 0$. Whenever $\bar{H}, \| \cdot \|$ satisfy $Q_\infty(s, \kappa)$, there exists an $m \times n$ matrix $H$ such that

$$\| \text{Col}_j [I_n - H^T A] \|_\infty \leq s^{-1} \kappa, 1 \leq j \leq n.$$ 

As a result, $H, \| \cdot \|_\infty$ satisfy $Q_\infty(s, \kappa)$. Besides this,

$$\| H^T \eta \|_\infty \leq \| \bar{H}^T \eta \| \forall \eta \in \mathbb{R}^m.$$ 

In addition, $m \times n$ contrast matrix $H$ such that $H, \| \cdot \|_\infty$ satisfy $Q_\infty(s, \kappa)$ with as small $\kappa$ as possible can be found as follows: we consider $n$ LP programs

$$\text{Opt}_i = \min_{\nu, h} \left\{ \nu : \| A^T h - e^i \|_\infty \leq \nu \right\},$$  

(\#_i)

where $e^i$ is $i$-th basic orth in $\mathbb{R}^n$, find optimal solutions $\text{Opt}_i, h_i$ to these problems, and make $h_i, i = 1, \ldots, n$, the columns of $H$; the corresponding value of $\kappa$ is

$$\kappa_* = s \max_i \text{Opt}_i.$$ 

Finally, there exists a transparent alternative description of the quantities $\text{Opt}_i$ (and thus – of $\kappa_*$):

$$\text{Opt}_i = \max_x \left\{ x_i : \| x \|_1 \leq 1, Ax = 0 \right\}.$$ 

1.34
Let $A$ be an $m \times n$ sensing matrix, $s$ be a sparsity level, and $\kappa \geq 0$. Whenever $\overline{H}$, $\| \cdot \|$ satisfy $Q_\infty(s, \kappa)$, there exists an $m \times n$ matrix $H$ such that

$$\| \text{Col}_j [I_n - H^T A] \|_\infty \leq s^{-1} \kappa, \ 1 \leq j \leq n,$$

As a result, $H$, $\| \cdot \|_\infty$ satisfy $Q_\infty(s, \kappa)$. Besides this,

$$\| H^T \eta \|_\infty \leq \| \overline{H}^T \eta \| \ \forall \eta \in \mathbb{R}^m.$$

Proof uses Basic fact of Convex Geometry: A norm $\| \cdot \|$ on $\mathbb{R}^N$ induces the conjugate norm

$$\| f \|_* = \max_{h: \| h \| \leq 1} f^T h.$$

One always has $|f^T h| \leq \| f \|_* \| h \| \ \& \ |h| = \max_{f: \| f \| \leq 1} f^T h$

Now,

$$i \leq n \Rightarrow x_i \leq \| x \|_{s, \infty} \leq \| \overline{H}^T A x \| + \kappa s^{-1} \| x \|_1 \ \forall x \ [\text{by } Q_\infty(s, \kappa)]$$

$$\Rightarrow \max_x \{ x_i - \| \overline{H}^T A x \| : \| x \|_1 \leq 1 \} \leq s^{-1} \kappa$$

$$\Leftrightarrow \max \min_{x: \| x \|_1 \leq 1, \| f \| \leq 1} \left[ [e^i]^T x - f^T \overline{H}^T A x \right] \leq s^{-1} \kappa$$

$$\Leftrightarrow \min \max_{f: \| f \| \leq 1} \left[ [e^i - A^T \overline{H} f]^T x \right] \leq s^{-1} \kappa$$

$$\Rightarrow \forall i \leq n \exists f_i \in \mathbb{R}^N: \| e^i - A^T \overline{H} f_i \|_\infty \leq s^{-1} \kappa \ \& \ \| f_i \|_* \leq 1.$$
∀i ≤ n ∃f_i ∈ ℝ^N : \|e^i - A^T \bar{H} f_i\|_∞ ≤ \kappa \& \|f_i\|* ≤ 1.

Let h_i = \bar{H} f_i and H = [h_1, ..., h_n]. Then

\[[I_n - H^T A]_{ij} = [I_n - A^T H]_{ij} = [e^i - A^T h_i]_j = [e^i - A^T \bar{H} f_i]_j \]

⇒ \max_{i,j} |[I_n - H^T A]_{ij}| ≤ \max_i \max_j |[e^i - A^T \bar{H} f_i]_j|

≤ \max_i \|e^i - A^T \bar{H} f_i\|_∞ ≤ s^{-1}\kappa

⇒ \|\text{Col}_i[I_n - H^T A]\|_∞ ≤ s^{-1}\kappa \forall i

Further,

\|\text{Col}_i[I_n - H^T A]\|_∞ ≤ s^{-1}\kappa \forall i

⇒ \|[I_n - H^T A]x\|_∞ ≤ s^{-1}\kappa \|x\|_1 \forall x ∈ ℝ^n

⇒ \|x\|_∞ - \|H^T Ax\|_∞ ≤ s^{-1}\kappa \|x\|_1 \forall x ∈ ℝ^n

⇒ H, \|\cdot\|_∞ satisfy Q_∞(s, \kappa)

In addition,

\|H^T \eta\|_∞ = \max_i |h_i^T \eta| = \max_i |f_i^T \bar{H} \eta| ≤ \max_i \|f_i\|* \|\bar{H}^T \eta\|$

\leq \|\bar{H}^T \eta\| \forall \eta.
... In addition, \( m \times n \) contrast matrix \( H \) such that \( H, \| \cdot \|_\infty \) satisfy \( Q_\infty(s, \kappa) \) with as small \( \kappa \) as possible can be found as follows: we consider \( n \) LP programs

\[
\text{Opt}_i = \min_{\nu, h} \left\{ \nu : \| A^T h - e^i \|_\infty \leq \nu \right\},
\]

where \( e^i \) is \( i \)-th basic orth in \( \mathbb{R}^n \), find optimal solutions \( \text{Opt}_i, h_i \) to these problems, and make \( h_i, i = 1, \ldots, n \), the columns of \( H \); the corresponding value of \( \kappa \) is \( \kappa^* = s \max_i \text{Opt}_i \).

**Proof:** By the above reasoning, if \( H, \| \cdot \| \) satisfy \( Q_\infty(s, \kappa) \), then \( \forall (i \leq n) \exists h_i : \| e^i - A^T h_i \|_\infty \leq s^{-1} \kappa \), and if \( h_i, i \leq n \), satisfy \( \| e^i - A^T h_i \|_\infty \leq s^{-1} \kappa \) for some \( \kappa \), then \( H := [h_1, \ldots, h_n], \| \cdot \|_\infty \) satisfy \( Q_\infty(s, \kappa) \).

... Finally, there exists a transparent alternative description of the quantities \( \text{Opt}_i \) (and thus – of \( \kappa^* \));

\[
\text{Opt}_i = \max_x \left\{ x_i : \| x \|_1 \leq 1, Ax = 0 \right\}.
\]

**Proof:**

\[
\text{Opt}_i = \min \left\{ t : -t \leq e^i_j - [A^T h]_j \leq t, \forall j \right\}
\]

\[
= \max_{\lambda, \mu} \left\{ [\lambda - \mu]_i : \sum_i \lambda_i + \sum_i \mu_i = 1 \right\} \quad \text{[LP duality]}
\]

\[
= \max_x \left\{ x_i : A^T x = 0, \| x \|_1 \leq 1 \right\}
\]
Illustration

♠ $k$-th Hadamard matrix $\mathcal{H}_k$ is $n_k \times n_k$ matrix, $n_k = 2^k$, with entries $\pm 1$ given by the recurrence

$$
\mathcal{H}_0 = [1]; \mathcal{H}_{k+1} = \begin{bmatrix} \mathcal{H}_k & \mathcal{H}_k \\ \mathcal{H}_k & -\mathcal{H}_k \end{bmatrix}
$$

Note: $\mathcal{H}_k$ is symmetric and is proportional to orthogonal matrix: $\mathcal{H}_k^T \mathcal{H}_k = n_k I_{n_k}$

$\Rightarrow$ When $k > 0$, the only eigenvalues of $\mathcal{H}_k$ are $\sqrt{n_k}$ and $-\sqrt{n_k}$ with multiplicities $n_k/2$ each.

• Let $k > 1$, $m_k = n_k/2 = 2^{k-1}$, and let $a_1, \ldots, a_{m_k}$ be an orthonormal system of eigenvectors of $\mathcal{H}_k$ with eigenvalue $\sqrt{n_k}$. Let $A_k$ be the $m_k \times n_k$ matrix with the rows $a_T^1, \ldots, a_{m_k}^T$.

Fact: Let $s < \frac{1}{2} \sqrt{n_k} = 2^{k-1}$. Then the matrix $A_k$ is $s$-good. Moreover, there exists (and can be efficiently computed) contrast matrix $H_k$ such that $(H_k, \| \cdot \|_\infty)$ satisfies the condition $Q_\infty(s, \kappa_s = s/\sqrt{n_k})$, and $\|\text{Col}_i[H_k]\|_2 \leq \sqrt{2 + 2/\sqrt{n_k}}$ for all $j$. 

1.38
Verifiable Sufficient condition for satisfiability of \( Q_q(s, \kappa) \): Let \( m \times n \) matrix \( H \) satisfy the condition

\[
\| \text{Col}_j [I_n - H^T A] \|_{s,q} \leq s^{1/q - 1} \kappa, \quad 1 \leq j \leq n
\]

Then \( H, \| \cdot \| \) satisfy \( Q_q(s, \kappa) \).

Proof:

\[
(!) \Rightarrow \| [I_n - H^T A] x \|_{s,q} \leq s^{1/q - 1} \kappa \| x \|_1 \quad \forall x
\]

\[
\Rightarrow \| x \|_{s,q} - \| H^T Ax \|_{s,q} \leq s^{1/q - 1} \kappa \| x \|_1 \quad \forall x
\]

\[
\Rightarrow \| x \|_{s,q} \leq \| H^T Ax \|_{s,q} + s^{1/q - 1} \kappa \| x \|_1
\]

\[
\Rightarrow \| x \|_{s,q} \leq s^{1/q} \| H^T Ax \|_\infty + s^{1/q - 1} \kappa \| x \|_1 \quad \forall x
\]

Note: (!) is an explicit system of convex constraints on \( H \)

\( \Rightarrow \) The sufficient condition (!) for \( H, \| \cdot \|_\infty \) to satisfy \( Q_q(s, \kappa) \) is computationally tractable.

Note: When \( q = \infty \), feasibility of (!) is necessary and sufficient for satisfiability of \( Q_\infty(s, \kappa) \): \( (H \in \mathbb{R}^{m \times n}, \| \cdot \|_\infty) \) satisfies \( Q_\infty(s, \kappa) \) if and only if

\[
\| \text{Col}_j [I_n - H^T A] \|_\infty \leq s^{-1} \kappa \quad \forall j.
\]
Let $m \times n$ matrix $H$ satisfy the condition

$$\|\text{Col}_j[I_n - H^T A]\|_{s,q} \leq s^{q-1} \kappa, \quad 1 \leq j \leq n$$

Then $H, \| \cdot \|$ satisfy $Q_q(s, \kappa)$.

The above statement, whatever simple, has an instructive origin. Consider the following problem:

(?) Given a convex function $\phi(x) : \mathbb{R}^n \to \mathbb{R}$ and a convex set

$$X = \{ x \in \text{Conv}\{f_1, \ldots, f_N\} : Ax = 0 \}$$

we want to compute/upper-bound efficiently the quantity

$$\phi_* = \max_{x \in X} \phi(x).$$

Example: Verifying the Nullspace Property of matrix $A$ reduces to checking whether the quantity

$$\phi_* := \max_{x \in X} \left[ \phi(x) := \| x \|_{s,1} \right],$$

$$X = \{ x \in \text{Conv}\{\pm e_1, \pm e_2, \ldots, \pm e_n\} : Ax = 0 \}$$

is or is not $< 1/2$. 

1.40
\( \phi_* = \max_{x \in X} \phi(x), \ X = \{x \in \text{Conv}\{f_1, ..., f_N\} : Ax = 0\} \)

- \( \phi_* \) is the maximum of a convex function over a bounded polyhedral set and as such is in general NP-hard to compute. However, we can point out a simple scheme for efficient upper-bounding \( \phi_* \):

\[
\forall H \in \mathbb{R}^{m \times n} : \\
\phi_* = \max_x \{\phi(x) : x \in \text{Conv}\{f_1, ..., f_N\}, Ax = 0\} \\
= \max_x \{\phi([I - H^T A]x) : x \in \text{Conv}\{f_1, ..., f_N\}, Ax = 0\} \\
\leq \max_x \{\phi([I - H^T A]x) : x \in \text{Conv}\{f_1, ..., f_N\}\} \\
= \max_{j \leq N} \phi([I - H^T A]f_j),
\]

\[ \Rightarrow [\phi_* \leq \bar{\phi} := \min_H \left[ \max_{j \leq N} \phi([I - H^T A]f_j) \right] \]

and \( \bar{\phi} \) is efficiently computable (as the optimal value in a convex problem).

- **Note:** As applied to

\[ \phi(x) = \|x\|_{s,1}, \ X = \{x \in \text{Conv}\{\pm e_1, ..., \pm e_n\} : Ax = 0\}, \]

the above bounding scheme results in the verifiable sufficient condition

\[ \exists (\kappa < 1/2, H) : \|\text{Col}_j[I - H^T A]\|_{s,1} \leq \kappa, \ 1 \leq j \leq n \]

for \( s \)-goodness of \( A \). This hint leads to the verifiable sufficient conditions for \( Q_q(s, \kappa) \).

1.41
Bad news: When \( m \times n \) sensing matrix \( A \) is “essentially non-square”, namely, \( n \geq 2m \), the above verifiable sufficient conditions for the validity of \( Q_q(s, \kappa) \) can be satisfiable only in the range

\[
s \leq \sqrt{2m}
\]

which is much less than the range

\[
s \leq O(1)\frac{m}{\ln(n/m)}
\]

where random Gaussian/Rademacher \( m \times n \) sensing matrices satisfy RIP(\( \frac{1}{4}, 2s \)) with overwhelming probability, thus implying satisfiability of \( Q_2(s, \frac{1}{3}) \).

Note:

A. No series of individual essentially non-square \( m \times n \) sensing matrices \( A \) with \( m, n \to \infty \) which are provably \( s \)-good for \( s \geq O(1)\sqrt{m} \) are known.

B. For \( k = 1, 2, \ldots \) one can easily point out individual \( 2^{k-1} \times 2^k \) sensing matrices for which condition \( Q_\infty(s, \frac{1}{3}) \) is satisfiable whenever \( s \leq \frac{\sqrt{2m}}{3} \).

C. Whenever \( A \) satisfies RIP(\( \delta, 2k \)) and \( s \leq \frac{1-\delta}{3\delta} \sqrt{k} \), the pair \( (H = \sqrt{k}A, \| \cdot \|_\infty) \) satisfies \( Q_\infty(s, \frac{1}{3}) \).

D. For properly selected \( C > 0 \) and every \( m, n \), one can point out individual \( m \times n \) sensing matrix which is \( C\sqrt{m} \)-good.
Mutual Incoherence. Let $A$ be $m \times n$ sensing matrix without zero columns. Mutual Incoherence of $A$ is the quantity

$$\mu(A) = \max_{i \neq j} \frac{\text{Col}_i^T[A]\text{Col}_j[A]}{\text{Col}_i^T[A]\text{Col}_i[A]}$$

Observation: The $m \times n$ matrix $H$ with columns $\frac{\text{Col}_j[A]}{\text{Col}_j^T[A]\text{Col}_j[A]}$, $j = 1, \ldots, n$, satisfies

$$\forall j : \|\text{Col}_j[I_n - H^T A]\|_\infty \leq \frac{\mu(A)}{1 + \mu(A)}$$

$\implies H, \| \cdot \|_\infty$ satisfy $Q_\infty(s, \frac{s\mu(A)}{1 + \mu(A)})$ for every $s$. In particular, $A$ is $s$-good, provided that

$$\frac{2\mu(A)}{1 + \mu(A)} < \frac{1}{s}.$$
HYPOTHESIS TESTING, I

- Preliminaries
  - Tests & Risks
  - Repeated Observations
  - 2-Point Lower Risk Bound
- Pairwise Tests via Euclidean Separation
- From Pairwise to Multiple Hypothesis Testing
Hypothesis Testing Problem: Given
• observation space $\Omega$ where our observations take values,
• $L$ families $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_L$ of probability distributions on $\Omega$, and
• an observation $\omega$ – a realization of random variable with unknown probability distribution $P$ known to belong to one of the families $\mathcal{P}_\ell$: $P \in \bigcup_{\ell=1}^{L} \mathcal{P}_\ell$,

we want to decide to which one of the families $\mathcal{P}_\ell$ the distribution $P$ belongs.

Equivalent wording: Given the outlined data, we want to decide on $L$ hypotheses $H_1, \ldots, H_L$, with $\ell$-th hypothesis $H_\ell$ stating that $P \in \mathcal{P}_\ell$.

A test is a function $\mathcal{T}(\cdot)$ on $\Omega$. The value $\mathcal{T}(\omega)$ of this function at a point $\omega \in \Omega$ is a subset of the set $\{1, \ldots, L\}$.
• relation $\ell \in \mathcal{T}(\omega)$ is interpreted as “given observation $\omega$, the test accepts the hypothesis $H_\ell$”
• relation $\ell \notin \mathcal{T}(\omega)$ is interpreted as “given observation $\omega$, the test rejects the hypothesis $H_\ell$”

$\mathcal{T}$ is called simple, if $\mathcal{T}(\omega)$ is a singleton for every $\omega \in \Omega$. 

2.1
For a simple test $\mathcal{T}$, its *risks* are defined as follows:

- **ℓ-th partial risk** of $\mathcal{T}$ is the (worst-case) probability to reject $\ell$-th hypothesis when it is true:

  $$\text{Risk}_\ell(\mathcal{T}|H_1, \ldots, H_L) = \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{\omega \sim P}\{\ell \notin \mathcal{T}(\omega)\}$$

- **total risk** of $\mathcal{T}$ is the sum of all partial risks:

  $$\text{Risk}_{\text{tot}}(\mathcal{T}|H_1, \ldots, H_L) = \sum_{1 \leq \ell \leq L} \text{Risk}_\ell(\mathcal{T}|H_1, \ldots, H_L).$$

- **risk** of $\mathcal{T}$ is the maximum of all partial risks:

  $$\text{Risk}(\mathcal{T}|H_1, \ldots, H_L) = \max_{1 \leq \ell \leq L} \text{Risk}_\ell(\mathcal{T}|H_1, \ldots, H_L).$$
Note: What was called test is in fact a deterministic test.

A randomized test is a deterministic function $T(\omega, \eta)$ of observation $\omega$ and independent of $\omega$ random variable $\eta \sim P_\eta$ with once for ever fixed distribution (say, $P_\eta = \text{Uniform}[0, 1]$). The values $T(\omega, \eta)$ of $T$ are subsets of $\{1, \ldots, L\}$ (singletons for a simple test).

- Given observation $\omega$, we “flip a coin” (draw a realization of $\eta$), accept hypotheses $H_\ell$, $\ell \in T(\omega, \eta)$, and reject all other hypotheses.
- Partial risks of randomized test are

$$\text{Risk}_\ell(T|H_1, \ldots, H_L) = \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{(\omega, \eta) \sim P \times P_\eta} \{\ell \notin T(\omega, \eta)\}.$$ 

Exactly as above, these risks give rise to the total risk and risk of $T$. 

2.3
Testing from repeated observations. There are situations where an inference can be based on several observations $\omega_1, \ldots, \omega_K$ rather than on a single observation. Our related setup is as follows:

We are given $L$ families $\mathcal{P}_\ell$, $\ell = 1, \ldots, L$, of probability distributions on observation space $\Omega$ and a collection

$$\omega^K = (\omega_1, \ldots, \omega_K)$$

and want to make conclusions on how the distribution of $\omega^K$ “is positioned” w.r.t. the families $\mathcal{P}_\ell$, $1 \leq \ell \leq L$. Specifically, we are interested in three situations of type:

A. Stationary $K$-repeated observations: $\omega_1, \ldots, \omega_K$ are independently of each other drawn from a distribution $P$. Our goal is to decide, given $\omega^K$, on the hypotheses $P \in \mathcal{P}_\ell$, $\ell = 1, \ldots, L$.

Equivalently: Families $\mathcal{P}_\ell$ give rise to the families

$$\mathcal{P}^{\ominus,K}_\ell = \{ P^K = \underbrace{P \times \ldots \times P}_K : P \in \mathcal{P}_\ell \}$$

of probability distributions on $\Omega^K = \underbrace{\Omega \times \ldots \times \Omega}_K$ – direct powers of families $\mathcal{P}_\ell$.

Given observation $\omega^K \in \Omega^K$, we want to decide on the hypotheses

$$H^{\ominus,K}_\ell : \omega^K \sim P^K \in \mathcal{P}^{\ominus,K}_\ell, \ 1 \leq \ell \leq L.$$
B. Semi-stationary $K$-repeated observations: "The nature" selects somehow a sequence $P_1, ..., P_K$ of distributions on $\Omega$, and then draws, independently across $k$, observations $\omega_k$ from these distributions:

$$\omega_k \sim P_k$$ are independent across $k \leq K$

Our goal is to decide, given $\omega^K = (\omega_1, ..., \omega_K)$, on the hypotheses $\{P_k \in \mathcal{P}_\ell, 1 \leq k \leq K\}$, $\ell = 1, ..., L$.

Equivalently: Families $\mathcal{P}_\ell$ give rise to the families

$$\mathcal{P}^{\oplus,K}_\ell = \bigoplus_{k=1}^{K} \mathcal{P}_\ell := \{P^K = P_1 \times ... \times P_K : P_k \in \mathcal{P}_\ell, 1 \leq k \leq K\}$$

of probability distributions on $\Omega^K = \underbrace{\Omega \times ... \times \Omega}_K$ – direct products of families $\mathcal{P}_\ell$.

Given observation $\omega^K \in \Omega^K$, we want to decide on the hypotheses

$$H^{\oplus,K}_\ell : \omega^K \sim P^K \in \mathcal{P}^{\oplus,K}_\ell, 1 \leq \ell \leq L.$$
C. Quasi-stationary $K$-repeated observations: We observe random sequence \( \omega^K = (\omega_1, \ldots, \omega_K) \) generated as follows:

There exists a random sequence \( \zeta_1, \ldots, \zeta_K \) of driving factors such that for \( 1 \leq k \leq K \):

- \( \omega_k \) is a deterministic function of \( \zeta^k = (\zeta_1, \ldots, \zeta_k) \)
- conditional, \( \zeta^{k-1} \) given, distribution of \( \omega_k \) always belongs to \( \mathcal{P}_\ell \).

Our goal is to decide, given \( \omega^K \), on the underlying \( \ell \).

**Equivalently:** Families \( \mathcal{P}_\ell \) of probability distributions on \( \Omega \), \( 1 \leq \ell \leq L \), give rise to the quasi-direct products \( \mathcal{P}^{\otimes,K}_\ell = \bigotimes_{k=1}^K \mathcal{P}_\ell \) of families \( \mathcal{P}_\ell \). The family \( \bigotimes_{k=1}^K \mathcal{P}_\ell \) is comprised of all probability distributions on \( \Omega^K = \underbrace{\Omega \times \ldots \times \Omega}_K \) which can be obtained from \( \mathcal{P}_\ell \) via the above “driving factors” mechanism.

Given observation \( \omega^K \in \Omega^K \), we want to decide on the hypotheses

\[
H^{\otimes,K}_\ell : \omega^K \sim P^K \in \mathcal{P}^{\otimes,K}_\ell, \ 1 \leq \ell \leq L.
\]
Important fact: 2-point lower risk bound. Consider simple pairwise test deciding on two simple hypotheses on the distribution $P$ of observation $\omega \in \Omega$:

$$H_1 : P = P_1, \quad H_2 : P = P_2.$$ Let $P_1, P_2$ have densities $p_1, p_2$ w.r.t. some reference measure $\Pi$ on $\Omega$. Then the total risk of every test $T$ deciding on $H_1, H_2$ admits lower bound as follows:

$$\text{Risk}_{\text{tot}}(T|H_1, H_2) \geq \int_{\Omega} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega).$$

As a result,

$$\text{Risk}(T|H_1, H_2) \geq \frac{1}{2} \int_{\Omega} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega). \quad (*)$$

Note: The bound does not depend on the choice of $\Pi$ (for example, we can always take $\Pi = P_1 + P_2$).
\[
\text{Risk}(\mathcal{T}|H_1, H_2) \geq \frac{1}{2} \int_{\Omega} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega). \tag{?}
\]

**Proof** (for deterministic test). Simple test deciding on \(H_1, H_2\) must accept \(H_1\) and reject \(H_2\) on some subset \(\Omega_1\) of \(\Omega\) and must reject \(H_1\) and accept \(H_2\) on the complement \(\Omega_2 = \Omega \setminus \Omega_1\) of this set. We have

\[
\begin{align*}
\text{Risk}_1(\mathcal{T}|H_1, H_2) &= \int_{\Omega_2} p_1(\omega) \Pi(d\omega) \geq \int_{\Omega_2} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega) \\
\text{Risk}_2(\mathcal{T}|H_1, H_2) &= \int_{\Omega_1} p_2(\omega) \Pi(d\omega) \geq \int_{\Omega_1} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega) \\
\Rightarrow \text{Risk}_{\text{tot}}(\mathcal{T}|H_1, H_2) &\geq \int_{\Omega_2} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega) + \int_{\Omega_1} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega) \\
&= \int_{\Omega} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega) \quad \square
\end{align*}
\]
Corollary. Consider $L$ hypotheses $H_\ell : P \in \mathcal{P}_\ell$, $\ell = 1, 2, \ldots, L$, on the distribution $P$ of observation $\omega \in \Omega$, let $\ell \neq \ell'$ and let $P_\ell \in \mathcal{P}_\ell$, $P_{\ell'} \in \mathcal{P}_{\ell'}$. The risk of any simple test $\mathcal{T}$ deciding on $H_1, \ldots, H_L$ can be lower-bounded as

$$\text{Risk}(\mathcal{T}|H_1, \ldots, H_L) \geq \frac{1}{2} \int_{\Omega} \min[P_\ell(d\omega), P_{\ell'}(d\omega)],$$

where, by convention, the integral in the right hand side is

$$\int_{\Omega} \min[p_\ell(\omega), p_{\ell'}(\omega)] \Pi(d\omega),$$

with $p_\ell, p_{\ell'}$ being the densities of $P_\ell, P_{\ell'}$ w.r.t. $\Pi = P_\ell + P_{\ell'}$.

Indeed, risk of $\mathcal{T}$ cannot be less than the risk of the naturally induced by $\mathcal{T}$ simple test deciding on two simple hypotheses $P = P_\ell$, $P = P_{\ell'}$, specifically, the simple test which, given observation $\omega$ accepts the hypothesis $P = P_1$ whenever $\ell \in \mathcal{T}(\omega)$ and accepts the hypothesis $P = P_{\ell'}$ otherwise.
Pairwise Hypothesis Testing via Euclidean Separation

Situation: Let $\Omega = \mathbb{R}^d$, and let our observation be

$$\omega = x + \xi$$

where the deterministic vector $x$ is the signal of interest, and $\xi$ is random observation noise with probability density $p(\cdot)$ of the form

$$p(u) = f(\|u\|_2)$$

where $f(\cdot)$ is a strictly monotonically decreasing function on the nonnegative ray.

Simple example: standard (zero mean, unit covariance) Gaussian noise: $p(u) = (2\pi)^{-d/2}e^{-u^T u/2}$.

Our goal is to decide on two simple hypotheses on the signal underlying observation, the first stating that $x = x^1$, and the second stating that $x = x^2$, where $x^1$, $x^2$ are two given points.

Equivalent wording: We are given two probability distributions, $P_1$ and $P_2$, on $\mathbb{R}^d$, with densities $p_1(u) = p(u - x^1)$ and $p_2(u) = p(u - x^2)$, and want to decide on two simple hypotheses $H_1 : P = P_1$, $H_2 : P = P_2$ on the distribution $P$ of our observation.
Assuming \( x^1 \neq x^2 \), let \( 2\delta = \|x^1 - x^2\|_2 \), \( e = \frac{x^1 - x^2}{\|x^1 - x^2\|_2} \),

\[
\Pi = \{ \omega : \|\omega - x^1\|_2 = \|\omega - x^2\|_2 \} = \{ \omega : \phi(\omega) = 0 \}, \quad \phi(\omega) = e^T \omega - \frac{1}{2}e^T[x^1 + x^2]
\]

Consider test \( T \) which, given observation \( \omega = x + \xi \), accepts the hypothesis \( H_1 : P = P_1 \) (i.e., \( x = x^1 \)) when \( \phi(\omega) \geq 0 \), and accepts the hypothesis \( H_2 : P = P_2 \) (i.e., \( x = x^2 \)) otherwise. We have

\[
\text{Risk}_1(T|H_1, H_2) = \int_{\omega : \phi(\omega) < 0} p_1(\omega) d\omega = \int_{u : e^T u \geq \delta} f(\|u\|_2) du
\]

\[
= \int_{\omega : \phi(\omega) \geq 0} p_2(\omega) d\omega = \text{Risk}_2(T|H_1, H_2)
\]

Since \( p(u) \) is strictly decreasing function of \( \|u\|_2 \), we have \( \min[p_1(u), p_2(u)] = \begin{cases} p_1(u), & \phi(u) \geq 0 \\ p_2(u), & \phi(u) \leq 0 \end{cases} \), whence

\[
\text{Risk}_1(T|H_1, H_2) + \text{Risk}_2(T|H_1, H_2) = \int_{\omega : \phi(\omega) < 0} p_1(\omega) d\omega + \int_{\omega : \phi(\omega) \geq 0} p_2(\omega) d\omega = \int_{\mathbb{R}^d} \min[p_1(u), p_2(u)] du
\]

\( \Rightarrow \) Test \( T \) is the minimum risk simple test deciding on \( H_1, H_2 \).
♣ Extension: Given observation $\omega = x + \xi$ with observation noise $\xi$ possessing probability density

$$p(u) = f(\|u\|_2),$$

where $f(\cdot)$ is a strictly decreasing function on the nonnegative ray, we want to decide on two composite hypotheses $H_1, H_2$:

$$H_1 : x \in X_1, \quad H_2 : x \in X_2,$$

where $X_1, X_2$ are nonempty nonintersecting, closed and convex sets, and one of the sets is bounded.

♠ Elementary fact: With $X_1, X_2$ as above, consider the convex minimization problem

$$\text{Opt} = \min_{x^1 \in X_1, x^2 \in X_2} \frac{1}{2}\|x^1 - x^2\|_2.$$

The problem is solvable. Let $(x^1_*, x^2_*)$ be an optimal solution, and let

$$\phi(\omega) = e^T \omega - c, \quad e = \frac{x^1_* - x^2_*}{\|x^1_* - x^2_*\|_2}, \quad c = \frac{1}{2} e^T [x^1_* + x^2_*].$$

Then the stripe $\{\omega : -\text{Opt} \leq \phi(\omega) \leq \text{Opt}\}$ separates $X_1$ and $X_2$:

$$\phi(x^1) \geq \phi(x^1_*) = \text{Opt} \forall x^1 \in X_1,$$

$$\phi(x^2) \geq \phi(x^2_*) = -\text{Opt} \forall x^2 \in X_2.$$
We have associated with two non-intersecting closed convex $X_1$, $X_2$, one of the sets being bounded,
— convex optimization problem
\[
\text{Opt} = \min_{x^1 \in X_1, x^2 \in X_2} \frac{1}{2} \|x^1 - x^2\|_2
\]
— linear function
\[
\phi(\omega) = e^T \omega - \frac{1}{2} e^T [x^1_* + x^2_*], \quad e = \frac{1}{2 \text{Opt}} [x^1_* - x^2_*]
\]
where $[x^1_*, x^2_*]$ is an optimal solution to the above problem. While this solution not necessarily is uniquely defined by $X_1$, $X_2$, $\phi(\cdot)$ is uniquely defined by $X_1$, $X_2$. 
Given $\delta_1 \geq 0, \delta_2 \geq 0$ with $\delta_1 + \delta_2 = 2\text{Opt}$, $\phi(\cdot)$ specifies simple Euclidean Separation Test $T$ induced by $X_1, X_2, \delta_1, \delta_2$:

$$T(\omega) = \begin{cases} 
\{1\}, & \phi(\omega) \geq \frac{1}{2}[\delta_2 - \delta_1] \\
\{2\}, & \text{otherwise}
\end{cases}$$

Fact: Let $\xi \sim p(\cdot)$, where $p(u) = f(\|u\|_2)$ with strictly decreasing $f(t), t \geq 0$. Given observation $\omega = x + \xi$ the Euclidean Separation Test $T$ decides on the hypotheses $H_1: x \in X_1, H_2: x \in X_2$ with risks satisfying

$$\text{Risk}_1(T|H_1, H_2) \leq \int_{\delta_1}^{\infty} \gamma(s)ds, \quad \text{Risk}_2(T|H_1, H_2) \leq \int_{\delta_2}^{\infty} \gamma(s)ds$$

where $\gamma(\cdot)$ is the univariate marginal density of $\xi$, that is, probability density of the scalar random variable $h^T \xi$, where $\|h\|_2 = 1$.

In addition, when $\delta_1 = \delta_2 = \text{Opt}$, $T$ is the minimum risk test deciding on $H_1, H_2$, and

$$\text{Risk}(T|H_1, H_2) = \int_{\text{Opt}}^{\infty} \gamma(s)ds.$$
Extension: Under the premise of Fact: the observation is $\omega = x + \xi$ with $\xi \sim p(\cdot) = f(\|\cdot\|_2)$, where

- $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly decreasing, and
- the hypotheses to be decided upon are $H_1 : x \in X_1, H_2 : x \in X_2$ with closed convex
  nonintersecting and nonempty $X_1, X_2$, one of the sets being bounded,

the risk bounds $\text{Risk}_\ell(T | H_1, H_2) \leq \int_{\delta_\ell}^{\infty} \gamma(s)ds$, $\ell = 1, 2$ for the Euclidean Separation Test stem from

the following observation:

Under the circumstances, for every half-space $E = \{u \in \mathbb{R}^d : e^T u \geq \delta\}$, where $\|e\|_2 = 1$ and $\delta \geq 0$, one has

$$\text{Prob}_{\xi \sim p(\cdot)}\{\xi \in E\} \leq \int_{\delta}^{\infty} \gamma(s)ds.$$
Given an even probability density \( \gamma(\cdot) \) on the axis such that \( \int_{\delta}^{\infty} \gamma(s) ds < \frac{1}{2} \) whenever \( \delta > 0 \), let us associate with it the family \( \mathcal{P}_d^\gamma \) of all probability distributions \( P \) on \( \mathbb{R}^d \) such that

**A:** distribution \( P \) possesses even density, and

**B:** whenever \( e \in \mathbb{R}^d, \|e\|_2 = 1 \), and \( \delta \geq 0 \), we have

\[
\Pr_{\xi \sim P}\{\xi : e^T \xi \geq \delta\} \leq \Gamma(\delta) := \int_{\delta}^{\infty} \gamma(s) ds
\]

By the same reasons as in Fact, we have the following

**Proposition.** Whenever the distribution \( P \) of noise \( \xi \) in observation \( \omega = x + \xi \) belongs to \( \mathcal{P}_d^\gamma \) and \( X_1, X_2 \) are non-intersecting closed convex sets, one of the sets being bounded, the risks of the Euclidean Separation Test \( \mathcal{T} \) induced by \( X_1, X_2 \) and \( \delta_1, \delta_2 \) can be upper-bounded as

\[
\text{Risk}_\ell(\mathcal{T}|H_1, H_2) \leq \Gamma(\delta_\ell) := \int_{\delta_\ell}^{\infty} \gamma(s) ds, \ \ell = 1, 2.
\]
Example: Gaussian mixtures. Let \( \eta \) be an \( d \)-dimensional Gaussian random vector with zero mean and covariance matrix \( \Theta \) (notation: \( \eta \sim \mathcal{N}(0, \Theta) \)). Let, further, \( Z \) be independent of \( \eta \) positive random variable. Gaussian mixture is the probability distribution of the random vector \( \xi = \sqrt{Z} \eta \). Examples of Gaussian mixtures are:

- Gaussian distribution \( \mathcal{N}(0, \Theta) \) (take \( Z \) identically equal to 1),
- multidimensional Student’s \( t \)-distribution with \( \nu \) degrees of freedom (\( \nu/Z \) has \( \chi^2 \)-distribution with \( \nu \) degrees of freedom)

Immediate Observations:

- Let \( Z \) be a random variable taking values in \([0, 1]\), let \( \eta \sim \mathcal{N}(0, \Theta) \) with \( \Theta \preceq I_d \) (i.e., the matrix \( I_d - \Theta \) is positive semidefinite) be independent of \( Z \), and let
  \[
  \gamma_g(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}
  \]
  be the standard (zero mean, unit variance) Gaussian density on the axis. Then the distribution of the Gaussian mixture \( \xi = \sqrt{Z} \eta \) belongs to the family \( \mathcal{P}_d^g \).
- With \( \gamma \) given by the distribution \( P_Z \) of \( Z \) according to
  \[
  \gamma_Z(s) = \int_{z>0} \frac{1}{\sqrt{2\pi z}} e^{-s^2/2z} P_Z(dz),
  \]
  the distribution of random variable \( \sqrt{Z} \eta \), with \( \eta \sim \mathcal{N}(0, \Theta) \), \( \Theta \preceq I_d \), independent of \( Z \), belongs to the family \( \mathcal{P}_d^\gamma_Z \).
From Euclidean Separation to Majority Test

Let $\gamma(\cdot)$, $\mathcal{P}_\gamma^d$, $X_1$, $X_2$ be as in Proposition, and assume we have access to semi-stationary $K$-repeated observations

$$\omega^K = \{\omega_k = x_k + \xi_k : 1 \leq k \leq K\}$$

where

- $\{x_k : 1 \leq k \leq K\}$ is a deterministic sequence of signals,
- $\xi_k \sim P_k$, $1 \leq k \leq K$, are independent across $k$ noises, and
- $\{P_k, 1 \leq k \leq K\}$ is a deterministic sequence of distributions from $\mathcal{P}_\gamma^d$.

Given $\omega^K$, we want to decide on the hypotheses

$$H^K_1 : x_k \in X_1, 1 \leq k \leq K \quad \text{and} \quad H^K_2 : x_k \in X_2, 1 \leq k \leq K.$$

Equivalently: The sets $X_\ell$, $\ell = 1, 2$, give rise to families $\mathcal{P}_\ell$ of probability distributions on $\Omega = \mathbb{R}^d$; $\mathcal{P}_\ell$ is comprised of distributions $P$ of random vectors of the form $x + \xi$, with deterministic $x \in X_\ell$ and with the distribution of noise $\xi$ belonging to $\mathcal{P}_\gamma^d$. The families $\mathcal{P}_\ell$, in turn, give rise to hypotheses

$$H^K_\ell = H^{\oplus,K}_\ell : P^K \in \mathcal{P}^{\oplus,K}_\ell, \ell = 1, 2,$$

on the distribution $P^K$ of $K$-repeated observation $\omega^K = (\omega_1, ..., \omega_K)$. Given $\omega^K$, we want to decide on the hypotheses $H^K_1$, $H^K_2$. 

2.18
\[ \omega^K = \{ \omega_k = x_k + \xi_k : 1 \leq k \leq K \} \]
\[ H^K_\ell : x_k \in X_\ell, 1 \leq k \leq K, \xi_k \sim P_k \in \mathcal{P}_d : \text{ independent across } k \]

Let us use the \textit{majority test} \( T^{\text{maj}}_K \) defined as follows:

- we build the Euclidean separator of \( X_1, X_2 \), thus arriving at the affine function

\[ \phi(\omega) = e^T \omega - c \quad [\|e\|_2 = 1] \]

such that the stripe

\[ \{ \omega : -\text{Opt} \leq \phi(\omega) \leq \text{Opt} \} \]

with

\[ \text{Opt} = \min_{x^1 \in X_1, x^2 \in X_2} \frac{1}{2}\|x^1 - x^2\|_2, \]

separates \( X_1, X_2 \);

- given \( (\omega_1, ..., \omega_K) \), we compute reals \( v_k = \phi(\omega_k) \), \( 1 \leq k \leq K \), and accept \( H^K_1 \) when the number of nonnegative \( v_k \)'s is at least \( K/2 \), otherwise we accept \( H^K_2 \).
Risk analysis. Assume that $H^K_1$ takes place, so that $\{x_k\}$ form some deterministic sequence of points from $X_1$, and $\xi_k$ are drawn, independently across $k$, from some distributions $P_k \in \mathcal{P}^d$. With $\{x_k\}$ and $\{P_k\}$ fixed, $v_k$ are independent across $k$, and probability for $v_k$ to be negative is, by our previous results, $\leq \epsilon_* := \Gamma(\text{Opt}) := \int_{\text{Opt}}^\infty \gamma(s)ds$, where

$$\text{Opt} = \min_{x^1 \in X_1, x^2 \in X_2} \frac{1}{2}\|x^1 - x^2\|_2.$$ 

Consequently, the probability to reject $H^K_1$ under the circumstances is

$$\leq \epsilon_K := \sum_{K/2 \leq k \leq K} \binom{K}{k} \epsilon_*^k (1 - \epsilon_*)^{K-k}.$$ 

By “symmetric” reasoning, the probability to reject $H^K_2$ when the hypothesis is true is $\leq \epsilon_K$ as well. We arrive at

Proposition. The risk of $T_K^{\text{maj}}$ can be upper-bounded as

$$\text{Risk}(T_K^{\text{maj}}|H^K_1, H^K_2) \leq \sum_{K/2 \leq k \leq K} \binom{K}{k} \epsilon_*^k (1 - \epsilon_*)^{K-k}$$

$$\left[\epsilon_* = \int_{\text{Opt}}^\infty \gamma(s)ds, \text{Opt} = \min_{x^1 \in X_1, x^2 \in X_2} \frac{1}{2}\|x^1 - x^2\|_2\right]$$

Fact: Conclusion remains true in the case of quasi-stationary observations.
\[
\text{Risk}(T_K^{\text{maj}}|H^K_1, H^K_2) \leq \sum_{K/2 \leq k \leq K} \binom{K}{k} \epsilon_k^*(1 - \epsilon_*)^{K-k} \\
\epsilon_* = \int_0^\infty \gamma(s) ds, \ \text{Opt} = \min_{x^1 \in X_1, x^2 \in X_2} \frac{1}{2} \|x^1 - x^2\|_2
\]

♣ Quiz: We have used “evident” observation as follows:

Let \(w_1, \ldots, w_K\) be independent random variables taking values 0 and 1, and let the probability for \(w_i\) to take value 1 be some \(p_i \in [0, 1]\). Then for every fixed \(M\) the probability of the event “at least \(M\) of \(w_1, \ldots, w_K\) are equal to 1” as a function of \(p_1, \ldots, p_K\) is nondecreasing in every one of \(p_i\)’s. (In our context, \(w_i\) were the signs of \(v_i\)).

Why this observation is true?
From Pairwise to Multiple Hypotheses Testing

Situation: We are given \(L\) families of probability distributions \(\mathcal{P}_\ell, 1 \leq \ell \leq L\), on observation space \(\Omega\), and observe a realization of random variable \(\omega \sim P\) taking values in \(\Omega\). Given \(\omega\), we want to decide on the \(L\) hypotheses

\[H_\ell : P \in \mathcal{P}_\ell, \ 1 \leq \ell \leq L.\]

Our ideal goal would be to find a low-risk simple test deciding on the hypotheses. However: It may happen that the “ideal goal” is not achievable, for example, when some pairs of families \(\mathcal{P}_\ell\) have nonempty intersections. When \(\mathcal{P}_\ell \cap \mathcal{P}_{\ell'} \neq \emptyset\) for some \(\ell \neq \ell'\), there is no way to decide on the hypotheses with risk \(< 1/2\).

But: Impossibility to decide reliably on all \(L\) hypotheses “individually” does not mean that no meaningful inferences can be done.
Example: Consider the 3 colored rectangles on the plane:

and 3 hypotheses, with $H_\ell$, $1 \leq \ell \leq 3$, stating that our observation is $\omega = x + \xi$ with deterministic “signal” $x$ belonging to $\ell$-th rectangle and $\xi \sim \mathcal{N}(0, \sigma^2 I_2)$.

Whatever small $\sigma$ be, no test can decide on the 3 hypotheses with risk $< 1/2$; e.g., there is no way to decide reliably on $H_1$ vs. $H_2$. However, we may hope that when $\sigma$ is small, an observation allows us to discard reliably some of the hypotheses. For example, if $H_1$ is true, we hopefully can discard $H_3$.

When handling multiple hypotheses which cannot be reliably decided upon “as they are,” it makes sense to speak about testing the hypotheses “up to closeness.”

2.23
Closeness relation $C$ on $L$ hypotheses $H_1, \ldots, H_L$ is defined as some set of pairs $(\ell, \ell')$ with $1 \leq \ell, \ell' \leq L$; we interpret the relation $(\ell, \ell') \in C$ as the fact that the hypotheses $H_\ell$ and $H_{\ell'}$ are close to each other.

We always assume that

- $C$ contains all “diagonal pairs” $(\ell, \ell)$, $1 \leq \ell \leq L$ (“every hypothesis is close to itself”)
- $(\ell, \ell') \in C$ if and only if $(\ell', \ell) \in C$ (“closeness is symmetric relation”)

Note: By symmetry of $C$, the relation $(\ell, \ell') \in \mathcal{T}$ is in fact a property of unordered pair $\{\ell, \ell'\}$. 
“Up to closeness” risks. Let $\mathcal{T}$ be a test deciding on $H_1, \ldots, H_L$; given observation $\omega$, $\mathcal{T}$ accepts all hypotheses $H_\ell$ with indexes $\ell \in \mathcal{T}(\omega)$ and rejects all other hypotheses. We say that $\ell$-th partial $C$-risk of test $\mathcal{T}$ is $\leq \varepsilon$, if whenever $H_\ell$ is true: $\omega \sim P \in \mathcal{P}_\ell$, the $P$-probability of the event

\[
\begin{align*}
&\mathcal{T} \text{ accepts } H_\ell: \ell \in \mathcal{T}(\omega) \\
&\text{and} \\
&\text{all hypotheses } H_\ell \text{ accepted by } \mathcal{T} \text{ are } C\text{-close to } H_\ell: (\ell, \ell') \in C \forall \ell' \in \mathcal{T}(\omega)
\end{align*}
\]

is at least $1 - \varepsilon$.

$\ell$-th partial $C$-risk of $\mathcal{T}$ is the smallest $\varepsilon$ with the outlined property:

\[
\begin{align*}
\text{Risk}_C^\ell (\mathcal{T}|H_1, \ldots, H_L) &= \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{\omega \sim P} \{[\ell \notin \mathcal{T}(\omega)] \text{ or } [\exists \ell' \in \mathcal{T}(\omega): (\ell, \ell') \notin C]\}
\end{align*}
\]

$C$-risk of $\mathcal{T}$ is the largest of the partial $C$-risks of the test:

\[
\text{Risk}_C^C (\mathcal{T}|H_1, \ldots, H_L) = \max_{1 \leq \ell \leq L} \text{Risk}_C^\ell (\mathcal{T}|H_1, \ldots, H_L).
\]
\[ \omega \sim P, \ H_\ell : P \in \mathcal{P}_\ell, \ 1 \leq \ell \leq L \]
\[ C : \text{closeness relation} \]

**♣ Multiple Hypothesis Testing via Pairwise Tests.** Assume that for every unordered pair \( \{\ell, \ell'\} \) with \((\ell, \ell') \notin C\) we are given a *simple* test \( T_{\{\ell, \ell'\}} \) deciding on \( H_\ell \) vs. \( H_{\ell'} \) via observation \( \omega \).

Our goal is to “assemble” the tests \( T_{\{\ell, \ell'\}}, \ (\ell, \ell') \notin C \), into a test \( T \) deciding on \( H_1, \ldots, H_L \) up to closeness \( C \).
The construction:

- For \((\ell, \ell') \not\in C\), so that \(\ell \neq \ell'\), we define function \(T_{\ell \ell'}(\omega)\) as follows:

\[
T_{\ell \ell'}(\omega) = \begin{cases} 
1, & T_{\{\ell, \ell'\}}(\omega) \equiv \{\ell\} \\
-1, & T_{\{\ell, \ell'\}}(\omega) \equiv \{\ell'\}
\end{cases}
\]

Note: \(T_{\{\ell, \ell'\}}\) is a simple test \(\Rightarrow T_{\ell \ell'}(\cdot)\) is well defined and takes values \(\pm 1\).

- For \((\ell, \ell') \in C\), we set \(T_{\ell \ell'}(\cdot) \equiv 0\).

Note: By construction, we have \(T_{\ell \ell'}(\omega) \equiv -T_{\ell' \ell}(\omega), 1 \leq \ell, \ell' \leq L\).

- The test \(T\) is as follows: \textit{given observation} \(\omega\), we build the \(L \times L\) matrix \(T(\omega) = [T_{\ell \ell'}(\omega)]\) and accept exactly those of the hypotheses \(H_\ell\) for which \(\ell\)-th row in \(T(\omega)\) is nonnegative, that is, all tests \(T_{\{\ell, \ell'\}}\) with \((\ell, \ell') \not\in C\) accept \(H_\ell\), observation being \(\omega\).
Example: \( L = 4 \)

- \( C = \{(1, 1), (2, 2), (3, 3), (4, 4), \{1, 2\}, \{2, 3\}, \{3, 4\}\} \)

Given tests \( T_{\{1,3\}}, T_{\{1,4\}}, T_{\{2,4\}} \) and observation \( \omega \)

♠ When \( T_{\{1,3\}} \) accepts \( H_1 \), \( T_{\{1,4\}} \) accepts \( H_1 \), \( T_{\{2,4\}} \) accepts \( H_4 \), we get

\[
T(\omega) = \begin{bmatrix}
0 & 0 & +1 & +1 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
-1 & +1 & 0 & 0 \\
\end{bmatrix}
\]

\( \Rightarrow \) Aggregated test \( T \) accepts \( H_1 \)
When $\mathcal{T}_{\{1,3\}}$ accepts $H_1$, $\mathcal{T}_{\{1,4\}}$ accepts $H_1$, $\mathcal{T}_{\{2,4\}}$ accepts $H_2$, we get

$$T(\omega) = \begin{bmatrix}
0 & 0 & +1 & +1 \\
0 & 0 & 0 & +1 \\
-1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0
\end{bmatrix}$$

$\Rightarrow$ Aggregated test $\mathcal{T}$ accepts $H_1$ and $H_2$
♠ **Observation:** When $T$ accepts some hypothesis $H_\ell$, all hypotheses accepted by $T$ are $C$-close to $H_\ell$.

Indeed, if $\ell$-th row in $T(\omega)$ is nonnegative and $\ell'$ is not $C$-close to $\ell$, we have $T_{\ell \ell'}(\omega) \geq 0$ and $T_{\ell \ell'}(\omega) \in \{-1, 1\}$

$\Rightarrow T_{\ell \ell'}(\omega) = 1$

$\Rightarrow T_{\ell' \ell}(\omega) = -T_{\ell \ell'}(\omega) = -1$

$\Rightarrow \ell'$-th row in $T(\omega)$ is not nonnegative

$\Rightarrow \ell'$ is not accepted.
**Risk analysis.** For \((\ell, \ell') \notin \mathcal{C}\), let
\[
\epsilon_{\ell \ell'} = \text{Risk}_1\left(\mathcal{T}_{\ell,\ell'} | H_{\ell}, H_{\ell'}\right) = \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{\omega \sim P}\{\ell \notin \mathcal{T}_{\ell,\ell'}(\omega)\}
\]
\[
= \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{\omega \sim P}\{T_{\ell \ell'}(\omega) = -1\} = \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{\omega \sim P}\{T_{\ell \ell'}(\omega) = 1\}
\]
\[
= \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{\omega \sim P}\{\ell' \in \mathcal{T}_{\ell,\ell'}(\omega)\}
\]
\[
= \text{Risk}_2\left(\mathcal{T}_{\ell,\ell'} | H_{\ell'}, H_{\ell}\right).
\]

**Proposition.** One has
\[
\text{Risk}_\mathcal{C}\ell\left(\mathcal{T} | H_1, ..., H_L\right) \leq \epsilon_\ell := \sum_{\ell' : (\ell, \ell') \notin \mathcal{C}} \epsilon_{\ell \ell'}.
\]

Indeed, let us fix \(\ell\), and let \(H_{\ell}\) be true. Let \(P \in \mathcal{P}_\ell\) be the distribution of observation \(\omega\), and let \(I = \{\ell' \leq L : (\ell, \ell') \notin \mathcal{C}\}\). For \(\ell' \in I\), let \(E_{\ell'}\) be the event \(\{\omega : T_{\ell \ell'}(\omega) = -1\}\). We have
\[
\text{Prob}_{\omega \sim P}(E_{\ell'}) \leq \epsilon_{\ell \ell'} \quad \text{(by definition of } \epsilon_{\ell \ell'}) \Rightarrow \text{Prob}_{\omega \sim P}\left(\bigcup_{\ell' \in I} E_{\ell'}\right) \leq \epsilon_\ell.
\]

When the event \(E\) does not take place, we have \(T_{\ell \ell'}(\omega) = 1\) for all \(\ell' \in I\)
\[
\Rightarrow T_{\ell \ell'}(\omega) \geq 0 \quad \text{for all } \ell', 1 \leq \ell' \leq L
\]
\[
\Rightarrow \ell \in \mathcal{T}(\omega)
\]
\[
\Rightarrow \text{(by Observation)} \{\ell \in \mathcal{T}(\omega)\} \& \{(\ell, \ell') \in \mathcal{C} \forall \ell' \in \mathcal{T}(\omega)\}.
\]
By definition of partial \(\mathcal{C}\)-risk, we get
\[
\text{Risk}_\mathcal{C}\ell\left(\mathcal{T} | H_1, ..., H_L\right) \leq \text{Prob}_{\omega \sim P}(E) \leq \epsilon_\ell.
\]
\[
\square
\]
Testing Multiple Hypotheses via Euclidean Separation

\textbf{Situation:} We are given $L$ nonempty, closed and bounded convex sets $X_\ell \subset \mathbb{R}^d$, $1 \leq \ell \leq L$, and a family $\mathcal{P}^d_\gamma$ of noise distributions, a closeness $C$, and semi-stationary $K$-repeated observation

$$\omega^K = \{\omega_k = x_k + \xi_k, 1 \leq k \leq K\},$$

so that

- $\{x_k, 1 \leq k \leq K\}$, is a deterministic sequence of signals,
- $\xi_k \sim P_k$, $1 \leq k \leq K$, are independent across $k$ noises, and
- $\{P_k, 1 \leq k \leq K\}$, is a deterministic sequence of distributions from $\mathcal{P}^d_\gamma$.

Given $\omega^K$, we want to decide up to closeness $C$ on $L$ hypotheses

$$H_\ell : \{x_k \in X_\ell, 1 \leq k \leq K\}.$$
Given $\omega^K$, we want to decide up to closeness $C$ on $L$ hypotheses

$$H_\ell : \{x_k \in X_\ell, 1 \leq k \leq K\}.$$

**Equivalently:** The sets $X_\ell \subset \mathbb{R}^d$ along with $\mathcal{P}_\gamma^d$ specify $L$ families of distributions $\mathcal{P}_\ell$, $1 \leq \ell \leq L$; specifically, $\mathcal{P}_\ell$ is comprised of probability distributions of random variables $x + \xi$, where deterministic $x$ belongs to $X_\ell$, and the distribution of random noise $\xi$ belongs to $\mathcal{P}_\gamma^d$. Given $\omega^K$, we want to decide, up to closeness $C$, on $L$ hypotheses

$$H_\ell : P^K \in \mathcal{P}_\ell^{\oplus,K}, 1 \leq \ell \leq L$$

on the distribution $P^K$ of observation $\omega^K$.

♠ **Standing Assumption:** Whenever $\ell, \ell'$ are not $C$-close: $(\ell, \ell') \notin C$, the sets $X_\ell, X_{\ell'}$ do not intersect.

♠ **Strategy:** We intend to assemble pairwise Euclidean separation tests.
Building blocks. For \((\ell, \ell') \not\in C\), we solve convex optimization problems
\[
\text{Opt}_{\ell\ell'} = \min_{u \in X_\ell, v \in X_{\ell'}} \frac{1}{2} \|u - v\|_2.
\]

Note: By Standing Assumption, \(\text{Opt}_{\ell\ell'} > 0\). Optimal solution \((u_*, v_*)\) to \((P_{\ell\ell'})\) defines affine functions
\[
\phi_{\ell\ell'}(\omega) = e^{T}_{\ell\ell'} \omega - c_{\ell\ell'},
\]
\[
e_{\ell\ell'} = \frac{u_* - v_*}{\|u_* - v_*\|_2}, \quad c_{\ell\ell'} = \frac{1}{2} e^{T}_{\ell\ell'}[u_* + v_*].
\]

Note: We have \(\phi_{\ell\ell'}(\cdot) \equiv -\phi_{\ell'\ell}(\cdot)\) for all \((\ell, \ell') \not\in C\).

As we know, whenever \(\delta_{\ell\ell'} \geq 0, \delta_{\ell'\ell} \geq 0\) satisfy
\[
2\text{Opt}_{\ell\ell'} = \delta_{\ell\ell'} + \delta_{\ell'\ell}
\]
it holds
\[
\forall (u \in X_\ell, P \in P^d_\gamma) : \quad \text{Prob}_{\xi \sim P} \{\phi(u + \xi) < \frac{1}{2}[\delta_{\ell'\ell} - \delta_{\ell\ell'}]\} \leq \Gamma(\delta_{\ell\ell'}) := \int^{\delta_{\ell\ell'}}_{\delta_{\ell\ell'}} \gamma(s) ds.
\]
\[
\forall (v \in X_{\ell'}, P \in P^d_\gamma) : \quad \text{Prob}_{\xi \sim P} \{\phi(u + \xi) \geq \frac{1}{2}[\delta_{\ell'\ell} - \delta_{\ell\ell'}]\} \leq \Gamma(\delta_{\ell\ell'}) := \int^{\delta_{\ell\ell'}}_{\delta_{\ell\ell}} \gamma(s) ds.
\]
\[
\ell, \ell' : (\ell, \ell') \not\in C
\]
\[
\Rightarrow \quad \text{Opt}_{\ell\ell'} = \min_{u \in X_\ell, v \in X_{\ell'}} \frac{1}{2} \|u - v\|_2 > 0 = \text{Opt}_{\ell\ell}
\]
\[
\Rightarrow u_*, v_*, \phi_{\ell\ell'}(\omega) = e^T_{\ell\ell'} \omega - c_{\ell\ell'} \equiv -\phi_{\ell\ell}(\omega) \quad \left[ e_{\ell\ell'} = \frac{u_* - v_*}{\|u_* - v_*\|_2}, \quad c_{\ell\ell'} = \frac{1}{2} e^T_{\ell\ell'} [u_* + v_*] \right]
\]
\[
\delta_{\ell'\ell} \geq 0, \delta_{\ell\ell'} \geq 0, 2 \text{Opt}_{\ell\ell'} = \delta_{\ell'\ell} + \delta_{\ell\ell'} \quad (\ast)
\]
\[
\forall (u \in X_\ell, P \in \mathcal{P}_\gamma^d) : \quad \text{Prob}_{\xi \sim P} \left\{ \phi(u + \xi) < \frac{1}{2} [\delta_{\ell'\ell} - \delta_{\ell\ell'}] \right\} \leq \Gamma(\delta_{\ell\ell'}) := \int_{\delta_{\ell'\ell}}^{\infty} \gamma(s) ds
\]
\[
\forall (v \in X_{\ell'}, P \in \mathcal{P}_\gamma^d) : \quad \text{Prob}_{\xi \sim P} \left\{ \phi(v + \xi) \geq \frac{1}{2} [\delta_{\ell'\ell} - \delta_{\ell\ell'}] \right\} \leq \Gamma(\delta_{\ell\ell'}) := \int_{\delta_{\ell\ell'}}^{\infty} \gamma(s) ds \quad (!)
\]

\[\text{Assembling building blocks, case of } K = 1.\]

- For \( \ell, \ell' \) with \( (\ell, \ell') \not\in C \) we select \( \delta_{\ell\ell'} \) satisfying \((\ast)\), thus arriving at pairwise simple tests

\[
\mathcal{T}_{\{\ell, \ell'\}}(\omega) = \left\{ \begin{array}{ll}
\{\ell\}, & \phi_{\ell\ell'}(\omega) \geq \frac{1}{2} [\delta_{\ell'\ell} - \delta_{\ell\ell'}] \\
\{\ell'\}, & \phi_{\ell\ell'}(\omega) < \frac{1}{2} [\delta_{\ell'\ell} - \delta_{\ell\ell'}]
\end{array} \right.
\]

- Further, we use out general construction to assemble pairwise tests \( \{\mathcal{T}_{\{\ell, \ell'\}} : (\ell, \ell') \not\in C\} \) into single-observation test \( \mathcal{T} \) deciding on \( H_1, ..., H_L \)

**Note:** By (!), the associated with tests \( \mathcal{T}_{\{\ell, \ell'\}} \) quantities \( \epsilon_{\ell\ell'} \) satisfy the relations

\[
\epsilon_{\ell\ell'} \leq \Gamma(\delta_{\ell\ell'}) := \int_{\delta_{\ell\ell'}}^{\infty} \gamma(s) ds, \quad \text{whence } \text{Risk}^C_{\ell}(\mathcal{T} | H_1, ..., H_L) \leq \sum_{\ell' : (\ell, \ell') \not\in C} \Gamma(\delta_{\ell\ell'}). \]
\[ \ell, \ell' : (\ell, \ell') \notin C \Rightarrow \text{Opt}_{\ell \ell'} = \min_{u \in X_{\ell}, v \in X_{\ell'}} \frac{1}{2} \|u - v\|_2 \]

\[ \Rightarrow \quad \delta_{\ell \ell'} \geq 0, \delta_{\ell' \ell} \geq 0, 2\text{Opt}_{\ell \ell'} = \delta_{\ell \ell'} + \delta_{\ell' \ell} \]

\[ \Rightarrow \quad T : \text{Risk}_C^C (T|H_1, \ldots, H_L) \leq \sum_{\ell' : (\ell, \ell') \notin C} \Gamma (\delta_{\ell \ell'}) \quad [\Gamma (\delta) = \int_{\delta}^{\infty} \gamma (s) ds] \]

♠ Single-observation case \( K = 1 \): optimizing the construction over the “free parameters” \( \delta_{\ell \ell'}, (\ell, \ell') \notin C \), of the construction.

♥ A natural model here is as follows: given nonnegative weight matrix \( W \) and non-negative vectors \( \alpha, \beta \), we want to minimize “scale factor” \( t \) under the constraint

\[ W \cdot [\text{Risk}_C^C (T|H_1, \ldots, H_L)]_{\ell=1}^L \leq \alpha + t\beta \]

This problem can be safely approximated by the optimization problem

\[ \min_{\{\delta, t\}} \left\{ t : W \cdot \left[ \sum_{\ell' : (\ell, \ell') \notin C} \Gamma (\delta_{\ell \ell'}) \right]_{\ell=1}^L \leq \alpha + t\beta \quad \delta_{\ell \ell} \geq 0, \delta_{\ell \ell'} + \delta_{\ell' \ell} = 2\text{Opt}_{\ell \ell'}, (\ell, \ell') \notin C \right\} \quad (\#) \]

**Note:** Assuming \( \gamma (\cdot) \) nonincreasing on \( \mathbb{R}_+ \) (as is the case, e.g., for Gaussian mixtures), function \( \Gamma (\delta) = \int_{\delta}^{\infty} \gamma (s) ds \) is convex on \( \mathbb{R}_+ \)

\[ \Rightarrow (\#) \text{ is an explicit Convex Programming problem!} \]
\(\ell, \ell' : (\ell, \ell') \notin C\)

\[\Rightarrow \quad \text{Opt}_{\ell\ell'} = \min_{u \in X_\ell, v \in X_{\ell'}} \frac{1}{2} \|u - v\|_2 > 0 = \text{Opt}_{\ell\ell}\]

\[\Rightarrow \quad u_*, v_*, \phi_{\ell\ell'}(\omega) = e_{\ell\ell}^T \omega - c_{\ell\ell} \equiv -\phi_{\ell\ell'}(\omega)\]

\[e_{\ell\ell} = \frac{u_* - v_*}{\|u_* - v_*\|_2}, \quad c_{\ell\ell} = \frac{1}{2} e_{\ell\ell}^T [u_* + v_*]\]

\[\forall (u \in X_\ell, P \in \mathcal{P}_\gamma) : \text{Prob}_{\xi \sim P} \{\phi(u + \xi) < 0\} \leq \Gamma(\text{Opt}_{\ell\ell'})\]

\[\forall (v \in X_{\ell'}, P \in \mathcal{P}_\gamma) : \text{Prob}_{\xi \sim P} \{\phi(v + \xi) \geq 0\} \leq \Gamma(\text{Opt}_{\ell\ell'})\]

\[\Gamma(\delta) := \int_{\delta}^{\infty} \gamma(s) ds\] (1)

\[\begin{aligned}
\top_{\{\ell, \ell'\}}(\omega^K) &= \begin{cases} 
\{\ell\}, & \text{Card}\{k \leq K : \phi_{\ell\ell'}(\omega_k) \geq 0\} \geq K/2 \\
\{\ell'\}, & \text{otherwise}
\end{cases}
\end{aligned}\]

\[\begin{aligned}
\top_{\{\ell, \ell'\}} : (\ell, \ell') \notin C
\end{aligned}\]

\[\text{into test } \top_K \text{ deciding on } H_{1}^{K}, \ldots, H_{L}^{K} \text{ via observation } \omega^K\]

\[\textbf{Case of } K\text{-repeated observations, } K > 1. \text{ In the case of semi-stationary } K\text{-repeated observations } \omega^K = (\omega_1, \ldots, \omega_K), \text{ we act as follows:}\]

\[\begin{itemize}
\item For \((\ell, \ell') \notin C\), we build majority tests

\[\top_{\{\ell, \ell'\}}(\omega^K) = \begin{cases} 
\{\ell\}, & \text{Card}\{k \leq K : \phi_{\ell\ell'}(\omega_k) \geq 0\} \geq K/2 \\
\{\ell'\}, & \text{otherwise}
\end{cases}\]

\item Further, we use out general construction to assemble simple tests

\[\{\top_{\{\ell, \ell'\}} : (\ell, \ell') \notin C\}
\]

\[\text{into test } \top_K \text{ deciding on } H_{1}^{K}, \ldots, H_{L}^{K} \text{ via observation } \omega^K\]

2.37
Note: By our results on majority tests, the associated with tests $T_{\{\ell, \ell'\}}$ quantities $\epsilon_{\ell\ell'}$ satisfy the relations

$$\epsilon_{\ell\ell'} \leq \sum_{K/2 \leq k \leq K} \binom{K}{k} \left[\Gamma(Opt_{\ell\ell'})\right]^k [1 - \Gamma(Opt_{\ell\ell'})]^{K-k}$$

whence

$$\text{Risk}^C_\ell(T_K|H_1, \ldots, H_L) \leq \sum_{\ell': (\ell, \ell') \notin C} \sum_{K/2 \leq k \leq K} \binom{K}{k} \left[\Gamma(Opt_{\ell\ell'})\right]^k [1 - \Gamma(Opt_{\ell\ell'})]^{K-k}.$$ 

Note: By Standing Assumption, $Opt_{\ell\ell'} > 0$ when $(\ell, \ell') \notin C \Rightarrow \Gamma(Opt_{\ell\ell'}) < 1/2 \Rightarrow \text{Risks} \text{ Risk}^C_\ell(T_K|H_1, \ldots, H_L)$ go to 0 exponentially fast as $K \to \infty.$
HYPOTHESIS TESTING, II

- Detector-Based Tests
  - Detectors & Detector-Based Pairwise Tests
  - Testing “up to Closeness”
  - Simple Observation Schemes
    - Minimum Risk Detectors
    - Near-Optimal Tests
    - Sequential Hypothesis Testing
    - Measurement Design
- Recovering linear forms in Simple o.s.
Detectors & Detector-Based Pairwise Tests

♦ Situation: Given two families $\mathcal{P}_1$, $\mathcal{P}_2$ of probability distributions on a given observation space $\Omega$ and an observation $\omega \sim P$ with $P$ known to belong to $\mathcal{P}_1 \cup \mathcal{P}_2$, we want to decide whether $P \in \mathcal{P}_1$ (hypothesis $H_1$) or $P \in \mathcal{P}_2$ (hypothesis $H_2$).

♦ Detectors. A detector is a function $\phi : \Omega \to \mathbb{R}$. Risks of a detector $\phi$ w.r.t. $\mathcal{P}_1$, $\mathcal{P}_2$ are defined as

$$\text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2] = \sup_{P \in \mathcal{P}_1} \int \Omega e^{-\phi(\omega)} P(d\omega), \quad \text{Risk}_2[\phi|\mathcal{P}_1, \mathcal{P}_2] = \sup_{P \in \mathcal{P}_2} \int \Omega e^{\phi(\omega)} P(d\omega)$$

$$\text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2] = \text{Risk}_2[-\phi|\mathcal{P}_2, \mathcal{P}_1]$$

♦ Simple test $T_\phi$ associated with detector $\phi$, given observation $\omega$,

• accepts $H_1$ when $\phi(\omega) \geq 0$,
• accepts $H_2$ when $\phi(\omega) < 0$.

♦ Immediate observation:

$$\text{Risk}_1(T_\phi|H_1, H_2) \leq \text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2]$$
$$\text{Risk}_2(T_\phi|H_1, H_2) \leq \text{Risk}_2[\phi|\mathcal{P}_1, \mathcal{P}_2]$$

Reason: $\text{Prob}_{\omega \sim P} \{\omega : \psi(\omega) \geq 0\} \leq \int \mathbb{E}^{\psi(\omega)} P(d\omega)$. 

3.1
Elementary Calculus of Detectors

\[
\begin{align*}
\text{Risk}_1[\phi | \mathcal{P}_1, \mathcal{P}_2] &= \sup_{P \in \mathcal{P}_1} \int \Omega e^{-\phi(\omega)} P(d\omega), \\
\text{Risk}_2[\phi | \mathcal{P}_1, \mathcal{P}_2] &= \sup_{P \in \mathcal{P}_2} \int \Omega e^{\phi(\omega)} P(d\omega)
\end{align*}
\]

♦ Detectors admit simple “calculus:”

♦ **Renormalization:** \( \phi(\cdot) \Rightarrow \phi_a(\cdot) = \phi(\cdot) - a \)

\[
\Rightarrow \begin{cases} 
\text{Risk}_1[\phi_a | \mathcal{P}_1, \mathcal{P}_2] = e^a \text{Risk}_1[\phi | \mathcal{P}_1, \mathcal{P}_2] \\
\text{Risk}_2[\phi_a | \mathcal{P}_1, \mathcal{P}_2] = e^{-a} \text{Risk}_2[\phi | \mathcal{P}_1, \mathcal{P}_2]
\end{cases}
\]

⇒ **What matters, is the product**

\[
[\text{Risk}[\phi | \mathcal{P}_1, \mathcal{P}_2]]^2 := \text{Risk}_1[\phi | \mathcal{P}_1, \mathcal{P}_2] \text{Risk}_2[\phi | \mathcal{P}_1, \mathcal{P}_2]
\]

of partial risks of a detector. **Shifting the detector by constant, we can distribute this product between factors as we want, e.g., always can make the detector balanced:**

\[
\text{Risk}[\phi | \mathcal{P}_1, \mathcal{P}_2] = \text{Risk}_1[\phi | \mathcal{P}_1, \mathcal{P}_2] = \text{Risk}_2[\phi | \mathcal{P}_1, \mathcal{P}_2].
\]
Detectors are well-suited to passing to multiple observations. For $1 \leq k \leq K$, let

- $\mathcal{P}_{1,k}, \mathcal{P}_{2,k}$ be families of probability distributions on observation spaces $\Omega_k$,
- $\phi_k$ be detectors on $\Omega_k$.

Families $\{\mathcal{P}_{1,k}, \mathcal{P}_{2,k}\}_{k=1}^K$ give rise to families of product distributions on $\Omega^K = \Omega_1 \times ... \times \Omega_K$:

$$\mathcal{P}^K_\chi = \{P^K = P_1 \times ... \times P_K : P_k \in \mathcal{P}_{\chi,k}, 1 \leq k \leq K\}, \chi = 1, 2,$$

and detectors $\phi_1, .., \phi_K$ give rise to detector $\phi^K$ on $\Omega^K$:

$$\phi^K(\omega_1, ..., \omega_K) = \sum_{k=1}^K \phi_k(\omega_k).$$

Observation: For $\chi = 1, 2$, we have

$$\text{Risk}_\chi[\phi^K | \mathcal{P}^K_1, \mathcal{P}^K_2] = \prod_{k=1}^K \text{Risk}_\chi[\phi_k | \mathcal{P}_{1,k}, \mathcal{P}_{2,k}].$$

(!!)
In the sequel, we refer to families $\mathcal{P}^K_X$ as to \textit{direct products} of families of distributions $\mathcal{P}_{X,k}$ over $1 \leq k \leq K$:

$$\mathcal{P}^K_X = \mathcal{P}^\oplus_K = \bigoplus_{k=1}^K \mathcal{P}_{X,k} := \{P^K = P_1 \times \ldots \times P_K : P_k \in \mathcal{P}_{X,k}, 1 \leq k \leq K\}.$$ 

We can define also \textit{quasi-direct products}

$$\mathcal{P}^\otimes_K = \bigotimes_{k=1}^K \mathcal{P}_{X,k}$$

of the families $\mathcal{P}_{X,k}$ over $1 \leq k \leq K$. By definition, $\mathcal{P}^\otimes_K$ is comprised of all distributions $P^K$ of random sequences $\omega^K = (\omega, \ldots, \omega_K)$, $\omega_k \in \Omega_k$, which can be generated as follows: in the nature there exists a random sequence $\zeta^K = (\zeta_1, \ldots, \zeta_K)$ of “driving factors” such that for every $k \leq K$, $\omega_k$ is a deterministic function of $\zeta^k = (\zeta_1, \ldots, \zeta_k)$, and the conditional, $\zeta^{k-1}$ being fixed, distribution of $\omega_k$ always belongs to $\mathcal{P}_{X,k}$.

\textit{It is immediately seen that for $X = 1, 2$ it holds}

$$\text{Risk}_X[\phi^K|\mathcal{P}^\oplus_K, \mathcal{P}^\oplus_2] = \prod_{k=1}^K \text{Risk}_X[\phi_k|\mathcal{P}_{1,k}, \mathcal{P}_{2,k}].$$
From pairwise detectors to detectors for unions. Assume that we are given an observation space \( \Omega \) along with

- \( R \) families \( \mathcal{R}_r, r = 1, \ldots, R \) of “red” probability distributions on \( \Omega \),
- \( B \) families \( \mathcal{B}_b, b = 1, \ldots, B \) of “brown” probability distributions on \( \Omega \),
- detectors \( \phi_{rb}(\cdot), 1 \leq r \leq R, 1 \leq b \leq B \).

Let us aggregate the red and the brown families as follows

\[
\mathcal{R} = \bigcup_{r=1}^{R} \mathcal{R}_r, \quad \mathcal{B} = \bigcup_{b=1}^{B} \mathcal{B}_b
\]

and assemble detectors \( \phi_{rb} \) into a single detector

\[
\phi(\omega) = \max_{r \leq R} \min_{b \leq B} \phi_{rb}(\omega).
\]

Observation: We have

\[
\begin{align*}
\text{Risk}_1[\phi|\mathcal{R}, \mathcal{B}] & \leq \max_{r \leq R} \sum_{b \leq B} \text{Risk}_1[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b], \\
\text{Risk}_2[\phi|\mathcal{R}, \mathcal{B}] & \leq \max_{b \leq B} \sum_{r \leq R} \text{Risk}_2[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b].
\end{align*}
\]
Observation: We have

\[
\begin{align*}
\text{Risk}_1[\phi|\mathcal{R}, \mathcal{B}] & \leq \max_{r \leq R} \sum_{b \leq B} \text{Risk}_1[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b], \\
\text{Risk}_2[\phi|\mathcal{R}, \mathcal{B}] & \leq \max_{b \leq B} \sum_{r \leq R} \text{Risk}_2[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b].
\end{align*}
\]

Indeed,

\[
\begin{align*}
P \in \mathcal{R}_r \Rightarrow \quad & \int e^{-\left[\max_r \min_b \phi_{rb}(\omega)\right]} P(d\omega) = \int e^{\min_r \max_b [-\phi_{rb}(\omega)]} P(d\omega) \\
& \leq \int e^{\max_b [-\phi_{rb}(\omega)]} P(d\omega) \leq \sum_b \int e^{-\phi_{rb}(\omega)} P(d\omega) \leq \sum_b \text{Risk}_1[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b] \\
\Rightarrow \quad & \text{Risk}_1[\phi|\mathcal{R}, \mathcal{B}] \leq \max_{r \leq R} \sum_{b \leq B} \text{Risk}_1[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b]; \\
P \in \mathcal{B}_b \Rightarrow \quad & \int e^{\max_r \min_b \phi_{rb}(\omega)} P(d\omega) \leq \int e^{\max_r \phi_{rb}*_{b}(\omega)} P(d\omega) \\
& \leq \sum_r \int e^{\phi_{rb}*_{b}(\omega)} P(d\omega) \leq \sum_r \text{Risk}_2[\phi_{rb}*_{b}|\mathcal{R}_r, \mathcal{B}_b] \\
\Rightarrow \quad & \text{Risk}_2[\phi|\mathcal{R}, \mathcal{B}] \leq \max_{b \leq B} \sum_{r \leq R} \text{Risk}_2[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b].
\end{align*}
\]
\textbf{Refinement:} W.l.o.g. we can assume that the detectors $\phi_{rb}$ are balanced:

$$\epsilon_{rb} \coloneqq \text{Risk}[\phi_{rb}|R_r, B_b] = \text{Risk}_1[\phi_{rb}|R_r, B_b] = \text{Risk}_2[\phi_{rb}|R_r, B_b].$$

Consider matrices

$$E = \begin{bmatrix}
\epsilon_{1,1} & \cdots & \epsilon_{1,B} \\
\vdots & \ddots & \vdots \\
\epsilon_{R,1} & \cdots & \epsilon_{R,B}
\end{bmatrix}, \quad F = \begin{bmatrix} E \\ E^T \end{bmatrix}$$

\text{❤️} The maximal eigenvalue $\theta$ of $F$ is the spectral norm $\|E\|_{2,2}$ of $E$, and the leading eigenvector $[g; f]$ can be selected to be positive (Perron-Frobenius Theorem).

\textbf{Note:} $\theta g = Ef$ & $\theta f = E^T g$

\text{❤️} Let us pass from the detectors $\phi_{rb}$ to shifted detectors $\psi_{rb} = \phi_{rb} - \ln(f_b/g_r)$ and assemble the shifted detectors into the detector

$$\psi(\omega) = \max_{r \leq R} \min_{b \leq B} \psi_{rb}(\omega)$$

By previous observation

$$\text{Risk}_1(\psi|\mathcal{R}, \mathcal{B}) \leq \max_r \sum_b \text{Risk}_1[\psi_{rb}|\mathcal{R}_r, \mathcal{B}_b] = \max_r \sum_b \epsilon_{rb}(f_b/g_r)$$
$$= \max_r [(Ef)_r/g_r] = \|E\|_{2,2}$$

$$\text{Risk}_2(\psi|\mathcal{R}, \mathcal{B}) \leq \max_b \sum_r \text{Risk}_2[\psi_{rb}|\mathcal{R}_r, \mathcal{B}_b] = \max_b \sum_r \epsilon_{rb}(g_r/f_b)$$
$$= \max_b [(E^T g)_b/f_b] = \|E\|_{2,2}$$

\Rightarrow \text{Partial risks of detector $\psi$ on aggregated families $\mathcal{R}, \mathcal{B}$ are $\leq \|E\|_{2,2}$.}

3.7
Detector-Based Tests "Up to Closeness"

♠ Situation: We are given
- $L$ families of probability distributions $\mathcal{P}_\ell$, $\ell = 1, \ldots, L$, on observation space $\Omega$, giving rise to $L$ hypotheses $H_\ell$, on the distribution $P$ of random observation $\omega \in \Omega$:
  \[ H_\ell : P \in \mathcal{P}_\ell, \ 1 \leq \ell \leq L; \]
- closeness relation $\mathcal{C}$;
- system of balanced detectors
  \[ \left\{ \phi_{\ell \ell'} : \ell < \ell', (\ell, \ell') \notin \mathcal{C} \right\} \]
along with upper bounds $\epsilon_{\ell \ell'}$ on detectors’ risks:
\[ \forall (\ell, \ell' : \ell < \ell', (\ell, \ell') \notin \mathcal{C}) : \left\{ \begin{array}{l} \int_{\Omega} e^{-\phi_{\ell'}(\omega)} P(d\omega) \leq \epsilon_{\ell \ell'} \forall P \in \mathcal{P}_\ell \\ \int_{\Omega} e^{\phi_{\ell'}(\omega)} P(d\omega) \leq \epsilon_{\ell \ell'} \forall P \in \mathcal{P}_{\ell'} \end{array} \right\} \]
- Our goal is to build single-observation test deciding on hypotheses $H_1, \ldots, H_L$ up to closeness $\mathcal{C}$. 

3.8
Construction: Let us set
\[ \phi_{\ell\ell}(\omega) = \begin{cases} -\phi_{\ell'\ell}(\omega), & \ell > \ell', (\ell, \ell') \not\in C \\ 0, & (\ell, \ell') \not\in C \end{cases}, \quad \epsilon_{\ell\ell} = \begin{cases} \epsilon_{\ell\ell}, & \ell > \ell', (\ell, \ell') \not\in C \\ 1, & (\ell, \ell') \not\in C \end{cases}, \]
thus ensuring that
\[ \phi_{\ell\ell'}(\cdot) \equiv -\phi_{\ell'\ell}(\cdot), \quad \epsilon_{\ell\ell'} = \epsilon_{\ell'\ell}, \quad 1 \leq \ell, \ell' \leq L \]
\[ \int_{\Omega} e^{-\phi_{\ell\ell'}(\omega)} P(d\omega) \leq \epsilon_{\ell\ell'} \forall (P \in \mathcal{P}_\ell, 1 \leq \ell, \ell' \leq L) \]

Given shifts \( a_{\ell\ell'} = -a_{\ell'\ell} \), we specify test \( \mathcal{T} \) as follows: Given observation \( \omega \), \( \mathcal{T} \) accepts all hypotheses \( H_{\ell} \) such that
\[ \phi_{\ell\ell'}(\omega) > a_{\ell\ell'} \forall (\ell' : (\ell, \ell') \not\in C) \]
and rejects all other hypotheses.

Proposition. The \( C \)-risk of \( \mathcal{T} \) can be upper-bounded as
\[ \text{Risk}^C(\mathcal{T}|H_1, \ldots, H_L) \leq \max_{\ell \leq L} \sum_{\ell' : (\ell, \ell') \not\in C} \epsilon_{\ell\ell'} e^{a_{\ell\ell'}} \]
Optimal shifts: Consider the symmetric nonnegative matrix

\[ E = \begin{bmatrix} \epsilon_{\ell\ell'} \chi_{\ell\ell} \end{bmatrix}_{\ell,\ell'}^{L}, \quad \chi_{\ell\ell'} = \begin{cases} 1, & (\ell, \ell') \notin C \\ 0, & (\ell, \ell') \in C \end{cases}, \]

and let \( \theta = \|E\|_{2,2} \) be the spectral norm of \( E \), or, which is the same under the circumstances, the largest eigenvalue of \( E \). By Perron-Frobenius Theorem, for every \( \theta' > \theta \) there exists a positive vector \( f \) such that

\[ Ef \leq \theta' f; \]

the same holds true when \( \theta = \theta' \), provided the leading eigenvector of \( E \) (which always can selected to be nonnegative) is positive.

Fact: With \( \alpha_{\ell\ell'} = \ln(f_{\ell'}/f_\ell) \), the risk bound from Proposition reads

\[ \text{Risk}^C(\mathcal{T}|H_1, ..., H_L) \leq \theta'. \]

Thus, assembling the detectors \( \phi_{\ell\ell'} \) appropriately, one can get a test with \( C \)-risk arbitrarily close to \( \|E\|_{2,2} \).
Utilizing repeated observations. Assuming $K$-repeated observations allowed, we can apply the above construction to

- $K$-repeated observation $\omega^K = (\omega_1, \ldots, \omega_K)$ in the role of $\omega$,
- quasi-direct powers $P_{\ell}^{\otimes, K} = \bigotimes_{k=1}^{K} P_{\ell}$ in the role of families $P_{\ell}$, and respective hypotheses $H_{\ell}^{\otimes, K}$ in the role of hypotheses $H_{\ell}$, $\ell = 1, \ldots, L$,
- detectors $\phi_{\ell \ell'}^{(K)}(\omega^K) = \sum_{k=1}^{K} \phi_{\ell \ell'}(\omega_k)$ in the role of detectors $\phi_{\ell \ell'}$, which allows to replace $\epsilon_{\ell \ell'}$ with $\epsilon_{K_{\ell \ell'}}$.

As a result, we get $K$-observation test $T^K$ such that

$$\text{Risk}^C(T^K|H_{1}^{\otimes, K}, \ldots, H_{L}^{\otimes, K}) \leq \theta'_K$$

where $\theta'_K$ can be made arbitrarily close (under favorable circumstances, even equal) to the quantity

$$\theta_K = \left\| \left[ \epsilon_{\ell \ell'} \chi_{\ell \ell'} \right]_{\ell \ell'=1}^{K} \right\|_{2,2}, \chi_{\ell \ell'} = \begin{cases} 1, & (\ell, \ell') \notin C \\ 0, & (\ell, \ell') \in C \end{cases}$$

In particular, in the case when $\epsilon_{\ell \ell'} < 1$ whenever $(\ell, \ell') \notin C$, we can ensure that the $C$-risk of $T^K$ converges to 0 exponentially fast as $K \to \infty$. 

3.11
“Universality” of detector-based tests. Let \( P_\chi, \chi = 1, 2, \) be two families of probability distributions on observation space \( \Omega, \) and let \( H_\chi, \chi = 1, 2, \) be associated hypotheses on the distribution of an observation.

Assume that there exists a simple deterministic or randomized test \( T \) deciding on \( H_1, H_2 \) with risk \( \leq \epsilon \in (0, 1/2). \) Then there exists a detector \( \phi \) with

\[
\text{Risk}[\phi|P_1, P_2] \leq \epsilon_+ := 2\sqrt{\epsilon[1-\epsilon]} < 1.
\]

Indeed, let \( T \) be deterministic, let \( \Omega_\chi = \{\omega \in \Omega : T(\omega) = \{\chi}\}, \chi = 1, 2, \) and let

\[
\phi(\omega) = \begin{cases} \frac{1}{2} \ln \left(\frac{[1-\epsilon]/\epsilon}{\epsilon} \right), & \omega \in \Omega_1 \\ \frac{1}{2} \ln \left(\frac{\epsilon}{[1-\epsilon]} \right), & \omega \in \Omega_2 \end{cases}
\]

Then

\[
\begin{align*}
P \in P_1, \epsilon' &= \int_{\Omega_2} P(d\omega) \leq \epsilon \implies \\
\int e^{-\phi(\omega)} P(d\omega) &= \sqrt{\epsilon/[1-\epsilon]}(1-\epsilon') + \sqrt{[1-\epsilon]/\epsilon\epsilon'} \\
&= \sqrt{\epsilon/[1-\epsilon]} + \left[\sqrt{[1-\epsilon]/\epsilon} - \sqrt{\epsilon/[1-\epsilon]}\right] \epsilon' \geq 0 \\
&\leq \sqrt{\epsilon/[1-\epsilon]} + \left[\sqrt{[1-\epsilon]/\epsilon} - \sqrt{\epsilon/[1-\epsilon]}\right] \epsilon = 2\sqrt{\epsilon[1-\epsilon]} \\
P \in P_2, \epsilon' &= \int_{\Omega_1} P(d\omega) \leq \epsilon \implies \\
\int e^{\phi(\omega)} P(d\omega) &= \sqrt{\epsilon/[1-\epsilon]}(1-\epsilon') + \sqrt{[1-\epsilon]/\epsilon\epsilon'} \leq 2\sqrt{\epsilon[1-\epsilon]} \\
\Rightarrow \text{Risk}_\chi[\phi|P_1, P_2] &\leq 2\sqrt{\epsilon[1-\epsilon]}.
\end{align*}
\]
Now let $\mathcal{T}$ be randomized. Setting $\mathcal{P}_\chi^+ = \{ P \times \text{Uniform}[0, 1] : P \in \mathcal{P}_\chi \}$, $\chi = 1, 2$, $\Omega^+ = \Omega \times [0, 1]$, by above there exists a bounded detector $\phi_+ : \Omega^+ \to \mathbb{R}$ such that

$$
\forall (P \in \mathcal{P}_1) : \int_\Omega \left[ \int_0^1 e^{-\phi_+(\omega, s)} ds \right] P(d\omega) ds \leq \epsilon_+ = 2\sqrt{\epsilon[1 - \epsilon]},
$$
$$
\forall (P \in \mathcal{P}_2) : \int_\Omega \left[ \int_0^1 e^{\phi_+(\omega, s)} ds \right] P(d\omega) \leq \epsilon_+,
$$

whence, setting $\phi(\omega) = \int_0^1 \phi(\omega, s) ds$ and applying Jensen’s Inequality,

$$
\forall (P \in \mathcal{P}_1) : \int_\Omega e^{-\phi(\omega)} P(d\omega) \leq \epsilon_+,
$$
$$
\forall (P \in \mathcal{P}_2) : \int_\Omega e^{\phi(\omega)} P(d\omega) \leq \epsilon_+.
$$

Ram 2\sqrt{\epsilon[1 - \epsilon]} of the detector-based test induced by simple test $\mathcal{T}$ is “much worse” than the risk $\epsilon$ of $\mathcal{T}$.

However: When repeated observations are allowed, we can compensate for risk deterioration $\epsilon \mapsto 2\sqrt{\epsilon[1 - \epsilon]}$ by passing in the detector-based test from a single observation to a moderate number of them.
\[
\inf_{\phi} \left\{ \text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2] = \min \left\{ \epsilon : \int_\Omega e^{-\phi(\omega)} P(d\omega) \leq \epsilon \quad \forall (P \in \mathcal{P}_1), \int_\Omega e^{\phi(\omega)} P(d\omega) \leq \epsilon \quad \forall (P \in \mathcal{P}_2) \right\} \right\} \tag{!}
\]

**Note:**
- The optimization problem specifying risk has constraints *convex* in \((\phi, \epsilon)\)
- When passing from families \(\mathcal{P}_\chi, \chi = 1, 2\), to their convex hulls, the risk of a detector remains intact.

♣ **Bottom line:** *It would be nice to be able to solve (!), thus arriving at the lowest risk detector-based tests.*

**But:** (!) is an optimization problem with *infinite-dimensional* decision “vector” and *infinitely many* constraints.

\(\Rightarrow \) (!) *in general is intractable.*

**Simple observation schemes:** A series of special cases where (!) is efficiently solvable via Convex Optimization.
Let $X \subset \mathbb{R}^n$, $\Lambda \subset \mathbb{R}^m$ be nonempty sets, and let $F(x, \lambda)$ be a real-valued function on $X \times \Lambda$. This function gives rise to two optimization problems

$$\text{Opt}(P) = \inf_{x \in X} \sup_{\lambda \in \Lambda} F(x, \lambda) \quad (P)$$

$$\text{Opt}(D) = \sup_{\lambda \in \Lambda} \inf_{x \in X} F(x, \lambda) \quad (D)$$
\[
\text{Opt}(P) = \inf_{x \in X} \sup_{\lambda \in \Lambda} F(x, \lambda) \quad (P)
\]
\[
\text{Opt}(D) = \sup_{\lambda \in \Lambda} \inf_{x \in X} F(x, \lambda) \quad (D)
\]

**Game interpretation:** Player I chooses \( x \in X \), player II chooses \( \lambda \in \Lambda \). With choices of the players \( x, \lambda \), player I pays to player II the sum of \( F(x, \lambda) \). What should the players do to optimize their wealth?

◊ If Player I chooses \( x \) first, and Player II knows this choice when choosing \( \lambda \), II will maximize his profit, and the loss of I will be \( \overline{F}(x) \). To minimize his loss, I should solve \( (P) \), thus ensuring himself loss \( \text{Opt}(P) \) or less.

◊ If Player II chooses \( \lambda \) first, and Player I knows this choice when choosing \( x \), I will minimize his loss, and the profit of II will be \( \underline{F}(\lambda) \). To maximize his profit, II should solve \( (D) \), thus ensuring himself profit \( \text{Opt}(D) \) or more.
Observation: For Player I, second situation seems better, so that it is natural to guess that his anticipated loss in this situation is \( \leq \) his anticipated loss in the first situation:

\[
\text{Opt}(D) \equiv \sup_{\lambda \in \Lambda} \inf_{x \in X} F(x, \lambda) \leq \inf_{x \in X} \sup_{\lambda \in \Lambda} F(x, \lambda) \equiv \text{Opt}(P).
\]

This indeed is true: assuming \( \text{Opt}(P) < \infty \) (otherwise the inequality is evident),

\[
\forall (\epsilon > 0) : \exists x_\epsilon \in X : \sup_{\lambda \in \Lambda} F(x_\epsilon, \lambda) \leq \text{Opt}(P) + \epsilon \\
\Rightarrow \forall \lambda \in \Lambda : \bar{F}(\lambda) = \inf_{x \in X} F(x, \lambda) \leq F(x_\epsilon, \lambda) \leq \text{Opt}(P) + \epsilon \\
\Rightarrow \text{Opt}(D) \equiv \sup_{\lambda \in \Lambda} \bar{F}(\lambda) \leq \text{Opt}(P) + \epsilon \\
\Rightarrow \text{Opt}(D) \leq \text{Opt}(P).
\]
\[
\begin{align*}
\text{Opt}(P) &= \inf_{x \in X} \sup_{\lambda \in \Lambda} F(x, \lambda) \\
\text{Opt}(D) &= \sup_{\lambda \in \Lambda} \inf_{x \in X} F(x, \lambda)
\end{align*}
\]

What should the players do when making their choices simultaneously? A “good case” when we can answer this question – \( F \) has a saddle point.

**Definition:** We call a point \((x_*, \lambda_*) \in X \times \Lambda\) a saddle point of \( F \), if

\[
F(x, \lambda_*) \geq F(x_*, \lambda_*) \geq F(x_*, \lambda) \quad \forall (x \in X, \lambda \in \Lambda).
\]

In game terms, a saddle point is an *equilibrium* – no one of the players can improve his wealth, provided the adversary keeps his choice unchanged.

**Proposition [Existence and Structure of saddle points]:** \( F \) has a saddle point if and only if both \((P)\) and \((D)\) are solvable with equal optimal values. In this case, the saddle points of \( F \) are exactly the pairs \((x_*, \lambda_*)\), where \( x_* \) is an optimal solution to \((P)\), and \( \lambda_* \) is an optimal solution to \((D)\).
\[
\begin{align*}
\text{Opt}(P) &= \inf_x \sup_{\lambda \in \Lambda} F(x, \lambda) \quad (P) \\
\text{Opt}(D) &= \sup_{\lambda \in \Lambda} \inf_x F(x, \lambda) \quad (D)
\end{align*}
\]

**Proof, \(\Rightarrow:\)** Assume that \((x_*, \lambda_*)\) is a saddle point of \(F\), and let us prove that \(x_*\) solves \((P)\), \(\lambda_*\) solves \((D)\), and \(\text{Opt}(P) = \text{Opt}(D)\).

Indeed, we have

\[
F(x, \lambda) \geq F(x_*, \lambda) \geq F(x_*, \lambda) \quad \forall (x \in X, \lambda \in \Lambda)
\]

whence

\[
\text{Opt}(P) \leq \overline{F}(x_*) = \sup_{\lambda \in \Lambda} F(x_*, \lambda) = F(x_*, \lambda_*)
\]

\[
\text{Opt}(D) \geq \underline{F}(\lambda_*) = \inf_x F(x, \lambda_*) = F(x_*, \lambda_*)
\]

Since \(\text{Opt}(P) \geq \text{Opt}(D)\), we see that all inequalities in the chain

\[
\text{Opt}(P) \leq \overline{F}(x_*) = F(x_*, \lambda_*) = \underline{F}(\lambda_*) \leq \text{Opt}(D)
\]

are equalities. Thus, \(x_*\) solves \((P)\), \(\lambda_*\) solves \((D)\) and \(\text{Opt}(P) = \text{Opt}(D)\).
\[
\begin{align*}
\text{Opt}(P) &= \inf_{x \in X} \sup_{\lambda \in \Lambda} F(x, \lambda) \\
\text{Opt}(D) &= \sup_{\lambda \in \Lambda} \inf_{x \in X} F(x, \lambda)
\end{align*}
\]

**Proof, \iff.** Assume that \((P), (D)\) have optimal solutions \(x_*, \lambda_*\) and \(\text{Opt}(P) = \text{Opt}(D)\), and let us prove that \((x_*, \lambda_*)\) is a saddle point. We have

\[
\begin{align*}
\text{Opt}(P) &= \overline{F}(x_*) = \sup_{\lambda \in \Lambda} F(x_*, \lambda) \\
&\geq F(x_*, \lambda_*) \\
\text{Opt}(D) &= \underline{F}(\lambda_*) = \inf_{x \in X} F(x, \lambda_*) \\
&\leq F(x_*, \lambda_*)
\end{align*}
\]

Since \(\text{Opt}(P) = \text{Opt}(D)\), all inequalities in (*) are equalities, so that

\[
\sup_{\lambda \in \Lambda} F(x_*, \lambda) = \overline{F}(x_*) = F(x_*, \lambda_*) = \inf_{x \in X} F(x, \lambda_*)
\]
Existence of Saddle Points

♣ **Theorem [Sion-Kakutani]** Let $X \subset \mathbb{R}^n$, $\Lambda \subset \mathbb{R}^m$ be nonempty convex closed sets and $F(x, \lambda) : X \times \Lambda \rightarrow \mathbb{R}$ be a continuous function which is convex in $x \in X$ and concave in $\lambda \in \Lambda$. Assume that $\Lambda$ is compact.

(i) "MinMax equals MaxMin:" One has

\[
\text{SadVal} := \inf_{x \in X} \sup_{\lambda \in \Lambda} F(x, \lambda) = \sup_{\lambda \in \Lambda} \min_{x \in X} F(x, \lambda)
\]

**Note:** SadVal is either real, or $-\infty$.

(ii) Assume that there exists $\bar{\lambda} \in \Lambda$ such that for every $a \in \mathbb{R}$ the set

\[
X_a : \{x \in X : F(x, \bar{\lambda}) \leq a\}
\]

is bounded (e.g., since $X$ is bounded).

Then SadVal is real, and $F$ possesses a saddle point on $X \times \Lambda$.  

3.21
Proof of Sion-Kakutani Theorem

**MinMax Lemma** [von Neumann] Let \( X \) be a nonempty convex compact set and \( f_1, \ldots, f_N \) be continuous convex functions on \( X \). then the quantity

\[
Opt = \min_{x \in X} \max [f_1(x), f_2(x), \ldots, f_N(x)]
\]

is the minimum over \( X \) of certain convex combination of \( f_i \):

\[
\exists \mu^* \in \mathbb{R}_+^N, \sum_i \mu_i^* = 1 : Opt = \min_x \sum_{i=1}^N \mu_i^* f_i + i(x).
\]

**Note:** for every collection of nonnegative weights \( \mu_i \) summing up to one we have \( \sum_i \mu_i f_i(x) \leq \max_i f_i(x) \) and therefore

\[
\min_X \sum_i \mu_i f_i(x) \leq Opt.
\]
Proof of MinMax Lemma: Assuming w.l.o.g. that $\text{Opt} = 0$ (replace $f_i$ with $f_i - \text{Opt}$!), consider two convex sets in $\mathbb{R}^N$:

$$S = \{0\}, \quad T = \{y \in \mathbb{R}^N : \exists x \in X : y \geq f(x) := [f_1(x); \ldots; f_N(x)]\}.$$ 

From convexity of $X$ and $f_i$'s it follows that $T$ is convex. Besides this, $T$ clearly possesses a nonempty interior.

We claim that $S = \{0\} \not\in \text{int} T$. Indeed, assuming the opposite, $T$ contains a negative vector, whence, by definition of $T$, $f_i(\bar{x}) < 0$ for some $\bar{x} \in X$ and all $i$, so that $\min_X \max_i f_i(x) < 0$, while we are in the case $\text{Opt} = 0$.

By Separation Theorem, the fact that $S = \{0\} \not\in \text{int} T \neq \emptyset$ implies that $S$ and $T$ can be separated: there exists $\lambda = [\lambda_1; \ldots; \lambda_N] \neq 0$ such that

$$0 = \max_{s \in S} \lambda^T s \leq \inf_{y \in Y} \sum_i \lambda_i y_i. \tag{*}$$

since $T$ contains all positive vectors with large enough entries, $(*)$ implies that $\lambda \geq 0$, and since $f(x) \in T$ for all $x \in X$, $(*)$ says that

$$\text{Opt} = 0 \leq \sum_i \lambda_i f_i(x) \quad \forall x \in X \tag{!}$$

Since $0 \neq \lambda \geq 0$, the weights $\mu_i^* = \lambda_i / \sum_j \lambda_j$ are well defined, nonnegative, sum up to 1, and by (!) we have

$$\text{Opt} = 0 \leq \sum_i \mu_i^* f_i(x) \quad \forall x \in X \quad \square$$

3.23
Proof of Sion-Kakutani Theorem: We should prove that problems

\[
\begin{align*}
\text{Opt}(P) &= \inf_{x\in X} \sup_{\lambda \in \Lambda} F(x, \lambda) \\
\text{Opt}(D) &= \sup_{\lambda \in \Lambda} \inf_{x\in X} F(x, \lambda)
\end{align*}
\]

are solvable with equal optimal values.

1°. Since \(X\) is compact and \(F(x, \lambda)\) is continuous on \(X \times \Lambda\), the function \(\overline{F}(\lambda)\) is continuous on \(\Lambda\). Besides this, the sets

\[
\Lambda^a = \{ \lambda \in \Lambda : \overline{F}(\lambda) \geq a \}
\]

are contained in the sets

\[
\Lambda_a = \{ \lambda \in \Lambda : \overline{F}(\bar{x}, \lambda) \geq a \}
\]

and therefore are bounded. Finally, \(\Lambda\) is closed, so that the continuous function \(\overline{F}(\cdot)\) with bounded level sets \(\Lambda^a\) attains it maximum on a closed set \(\Lambda\). Thus, \((D)\) is solvable.
Consider the sets

\[ X(\lambda) = \{ x \in X : F(x, \lambda) \leq \text{Opt}(D) \}. \]

These are closed convex subsets of a compact set \( X \). Let us prove that every finite collection of these sets has a nonempty intersection. Indeed, assume that \( X(\lambda^1) \cap \ldots \cap X(\lambda^N) = \emptyset \), so that

\[
\max_{j=1,\ldots,N} F(x, \lambda^j) > \text{Opt}(D) \quad \forall x \in X
\]

\[
\Rightarrow \min_{x \in X} \max_{j} F(x, \lambda^j) > \text{Opt}(D)
\]

by compactness of \( X \) and continuity of \( F \). By MinMax Lemma, there exist weights \( \mu_j \geq 0, \sum_j \mu_j = 1 \), such that

\[
\min_{x \in X} \left( \sum_j \mu_j F(x, \lambda^j) \right) > \text{Opt}(D),
\]

\[
\leq F(x, \sum \mu_j \lambda_j)
\]

since \( F \) is concave in \( \lambda \)

\[
\Rightarrow F(\sum \mu_j \lambda_j) := \min_{x \in X} F(x, \sum \mu_j \lambda_j) \geq \min_{x \in X} \sum_j \mu_j F(x, \lambda_j) > \text{Opt}(D),
\]

which is impossible.
3. Since every finite collection of closed convex subsets $X(\lambda)$ of the compact set $X$ has a nonempty intersection, all these sets have a nonempty intersection:

$$\exists x_* \in X : F(x_*, \lambda) \leq \text{Opt}(D) \ \forall \lambda.$$ 

Due to $\text{Opt}(P) \geq \text{Opt}(D)$, this is possible iff $x_*$ is optimal for $(P)$ and $\text{Opt}(P) = \text{Opt}(D)$.
Simple Observation Schemes

Simple Observation Scheme is a collection

\[ \mathcal{O} = ((\Omega, \Pi), \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F}), \]

where

- \((\Omega, \Pi)\) is a (complete separable metric) observation space \(\Omega\) with \((\sigma\text{-finite }\sigma\text{-additive})\) reference measure \(\Pi\),
  \[ \text{supp } \Pi = \Omega; \]
- \(\{p_\mu(\cdot) : \mu \in \mathcal{M}\}\) is a parametric family of probability densities, taken w.r.t. \(\Pi\), on \(\Omega\), and
  - \(\mathcal{M}\) is a relatively open convex set in some \(\mathbb{R}^n\)
  - \(p_\mu(\omega)\): positive and continuous in \(\mu \in \mathcal{M}, \omega \in \Omega\)
- \(\mathcal{F}\) is a finite-dimensional space of continuous functions on \(\Omega\) containing constants and such that
  \[ \ln(p_\mu(\cdot)/p_\nu(\cdot)) \in \mathcal{F} \text{ } \forall \mu, \nu \in \mathcal{M} \]
- For \(\phi \in \mathcal{F}\), the function
  \[ \mu \mapsto \ln \left( \int_{\Omega} e^{\phi(\omega)} p_\mu(\omega) P(d\omega) \right) \]
  is finite and concave in \(\mu \in \mathcal{M}\).
Example 1: Gaussian o.s.
- \((\Omega, \Pi) = (\mathbb{R}^d, \text{mes}_d)\) is \(\mathbb{R}^d\) with Lebesgue measure,
- \(\{p_\mu(\cdot) = \mathcal{N}(\mu, I_d) : \mu \in \mathbb{R}^d\}\),
- \(\mathcal{F} = \{\text{affine functions on } \Omega\} \Rightarrow \left\{ \begin{array}{l}
\ln(p_\mu(\cdot)/p_\nu(\cdot)) \in \mathcal{F}, \\
\ln\left(\int_{\Omega} e^{a^T \omega + b} p_\mu(\omega) \Pi(d\omega)\right) = a^T \mu + b + \frac{a^T a}{2} : \text{is concave in } \mu.
\end{array} \right\}
- Gaussian o.s. is the standard observation model in Signal Processing.

Example 2: Poisson o.s.
- \((\Omega, \Pi)\), is the nonnegative part \(Z^d_+\) of integer lattice in \(\mathbb{R}^d\) equipped with counting measure,
- \(\{p_\mu(\omega) = \prod_{i=1}^d \frac{\mu_i^{\omega_i} e^{-\mu_i}}{\omega_i!} : \mu \in \mathcal{M} := \mathbb{R}^d_{++}\}\) is the family of distributions of random vectors with independent across \(i\) Poisson entries \(\omega_i \sim \text{Poisson}(\mu_i)\),
- \(\mathcal{F} = \{\text{affine functions on } \Omega\} \Rightarrow \left\{ \begin{array}{l}
\ln(p_\mu(\cdot)/p_\nu(\cdot)) \in \mathcal{F}, \\
\ln\left(\int_{\Omega} e^{a^T \omega + b} p_\mu(\omega) \Pi(d\omega)\right) = b + \sum_i (e^{a_i} - 1) \mu_i \text{ is concave in } \mu.
\end{array} \right\}
- Poisson o.s. arises in Poisson Imaging, including
  - Positron Emission Tomography,
  - Large binocular Telescope,
  - Nanoscale Fluorescent Microscopy.
Example 3: Discrete o.s.

- $(\Omega, \Pi)$ is finite set $\{1, \ldots, d\}$ with counting measure,
- $\{p_{\mu}(\omega) = \mu_\omega, \mu \in \mathcal{M} = \{\mu > 0 : \sum_{\omega=1}^{d} \mu_\omega = 1\}\}$ is the set of non-vanishing probability distributions on $\Omega$,
- $\mathcal{F} = \{\text{all functions on } \Omega\} \ni \left\{ \ln\left(\frac{p_{\mu}(\cdot)}{p_\nu(\cdot)}\right) \in \mathcal{F}, \ln\left(\int_{\Omega} e^{\phi(\omega)} p_{\mu}(\omega) \Pi(d\omega)\right) = \ln\left(\sum_{\omega \in \Omega} e^{\phi(\omega)} \mu_\omega\right) \text{ is concave in } \mu. \right\}$

Example 4: Direct product of simple o.s.’s.

Simple o.s.’s

$O_k = ((\Omega_k, \Pi_k), \{p_{\mu_k, k}(\cdot) : \mu_k \in \mathcal{M}_k\}, \mathcal{F}_k), 1 \leq k \leq K$ give rise to their direct product $\bigotimes_{k=1}^{K} O_k$ defined as the o.s.

$((\Omega^K, \Pi^K), \{p_{\mu^K}(\cdot) : \mu^K \in \mathcal{M}_K\}, \mathcal{F}_K)$, where

- $\Omega^K = \Omega_1 \times \ldots \times \Omega_K, \Pi^K = \Pi_1 \times \ldots \times \Pi_K$
- $\mathcal{M}^K = \mathcal{M}_1 \times \ldots \times \mathcal{M}_K, p_{(\mu_1, \ldots, \mu_K)}(\omega_1, \ldots, \omega_K) = \prod_{k=1}^{K} p_{\mu_k, k}(\omega_k)$
- $\mathcal{F}^K = \{\phi(\omega_1, \ldots, \omega_K) = \sum_{k=1}^{K} \phi_k(\omega_k) : \phi_k \in \mathcal{F}_k, 1 \leq k \leq K\}$

Fact: Direct product of simple o.s.’s is a simple o.s.
♠ Example 5: Power of a simple o.s.

When all $K$ o.s.'s in direct product $\mathcal{O}^K = \bigotimes_{k=1}^K \mathcal{O}_k$ are identical to each other:

$$\mathcal{O}_k = \mathcal{O} := (\Omega, \Pi), \{p_\mu(\cdot) : \mu \in \mathcal{M}\}, \mathcal{F}, \text{ } 1 \leq k \leq K$$

we can “restrict $\mathcal{O}^K$ to its diagonal,” arriving at $K$-th power $\mathcal{O}^{(K)}$ of $\mathcal{O}$:

$$\mathcal{O}^{(K)} = \left( (\Omega^K, \Pi^K), \{p^{(K)}_\mu(\cdot) : \mu \in \mathcal{M}\}, \mathcal{F}^{(K)} \right),$$

$$p^{(K)}_\mu(\omega_1, ..., \omega_K) = \prod_{k=1}^K p_\mu(\omega_k), \quad \mathcal{F}^{(K)} = \{\phi^{(K)}(\omega^K) = \sum_{k=1}^K \phi(\omega_k) : \phi \in \mathcal{F}\}$$

❤️ Fact: Power of a simple o.s. is a simple o.s.
\( \varepsilon^*(\mathcal{P}_1, \mathcal{P}_2) = \min_{\phi(\cdot), \epsilon} \left\{ \epsilon : \begin{aligned} \int_{\Omega} e^{-\phi(\omega)} P(d\omega) &\leq \epsilon \forall (P \in \mathcal{P}_1) \\ \int_{\Omega} e^{\phi(\omega)} P(d\omega) &\leq \epsilon \forall (P \in \mathcal{P}_2) \end{aligned} \right\} \) \hspace{1cm} (1)

\[ \clubsuit \textbf{Main Result.} \] Let \( \mathcal{O} = ((\Omega, \Pi), \{p_\mu(\cdot) : \mu \in \mathcal{M}\}, \mathcal{F}) \) be a simple o.s., and let \( M_1, M_2 \) be two nonempty compact convex subsets of \( \mathcal{M} \). These subsets give rise to two families of probability distributions \( \mathcal{P}_1, \mathcal{P}_2 \) on \( \Omega \) and two hypotheses on the distribution \( P \) of random observation \( \omega \in \Omega \):

\[ \mathcal{P}_\chi = \{ P : \text{the density of } P \text{ is } p_\mu \text{ with } \mu \in \mathcal{M}_\chi \}, \ H_\chi : P \in \mathcal{P}_\chi, \ \chi = 1, 2. \]

Consider the function

\[ \Phi(\phi; \mu, \nu) = \frac{1}{2} \left[ \ln \left( \int_{\Omega} e^{-\phi(\omega)} p_\mu(\omega) \Pi(d\omega) \right) + \ln \left( \int_{\Omega} e^{\phi(\omega)} p_\nu(\omega) \Pi(d\omega) \right) \right] : \mathcal{F} \times [M_1 \times M_2] \to \mathbb{R}. \]

Then

A. \( \Phi(\phi; \mu, \nu) \) is continuous on its domain, convex in \( \phi \in \mathcal{F} \), concave in \( (\mu, \nu) \) on \( M_1 \times M_2 \) and possesses saddle point (min in \( \phi \), max in \( (\mu, \nu) \)):

\[ \exists (\phi_* \in \mathcal{F}, (\mu^*, \nu^*) \in M_1 \times M_2) : \]

\[ \Phi(\phi; \mu^*, \nu^*) \geq \Phi(\phi_*; \mu^*, \nu^*) \geq \Phi(\phi_*; \mu, \nu) \forall (\phi \in \mathcal{F}, (\mu, \nu) \in M_1 \times M_2) \]
\begin{align*}
\varepsilon^*(\mathcal{P}_1, \mathcal{P}_2) &= \min_{\phi(\cdot), \epsilon} \left\{ \epsilon : \begin{array}{c}
\int_{\Omega} e^{-\phi(\omega)} P(d\omega) \leq \epsilon \forall (P \in \mathcal{P}_1) \\
\int_{\Omega} e^{\phi(\omega)} P(d\omega) \leq \epsilon \forall (P \in \mathcal{P}_2)
\end{array} \right\} \\
\Phi(\phi; \mu, \nu) &= \frac{1}{2} \left[ \ln \left( \int_{\Omega} e^{-\phi(\omega)} p_\mu(\omega) \Pi(d\omega) \right) + \ln \left( \int_{\Omega} e^{\phi(\omega)} p_\nu(\omega) \Pi(d\omega) \right) \right] : \mathcal{F} \times [M_1 \times M_2] \to \mathbb{R}.
\end{align*}

**B.** The component \(\phi^*\) of a saddle point \((\phi^*, (\mu^*, \nu^*))\) of \(\Phi\) is an optimal solution to \((!)\), and

\[\varepsilon^*(\mathcal{P}_1, \mathcal{P}_2) = \exp\{\Phi(\phi^*; \mu^*, \nu^*)\}.\]
A saddle point \((\phi^*, (\mu^*, \nu^*))\) can be found as follows. We solve the optimization problem

\[
\text{SadVal} = \max_{\mu \in M_1, \nu \in M_2} \ln \left( \int_{\Omega} \sqrt{p_\mu(\omega)p_\nu(\omega)} \Pi(d\omega) \right);
\]

which is a solvable convex optimization problem, and take an optimal solution to the problem as \((\mu^*, \nu^*)\). We then set

\[
\phi^*(\omega) = \frac{1}{2} \ln \left( \frac{p_\mu^*(\omega)}{p_\nu^*(\omega)} \right),
\]

thus getting an optimal detector \(\phi^* \in \mathcal{F}\). For this detector and the associated simple test \(\mathcal{T}_{\phi^*}\),

\[
\text{Risk}(\mathcal{T}_{\phi^*} | H_1, H_2) \leq \text{Risk}[\phi^* | \mathcal{P}_1, \mathcal{P}_2] = \text{Risk}_1[\phi^* | \mathcal{P}_1, \mathcal{P}_2] = \text{Risk}_2[\phi^* | \mathcal{P}_1, \mathcal{P}_2] = \varepsilon^*(\mathcal{P}_1, \mathcal{P}_2) = e^{\text{SadVal}} = \int_{\Omega} \sqrt{p_\mu^*(\omega)p_\nu^*(\omega)} \Pi(d\omega).
\]

3.33
● **Gaussian o.s.** \( \mathcal{P}_\chi = \{ \mathcal{N}(\mu, I_d) : \mu \in M_\chi \}, \chi = 1, 2 \):

  - Problem \( \max_{\mu \in M_1, \nu \in M_2} \ln \left( \int \sqrt{p_\mu(\omega)p_\nu(\omega)} \prod(d\omega) \right) \) reads

\[
\max_{\mu \in M_1, \nu \in M_2} \left[ -\frac{1}{8} \| \mu - \nu \|_2^2 \right]
\]

  - The optimal balanced detector and its risk are given by

\[
\phi^*(\omega) = \frac{1}{2}[\mu^* - \nu^*] \omega - c,
\]

\[
(\mu^*, \nu^*) \in \text{Argmin}_{\mu \in M_1, \nu \in M_2} \| \mu - \nu \|_2^2
\]

\[
c = \frac{1}{4}[\mu^* - \nu^*]^T [\mu^* + \nu^*]
\]

\[
\varepsilon^*(\mathcal{P}_1, \mathcal{P}_2) = \exp \left\{ -\frac{\| \mu^* - \nu^* \|_2^2}{8} \right\}
\]

**Note:** We are in the “signal plus noise” model of observations with noise \( \sim \mathcal{N}(0, I_d) \). The test \( \mathcal{T}_{\phi^*} \) is nothing but the pairwise Euclidean separation test associated with \( X_\chi = M_\chi, \chi = 1, 2 \).
Poisson o.s. \( \mathcal{P}_\chi = \{ \otimes_{i=1}^d \text{Poisson}(\mu_i) : \mu = [\mu_1; \ldots; \mu_d] \in M_\chi \}, \chi = 1, 2 : \)

- Problem \( \max_{\mu \in M_1, \nu \in M_2} \ln \left( \int \sqrt{p_\mu(\omega)p_\nu(\omega)} \Pi(d\omega) \right) \) reads

\[
\max_{\mu \in M_1, \nu \in M_2} \left[ -\frac{1}{2} \sum_{i=1}^d (\sqrt{\mu_i} - \sqrt{\nu_i})^2 \right] \frac{\sum_i [\sqrt{\mu_i \nu_i} - \frac{1}{2} \mu_i - \frac{1}{2} \nu_i]}{\sum_i [\sqrt{\mu_i \nu_i} - \frac{1}{2} \mu_i - \frac{1}{2} \nu_i]}
\]

- The optimal balanced detector and its risk are given by

\[
\phi_*(\omega) = \frac{1}{2} \sum_{i=1}^d [\ln(\mu_i^*/\nu_i^*) \omega_i + \nu_i^* - \mu_i^*],
\]

\[
(\mu^*, \nu^*) \in \text{Argmax}_{\mu \in M_1, \nu \in M_2} \sum_i [\sqrt{\mu_i \nu_i} - \frac{1}{2} \mu_i - \frac{1}{2} \nu_i]
\]

\[
\varepsilon_*(\mathcal{P}_1, \mathcal{P}_2) = \exp \left\{ -\frac{1}{2} \sum_i \left( \sqrt{\mu_i^*} - \sqrt{\nu_i^*} \right)^2 \right\}
\]
Discrete o.s.

\[ \mathcal{P}_\chi = \{ \mu \in M_\chi \}, \quad M_\chi \subset \Delta^o_d = \{ \mu \in \mathbb{R}^d_+ : \sum \mu_\omega = 1, \mu > 0 \}, \chi = 1, 2 \]

- Problem \( \max_{\mu \in M_1, \nu \in M_2} \ln \left( \int \sqrt{p_\mu(\omega)p_\nu(\omega)} \prod(d\omega) \right) \) reads

\[
\max_{\mu \in M_1, \nu \in M_2} \sum_\omega \sqrt{\mu_\omega \nu_\omega}
\]

- The optimal balanced detector and its risk are given by

\[
\phi^*_\omega(\omega) = \frac{1}{2} \ln(\mu^*_\omega/\nu^*_\omega), \quad \omega \in \Omega = \{1, \ldots, d\}, \\
(\mu^*, \nu^*) \in \text{Argmin} \sum_\omega \sqrt{\mu_\omega \nu_\omega} \\
\varepsilon^*(\mathcal{P}_1, \mathcal{P}_2) = \sum_\omega \sqrt{\mu^*_\omega \nu^*_\omega}
\]
Direct product of simple o.s.’s. Let
\[ O_k = \left( (\Omega_k, \Pi_k), \{ p_{\mu_k,k}(\cdot) : \mu_k \in \mathcal{M}_k \}, \mathcal{F}_k \right), \quad 1 \leq k \leq K, \]
be simple o.s.’s, and \( M_{\chi,k} \subset \mathcal{M}_k, \chi = 1, 2, \) be nonempty convex compact sets. Consider the simple o.s.
\[ \left( (\Omega^K, \Pi^K), \{ p_{\mu^K} : \mu^K \in \mathcal{M}^K \}, \mathcal{F}^K \right) = \bigotimes_{k=1}^K O_k \]
along with two compact convex sets
\[ M_{\chi} = M_{\chi,1} \times \ldots \times M_{\chi,K}, \chi = 1, 2. \]

Question: What is the problem
\[
\max_{\mu^K \in M_1, \nu^K \in M_2} \ln \left( \int_{\Omega^K} \sqrt{p_{\mu^K}(\omega^K)p_{\nu^K}(\omega^K)} \Pi^K (d\omega^K) \right)
\]
responsible for the smallest risk detector for the families of distributions \( \mathcal{P}_1^{(K)}, \mathcal{P}_2^{(K)} \) associated in \( O^K \) with the sets \( M_1, M_2 \)?

Answer: This is the separable problem
\[
\max_{\{\mu_k \in M_{1,k}, \nu_k \in M_{2,k}\}_{k=1}^K} \sum_{k=1}^K \ln \left( \int_{\Omega_k} \sqrt{p_{\mu_k,k}(\omega_k)p_{\nu_k,k}(\omega_k)} \Pi_k (d\omega_k) \right)
\]
Minimum risk balanced detector for $\mathcal{P}_1^{(K)}$, $\mathcal{P}_2^{(K)}$ can be chosen as

$$
\phi^K_*(\omega_1, \ldots, \omega_K) = \sum_{k=1}^{K} \phi^*_{*,k}(\omega_k),
\phi^*_{*,k}(\omega_k) = \frac{1}{2} \ln \left( \frac{p\mu^*_k,k(\omega)/p\nu^*_k,k(\omega)}{p\mu^*_k,k(\omega)/p\nu^*_k,k(\omega)} \right)
$$

$$
(\mu^*_k, \nu^*_k) \in \operatorname{Argmax}_{\mu_k \in M_{1,k}, \nu_k \in M_{2,k}} \ln \left( \int_{\Omega_k} \sqrt{p\mu_k,k(\omega_k)p\nu_k,k(\omega_k)} \prod_k d\omega_k \right)
$$

and

$$
\varepsilon^*_*(\mathcal{P}_1^{(K)}, \mathcal{P}_2^{(K)}) = \prod_{k=1}^{K} \varepsilon^*_*(\mathcal{P}_{1,k}, \mathcal{P}_{2,k}),
$$

where $\mathcal{P}_{\chi,k}$ are the families of distributions associated in $\mathcal{O}_k$ with $M_{\chi,k}$, $\chi = 1, 2$. 

3.38
\textbf{Remark:} The families of distributions $\mathcal{P}_\chi^{(K)}$ are direct products of the families $\mathcal{P}_\chi, k$ over $k = 1, \ldots, K$. From Detector Calculus, extending $\mathcal{P}_\chi^{(K)}$ to families $\mathcal{P}_\chi^{\otimes, K}$ of quasi-direct products of families $\mathcal{P}_\chi, k, k = 1, \ldots, K$, we still have

\[
\text{Risk}\left[\phi^K | \mathcal{P}_1^{\otimes, K}, \mathcal{P}_2^{\otimes, K}\right] \leq \prod_{k=1}^{K} \varepsilon_* (\mathcal{P}_{1,k}, \mathcal{P}_{2,k}),
\]

whence also $\varepsilon_K = \varepsilon_* (\mathcal{P}_1^K, \mathcal{P}_2^K) \leq \varepsilon_* (\mathcal{P}_1^{\otimes, K}, \mathcal{P}_2^{\otimes, K}) \leq \varepsilon_K$

\[
\Rightarrow \varepsilon_* (\mathcal{P}_1^{\otimes, K}, \mathcal{P}_2^{\otimes, K}) = \prod_{k=1}^{K} \varepsilon_* (\mathcal{P}_{1,k}, \mathcal{P}_{2,k}).
\]
Power of a simple o.s. Let

\[ \mathcal{O} = (((\Omega, \Pi), \{p_{\mu}(\cdot) : \mu \in \mathcal{M}\}, \mathcal{F}) \]

be a simple o.s., and \( M_{\chi} \subset \mathcal{M}, \chi = 1, 2 \), be nonempty convex compact sets. Consider the \( K \)-th power of \( \mathcal{O} \), that is, the simple o.s.

\[ \mathcal{O}^{(K)} = \left( (\Omega^K, \Pi^K), \{p_{\mu}^{(K)}(\omega_1, \ldots, \omega_K) = \prod_{k=1}^{K} p_{\mu}(\omega_k) : \mu \in \mathcal{M}\}, \mathcal{F}^{(K)} \right) \]

Question: What is the problem responsible for the smallest risk detector for the families of distributions \( \mathcal{P}^{K}_{\chi} \) associated in \( \mathcal{O}^{(K)} \) with the sets \( M_{\chi}, \chi = 1, 2 \) ?

Answer: This is the separable problem

\[
\max_{\mu \in M_1, \nu \in M_2} \sum_{k=1}^{K} \ln \left( \int_{\Omega^K} \sqrt{p_{\mu}(\omega_k)p_{\nu}(\omega_k)} \prod_{k=1}^{K} (d\omega_k) \right)
\]

\[
K \ln \left( \int_{\Omega} \sqrt{p_{\mu}(\omega)p_{\nu}(\omega)} \prod (d\omega) \right)
\]
Minimum risk balanced detector for \( \mathcal{P}_1^K, \mathcal{P}_2^K \) can be chosen as

\[
\phi_*(\omega_1, \ldots, \omega_K) = \sum_{k=1}^K \phi_*(\omega_k),
\]

\[
\phi_*(\omega_k) = \frac{1}{2} \ln \left( \frac{p_{\mu^*}(\omega)}{p_{\nu^*}(\omega)} \right)
\]

\[
[\mu^*, \nu^*] \in \operatorname{Argmax}_{\mu \in M_1, \nu \in M_2} \ln \left( \int_{\Omega} \sqrt{p_{\mu}(\omega)p_{\nu}(\omega)} \Pi(d\omega) \right)
\]

and

\[
\varepsilon_*\left( \mathcal{P}_1^K, \mathcal{P}_2^K \right) = [\varepsilon_*\left( \mathcal{P}_1, \mathcal{P}_2 \right)]^K,
\]

where \( \mathcal{P}_\chi \) are the families of distributions associated in \( \mathcal{O} \) with \( M_\chi, \chi = 1, 2 \).
Remark: The families of distributions $\mathcal{P}_\chi^K$ are direct powers $\mathcal{P}_\chi^{\oplus,K}$ of the families $\mathcal{P}_\chi$. From Detector Calculus, extending $\mathcal{P}_\chi^K$ to families $\mathcal{P}_\chi^{\otimes,K}$ of quasi-direct powers of families $\mathcal{P}_\chi$, we still have

$$\text{Risk}[\phi^*(K)|\mathcal{P}_1^{\oplus,K}, \mathcal{P}_2^{\oplus,K}] \leq [\varepsilon_*(\mathcal{P}_1, \mathcal{P}_2)]^K,$$

whence also $\varepsilon_K = \varepsilon_*(\mathcal{P}_1^K, \mathcal{P}_2^K) \leq \varepsilon_*(\mathcal{P}_1^{\otimes,K}, \mathcal{P}_2^{\otimes,K}) \leq \varepsilon_K$

$$\Rightarrow \varepsilon_*(\mathcal{P}_1^{\otimes,K}, \mathcal{P}_2^{\otimes,K}) = [\varepsilon_*(\mathcal{P}_1, \mathcal{P}_2)]^K.$$
Proposition A. Let 
\[ \mathcal{O} = (\Omega, \Pi), \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F} \]
be a simple o.s., and \( M_\chi \subset \mathcal{M}, \chi = 1, 2 \), be nonempty convex compact sets, giving rise to families of distributions 
\[ \mathcal{P}_\chi = \{ P : P \text{ has density } p_\mu(\cdot) \text{ w.r.t. } \Pi \text{ with } \mu \in M_\chi \}, \chi = 1, 2, \]
hypotheses 
\[ H_\chi : P \in \mathcal{P}_\chi, \chi = 1, 2, \]
on the distribution of a random observation \( \omega \in \Omega \), and minimum risk detector \( \phi^* \) for \( \mathcal{P}_1, \mathcal{P}_2 \).

Assume that in the nature there exists a simple single-observation test, deterministic or randomized, \( T \) with 
\[ \text{Risk}(T|H_1, H_2) \leq \epsilon < 1/2. \]
Then the risk of the simple test \( T_{\phi^*} \) accepting \( H_1 \) when \( \phi^*(\omega) \geq 0 \) and accepting \( H_2 \) otherwise “is comparable” to \( \epsilon \):
\[ \text{Risk}(T_{\phi^*}|H_1, H_2) \leq \epsilon_+ := 2\sqrt{\epsilon(1 - \epsilon)} < 1. \]
Proof. From what we called “universality” of detector-based tests, there exists a detector $\phi$ with $\text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2] \leq \epsilon_+$, and $\text{Risk}[\phi_\ast|\mathcal{P}_1, \mathcal{P}_2]$ can be only less than $\text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2]$. □
Proposition B. Let $\mathcal{O} = ((\Omega, \Pi), \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F})$ be a simple o.s., and $M_\chi \subset \mathcal{M}$, $\chi = 1, 2$, be nonempty convex compact sets, giving rise to families of distributions

$$P_\chi = \{P : P \text{ has density } p_\mu(\cdot) \text{ w.r.t. } \Pi \text{ with } \mu \in M_\chi\}, \chi = 1, 2$$

their direct powers

$$P_\chi^{\otimes, K} = \{P \times \ldots \times P : P \in P_\chi\}, \chi = 1, 2, K = 1, 2, \ldots$$

hypotheses $H^K_\chi : P \in P_\chi^{\otimes, K}, \chi = 1, 2, K = 1, 2, \ldots$ on the distribution $P$ of random $K$-repeated observation $\omega^K = (\omega_1, \ldots, \omega_K) \in \Omega^K$, and minimum risk detector $\phi_*$ for $P_1, P_2$.

Assume that in the nature there positive integer $K_*$ and a simple $K_*$-observation test, deterministic or randomized, $T_{K_*}$ capable to decide on the hypotheses $H^K_\chi$, $\chi = 1, 2$, with risk $\leq \epsilon < 1/2$. Then the test $T_{\phi_*, K}$ deciding on $H^K_\chi$, $\chi = 1, 2$, by accepting $H^K_1$ whenever $\phi(K)(\omega^K) := \sum_{k=1}^K \phi_*(\omega_k) \geq 0$ and accepting $H^K_2$ otherwise, satisfies

$$\text{Risk}(T_{\phi_*, K} | H^K_1, H^K_2) \leq \epsilon \ \forall K \geq \hat{K}_* = \frac{2}{1 - \frac{\ln(4(1-\epsilon))}{\ln(1/\epsilon)}} K_*.$$

Moreover, this risk bound remains true when the hypotheses $H^K_\chi$ are extended to $H^{\otimes, K}_\chi$ stating that the distribution $P$ of $\omega^K$ belongs to the quasi-direct $K$-th power of $P_\chi, \chi = 1, 2$. Note that $\hat{K}_*/K_* \to 2$ as $\epsilon \to +0.$

3.44
**Proof.** As we know, \( K_\ast \)-th power \( \mathcal{O}^{(K)} \) of \( \mathcal{O} \) is simple o.s. along with \( \mathcal{O} \), and \( \phi^{(K)}_\ast \) is the minimum risk detector for the families \( \mathcal{P}_\chi^{\otimes,K_\ast}, \chi = 1, 2 \), the risk of this detector being \( [\varepsilon_\ast(\mathcal{P}_1, \mathcal{P}_2)]^{K_\ast} \). By Proposition A as applied to \( \mathcal{O}^{(K)} \) in the role of \( \mathcal{O} \), we have

\[
[\varepsilon_\ast(\mathcal{P}_1, \mathcal{P}_2)]^{K_\ast} \leq 2\sqrt{\epsilon(1-\epsilon)} \Rightarrow \varepsilon_\ast(\mathcal{P}_1, \mathcal{P}_2) \leq [2\sqrt{\epsilon(1-\epsilon)}]^{1/K_\ast} < 1.
\]

By Detector Calculus, it follows that for \( K = 1, 2, \ldots \) it holds

\[
\text{Risk}[\phi^{(K)}_\ast | \mathcal{P}_1^{\otimes,K}, \mathcal{P}_2^{\otimes,K}] \leq [2\sqrt{\epsilon(1-\epsilon)}]^{K/K_\ast}
\]

and the right hand side is \( \leq \epsilon \) whenever \( K \geq \hat{K}_\ast \).  \( \square \)
Situation: We are given a simple o.s.

\[ \mathcal{O} = ((\Omega, \Pi), \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F}) \]

and a collection of nonempty convex compact subsets \( M_\ell, 1 \leq \ell \leq L \) giving rise to

- Families \( \mathcal{P}_\ell = \{P : P \text{ admits density } p_\mu, \mu \in M_\ell \text{ w.r.t. } \Pi\}, \ell = 1, \ldots, L \), along with quasi-direct powers \( \mathcal{P}_\ell^{\otimes K} = \bigotimes_{k=1}^{K} \mathcal{P}_\ell \) of \( \mathcal{P}_\ell \) and hypotheses \( H_\ell^{\otimes K} : P \in \mathcal{P}_\ell^{\otimes K} \) on the distribution \( P \) of \( K \)-repeated observation \( \omega^K = (\omega_1, \ldots, \omega_K) \),
- minimum-risk balanced single-observation detectors \( \phi_{\ell\ell'}(\omega) \) for \( \mathcal{P}_\ell, \mathcal{P}_{\ell'} \) along with their risks \( \varepsilon_*(\mathcal{P}_\ell, \mathcal{P}_{\ell'}) \), \( 1 \leq \ell < \ell' \leq L \), and \( K \)-repeated versions

\[ \phi_{\ell\ell'}^{K}(\omega^K) = \sum_{k=1}^{K} \phi_{\ell\ell'}(\omega_k) \]

of \( \phi_{\ell\ell'} \) such that

\[ \text{Risk}[\pi_{\ell\ell'}^{(K)}|H_\ell^{\otimes K}, H_{\ell'}^{\otimes K}] \leq [\varepsilon_*(\mathcal{P}_\ell, \mathcal{P}_{\ell'})]^{K}. \]
Assume that in addition to the above data, we are given a closeness relation $C$ on \{1, ..., $L$\}. Applying Calculus of Detectors, for every positive integer $K$, setting

$$
\theta_K = \left\| \varepsilon^K_{\star} (P_\ell, P_{\ell'}) \cdot \left\{ \begin{array}{ll}
1, & (\ell, \ell') \not\in C \\
0, & (\ell, \ell') \in C 
\end{array} \right\}_{\ell, \ell'=1}^L \right\|_{2,2}
$$

we can assemble the outlined data, in a computationally efficient fashion, into a $K$-observation test $T^K$ deciding on $H_\ell^K$, $1 \leq \ell \leq L$, with $C$-risk upper-bounded as follows:

$$\text{Risk}^C(T^K | H_1^K, ..., H_L^K) \leq \kappa \theta_K$$

($\kappa > 1$ can be selected to be as close to 1 as we want).
Proposition. In the just described situation, assume that for some $\epsilon < 1/2$ and $K_*$ in the nature there exists a $K_*$-observation test $T$, deterministic or randomized, deciding on the hypotheses

$$H_{\ell}^{\otimes,K_*} : \omega^{K_*} = (\omega_1, \ldots, \omega_{K_*})$$

is an i.i.d. sample drawn from a $P \in \mathcal{P}_\ell$, $\ell = 1, \ldots, L$, with $C$-risk $\leq \epsilon$. Then the test $T^K$ with

$$K \geq 2 \left[ \frac{1 + \ln(\kappa L) / \ln(1/\epsilon)}{1 - \ln(4(1 - \epsilon))/\ln(1/\epsilon)} \right] K_*$$

$\rightarrow 1$ as $\epsilon \rightarrow +0$

decides on $H_{\ell}^{\otimes,K}$, $\ell = 1, \ldots, L$, with $C$-risk $\leq \epsilon$ as well.
Proof. Let us fix \( \ell, \ell' \) such that \((\ell, \ell') \notin C\), and let us convert \( T \) into a simple \( K_* \)-observation test \( \tilde{T} \) deciding on \( H^\odot_{\ell}, K_* \), \( H^\odot_{\ell'}, K_* \) as follows: whenever \( \ell \in T(\omega K) \), \( \tilde{T} \) accepts \( H^\odot_{\ell}, K_* \) and rejects \( H^\odot_{\ell'}, K_* \), otherwise the test accepts \( H^\odot_{\ell'}, K_* \) and rejects \( H^\odot_{\ell}, K_* \). It is immediately seen that

\[
\text{Risk}(\tilde{T} | H^\odot_{\ell}, K_* , H^\odot_{\ell'}, K_* ) \leq \epsilon.
\]

Indeed, let \( P^K_* = P \times \ldots \times P \) be the distribution of \( \omega K_* \). Whenever \( P^K_* \) obeys \( H^\odot_{\ell}, K_* \), \( T \) must accept the hypothesis with \( P^K_* \)-probability \( \geq 1 - \epsilon \), whence

\[
\text{Risk}_1 (\tilde{T} | H^\odot_{\ell}, K_* , H^\odot_{\ell'}, K_* ) \leq \epsilon.
\]

If \( P^K_* \) obeys \( H^\odot_{\ell'}, K_* \), the \( P^K_* \)-probability of the event "\( T \) accepts \( H^\odot_{\ell}, K_* \) and rejects \( H^\odot_{\ell'}, K_* \)" is \( \leq \epsilon \), since \( H^\odot_{\ell'}, K_* , H^\odot_{\ell}, K_* \) are not \( C \)-close to each other

\[\Rightarrow \text{P}^K_* \text{-probability to reject } H^\odot_{\ell}, K_* \text{ is at least } 1 - \epsilon\]

\[\Rightarrow \text{Risk}_2 (\tilde{T} | H^\odot_{\ell}, K_* , H^\odot_{\ell'}, K_* ) \leq \epsilon.
\]
\( H_{\ell}^{\ominus,K}, H_{\ell'}^{\ominus,K} \) can be decided upon by a simple test with risk \( \leq \epsilon \)

- \( H_{\ell}^{\ominus,K_*}, H_{\ell'}^{\ominus,K_*} \) can be decided upon with risk \( \leq \epsilon < 1/2 \)

\[ \Rightarrow \varepsilon_*(\mathcal{P}_{\ell}^{\ominus,K_*}, \mathcal{P}_{\ell'}^{\ominus,K_*}) \leq 2\sqrt{\epsilon(1-\epsilon)} < 1 \text{ (Calculus of Detectors)} \]

\[ \Rightarrow \varepsilon_*(\mathcal{P}_{\ell}, \mathcal{P}_{\ell'}) \leq \left[2\sqrt{\epsilon(1-\epsilon)}\right]^{1/K_*} < 1 \text{ (since } \mathcal{O} \text{ is a simple o.s.)} \]

\[ \Rightarrow \theta_K \leq \left[2\sqrt{\epsilon(1-\epsilon)}\right]^{K/K_*} L \]

\[ \Rightarrow \text{Risk}^C(\mathcal{T}^K|H_1^{\ominus,K}, ..., H_L^{\ominus,K}) \leq \kappa \theta_K \leq \epsilon \text{ when} \]

\[ K/K_* \geq 2 \frac{1 + \ln(\kappa L)/\ln(1/\epsilon)}{1 - \ln(4(1-\epsilon))/\ln(1/\epsilon)}. \]
How it works: Illustration I
Predicting Outcome of Elections via Opinion Polls

Situation: \( L \) candidates are running for office, with just one to be elected, and every voter has already decided whom to vote for in the forthcoming elections. We want to predict elections’ outcome via Opinion poll where \( K \) randomly selected voters reveal their choices. How large should be \( K \) in order to predict the winner with a given confidence?

Model: Assume that \( K \) voters to be interviewed are drawn from the population uniformly and independently of each other. Denoting by \( \mu_\ell \) the fraction of voters intending to vote for candidate \( \#\ell \) in the entire population, we get a probability distribution \( \mu \) on the \( L \)-element set of candidates.

Note: Outcomes of \( K \) interviews form \( K \)-element i.i.d. sample \( \omega^K \) drawn from \( \mu \).

♠ Given small “winning margin” \( \delta > 0 \) and assuming that the distribution \( \mu \) of voters’ preferences is not a “\( \delta \)-tie” — the difference between the largest and the second largest entries in \( \mu \) is at least \( \delta \) — predicting the winner can be modeled as deciding on \( L \) convex hypotheses

\[
H_\ell : \mu \in P_\ell := \{ p \in \Delta_L : p_\ell \geq \delta + \max_{j \neq \ell} p_j \}, \ \ell = 1, \ldots, L
\]

\[
[\Delta_L = \{ p \in \mathbb{R}_+^L : \sum_\ell p_\ell = 1 \}]
\]

in Discrete o.s. via stationary \( K \)-repeated observation.
\[ H_\ell : \mu \in \mathcal{P}_\ell := \{ p \in \Delta_L : p_\ell \geq \delta + \max_{j \neq \ell} p_j \}, \; \ell = 1, \ldots, L \]

\[ \because \text{Our machinery applies as follows:} \]

- We solve \( L(L-1)/2 \) convex optimization problems

\[ \epsilon_{ij} = \max_{\mu, \nu} \left\{ \sum_i \sqrt{\mu_i \nu_i} : \mu \in \mathcal{P}_i, \nu \in \mathcal{P}_j \right\}, \; 1 \leq i < j \leq L. \]

with optimal solutions \( \mu^{ij}, \nu^{ij} \) giving rise to detectors

\[ \phi_{ij}(\omega) = \frac{1}{2} \ln \left( \frac{\mu^{ij}_{\omega}/\nu^{ij}_{\omega}}{\mu^{ij}_{\omega}/\nu^{ij}_{\omega}} \right), \; \omega \in \Omega = \{1, 2, \ldots, L\} \]

We set also

\[ \epsilon_{ji} = \epsilon_{ij}, \; \phi_{ji}(\cdot) = -\phi_{ij}(\cdot), \; 1 \leq i < j \leq m, \; \epsilon_{ii} = 0, \; \phi_{ii}(\cdot) \equiv 0, \; i \leq m \]

- We build the symmetric matrix \( E = \left[ \epsilon_{ij}^{K} \right]_{i,j \leq L} \)

The Perron-Frobenius eigenvector \( f \) of \( E \) gives rise to the detectors

\[ \phi_{ij}^{(K)}(\omega^K) = \sum_{k=1}^{K} \phi_{ij}(\omega_k) + \ln(f_i/f_j) \]

and the test which accepts \( H_\ell \) if and only if \( p_{\ell j}^{(K)}(\omega^K) > 0 \) for all \( j \neq \ell \).

The risk of this test does not exceed the spectral norm of \( E \).

\[ \because \text{Given } \delta \text{ and upper bound } \epsilon \text{ on the risk of predicting elections' outcome, we can specify the smallest size } K \text{ of Opinion poll resulting in prediction of required quality.} \]
Results, confidence level \( 1 - \epsilon = 0.95 \):

<table>
<thead>
<tr>
<th>( L = 2 )</th>
<th>( L = 4 )</th>
<th>( L = 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10% )</td>
<td>( 5% )</td>
<td>( 2.5% )</td>
</tr>
<tr>
<td>( 166 \vee 597 )</td>
<td>( 664 \vee 2,394 )</td>
<td>( 2,657 \vee 9,584 )</td>
</tr>
<tr>
<td>( 166 \vee 815 )</td>
<td>( 664 \vee 3,272 )</td>
<td>( 2,657 \vee 13,098 )</td>
</tr>
<tr>
<td>( 166 \vee 984 )</td>
<td>( 664 \vee 3949 )</td>
<td>( 2,657 \vee 15,809 )</td>
</tr>
</tbody>
</table>

- upper bounds on poll sizes are given by our machinery
- lower bounds on poll sizes stem from lower bounding of pairwise risks

USA Presidential Elections-2016:

<table>
<thead>
<tr>
<th>State</th>
<th>Actual margin</th>
<th>Poll size, lower bound</th>
<th>Poll size, upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Georgia</td>
<td>5.1%</td>
<td>638</td>
<td>2,301</td>
</tr>
<tr>
<td>Wisconsin</td>
<td>0.77%</td>
<td>28,008</td>
<td>101,043</td>
</tr>
<tr>
<td>Pennsylvania</td>
<td>0.72%</td>
<td>32,030</td>
<td>115,555</td>
</tr>
<tr>
<td>Michigan</td>
<td>0.23%</td>
<td>313,864</td>
<td>1,132,333</td>
</tr>
</tbody>
</table>

Note: the total number of Michigan voters participated in Presidential Elections-2016 was 4,799,284

3.53
How it Works: Illustration II
Selecting the Best in a Family of Estimates

♣ Problem:
• We are given a simple o.s. \( \mathcal{O} = ((\Omega, \Pi), \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F}) \) and have access to stationary \( K \)-repeated observations

\[
\omega_k \sim p_A(x_*)(\cdot), \quad k = 1, \ldots, K,
\]

of unknown signal \( x_* \) known to belong to a given convex compact set \( X \subset \mathbb{R}^n \).

• \( x \mapsto A(x) \): affine mapping such that \( A(X) \subset \mathcal{M} \).

• We are given \( M \) candidate estimates \( x_i \in \mathbb{R}^n, \ 1 \leq i \leq M \), of \( x_* \), a norm \( \| \cdot \| \) on \( \mathbb{R}^n \), and a reliability tolerance \( \epsilon \in (0, 1) \)

• Ideal Goal: Use observations \( \omega_1, \ldots, \omega_K \) to identify \((1 - \epsilon)\)-reliably the \( \| \cdot \| \)-closest to \( x_* \) point among \( x_1, \ldots, x_M \).

• Actual Goal:
Given \( \alpha \geq 1, \beta \geq 0 \) and a grid \( \Gamma = \{ r_0 > r_1 > \ldots > r_N > 0 \} \), use observations \( \omega_1, \ldots, \omega_K \) to identify \((1 - \epsilon)\)-reliably a point \( x_i(\omega_K) \) such that

\[
\|x_* - x_i(\omega_K)\| \leq \alpha \rho(x_*) + \beta
\]

\[
\rho(x) := \min\{ r : r \in \Gamma, r \geq \min_i \|x - x_i\| \}
\]

\( \rho(x) \) is grid approximation of \( \min_i \|x - x_i\| \)
A Solution:

• Introduce $M(N + 1)$ hypotheses

$$H_{ij} : \omega_k \sim p_A(x)(\cdot) \text{ for some } x \in X_{ij} := \{x \in X : \|x - x_i\| \leq r_j\}.$$  

• Define closeness $C = C_{\alpha,\beta}$: $ij$ $C$-close to $i'j'$ iff

$$\|x_i - x_i'\| \leq \bar{\alpha}(r_j + r_{j'}) + \beta$$  

[\bar{\alpha} = \frac{\alpha - 1}{2}]  

• Build minimum risk pairwise tests $\phi_{ij,i'j'}$, along with their risks $\epsilon_{ij,i'j'}$, for hypotheses $H_{ij}$.

  • $\epsilon_{ij,i'j'} = 1$ for some $(ij, i'j') \not\in C \Rightarrow$ reject $(\alpha, \beta)$
  
  • $\epsilon_{ij,i'j'} < 1$ for all $(ij, i'j') \not\in C \Rightarrow$

(a) find the smallest $K = K_\epsilon(\alpha, \beta)$ such that $K$-observation detector-based test $T^K$ decides on $\{H_{ij} : i \leq M, 0 \leq j \leq N\}$ with $C$-risk $\leq \epsilon$.

(b) applying $T^K$ to observation $\omega^K$, select among the accepted hypotheses $H_{ij}$ the one, $H_{i(\omega^K)j(\omega^K)}$ with the largest $j$. $i(\omega^K)$ is built.
Fact: In the situation in question, whenever \((\alpha, \beta)\) is not rejected, the resulting inference \(\omega^K \mapsto i(\omega^K)\) meets the design specifications:

\[
(x^*_\in \mathcal{X}, \omega_k \sim p_A(x^*_\in) (\cdot) \text{ are independent across } k \leq K) \Rightarrow \\
\text{Prob}\{\|x - x_{i(\omega^K)}\| \leq \alpha \rho(x^*_\in) + r\} \geq 1 - \epsilon.
\]

Fact: In the situation in question, assume that for some \(\epsilon \in (0, 1/2), a, b \geq 0\) and positive integer \(K\) in the nature there exists an inference \(\omega^{\bar{K}} \mapsto i^*_\in(\omega^{\bar{K}})\) such that

\[
\forall x^*_\in \mathcal{X} : \text{Prob}\{\|x^*_\in - x^*_i(\omega^{\bar{K}})\| \leq a \rho(x^*_\in) + b\} \geq 1 - \epsilon.
\]

Then the pair \((\alpha = 2a + 1, \beta = 2b)\) is not rejected by the above construction, and

\[
K_\epsilon(\alpha, \beta) \leq \text{Ceil} \left( \frac{2 \ln(M(N + 1))}{1 - \ln(4(1 - \epsilon))} \right) / \ln(1/\epsilon).
\]
**Numerical illustration:** Given noisy observation

\[ \omega = Ax + \sigma \xi, \quad \xi \sim \mathcal{N}(0, I_n) \]

of the “discretized primitive” \( Ax \) of a signal \( x = [x^1; \ldots; x^n] \in \mathbb{R}^n \):

\[ [Ax]_j = \frac{1}{n} \sum_{s=1}^{j} x^s, \quad 1 \leq j \leq n, \]

for \( i = 1, \ldots, \kappa \) we have built Least Squares polynomial, of order \( i - 1 \), approximations \( x_i \) of \( x \):

\[ x_i = \arg\min_{x \in \mathcal{X}_i} \| Ax - \omega \|_2^2 \]

\[ \mathcal{X}_i = \{ x = [x^1; \ldots; x^n] : x^s \text{ is polynomial, of degree } \leq i - 1, \text{ in } s \} \]

and now want to use \( K \) additional observations to identify the nearly closest to \( x_* \), in the norm

\[ \| u \| = \frac{1}{n} \sum_{i=1}^{n} |u_i| \]

on \( \mathbb{R}^n \), among the points \( x_i, 1 \leq i \leq \kappa \).
♠ Experiment $[\epsilon = 0.01, n = 128, \sigma = 0.01, \kappa = 5, \alpha = 3, \beta = 0.05]$  

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|x - x_i|$</td>
<td>0.5348</td>
<td>0.33947</td>
<td>0.23342</td>
<td>0.16313</td>
<td>0.16885</td>
</tr>
</tbody>
</table>

distances from $x_*$ to $x_i$

- Computation yielded $K = 3$. But

  — with $K = 3$, in sample of 1000 simulations, not a single case of wrong identification of the exactly closest to $x_*$ point was observed, i.e., we always got $\|x - x_{i(\omega^3)}\| = \rho(x_*)$, in spite of the theoretical guarantee as poor as $\|x_* - x_{i(\omega^3)}\| \leq 3\rho(x_*) + 0.05$ — the same was true when $K = 3$ was replaced with $K = 1$;

  — replacing $K = 3$ with $K = 1$ and increasing $\sigma$ from 0.01 to 0.05, the procedure started to make imperfect conclusions. However, the exactly closest to $x_*$ point $x_4$ was identified correctly in as many as 961 of 1000 simulations, and the empirical mean $E\{\|x_* - x_{i(\omega^1)}\| - \rho(x_*)\}$ was as small as 0.0024.

3.58
How it Works: Illustration III
Recovering Linear-Fractional Function of a Signal

♣ Problem: An unknown signal $x$ known to belong to a given convex compact set $X \subset \mathbb{R}^n$ is observed according to

$$\omega = Ax + \sigma \xi, \; \xi \sim \mathcal{N}(0, I_d)$$

Our goal is to recover the value at $x$ of a linear-fractional functional $F(z) = f^T z / e^T z$, with $e^T z > 0, z \in X$.

♠ Illustration: We are given noisy measurements of voltages $V_i$ at some nodes $i$ and currents $I_{ij}$ in some arcs $(i, j)$ of an electric circuit, and want to recover the resistance of a particular arc $(\hat{i}, \hat{j})$:

$$r_{\hat{i}\hat{j}} = \frac{V_{\hat{j}} - V_{\hat{i}}}{I_{\hat{i}\hat{j}}}$$
Circuit with 8 nodes and 11 arcs

\[ x = [\text{voltages at nodes}; \text{currents in arcs}] \]

\[ Ax = [\text{observable voltages}; \text{observable currents}] \]

- Currents are measured in blue arcs only
- Voltages are measured in magenta nodes only
- We want to recover resistance of red arc

\[ x : \begin{cases} \text{conservation of current, except for nodes ##1,8} \\ \text{zero voltage at node #1, nonnegative currents} \\ \text{current in red arc at least 1, total of currents at most 33} \\ \text{Ohm Law, resistances of arcs between 1 and 10} \end{cases} \]
Strategy: Given $L$,

- split the range $\Delta = [\min_{x \in X} F(x), \max_{x \in X} F(x)]$ into $L$ consecutive bins $\Delta_\ell$ of length $\delta_L = \text{length}(\Delta)/L$,
- define the convex compact sets
  $$X_\ell = \{x \in X : F(x) \in \Delta_\ell\}, \quad M_\ell = \{Ax : x \in X_\ell\}, \quad 1 \leq \ell \leq L$$

- decide on $L$ hypotheses $H_\ell : P = \mathcal{N}(\mu, \sigma^2 I)$, $\mu \in M_\ell$ on the distribution $P$ of observation $\omega = Ax + \sigma \xi$ up to closeness $C$ “$H_\ell$ is close to $H_{\ell'}$ iff $|\ell - \ell'| \leq 1$”
- estimate $F(x)$ by the center of masses of all accepted bins.

Fact: For the resulting test $T$, with probability $\geq 1 - \text{Risk}^C(T|H_1, ..., H_L)$ the estimation error does not exceed $\delta_L$. 

3.61
\textbf{Implementation and results:} Given target risk $\epsilon$ and $L$, we selected the largest $\sigma$ for which $\text{Risk}^C(T|H_1,\ldots,H_L)$ is $\leq \epsilon$.

- This is what we get in our Illustration for $\epsilon = 0.01$: $\Delta = [1, 10]$

<table>
<thead>
<tr>
<th>$L$</th>
<th>8</th>
<th>16</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_L$</td>
<td>$9/8 \approx 1.13$</td>
<td>$9/16 \approx 0.56$</td>
<td>$9/32 \approx 0.28$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.024</td>
<td>0.010</td>
<td>0.005</td>
</tr>
<tr>
<td>$\sigma_{\text{opt}}/\sigma \leq$</td>
<td>1.31</td>
<td>1.31</td>
<td>1.33</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.031</td>
<td>0.013</td>
<td>0.006</td>
</tr>
<tr>
<td>$\sigma_{\text{opt}}/\sigma \leq$</td>
<td>1.01</td>
<td>1.06</td>
<td>1.08</td>
</tr>
</tbody>
</table>

- $\sigma_{\text{opt}}$ – the largest $\sigma$ for which “in the nature” there exists a test deciding on $H_1,\ldots,H_L$ with $\bar{C}$-risk $\leq 0.01$

- \textbf{Red data:} Risks $\epsilon_{\ell\ell'}$ of pairwise tests are bounded via risks of optimal detectors, $C$-risk of $T$ is bounded by

$$\left\| \left[ \epsilon_{\ell\ell'} \cdot \chi_{(\ell,\ell') \not\in C} \right]_{\ell,\ell'=1}^L \right\|_{2,2} ;$$

- \textbf{Brown data:} Risks $\epsilon_{\ell\ell'}$ of pairwise tests are bounded via error function, $C$-risk of $T$ is bounded by

$$\max_{\ell} \sum_{\ell':(\ell,\ell') \not\in C} \epsilon_{\ell\ell'}.$$
Illustration II Revisited
Recovering $N$-Convex Functionals

♣ **Fact:** The construction used to recover linear-fractional function can be extended to recovering $N$-convex functionals.

♠ **Definition:** Let $X \subset \mathbb{R}^n$ be a convex compact set, $F : X \to \mathbb{R}$ be a continuous function, and $N$ be a positive integer. We say that $F$ is $N$-convex, if for every real $a$ the sets

$$X^{a,\geq} = \{x \in X : F(x) \geq a\}, \quad X^{a,\leq} = \{x \in X : F(x) \leq a\}$$

can be represented as the unions of at most $N$ convex compact sets.

**Examples:** A. Fractional-linear function $F(x) = \frac{e(x)}{d(x)}$ with positive on $X$ denominator is $1$-convex:

$$\{x \in X : F(x) \geq a\} = \{x \in X : e(x) - ad(x) \geq 0\}$$
B. If $F_\chi$ is $N_\chi$-convex on $X$, $\chi = 1, 2$, then $\max[F_1, F_2]$ and $\min[F_1, F_2]$ are $\max[N_1 + N_2, N_1N_2]$-convex on $X$:

$$
\begin{align*}
X_{\chi}^{a, \geq} &:= \{ x \in X : F_\chi(x) \geq a \} = \bigcup_{\nu = 1}^{N_\chi} U_{\nu, \chi}^a, \chi = 1, 2 \quad [U, V: \text{convex}] \\
X_{\chi}^{a, \leq} &:= \{ x \in X : F_\chi(x) \leq a \} = \bigcup_{\nu = 1}^{N_\chi} V_{\nu, \chi}^a
\end{align*}
$$

$$
\Rightarrow \quad \begin{cases}
\{ x \in X : \max[F_1(x), F_2(x)] \geq a \} = \bigcup_{\mu \leq N_1} U_{\mu, 1}^a \bigcup_{\nu \leq N_2} U_{\nu, 2}^a \\
\{ x \in X : \max[F_1(x), F_2(x)] \leq a \} = \bigcup_{\mu \leq N_1, \nu \leq N_2} [V_{\mu, 1}^a \cap V_{\nu, 2}^a]
\end{cases}
$$
C. Conditional quantile. Let a probabilistic vector \( 0 < p \in \mathbb{R}^n \) represent probability distribution on finite subset \( S = \{s_1 < s_2 < \ldots < s_n\} \) of the real axis.

Regularized \( \alpha \)-quantile \( q_\alpha[p] \) is defined as follows:

— we pass from \( p \) to the probability distribution \( P \) in \( \Delta = [s_1, s_n] \) by assigning probability mass \( p_1 \) to \( s_1 \) and uniformly spreading the probability masses \( p_i, i > 1 \), over the segments \([s_{i-1}, s_i]\)

— \( q_\alpha[p] \) is the usual \( \alpha \)-quantile of \( P \):

\[
q_\alpha[p] = \min \{ s \in \Delta : \text{Prob}_{\xi \sim P} \{ \xi \leq s \} \geq \alpha \}
\]

Fact: Let \( X = \{x(t, s) : t \in T, s \in S\} \) be a convex compact set comprised of nonvanishing probability distributions on 2D grid \( T \times S \), let \( t \in T \), and let \( \left\{ x|_t(s) = \frac{x(t, s)}{\sum_{s' \in S} x(t, s')} , s \in S \right\} \) be the conditional, given \( t \), probability distribution on \( S \) induced by \( x \in X \). Then

\[
f_{\alpha,t}(x) = q_\alpha[x|_t(\cdot)]
\]

is 1-convex function of \( x \in X \).
Problem of interest: Given
- convex compact set $X \subset \mathbb{R}^n$,
- $N$-convex functional $F : X \to \mathbb{R}$,
- a collection $X_j$, $\ell = 1, \ldots, J$, of convex compact subsets of $X$,
- stationary $K$-repeated observations $\omega_1, \ldots, \omega_K$ stemming, via simple o.s., from unknown signal $x \subset \bigcup_{j=1}^J X_j$,

we want to recover $F(x)$.

Strategy: Given $L$, we
- Split the range $\Delta = \left[ \min_{x \in X} F(x), \max_{x \in X} F(x) \right]$ into $L$ consecutive bins $\Delta_\ell$ of length $\delta_L = \text{length}(\Delta)/L$,
- Observe that by $N$-convexity of $F$ every one of the sets
  \[ \{ x \in \bigcup_{j=1}^J X_j : F(x) \in \Delta_\ell \} \]
  is the union of at most $N^2J$ convex compact sets $Y^\ell_s$
- Associate with the nonempty among the sets $Y^\ell_s$ the hypotheses “observation stems from a signal from $Y^\ell_s$”
- Define closeness $C$ on the resulting collection of hypotheses $H_1, \ldots, H_{L}$, $L \leq N^2JL$, by claiming $H_\mu$ and $H_\nu$ $C$-close iff both hypotheses stem from the same or from two consecutive bins $\Delta_\ell$
- Use our machinery for testing multiple convex hypotheses in simple o.s. to build a test $\mathcal{T}_K$ deciding on $H_1, \ldots, H_{L}$ up to closeness $C$ via $K$-repeated observation.
Apply the test $\mathcal{T}_K$ to observations $\omega_1, \ldots, \omega_K$ and take as the estimate of $F(x)$ the center of masses of all bins associated with the hypotheses accepted by the test.

Same as in the above fractional-linear example, it is immediately seen that

- The probability for the recovery error to be $> \delta_L$ is upper-bounded by the $\mathcal{C}$-risk of $\mathcal{T}_K$.

In addition, with our estimate, the number of observations $K$ required to ensure recovery error $\leq \delta_L$ with a given reliability $1 - \epsilon$, $\epsilon \ll 1$, is within logarithmic in $N, J, L$ factor off the “ideal” number of observations needed to achieve, with reliability $1 - \epsilon$, recovery error $\delta_L/2$. 

3.67
Sequential Hypothesis Testing

♠ Motivating example: Opinion polls. Recall the elections’ story:
• Population-wide elections with \( L \) candidates are to be held.
• Preferences of a voter are represented by \( L \)-dimensional basic orth with 1 in position \( \ell \) meaning voting for candidate \( \#\ell \).

Equivalently: Preference \( \omega \) of a voter is a vertex in the \( L \)-dimensional probabilistic simplex
\[
\Delta_L = \{ p \in \mathbb{R}^L : p \geq 0, \sum_{\ell} p_\ell = 1 \}.
\]
• The average \( \mu = [\mu_1; \ldots; \mu_L] \) of preferences of all voters “encodes” election’s outcome: \( \mu_\ell \) is the fraction of voters supporting \( \ell \)-th candidate, and the winner corresponds to the largest entry in \( \mu \) (assumed to be uniquely defined).

Note: \( \mu \) is a probabilistic vector: \( \mu \in \Delta_L \). We think of \( \mu \) as of a probability distribution on the \( L \)-element set \( \Omega = \text{Ext}(\Delta_L) \) of vertices of \( \Delta_L \).

• Our goal is to design opinion poll – to select \( K \) voters at random from the uniform distribution on the voters’ population and to observe their preferences, in order to predict, with reliability \( 1 - \epsilon \), election’s outcome.

♠ Poll’s model is drawing stationary \( K \)-repeated observation \( \omega^K = (\omega_1, \ldots, \omega_K) \), \( \omega_k \in \Omega \), from distribution \( \mu \).

3.68
We assume that the elections never end with “near tie,” that is, the fraction of votes for the winner is at least by a known margin \( \delta \) larger than the fraction of votes for every no-winner, and introduce \( L \) hypotheses on the distribution \( \mu \) from which \( \omega_1, \ldots, \omega_K \) are drawn:

\[
H_\ell : \mu \in \mathcal{P}_\ell = \{ \mu \in \Delta_L : \mu_\ell \geq \mu_{\ell'} + \delta, \forall \ell' \neq \ell \}, \ \ell = 1, \ldots, L
\]

Our goal is to specify \( K \) in a way which allows to decide on \( H_1, \ldots, H_L \) via stationary \( K \)-repeated observations with risk \( \leq \epsilon \).

We are in the case of Discrete o.s., and can use our machinery to build a near-optimal \( K \)-observation test deciding on \( H_1, \ldots, H_L \) up to trivial closeness \( C \) “\( H_\ell \) is close to \( H_{\ell'} \) iff \( \ell = \ell' \)” and then select the smallest \( K \) for which the \( C \)-risk of this test is \( \leq \epsilon \).
Illustration $L = 2$: In this case $\Omega$ is two-point set of basic orths in $\mathbb{R}^2$, the minimum risk single-observation detector is

$$
\phi_*(\omega) = \frac{1}{2} \ln \left( \frac{1 + \delta}{1 - \delta} \right) [\omega^1 - \omega^2] : \Omega \to \mathbb{R}
$$

and $\text{Risk}[\phi_*|\mathcal{P}_1, \mathcal{P}_2] = 1 - \delta^2$

$$
\Rightarrow K = \text{Ceil} \left( \frac{\ln(1/\epsilon)}{\ln(1/(1-\delta^2))} \right) \approx \frac{1}{\delta^2} \ln(1/\epsilon).
$$

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$0.3162$</th>
<th>$0.1000$</th>
<th>$0.0316$</th>
<th>$0.0100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K \lor K$</td>
<td>16 $\lor$ 57</td>
<td>166 $\lor$ 597</td>
<td>1,660 $\lor$ 5,989</td>
<td>16,607 $\lor$ 59,912</td>
</tr>
</tbody>
</table>

$K$: lower bound on optimal poll size

Poll sizes, $\epsilon = 0.05$
Bad news: Required size of opinion poll grows rapidly as “winning margin” decreases.

♠ Question: Can we do better?

♠ Partial remedy: Let us pass to sequential tests, where we attempt to make conclusion before all \( K \) respondents required by the worst-case-oriented analysis are interviewed.

Hope: If elections are about to be “landslide” (i.e., in unknown to us actual distribution \( \mu_* \) of voters’ preferences the winner beats all other candidates by margin \( \delta_* \gg \delta \)), the winner hopefully can be identified after a relatively small number of interviews.
Strategy. We select a number $S$ of attempts and associate with attempt $s$ number $K(s)$ of observations, $K(1) < \ldots < K(S)$.

$s$-th attempt to make inference is made when $K(s)$ observations are collected. When it happens, we apply to the collected so far observation $\omega^{K(s)} = (\omega_1, \ldots, \omega_{K(s)})$ a test $T_s$ which, depending on $\omega^{K(s)}$,

– either accepts exactly one of the hypotheses $H_1, \ldots, H_L$, in which case we terminate,

– or claims that information collected so far does not allow to make an inference, in which case we pass to collecting more observations (when $s < S$) or terminate (when $s = S$).

Specifications: We want the overall procedure to be

– conclusive: an inference should be made in one of the $S$ attempts (thus, when attempt $S$ is reached, making inference becomes a must);

– reliable: whenever the true distribution $\mu_*$ underlying observations obeys one of our $L$ hypotheses, the $\mu_*$-probability for this hypothesis to be eventually accepted should be $\geq 1 - \epsilon$, where $\epsilon \in (0, 1)$ is a given in advance risk bound.
An implementation:

- We select somehow the number of attempts $S$ and set $\delta_s = \delta^{s/S}$ so that $\delta_1 > \delta_2 > ... > \delta_S = \delta$.

Besides this, we split risk bound $\epsilon$ into $S$ parts $\epsilon_s$: $\epsilon_s > 0$, $s \leq S$ & $\sum_{s=1}^{S} \epsilon_s = \epsilon$.

- For $s < S$, we define $2L$ hypotheses

\[
H_{2\ell-1}^s = H_\ell : \mu \in \mathcal{P}_{2\ell-1}^s = \mathcal{P}_\ell := \{ \mu \in \Delta_L : \mu_\ell \geq \delta_s + \max_{\ell' \neq \ell} \mu_{\ell'} \} \\
\text{“weak hypothesis”}
\]

\[
H_{2\ell}^s = \{ \mu \in \mathcal{P}_{2\ell}^s := \{ \mu \in \Delta_L : \mu_\ell \geq \delta_s + \max_{\ell' \neq \ell} \mu_{\ell'} \} \subset \mathcal{P}_\ell \\
\text{“strong hypothesis”}
\]

$1 \leq \ell \leq L$, and assign $H_{2\ell-1}^s$ and $H_{2\ell}^s$ with color $\ell$, $1 \leq \ell \leq L$.

- For $s = S$ we introduce $L$ hypotheses $H_S^\ell = H_\ell$, $\ell = 1, ..., L$, with $H_S^\ell$ assigned color $\ell$.

- For $s < S$, we introduce closeness relation $C_s$ on the collection of hypotheses $H_1^s, ..., H_{2L}^s$ as follows:

  - the only hypotheses close to a strong hypothesis $H_{2\ell}^s$ are the hypotheses $H_{2\ell}^s$ and $H_{2\ell-1}^s$ of the same color;

  - the only hypotheses close to a weak hypothesis $H_{2\ell-1}^s$ are all weak hypotheses and the strong hypothesis $H_{2\ell}$ of the same color as $H_{2\ell-1}$.

- For $s = S$, the $C_s$-closeness is trivial: $H_S^\ell \equiv H_\ell$ is $C_S$-close to $H_S^{\ell'} \equiv H_{\ell'}$ if and only if $\ell = \ell'$.
3-candidate hypotheses in probabilistic simplex $\Delta_3$

- **[weak green] $M_1$** (dark green + light green): candidate A wins with margin $\geq \delta_s$
- **[strong green] $M_1^s$** (dark green): candidate A wins with margin $\geq \delta_s > \delta_S$
- **[weak red] $M_2$** (dark red + pink): candidate B wins with margin $\geq \delta_s > \delta_S$
- **[strong red] $M_2^s$** (dark red): candidate B wins with margin $\geq \delta_s > \delta_S$
- **[weak blue] $M_3$** (dark blue + light blue): candidate C wins with margin $\geq \delta_s$
- **[strong blue] $M_3^s$** (dark blue): candidate C wins with margin $\geq \delta_s > \delta_S$

- $H_{2\ell-1}^s: \mu \in M_\ell$ [weak hypothesis]
  - weak hypothesis $H_{2\ell-1}^s$ is $C_s$-close to itself, to all other weak hypotheses and to strong hypothesis $H_{2\ell}^s$ of the same color as $H_{2\ell-1}^s$

- $H_{2\ell}^s: \mu \in M_\ell^s$ [strong hypothesis]
  - strong hypothesis $H_{2\ell}^s$ is $S$-close only to itself and to weak hypothesis $H_{2\ell-1}^s$ of the same color as $H_{2\ell-1}^s$
• Note: We are in the case of stationary repeated observations in Discrete o.s., the hypotheses $H_j^s$ are of the form “i.i.d. observations $\omega_1, \omega_2, ...$ are drawn from distribution $\mu \in M_j^s$ with nonempty closed convex sets $M_j^s \subset \Delta_L$,” and sets $M_j^s, M_{j'}^s$ with $(j, j') \notin C_s$ do not intersect

⇒ the risks of the minimum-risk pairwise detectors for $P_j^s, P_{j'}^s, (j, j') \notin C_s$, are $< 1$

⇒ we can efficiently find out the smallest $K = K(s)$ for which our machinery produces a test $T = T_s$ deciding, via stationary $K(s)$-repeated observations, on the hypotheses ${H_j^s}_j$ with $C_s$-risk $\leq \epsilon_s$.

• It is easily seen that $K(1) < K(2) < ... < K(S - 1)$. In addition, discarding all attempts $s < S$ with $K(s) < K(S)$ and renumbering the remaining attempts, we may assume w.l.o.g. that $K(1) < K(2) < ... < K(S)$.

♦ Our inference routine works as follows: we observe $\omega_k, k = 1, 2, ..., K(S)$ (i.e., carry interviews with one by one randomly selected voters), and perform $s$-th attempt to make conclusion when $K(s)$ observations are acquired ($K(s)$ interviews are completed).

At $s$-th attempt, we apply the test $T_s$ to observation $\omega^{K(s)}$. If the test does accept some of the hypotheses $H_j^s$ and all accepted hypotheses have the same color $\ell$, we accept $\ell$-th of our original hypotheses $H_1, ..., H_L$ (i.e., predict that $\ell$-th candidate will be the winner) and terminate, otherwise we proceed to next observations (i.e., next interviews) (when $s < S$) or claim the winner to be, say, the first candidate and terminate (when $s = S$).

3.75
Facts:

- The risk of the outlined sequential hypothesis testing procedure is $\leq \epsilon$: whenever the distribution $\mu_*$ underlying observations obeys hypothesis $H_\ell$ for some $\ell \leq L$, the $\mu_*$-probability of the event "$H_\ell$ is the only accepted hypothesis" is at least $1 - \epsilon$.

- The worst-case volume of observations $K(S)$ is within logarithmic factor from the minimal number of observations allowing to decide on the hypotheses $H_1, \ldots, H_L$ with risk $\leq \epsilon$.

- Whenever the distribution $\mu_*$ underlying observations obeys strong hypothesis $H_{2\ell}^s$ for some $\ell$ and $s$ ("distribution $\mu_*$ of voters’ preferences corresponds to winning margin at least $\delta_s$"), the conclusion, with $\mu_*$-probability $\geq 1 - \epsilon$, will be made in course of the first $s$ attempts (i.e., in course of the first $K(s)$ interviews).

Informally: In landslide elections, the winner will be predicted reliably after a small number of interviews.
How it Works: 2-Candidate Elections

♠ Setup:
• # of candidates \( L = 2 \)
• # \( \delta_s = 10^{-s/4} \)
• range of # of attempts \( S \): \( 1 \leq S \leq 8 \)

♠ Numerical Results:

<table>
<thead>
<tr>
<th>( S )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta = \delta_s )</td>
<td>0.5623</td>
<td>0.3162</td>
<td>0.1000</td>
<td>0.0562</td>
<td>0.0316</td>
<td>0.0100</td>
</tr>
<tr>
<td>( K )</td>
<td>25</td>
<td>88</td>
<td>287</td>
<td>917</td>
<td>9206</td>
<td>92098</td>
</tr>
<tr>
<td>( K(S) )</td>
<td>25</td>
<td>152</td>
<td>1594</td>
<td>5056</td>
<td>16005</td>
<td>160118</td>
</tr>
</tbody>
</table>

Volume \( K \) of non-sequential test, number of attempts \( S \) and worst-case volume \( K(S) \) of sequential test as functions of winning margin \( \delta = \delta_s \). Risk \( \epsilon \) is set to 0.01.

Note: Worst-case volume of sequential test is essentially worse than the volume of non-sequential test.

But: When drawing the true distribution \( \mu^* \) of voters’ preferences at random from the uniform distribution on the set of \( \mu \)’s with winning margin \( \geq 0.01 \), the typical size of observations used by Sequential test with \( S = 8 \) prior to termination is \( \ll K(S) \):

Empirical Volume of Sequential test

<table>
<thead>
<tr>
<th>median</th>
<th>mean</th>
<th>60%</th>
<th>65%</th>
<th>75%</th>
<th>80%</th>
<th>85%</th>
<th>90%</th>
<th>95%</th>
<th>100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>177</td>
<td>9182</td>
<td>177</td>
<td>397</td>
<td>617</td>
<td>1223</td>
<td>1829</td>
<td>8766</td>
<td>87911</td>
<td>160118</td>
</tr>
</tbody>
</table>

Column “X%”: empirical X%-quantile of test’s volume. Data over 1,000 experiments.
Empirical risk: 0.01

3.77
Observation: In our Hypothesis Testing setup, observation scheme is our “environment” and is completely out of our control. However, there are situations where the observation scheme is under our partial control.

Example: Opinion Poll revisited. In our original Opinion Poll problem, a particular voter was represented by basic orth \( \omega = [0; \ldots; 0; 1; 0; \ldots; 0] \in \mathbb{R}^L \), with entry 1 in position \( \ell \) meaning that the voter prefers candidate \( \ell \) to all other candidates. Our goal was to predict the winner by observing preferences of respondents selected at random from uniform distribution on voters’ population.

However: Imagine we can split voters in \( I \) non-intersecting groups (say, according to age, education, gender, income, occupation,...) in such a way that we have certain a priori knowledge of the distribution of preferences within the groups. In this situation, our poll can be organized as follows:

- We assign the groups with nonnegative weights \( q_i \) summing up to 1
- To organize an interview, we first select at random one of the groups, with probability \( q_i \) to select group \( i \), and then select a respondent from \( i \)-th group at random, from uniform distribution on the group.
• We assign the groups with nonnegative weights $q_i$ summing up to 1

• To organize an interview, we first select at random one of the groups, with probability $q_i$ to select group $i$, and then select a respondent from $i$-th group at random, from uniform distribution on the group.

**Note:** When $q_i$ is equal to the fraction $\theta_i$ of group $i$ in the entire population, the above policy reduces to the initial one. It can make sense, however, to use $q_i$ different from $\theta_i$, with $q_i \ll \theta_i$ if a priori information about preferences of voters from $i$-th group is rich, and $q_i \gg \theta_i$ if this information is poor. Hopefully, this will allow us to make more reliable predictions with the same total number of interviews.
★ The model of outlined situation is as follows:

- We characterize distribution of preferences within group $i$ by vector $\mu^i \in \Delta_L$. for $1 \leq \ell \leq L$, $\ell$-th entry in $\mu^i$ is the fraction of voters in group $i$ voting for candidate $\ell$; **Note:** The population-wide distribution of voters’ preferences is $\mu = \sum_{i=1}^{I} \theta_i \mu^i$.

- A priori information on distribution of preferences of voters from group $i$ is modeled as the inclusion $\mu^i \in M^i$, for some known subset $M^i \subset \Delta_L$ which we assume to be nonempty convex compact set.

- Output of particular interview is pair $(i, j)$, where $i \in \{1, \ldots, I\}$ is selected at random according to probability distribution $q$, and $j$ is the candidate preferred by respondent selected from group $i$ at random, according to uniform distribution on the group.

$\Rightarrow$ Our observation (outcome of an interview) becomes

$$\omega := (i, \ell) \in \Omega = \{1, \ldots, I\} \times \{1, \ldots, L\}, \text{ Prob}\{\omega = (i, j)\} = p(i, j) := q_i \mu^i_j.$$ 

The hypotheses to be decided upon are

$$H_\ell[q]: p \in \mathcal{P}_\ell[q] := \left\{ p_{ij} = q_i \mu^i_j \right\}_{1 \leq i \leq I, 1 \leq j \leq L} : \left[ \sum_{i} \theta_i \mu^i \right] _\ell \geq \delta + \left[ \sum_{i} \theta_i \mu^i \right] _{\ell'} \forall (\ell' \neq \ell) \}

H_\ell[q]$, $\ell = 1, \ldots, L$, states that the “signal” $\bar{\mu} = [\mu^1; \ldots; \mu^I]$ underlying distribution $p$ of observations $\omega$ induces population-wide distribution $\sum_i \theta_i \mu^i$ of votes resulting in electing candidate $\ell$ with winning margin $\geq \delta$.  

3.80
\[ H_\ell[q] : p \in \mathcal{P}_\ell[q] := \left\{ p_{ij} = q_i \mu^i_j \right\}_{1 \leq i \leq I, 1 \leq j \leq L} : \sum_{i} \theta_i \mu^i \geq \delta + \sum_{i} \theta_i \mu^i \right\}_{\ell'} \forall (\ell' \neq \ell) \]

\[ \mu^i \in M^i \forall i, \]

\[ \sum_{i} \theta_i \mu^i \right\}_{\ell} \geq \delta + \sum_{i} \theta_i \mu^i \right\}_{\ell'} \forall (\ell' \neq \ell) \]

\[ H_\ell[q] = \{ p = A[q] \mu : \mu = [\mu^1; \ldots; \mu^L] \in M^\ell \}, \]

\[ [A[q] \mu]_{ij} = q_i \mu^i_j, \]

where \( M^\ell, \ell = 1, \ldots, L, \) are nonempty nonintersecting convex compact subsets in \( \Delta_L \times \ldots \times \Delta_L \)

\[ \Delta_L \times \ldots \times \Delta_L \]

Note: Opinion Poll with \( K \) interviews corresponds to stationary \( K \)-repeated observation in Discrete o.s. with \((IL)\)-element observation space \( \Omega \)

\[ \Rightarrow \text{Given } K, \text{ we can use our machinery to design a near-optimal detector-based test } T_K \text{ deciding via stationary } K \text{-repeated observation (i.e., via the outcomes of } K \text{ interviews) on hypotheses } H_\ell[q], \ell = 1, \ldots, L \text{ up to trivial closeness } "H_\ell[q] \text{ is close to } H_{\ell'}[q] \text{ iff } \ell = \ell'." \text{ This test will predict the winner with reliability } 1 - \text{Risk}(T_K|H_1[q], \ldots, H_L[q]). \]
\[ H_{\ell}[q] = \{ p = A[q]\vec{\mu} : \vec{\mu} := [\mu^1; \ldots; \mu^L] \in \mathcal{M}^{\ell} \}, \]

\[ [A[q]\vec{\mu}]_{ij} = q_i\mu_j^i, \]

\[ \star \text{By our theory, setting } \chi_{\ell\ell'} = \begin{cases} 0, & \ell = \ell' \\ 1, & \ell \neq \ell' \end{cases}, \text{ we have} \]

\[ \text{Risk}(\mathcal{T}_K|H_1[q], \ldots, H_L[q]) \leq \epsilon_K[q] := \left\| \epsilon_{\ell\ell'}[q]\chi_{\ell\ell'} \right\|_{2,2}, \]

\[ \epsilon_{\ell\ell'}[q] = \max_{\vec{\mu} \in \mathcal{M}^{\ell'}, \vec{\nu} \in \mathcal{M}^{\ell'}} \sum_{i,j} \sqrt{[A[q]\vec{\mu}]_{ij}[A[q]\vec{\nu}]_{ij}} \]

\[ = \max_{\vec{\mu} \in \mathcal{M}^{\ell'}, \vec{\nu} \in \mathcal{M}^{\ell'}} \sum_{i=1}^I q_i \left[ \sum_{j=1}^L \sqrt{\mu_j^i\nu_j^i} \right] \]

\[ \Phi(q; \vec{\mu}, \vec{\nu}) \]

\textbf{Note: } \Phi(q; \vec{\mu}, \vec{\nu}) \text{ is linear in } q.
Let us carry out Measurement Design – optimization of $\epsilon_K[q]$ in $q$.

Main observation: $\epsilon_K[q] = \Gamma(\Psi(q))$, where

- $\Gamma(Q) = \|[(Q_{\ell\ell'})^K\chi_{\ell\ell'}]^L_{\ell,\ell' = 1}\|_{2,2}$ is efficiently computable convex and entrywise nondecreasing function on the space of nonnegative $L \times L$ matrices
- $\Psi(q)$ is matrix-valued function with efficiently computable convex in $q$ and non-negative entries

$\Psi_{\ell\ell'}(q) = \max_{\vec{\mu} \in \mathcal{M}_\ell, \vec{\nu} \in \mathcal{M}_{\ell'}} \Phi(q; \vec{\mu}, \vec{\nu})$

$\Rightarrow$ Optimal selection of $q_i$'s reduces to solving explicit convex problem

$$\min_q \{ \Gamma(\Psi(q)) : q = [q_1; \ldots; q_I] \geq 0, \sum_{i=1}^I q_i = 1 \}$$
How it Works: Measurement Design in Election Polls

♣ Setup:
- Opinion Poll problem with \( L \) candidates and winning margin \( \delta = 0.05 \)
- Reliability \( \epsilon = 0.01 \)
- A priori information on voters’ preferences in groups:

\[
M^i = \{ \mu^i \in \Delta_L : p^i_\ell - u_i \leq \mu^i_\ell \leq p^i_\ell + u_i, \ell \leq L \}
\]

- \( p^i \): randomly selected probabilistic vector
- \( u_i \): uncertainty level

♣ Sample of results:

<table>
<thead>
<tr>
<th>( L )</th>
<th>( I )</th>
<th>( \theta ) Group sizes ( \theta )</th>
<th>Uncertainty levels ( u )</th>
<th>( K_{\text{ini}} )</th>
<th>( q_{\text{opt}} )</th>
<th>( K_{\text{opt}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>( \theta = [0.50; 0.50] ) ( u = [0.03; 1.00] )</td>
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<td>( [0.44; 0.56] )</td>
<td>1194</td>
<td></td>
</tr>
<tr>
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<td>2</td>
<td>( [0.50; 0.50] ) ( [0.02; 1.00] )</td>
<td>2699</td>
<td>( [0.00; 1.00] )</td>
<td>1948</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>( [0.33; 0.33; 0.33] ) ( [0.02; 0.03; 1.00] )</td>
<td>3177</td>
<td>( [0.00; 0.46; 0.54] )</td>
<td>2726</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>( [0.25; 0.25; 0.25; 0.25] ) ( [0.02; 0.02; 0.03; 1.00] )</td>
<td>2556</td>
<td>( [0.00; 0.13; 0.32; 0.55] )</td>
<td>2086</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>( [0.25; 0.25; 0.25; 0.25] ) ( [1.00; 1.00; 1.00; 1.00] )</td>
<td>4788</td>
<td>( [0.25; 0.25; 0.25; 0.25] )</td>
<td>4788</td>
<td></td>
</tr>
</tbody>
</table>

Effect of measurement design. \( K_{\text{ini}} \) and \( K_{\text{opt}} \) are the poll sizes required for 0.99-reliable prediction of the winner when \( q_i = \theta_i \) and \( q = q_{\text{opt}} \), respectively.

Note: Uncertainty \( = 1.00 \) \( \Leftrightarrow \) No a priori information
In numerous situations, we do have partial control of observation scheme and thus can look for optimal Measurement Design. However: the situations where optimal Measurement Design can be found efficiently, like in design of Election Polls, are rare. Additional examples of these rare situations are Poisson o.s. and Gaussian o.s. with time control.
Poisson o.s. with time control. Typical models where Poisson o.s. arises are as follows:

- “in the nature” there exists a “signal” $x$ known to belong to some convex compact set $\subset \mathbb{R}^n$

For example, in Positron Emission Tomography, $x$ is (discretized) density of radioactive tracer administered to patient

- We observe random vector $\omega \in \mathbb{R}^m$ with independent entries $\omega_i \sim \text{Poisson}(a_i^T x)$, and want to make inferences on $x$.

For example, in PET, tracer disintegrates, and every disintegration act results in pair of gamma-quants flying in opposite directions along a randomly oriented line passing through disintegration point. This line is registered when two detector cells are (nearly) simultaneously hit:
The data acquired in PET study are the numbers $\omega_i$ of lines registered in bins (pairs of detector cells) $i = 1, \ldots, m$ over a time horizon $T$, and

$$\omega_i \sim \text{Poisson}(T \sum_{j=1}^{n} p_{ij} x_j)$$

$[p_{ij}$: probability for line emanated from voxel $j = 1, \ldots, n$ to cross pair $i = 1, \ldots, m$ of detector cells$] \Rightarrow A = T [p_{ij}]_{i \leq m, j \leq n}$
\[ \omega = \{ \omega_i \sim \text{Poisson}([Ax]_i) \}_{i \leq m} \]

- In some situations, the sensing matrix \( A \) can be partially controlled:

\[
A = A[q] := \text{Diag}\{q\}A_*
\]

- \( A_* \): given \( m \times n \) matrix; \( q \in Q \): vector of control parameters.

For example, in a full body PET scan the position of the patient w.r.t. the scanner is updated several times to cover the entire body.

The data acquired in position \( i \) form subvector \( \omega^i \) in the entire observation \( \omega = [\omega^1; \ldots; \omega^I] \):

\[
\omega^i \sim \text{Poisson}([t_i A^i x]_i), \ 1 \leq i \leq \bar{m} = m/I
\[
\left[ A^i : \text{given matrices}; \ t_i : \text{duration of study in position } i \right]
\]

implying that \( \omega = \text{Diag}\{q\}A_* \) with properly selected \( A_* \) and \( q \) of the form

\[
q = \left[ \underbrace{t_1; \ldots; t_1}_{\bar{m}}; \ldots; \underbrace{t_I; \ldots; t_I}_{\bar{m}} \right]
\]
$H_q^\ell : \omega_i \sim \text{Poisson}([A[q]x]_i)$ are independent across $i \leq m$ and $x \in X_\ell$

$A = A[q] := \text{Diag}\{q\}A_*$

- $A_*$: given $m \times n$ matrix; • $q \in Q$: control parameters.

Let our goal be to decide, up to a given closeness $C$, on $L$ hypotheses on the distribution of Poisson observation $\omega$:

$$H_q^\ell : \omega \sim \text{Poisson}([A[q]x]_1) \times ... \times \text{Poisson}([A[q]x]_m) \& x \in X_\ell$$

$X_\ell$: given convex compact sets, $1 \leq \ell \leq L$.

By our theory, the (upper bound on the) $C$-risk of near-optimal test deciding on $H_q^\ell$, $\ell = 1, ..., L$, is $\epsilon(q) = \left\| [\exp\{\text{Opt}_{\ell\ell'}(q)\}X_{\ell\ell'}]^{L}_{\ell,\ell' = 1} \right\|_{2,2}$ where

$$X_{\ell\ell'} = \begin{cases} 0, & (\ell, \ell') \in C \\ 1, & (\ell, \ell') \not\in C \end{cases}, \text{Opt}_{\ell\ell'}(q) = \max_{u \in X_\ell, v \in X_{\ell'}} -\frac{1}{2} \sum_{i=1}^{m} \left( [A[q]u]_i - [A[q]v]_i \right)^2$$

As in Opinion Polls, $\epsilon(q) = \Gamma(\Psi(q))$, where

- $\Gamma(Q) = \left\| [\exp\{Q_{\ell\ell'}\}X_{\ell\ell'}]^{L}_{\ell,\ell' = 1} \right\|_{2,2}$ is a convex entrywise nondecreasing function of $Q \in \mathbb{R}^{L \times L}_+$

- $[\Psi(q)]_{\ell\ell'} = \exp \left\{ \max_{u \in X_\ell, v \in X_{\ell'}} \sum_{i=1}^{m} q_i \left( [A_*u]_i [A_*v]_i - \frac{1}{2} [A_*u]_i - \frac{1}{2} [A_*v]_i \right) \right\}$ is efficiently computable and convex in $q$

$\Rightarrow$ Assuming the set $Q \subset \mathbb{R}^m_+$ of allowed controls $q$ convex, optimizing $\epsilon(q)$ over $q \in Q$ is an explicitly given convex optimization problem.
An efficiently solvable Measurement Design problem in Gaussian o.s.

\[ \omega = A[q]x + \xi, \xi \sim \mathcal{N}(0, I_m) \]

[• \( A[q] \) partially controlled sensing matrix; • \( q \in \mathcal{Q} \): control parameters.]

is the one where

\[ A[q] = \text{Diag}\{\sqrt{q_1}, ..., \sqrt{q_m}\} \& \mathcal{Q} \subset \mathbb{R}_+^m \text{ is a convex compact set} \]

In this case, minimizing \( \mathcal{Q} \)-risk of test deciding up to closeness \( C \) on \( L \) hypotheses

\[ H_q^\ell : \omega \sim \mathcal{N}(A[q]x, I_m), x \in X_\ell, \ 1 \leq \ell \leq L \]

associated with nonempty convex compact sets \( X_\ell \) reduces to solving convex problem

\[ \min_{q \in \mathcal{Q}} \Gamma(\Psi(q)) \]

where

\[ \Gamma(Q) = \| \left[ \exp\left\{ Q_{\ell\ell'}/8 \right\} X_{\ell\ell'} \right]_{\ell,\ell' \leq L} \|_{2,2} \]

is convex entrywise nondecreasing function of \( L \times L \) matrix \( Q \), and

\[ [\Psi(q)]_{\ell\ell'} = \max_{u \in X_\ell, v \in X_{\ell'}} \left[ -\| A[q](u - v) \|_2^2 \right] \]

\[ = -\min_{u \in X_\ell, v \in X_{\ell'}} (u - v)^T A^*_T \text{Diag}\{q\} A_*(u - v) \]

is efficiently computable convex function of \( q \in \mathcal{Q} \).

3.90
Illustration. In some applications, “the physics” beyond Gaussian o.s. \( \omega = Ax + \xi \) is as follows. There are \( \ell \) sensors measuring analogous vector-valued continuous time signal \( x \) (nearly constant on the observation horizon) in the presence of noise. The output of sensor \( \neq i \) is

\[
\omega_i = \frac{1}{|\Delta_i|} \int_{\Delta_i} [a_{i,\ast}^T x + B_i(t)] dt
\]

- \( \Delta_i \): continuous time interval
- \( B_i(t) \): "Brownian motion": \( \frac{1}{|\Delta|} \int_{\Delta} B_i(t) dt \sim \mathcal{N}(0, |\Delta|^{-1}) \), \( \int_{\Delta} B_i(t) dt, \int_{\Delta'} B_i(t) dt \) are independent when \( \Delta \cap \Delta' = \emptyset \)
- Brownian motions \( B_i(t) \) are independent across \( i \)

When all sensors work in parallel for unit time, we arrive at the standard Gaussian o.s. \( \omega = A_{\ast} x + \xi, \xi \sim \mathcal{N}(0, I_m) \).

When sensors work on consecutive segments \( \Delta_1, \ldots, \Delta_m \) of durations \( q_i = |\Delta_i| \), we arrive at

\[
\omega_i = a_{i,\ast}^T x + q_i^{-1/2} \xi_i, \xi_i \sim \mathcal{N}(0, 1) \text{ are independent across } i
\]

Rescaling observations: \( \omega_i \mapsto \sqrt{q_i} \omega_i \), we arrive at partially controlled o.s.

\[
\omega = \text{Diag}\{\sqrt{q_1}, \ldots, \sqrt{q_m}\} A_{\ast} x + \xi, \xi \sim \mathcal{N}(0, I_m)
\]

A natural selection of \( Q \) is, e.g.,

\( Q = \{q \geq 0 : \sum_i q_i = n\} \) – setting “time budget” to the same value as in the case of consecutive observations of duration 1 each.

3.91
Situation: Given are:

- Simple o.s. $\mathcal{O} = ((\Omega, \Pi), \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F})$
- Convex compact set $X \subset \mathbb{R}^n$ and affine mapping $x \mapsto A(x) : X \to \mathcal{M}$
- Linear function $g^T x$ on $\mathbb{R}^n$
- Positive integer $K$

Given observation

$$\omega \sim p_A(x)$$

stemming from unknown signal $x$ known to belong to $X$, we want to recover $g^T x$.

Given reliability tolerance $\epsilon \in (0, 1)$, we quantify performance of a candidate estimate $\hat{g}(\cdot) : \Omega \to \mathbb{R}$ by its $\epsilon$-risk

$$\text{Risk}_\epsilon[\hat{g} | X] = \min \left\{ \rho : \text{Prob}_{\omega \sim p_A(x)} \left\{ |\hat{g}(\omega) - g^T x| > \rho \right\} \leq \epsilon \forall x \in X \right\}.$$

We intend to build, in a computationally efficient manner, a provably near-optimal in terms of its $\epsilon$-risk estimate of the form

$$\hat{g}(\omega) = \phi(\omega) + \kappa$$

with $\phi \in \mathcal{F}$.

3.92
Construction: Let us set

$$
\Phi(\phi; \mu) = \ln \left( E_{\omega \sim p_{\mu}} \{ \exp\{\phi(\omega)\} \} \right)
$$

Recall that $\Phi$ is continuous real-valued convex-concave function on $\mathcal{F} \times \mathcal{M}$.

Main observation: Let $\psi \in \mathcal{F}$ and $\alpha > 0$. Then for $x, y \in \mathcal{X}$ one has

$$
\ln \left( \operatorname{Prob}_{\omega \sim p_{A(x)}} \{ \psi(\omega) > g^T x + \rho \} \right) \leq \Phi(\psi/\alpha; A(x)) - \frac{\rho + g^T x}{\alpha} \quad (a)
$$
$$
\ln \left( \operatorname{Prob}_{\omega \sim p_{A(y)}} \{ \psi(\omega) < g^T y - \rho \} \right) \leq \Phi(-\psi/\alpha; A(y)) - \frac{\rho - g^T y}{\alpha} \quad (b)
$$

As a result, for every $\psi \in \mathcal{F}$ and $\alpha > 0$, setting

$$
\Psi_+(\alpha, \psi) = \max_{x \in \mathcal{X}} \left[ \alpha \Phi(\psi/\alpha; A(x)) - g^T x + \alpha \ln(2/\epsilon) \right],
$$
$$
\Psi_-(\alpha, \psi) = \max_{y \in \mathcal{X}} \left[ \alpha \Phi(-\psi/\alpha; A(y)) + g^T y + \alpha \ln(2/\epsilon) \right],
$$
$$
\kappa = \frac{1}{2} \left[ \Psi_-(\alpha, \psi) - \Psi_+(\alpha, \psi) \right],
$$

for the estimate $\phi(\omega) = \psi(\omega) + \kappa$ we have

$$
\operatorname{Risk}_\epsilon[\phi(\cdot)|\mathcal{X}] \leq \frac{1}{2} \left[ \Psi_+(\alpha, \psi) + \Psi_-(\alpha, \psi) \right]
$$
\[ \Phi(\phi; \mu) = \ln \left( E_{\omega \sim p} \{ \exp \{ \phi(\omega) \} \} \right) \]

**Claim:** For every \( \psi \in \mathcal{F} \), \( \alpha > 0 \) and all \( x, y \in X \) one has

\[
\begin{align*}
\ln \left( \text{Prob}_{\omega \sim p_{A(x)}} \{ \psi(\omega) > g^T x + \rho \} \right) & \leq \Phi(\psi/\alpha; A(x)) - \frac{\rho + g^T x}{\alpha} \quad (a) \\
\ln \left( \text{Prob}_{\omega \sim p_{A(y)}} \{ \psi(\omega) < g^T y - \rho \} \right) & \leq \Phi(-\psi/\alpha; A(y)) - \frac{\rho - g^T y}{\alpha} \quad (b)
\end{align*}
\]

Indeed,

\[
\exp \{ \Phi(\psi/\alpha; A(x)) \} = E_{\omega \sim p_{A(x)}} \{ \exp \{ \psi(\omega)/\alpha \} \} = E_{\omega \sim p_{A(x)}} \left\{ \exp \left\{ \frac{\psi(\omega) - g^T x - \rho}{\alpha} \right\} \right\} \exp \{ \frac{g^T x + \rho}{\alpha} \} \\
\geq \text{Prob}_{\omega \sim p_{A(x)}} \{ \psi(\omega) > g^T x + \rho \geq 0 \} \exp \{ \frac{g^T x + \rho}{\alpha} \} \Rightarrow (a); \\
\exp \{ \Phi(-\psi/\alpha; A(y)) \} = E_{\omega \sim p_{A(y)}} \{ \exp \{ -\psi(\omega)/\alpha \} \} = E_{\omega \sim p_{A(y)}} \left\{ \exp \left\{ -\frac{\psi(\omega) + g^T y - \rho}{\alpha} \right\} \right\} \exp \{ -\frac{g^T y + \rho}{\alpha} \} \\
\geq \text{Prob}_{\omega \sim p_{A(y)}} \{ \psi(\omega) < g^T y - \rho \geq 0 \} \exp \{ -\frac{g^T y + \rho}{\alpha} \} \Rightarrow (b).
\]
\[ \ln \left( \Prob_{\omega \sim p_{A(x)}} \{ \psi(\omega) > g^T x + \rho \} \right) \leq \Phi\left( \frac{\psi}{\alpha}; A(x) \right) - \frac{\rho + g^T x}{\alpha} \quad (a) \]
\[ \ln \left( \Prob_{\omega \sim p_{A(y)}} \{ \psi(\omega) < g^T y - \rho \} \right) \leq \Phi\left( -\frac{\psi}{\alpha}; A(y) \right) - \frac{\rho - g^T y}{\alpha} \quad (b) \]

**Claim:** For every \( \psi \in F \) and \( \alpha > 0 \), setting
\[
\Psi_+(\alpha, \psi) = \max_{x \in X} \left[ \frac{\alpha}{2} \Phi\left( \frac{\psi}{\alpha}; A(x) \right) - g^T x + \alpha \ln(2/\epsilon) \right],
\]
\[
\Psi_-(\alpha, \psi) = \max_{y \in X} \left[ \frac{\alpha}{2} \Phi\left( -\frac{\psi}{\alpha}; A(y) \right) + g^T y + \alpha \ln(2/\epsilon) \right],
\]
\[
\kappa = \frac{1}{2} [\Psi_-(\alpha, \psi) - \Psi_+(\alpha, \psi)]
\]
we have
\[
\operatorname{Risk}_\epsilon[\psi(\cdot) + \kappa | X] \leq \frac{1}{2} [\Psi_+(\alpha, \psi) + \Psi_-(\alpha, \psi)] \tag{*}
\]

Indeed, given \( \psi \in F, \alpha > 0, z \in X \), let \( \Psi = \Psi_+ + \Psi_- \). We have
\[
\Prob_{\omega \sim p_{A(z)}} \left\{ \psi(\omega) + \kappa > g^T z + \Psi \right\} = \Prob_{\omega \sim p_{A(z)}} \left\{ \psi(\omega) > g^T z + \Psi_+ \right\}
\]
\[
\leq \exp \left\{ \Phi\left( \frac{\psi}{\alpha}; A(z) \right) - (\Psi_+ + g^T z) / \alpha \right\} \quad \text{[by (a)]}
\]
\[
\leq \exp \left\{ \Phi\left( \frac{\psi}{\alpha}; A(z) \right) - \left( \frac{\alpha}{2} \Phi\left( \frac{\psi}{\alpha}; A(z) \right) - g^T z + \alpha \ln(2/\epsilon) + g^T z \right) / \alpha \right\} = \epsilon / 2
\]
and
\[
\Prob_{\omega \sim p_{A(z)}} \left\{ \psi(\omega) + \kappa < g^T z - \Psi \right\} = \Prob_{\omega \sim p_{A(z)}} \left\{ \psi(\omega) < g^T z - \Psi_- \right\}
\]
\[
\leq \exp \left\{ \Phi\left( -\frac{\psi}{\alpha}; A(z) \right) - (\Psi_- - g^T z) / \alpha \right\} \quad \text{[by (b)]}
\]
\[
\leq \exp \left\{ \Phi\left( -\frac{\psi}{\alpha}; A(z) \right) - \left( \frac{\alpha}{2} \Phi\left( -\frac{\psi}{\alpha}; A(z) \right) + g^T z + \alpha \ln(2/\epsilon) - g^T z \right) / \alpha \right\} = \epsilon / 2
\]
and (*) follows.
**Result:** We have justified the first claim in the following

**Theorem [Ju&N’09]** *In the situation in question, consider convex (due to convexity-concavity of $\Phi$) optimization problem*

$$\text{Opt} = \inf_{\alpha > 0, \psi \in \mathcal{F}} \left\{ \Psi(\alpha, \omega) := \frac{1}{2} \left[ \Psi_+(\alpha, \psi) + \Psi_-(\alpha, \psi) \right] \right\}.$$  

*A feasible solution $\alpha, \psi$ to this problem gives rise to estimate $\phi(\omega) = \psi(\omega) + \kappa$ such that*

$$\text{Risk}_\epsilon[\phi|X] \leq \Psi(\alpha, \omega).$$  

*and the right hand side in this bound can be made arbitrarily close to Opt.*

*In addition, when $\epsilon < 1/2$, Opt is within moderate factor of the minimax optimal $\epsilon$-risk*

$$\text{Risk}_{\text{Opt}}\epsilon[X] = \inf \text{Risk}_\epsilon[\hat{g}|X],$$

*specifically,*

$$\text{Opt} \leq \frac{2 \ln(2/\epsilon)}{\ln \left( \frac{1}{4\epsilon(1-\epsilon)} \right)} \text{Risk}_{\text{Opt}}\epsilon[X].$$

**Note:** The “Gaussian o.s.” version of this result is due to D. Donoho (1994).
Note: The above scheme is applicable to every simple o.s., in particular, to $K$-th degree of simple o.s. $\mathcal{O} = ((\Omega, \Pi), \{ p_\mu : \mu \in \mathcal{M}\}, \mathcal{F})$, that is, to the case where instead of estimation via single observation $\omega$ we speak about estimating via stationary $K$-repeated observation $\omega^K = (\omega_1, ..., \omega_K)$ with $\omega_1, ..., \omega_K$ supplied by $\mathcal{O}$.

In terms of $\mathcal{O}$, our Main Observation reads:

Let $\psi \in \mathcal{F}$, $\alpha > 0$, and $\psi^K(\omega^K) = \sum_{k=1}^K \psi(\omega_k)$. Then for $x, y \in X$ one has

$$\ln \left( \text{Prob}_{\omega^K \sim p_A^K} \left\{ \psi^K(\omega^K) > g^T x + \rho \right\} \right) \leq K \Phi(\psi/\alpha; A(x)) - \frac{\rho + g^T x}{\alpha} \quad (a)$$

$$\ln \left( \text{Prob}_{\omega^K \sim p_A^K} \left\{ \psi^K(\omega^K) < g^T y - \rho \right\} \right) \leq K \Phi(-\psi/\alpha; A(y)) - \frac{\rho - g^T y}{\alpha} \quad (b)$$

As a result, for every $\psi \in \mathcal{F}$ and $\alpha > 0$, setting

$$\Psi_+(\alpha, \psi) = \max_{x \in X} \left[ K \alpha \Phi(\psi/\alpha; A(x)) - g^T x + \alpha \ln(2/\epsilon) \right],$$

$$\Psi_-(\alpha, \psi) = \max_{y \in X} \left[ K \alpha \Phi(-\psi/\alpha; A(y)) + g^T y + \alpha \ln(2/\epsilon) \right],$$

$$\kappa = \frac{1}{2} \left[ \Psi_-(\alpha, \psi) - \Psi_+(\alpha, \psi) \right],$$

for the estimate $\phi(\omega^K) = \sum_{k=1}^K \psi(\omega_K) + \kappa$ we have

$$\text{Risk}_\epsilon[\phi(\cdot)|X] \leq \frac{1}{2} \left[ \Psi_+(\alpha, \psi) + \Psi_-(\alpha, \psi) \right]$$

3.97
**Example: Gaussian o.s.** Here \( \mathcal{F} = \{ \phi(\omega) = \psi_0 + \psi^T \omega \} \); on a close inspection, we lose nothing when setting \( \psi_0 = 0 \).

\[
\Rightarrow \Phi(\psi, \mu) = \ln \left( c_n \int e^{\psi^T \omega - (\omega - \mu)^T (\omega - \mu) / 2} d\omega \right) = \{ \psi^T \mu + \frac{1}{2} \psi^T \psi \}
\]

\[
\Rightarrow \begin{cases}
\Psi_+ (\alpha, \psi) = \max_{x \in X} \left[ \psi^T A(x) - g^T x \right] + \left[ K \frac{\psi^T \psi}{2\alpha} + \alpha \ln(2/\epsilon) \right] \\
\Psi_- (\alpha, \psi) = \max_{x \in X} \left[ g^T y - \psi^T A(y) \right] + \left[ K \frac{\psi^T \psi}{2\alpha} + \alpha \ln(2/\epsilon) \right]
\end{cases}
\]

\[
\Rightarrow \text{The optimization problem } \min_{\alpha > 0, \psi} \frac{1}{2} \left[ \Psi_+ (\alpha, \psi) + \Psi_- (\alpha, \psi) \right] \text{ responsible for good estimates admits analytical elimination of } \alpha \text{ and results in the optimization problem}
\]

\[
\min_{\psi} \left\{ \frac{1}{2} \max_{x \in X} \left[ \psi^T A(x) - g^T x \right] + \frac{1}{2} \max_{y \in X} \left[ g^T y - \psi^T A(y) \right] + \sqrt{2K \ln(2/\epsilon)} \| \phi \|_2 \right\}
\]

in \( \psi \)-variable only.
Numerical Illustration

Covering story: At the North-bound part of a highway leaving Atlanta there at \( n + 1 \) crossings where cars traveling North enter/exit the highway.

- Arrivals of cars traveling North and entering the highway at crossing \( j \), \( j = 0, 1, \ldots, n - 1 \), form Poisson process with (unknown) parameter \( x_j \leq 1 \); the arrival processes are mutually independent
- A car on a highway traveling North and approaching a crossing exits the highway at this crossing with given probability \( p \)
- For \( i = 1, \ldots, n \), we observe the total number \( \omega_i \) of cars traveling North and exiting the highway at crossing \( i \) on time horizon \([0, T]\) and want to recover \( x_j \) for a particular value of \( j \).

Model: Observation \( \omega = [\omega_1; \ldots; \omega_n] \) is collection of independent of each other Poisson random variables; the vector of their Poisson parameters is \( Ax \), with

\[
A = \begin{bmatrix}
p & p(1-p) & p(1-p)^2 & \cdots & p(1-p)^{n-1} \\
p(1-p) & p(1-p) & p(1-p)^2 & \cdots & p(1-p)^{n-2} \\
p(1-p)^2 & p(1-p)^2 & \ddots & \ddots & p(1-p)^{n-3} \\
p(1-p)^{n-1} & p(1-p)^{n-2} & \ldots & p(1-p) & p \\
\end{bmatrix}
\]

⇒ Our problem is to recover linear form of signal

\[
x \in X = \{ x \in \mathbb{R}^n : 0 \leq x_j \leq 1, 0 \leq j < n \}
\]

observed via Poisson o.s.

3.99
Note: empirical risks are at most by 5% worse than lower bounds on minimax optimal 0.01-risks
Extension: Recovering Linear Form on Union of Convex Sets

♣ **Situation:** Given are:
- Simple o.s. $\mathcal{O} = ((\Omega, \Pi), \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F})$
- Convex compact sets $X_i \subset \mathbb{R}^n, i \leq I$, and affine mappings $x \mapsto A_i(x) : X_i \to \mathcal{M}$
- Linear function $g^T x$ on $\mathbb{R}^n$

Given stationary $K$-repeated observation $\omega^K = (\omega_1, ..., \omega_K)$, with

$$\omega_k \sim p_{A_i(x)}, \ 1 \leq k \leq K,$$

stemming from *unknown* signal $x$ known to belong to $X_i$ with some *unknown* $i \leq I$, we want to recover $g^T x$. 

3.101
**Construction:**

A. Given “reliability tolerance” $0 < \epsilon < 1$, for $1 \leq i, j \leq I$, let

$$\Phi_{ij}(\alpha, \phi; x, y) = \frac{1}{2}K\alpha [\Phi_O(\phi/\alpha; A_i(x)) + \Phi_O(-\phi/\alpha; A_j(y))] + \frac{1}{2}g^T[y - x] + \alpha \ln(2I/\epsilon) : \{ \alpha > 0, \phi \in F \} \times [X_i \times X_j] \to \mathbb{R},$$

$$\Psi_{ij}(\alpha, \phi) = \max_{x \in X_i, y \in X_j} \Phi_{ij}(\alpha, \phi; x, y) = \frac{1}{2} [\Psi_{i, +}(\alpha, \phi) + \Psi_{j, -}(\alpha, \phi) : \{ \alpha > 0 \} \times F \to \mathbb{R},$$

where

$$\Psi_{i, +}(\alpha, \psi) = \max_{x \in X_i} [K\alpha \Phi_O(\psi/\alpha; A_i(x)) - g^T x + \alpha \ln(2I/\epsilon)] : \{ \alpha > 0, \psi \in F \} \to \mathbb{R},$$

$$\Psi_{i, -}(\alpha, \psi) = \max_{x \in X_i} [K\alpha \Phi_O(-\psi/\alpha; A_i(x)) + g^T x + \alpha \ln(2I/\epsilon)] : \{ \alpha > 0, \psi \in F \} \to \mathbb{R},$$

and $\Phi_O(\phi; \mu) = \ln \left( \int_\Omega e^{\phi(\omega)}p_{\mu}(\omega)\Pi(d\omega) \right)$

**Comment:** It is easy to verify that whenever $\alpha_{ij} > 0, \phi_{ij} \in F$, setting

$$\rho_{ij} = \Psi_{ij}(\alpha_{ij}, \phi_{ij}) = \frac{1}{2} [\Psi_{i, +}(\alpha_{ij}, \phi_{ij}) + \Psi_{j, -}(\alpha_{ij}, \phi_{ij})]$$

$$\chi_{ij} = \frac{1}{2} [\Psi_{j, -}(\alpha_{ij}, \phi_{ij}) - \Psi_{i, +}(\alpha_{ij}, \phi_{ij})]$$

$$g_{ij}(\omega^K) = \sum_{k=1}^K \phi_{ij}(\omega_k) + \chi_{ij}$$

we ensure that

$$x \in X_i, \omega^K \sim p_{A_i}(x) \Rightarrow \text{Prob}\{g_{ij}(\omega^K) > g^T x + \rho_{ij}\} \leq \frac{\epsilon}{2I}$$

$$y \in X_j, \omega^K \sim p_{A_j}(x) \Rightarrow \text{Prob}\{g_{ij}(\omega^K) < g^T y - \rho_{ij}\} \leq \frac{\epsilon}{2I}$$

3.102
B. For $1 \leq i, j \leq I$, we find feasible near-optimal solutions $\alpha_{ij}, \phi_{ij}$ to (convex by their origin) optimization problems

$$\text{Opt}_{ij} = \min_{\alpha > 0, \phi \in \mathcal{F}} \Psi_{ij}(\alpha, \phi),$$

and set

$$\rho_{ij} = \Psi_{ij}(\alpha_{ij}, \phi_{ij}), \quad \kappa_{ij} = \frac{1}{2} \left[ \Psi_{j,-}(\alpha_{ij}, \phi_{ij}) - \Psi_{i,+}(\alpha_{ij}, \phi_{ij}) \right]$$

$$g_{ij}(\omega^K) = \sum_{k=1}^{K} \phi_{ij}(\omega_k) + \kappa_{ij}$$

Given observation $\omega^K$, we set

$$G = [g_{ij}(\omega^K)]_{i \leq I, \; j \leq I}, \quad r_i = \max_j g_{ij}(\omega^K), \quad c_j = \min_i g_{ij}(\omega^K)$$

and take the quantity

$$\hat{g}(\omega^K) = \frac{1}{2} \left[ \min_i \rho_i + \max_j c_j \right]$$

as the estimate of $g^T x$.

\begin{enumerate}
\item Prop: $\epsilon$-risk of the estimate $\hat{g}$ does not exceed $\rho = \max_{i,j} \rho_{ij}$, i.e., whenever $\ell \leq I$ and $x \in X_\ell$, the $p_A^K(x)$-probability of the event $|g^T x - \hat{g}(\omega^K)| > \rho$ is $\leq \epsilon$.
\end{enumerate}

Note that $\rho$ can be made arbitrarily close to $\text{Opt}(K) = \max_{i,j} \text{Opt}_{ij}$.

3.103
Sketch of the proof: Let $\omega^K \sim p^K_{A\ell}(x)$. From comment to $A$ it follows that the $p^K_{A\ell}(x)$-probability of the event

$$\forall i, j : g_{\ell j} \leq g^T x + \rho_{\ell j} \& g_{i\ell} \leq g^T x - \rho_{i\ell} \quad [g_{ij} = g_{ij}(\omega^K)]$$

is at least $1 - \epsilon$.

When this event takes place, we have

- all entries in $\ell$-th row of $G = [g_{ij}]$ by magenta inequalities are $\leq g^T x + \rho$,
- all entries in $\ell$-to column of $G$, by red inequalities, are $\geq g^T x - \rho$
- $r_i = \max_j g_{ij}$, $c_j = \min_i g_{ij}$ (by definition of $r_i$ and $c_j$)

$$\Rightarrow f^T x - \rho \leq \min_i g_{i\ell} \leq \min_i r_i \leq r_\ell \leq g^T x + \rho \Rightarrow f^T x \in [\min_i r_i - \rho, \min_i r_i + \rho]$$

and similarly $f^T x \in [\max_j c_j - \rho, \max_j c_j + \rho]$
Near-Optimality: Let $\epsilon \in (0, 1/2)$ and $K_*$ be a positive integer, and let $\text{Risk}_\epsilon^*(K_*)$ be the minimax optimal $\epsilon$-risk, the number of observations being $K_*$ (that is, the infimum, over all Borel $K_*$-observation estimates, of $\epsilon$-risks of the estimates) Then for every integer $K$ satisfying

$$K > \frac{2 \ln(2I/\epsilon)}{\ln([4\epsilon(1-\epsilon)]^{-1})} K_*$$

one has

$$\text{Opt}(K) \leq \text{Risk}_\epsilon^*(K_*).$$

In addition, assuming that every $i, j$ there exists $\bar{x}_{ij} \in X_i \cap X_j$ such that $A_i(\bar{x}_{ij}) = A_j(\bar{x}_{ij})$ one has

$$K \geq K_* \Rightarrow \text{Opt}(K) \leq \frac{2 \ln(2I/\epsilon)}{\ln([4\epsilon(1-\epsilon)]^{-1})} \text{Risk}_\epsilon^*(K_*).$$
Sketch of the proof [first claim only]: Since \( \text{Opt}(K) = \max_{i,j} \text{Opt}_{ij}(K) \), all we need to verify is that when

\[
K > \frac{2 \ln(2I/\epsilon)}{\ln([4\epsilon(1-\epsilon)]^{-1})} K_*
\]

we have \( \text{Opt}_{ij}(K) \leq \text{Risk}_\epsilon^*(K_*) \) for every \( i, j \).

- Recall that \( \text{Opt}_{ij}(K) = \inf_{\alpha > 0, \phi \in \mathcal{F}} \left[ \Psi_{ij}(\alpha, \phi) := \max_{x \in X_i, y \in X_j} \Phi_{ij}(\alpha, \phi; x, y) \right] \) and by its origin, \( \Phi_{ij} \) is convex in \( \alpha, \phi \) and concave in \( x, y \), whence

\[
\text{Opt}_{ij}(K) = \max_{x \in X_i, y \in X_j} \inf_{\alpha > 0, \phi \in \mathcal{F}} \Phi_{ij}(\alpha, \phi; x, y)
\]


(!) given by straightforward computation,

Assuming, on the contrary to what should be proved, that \( \text{Opt}_{ij}(K) > \text{Risk}_\epsilon^*(K_*) \), we can find \( \bar{x} \in X_i, \bar{y} \in X_j \) such that with \( \mu = A_i(\bar{x}), \nu = A_j(\bar{x}) \) it holds

\[
\frac{1}{2} g^T [\bar{y} - x] > \text{Risk}_\epsilon^*(K_*) \& \left[ \int \sqrt{p_{\mu}(\omega)p_{\nu}(\omega)} \Pi(d\omega) \right]^K \geq \frac{\epsilon}{2I}
\]

By first relation in (!), two simple hypotheses stating that the distributions of \( \omega^{K_*} \) is \( p_{\mu}^{K_*} \), resp., \( p_{\nu}^{K_*} \) can be decided upon with risk \( \leq \epsilon \), whence by elementary results about Hellinger affinity,

\[
2\sqrt{\epsilon(1-\epsilon)} \geq \int \sqrt{p_{\mu}^{K_*}(\omega^{K_*})p_{\nu}^{K_*}(\omega^{K_*})} \Pi^K(d\omega^{K_*}) = \left[ \int \sqrt{p_{\mu}(\omega)p_{\nu}(\omega)} \Pi(d\omega) \right]^{K_*}
\]

This combines with (*) to imply the inequality opposite to (!), which is a desired contradiction. \( \square \)
Toy Illustration: Recovering Origin-Destination Traffics

✿ Covering story: Nodes in the network represent five villages (magenta dots) and crossing with no population (cyan dot), and arcs represent road segments.

• There are two states of the road net:
  — normal: some normal traveling times in all segments,
  — abnormal: normal traveling times in magenta segments and much larger than normal traveling times in blue segments.

• There are $L = 7$ origin-destination pairs, $\ell$-th with its own traffic $x_\ell$. The travelers know normal and abnormal traveling times of the arcs and the state of the network and select the fastest routes between their origins and destinations. As a result, the total traffic in arc $\gamma$ is $\sum_\ell A^\chi_{\gamma\ell}x_\ell$ where $\chi \in \{\text{normal, abnormal}\}$ is the state of the network.

• We do not know network’s state and traffics in origin-destination pairs. All we know are
  — the number $L$ of origin-destination pairs and an upper bound $T$ on the total traffic $\sum_\ell x_\ell$
  — the sensing matrices $A^\chi = [A^\chi_{\gamma\ell}]_{\gamma \in \Gamma, \ell \leq L}$, where $\chi \in \{\text{normal, abnormal}\}$, and $\Gamma$ is the set of $M = 29$ arcs where we measure traffic.

• Given noisy measurements of traffics in the arcs of $\Gamma$: $y_\gamma = [A^\chi x]_\gamma + \sigma \xi_\gamma$, with independent across $\gamma$ noises $\xi_\gamma \sim \mathcal{N}(0, 1)$ and known $\sigma$, we want to recover origin-destination traffics $x_\ell$, $\ell \leq L$. 

3.107
Model: The unknown signal $x$ lives in $X = \{x \in \mathbb{R}^L_+ : \sum_\ell x_\ell \leq T\}$. We set $X^1 = X^2 = X$ and

$$A_1(x) = A^{\text{normal}} x, \quad A_2(x) = A^{\text{abnormal}} x.$$ 

⇒ The problem of recovering $x_\ell$ for a particular $\ell$ is covered by the Gaussian case of our setup, and we can use the above machinery to recover $x_\ell$'s one by one.

<table>
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<th>median</th>
<th>maximal</th>
<th>computed upper bound on 0.01 risk</th>
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<td>0.017</td>
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<td>$2^{-11}$</td>
<td>0.002</td>
<td>0.001</td>
<td>0.005</td>
<td>0.003</td>
</tr>
</tbody>
</table>

Numerical results over 100 simulations

• Pay attention to clear “numerical consistency.”
Note: For every $\sigma$, our estimate is a “nonlinear aggregation” of 4 estimates which are affine in observations. In the reported instance, this estimate is consistent.

In contrast: In the same instance, even in the noiseless case, the worst-case recovery error for every affine estimate of $x_2$ is $\geq 0.25$.

Explanation: We are observing in Gaussian noise either $Ax$, or $Bx$, with unknown $x$ belonging to the known signal set $X = \{x \in \mathbb{R}_+^7 : \sum_\ell x_\ell \leq T\}$. We do know $A$ and $B$, but do not know from which one of the matrices $A$ , $B$ the observation comes. In this situation, the ultimate obstacle for high-accuracy recovering $g^T x$ in the low-noise case is

— for our estimate – the fact that $g^T x - g^T y$ is not identically zero on the intersection of $X \times X$ and the linear subspace $L = \{[x; y] : Ax = By\}$ of pairs $(x, y)$ of ”nondistinguishable signals.” In the reported instance, this obstacle is absent – the only common point of $L$ and $X \times X$ is the origin.

— for an affine estimate – the fact that the vector $[g; -g]$ is not orthogonal to $L$.

In the reported instance this obstacle is present – the vector $[e_2; -e_2]$ is far from being orthogonal to $L$.  

3.109
**Another Illustration**

**SetUp:** Given $J = 100$ points $x_j \in \mathbb{R}^{20}$ and stationary $K$-repeated observation

$$\omega^K = (\omega_1, .., \omega_K), \ \omega_k \sim \mathcal{N}(Ax, I_{20})$$

of one of the points (we do not know which one!), we want to recover the first entry of the point.

- $A$: randomly generated matrix
- $\epsilon = 0.01$.

**Note:** we are in the situation where $X_i = \{x_i\}$ are singletons.

**Results:**

Recovery error vs. $K$, data over 20 randomly generated collections $\{x_i\}_{i=1}^{100}$

3.110
HYPOTHESIS TESTING, III

- Beyond simple observation schemes
Observation: A “common denominator” of minimum risk detectors for simple o.s.’s is their affinity in observations:

- the optimal detectors in Gaussian and Poisson o.s.’s are affine “as they are”
- encoding observation space \( \Omega = \{1, \ldots, d\} \) of Discrete o.s. by vertices \( e_i \), \( i = 1, \ldots, d \), of the standard simplex \( \Delta_d = \{x \in \mathbb{R}^d : x \geq 0, \sum_j x_j = 1\} \), every function on \( \Omega \) becomes affine
  \[ \Rightarrow \text{we can treat optimal detector in Discrete o.s. as affine function on } \mathbb{R}^d. \]
- operations with optimal detectors induced by taking direct products of basic simple o.s.’s or passing to repeated observations preserves affinity.

Fact: “Reasonable” (perhaps, non-optimal) affine detectors can be found, in a computationally efficient way, in many important situations which are beyond simple o.s.’s.
Setup

Given an observation space $\Omega = \mathbb{R}^d$, consider a triple $\mathcal{H}, \mathcal{M}, \Phi$, where

- $\mathcal{H}$ is a nonempty closed convex set in $\Omega$ symmetric w.r.t. the origin,
- $\mathcal{M}$ is a compact convex set in some $\mathbb{R}^n$,
- $\Phi(h; \mu) : \mathcal{H} \times \mathcal{M} \to \mathbb{R}$ is a continuous function convex in $h \in \mathcal{H}$ and concave in $\mu \in \mathcal{M}$.

$\mathcal{H}, \mathcal{M}, \Phi$ specify a family $S[\mathcal{H}, \mathcal{M}, \Phi]$ of probability distributions on $\Omega$. A probability distribution $P$ belongs to the family iff there exists $\mu \in \mathcal{M}$ such that

$$\ln \left( \int_{\Omega} e^{h^T \omega} P(d\omega) \right) \leq \Phi(h; \mu) \ \forall h \in \mathcal{H} \quad (*)$$

We refer to $\mu$ ensuring $(*)$ as to parameter of distribution $P$.

- **Warning:** A distribution $P$ may have many different parameters!

We refer to triple $\mathcal{H}, \mathcal{M}, \Phi$ satisfying the above requirements as to regular data, and to $S[\mathcal{H}, \mathcal{M}, \Phi]$ – as to the simple family of distributions induced by these data.
Example 1: Gaussian and sub-Gaussian distributions. When

- \( \mathcal{M} = \{(u, \Theta)\} \subset \mathbb{R}^d \times \text{int } S^d_+ \) is a convex compact set such that \( \Theta \succ 0 \) for all \( (u, \Theta) \in \mathcal{M} \),
- \( \mathcal{H} = \mathbb{R}^d \),
- \( \Phi(h; u, \Theta) = h^T y + \frac{1}{2} h^T \Theta h \),

\( S = S[\mathcal{H}, \mathcal{M}, \Phi] \) contains all probability distributions \( P \) which are sub-Gaussian with parameters \((u, \Theta)\), meaning that

\[
\ln \left( \int_{\Omega} e^{h^T \omega} P(d\omega) \right) \leq h^T u + \frac{1}{2} h^T \Theta h \quad \forall h, \tag{1}
\]

and, in addition, the “parameter” \((u, \Theta)\) belongs to \( \mathcal{M} \).

Note: Whenever \( P \) is sub-Gaussian with parameters \((u, \Theta)\), \( u \) is the expectation of \( P \).

Note: \( \mathcal{N}(u, \Theta) \in S \) whenever \((u, \Theta) \in \mathcal{M} \); for \( P = \mathcal{N}(u, \Theta) \), (1) is an identity.
Example 2: **Poisson distributions.** When

- $\mathcal{M} \subset \mathbb{R}^d_+$ is a convex compact set,
- $\mathcal{H} = \mathbb{R}^d$,
- $\Phi(h; \mu) = \sum_{i=1}^{d} \mu_i (e^{h_i} - 1)$,

$S = S[\mathcal{H}, \mathcal{M}, \Phi]$ contains distributions of all $d$-dimensional random vectors $\omega_i$ with independent across $i$ entries $\omega_i \sim \text{Poisson}(\mu_i)$ such that $\mu = [\mu_1; \ldots; \mu_d] \in \mathcal{M}$.

Example 3: **Discrete distributions.** When

- $\mathcal{M} = \{ \mu \in \mathbb{R}^d : \mu \geq 0, \sum_{j} \mu_j = 1 \}$ is the probabilistic simplex in $\mathbb{R}^d$,
- $\mathcal{H} = \mathbb{R}^d$,
- $\Phi(h; \mu) = \ln \left( \sum_{i=1}^{d} \mu_i e^{h_i} \right)$,

$S = S[\mathcal{H}, \mathcal{M}, \Phi]$ contains all discrete distributions supported on the vertices of the probabilistic simplex.
Example 4: Distributions with bounded support. Let $X \subset \mathbb{R}^d$ be a nonempty convex compact set with support function $\phi_X(\cdot)$:

$$\phi_X(y) = \max_{x \in X} y^T x : \mathbb{R}^d \to \mathbb{R}^d.$$ 

When $\mathcal{M} = X$, $\mathcal{H} = \mathbb{R}^d$ and

$$\Phi(h; \mu) = h^T \mu + \frac{1}{8} [\phi_X(h) + \phi_X(-h)]^2,$$

$S = S[\mathcal{H}, \mathcal{M}, \Phi]$ contains all probability distributions supported on $X$, and for such a distribution $P$, $\mu = \int_X \omega P(d\omega)$ is a parameter of $P$.

Note: When $G$, $0 \in G$, is a convex compact set, the conclusion in Example 4 remains valid when function (2) is replaced with the smaller function

$$\Phi(h; \mu) = \min_{g \in G} \left[ \mu^T (h - g) + \frac{1}{8} [\phi_X(h - g) + \phi_X(g - h)]^2 + \phi_X(g) \right].$$
Fact: Simple families of probability distributions admit “calculus:”

For $1 \leq \ell \leq L$, let $\lambda_\ell$ be reals, and let $\mathcal{H}_\ell, \mathcal{M}_\ell, \Phi_\ell$ be regular data with common observation space: $\mathcal{H}_\ell \subset \Omega = \mathbb{R}^d$. Setting

$$
\mathcal{H} = \{ h \in \mathbb{R}^d : \lambda_\ell h \in \mathcal{H}_\ell, 1 \leq \ell \leq L \},
\mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_L,
\Phi(h; \mu_1, \ldots, \mu_L) = \sum_{\ell=1}^L \Phi_\ell(\lambda_\ell h; \mu),
$$

we get regular data with the following property:

Whenever random vectors $\xi_\ell \sim P_\ell \in S[\mathcal{H}_\ell, \mathcal{M}_\ell, \Phi_\ell], 1 \leq \ell \leq L$, are independent across $\ell$, the distribution $P$ of the random vector $\xi = \sum_{\ell=1}^L \lambda_\ell \xi_\ell$ belongs to $S[\mathcal{H}, \mathcal{M}, \Phi]$. Denoting by $\mu_\ell$ parameters of $P_\ell$, $\mu = [\mu_1; \ldots; \mu_L]$ can be taken as parameter of $P$. 


[direct product] For $1 \leq \ell \leq L$, let $\mathcal{H}_\ell, \mathcal{M}_\ell, \Phi_\ell$ be regular data with observation spaces $\Omega_\ell = \mathbb{R}^{d_\ell}$. Setting

$$
\mathcal{H} = \mathcal{H}_1 \times \ldots \times \mathcal{H}_L \subset \Omega = \mathbb{R}^{d_1} + \ldots + d_L. \mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_L,$n
$$

$$
\Phi(h_1, \ldots, h_L; \mu_1, \ldots, \mu_L) = \sum_{\ell=1}^{L} \Phi_\ell(h_\ell; \mu_\ell),
$$

we get regular data with the following property:

Whenever $P_\ell \in S[\mathcal{H}_\ell, \mathcal{M}_\ell, \Phi_\ell], 1 \leq \ell \leq L$, the direct product distribution $P = P_1 \times \ldots \times P_L$ belongs to $S[\mathcal{H}, \mathcal{M}, \Phi]$. Denoting by $\mu_\ell$ parameters of $P_\ell, \mu = [\mu_1; \ldots; \mu_L]$ can be taken as parameter of $P$. 

4.7
[marginal distribution] Let $\mathcal{H}, \mathcal{M}, \Phi$ be regular data with observation space $\mathbb{R}^d$, and let $\omega \mapsto A\omega + a : \mathbb{R}^d \mapsto \Omega = \mathbb{R}^\delta$. Setting

$$\bar{\mathcal{H}} = \{ h \in \mathbb{R}^\delta : A^T h \in \mathcal{H} \}, \quad \bar{\Phi}(h; \mu) = h^T a + \Phi(A^T h; \mu),$$

we get regular data $\bar{\mathcal{H}}, \bar{\mathcal{M}}, \bar{\Phi}$ with the following property:

Whenever $\xi \sim P \in S[\mathcal{H}, \mathcal{M}, \Phi]$, the distribution $\bar{P}$ of the random variable $\omega = A\xi + a$ belongs to the simple family $S[\bar{\mathcal{H}}, \bar{\mathcal{M}}, \bar{\Phi}]$, and parameter of $P$ is a parameter of $\bar{P}$ as well.
Main observation: When deciding on simple families of distributions, affine tests and their risks can be efficiently computed via Convex Programming:

Theorem. Let $\mathcal{H}_\chi, \mathcal{M}_\chi, \Phi_\chi, \chi = 1, 2$, be two collections of regular data with compact $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{H}_1 = \mathcal{H}_2 =: \mathcal{H}$, and let

$$\Psi(h) = \max_{\mu_1 \in \mathcal{M}_1, \mu_2 \in \mathcal{M}_2} \left\{ \frac{1}{2} \Phi(h; \mu_1) + \Phi_1(-h; \mu_1) + \Phi_2(h, \mu_2) \right\} : \mathcal{H} \to \mathbb{R}$$

Then $\Psi$ is efficiently computable convex function, and for every $h \in \mathcal{H}$, setting

$$\phi(\omega) = h^T \omega + \frac{1}{2} \left[ \max_{\mu_1 \in \mathcal{M}_1} \Phi_1(-h; \mu_1) - \max_{\mu_2 \in \mathcal{M}_2} \Phi_2(h; \mu_2) \right],$$

one has

$$\text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2] \leq \exp\{\Psi(h)\} \quad [\mathcal{P}_\chi = S[\mathcal{H}, \mathcal{M}_\chi, \Phi_\chi]]$$

In particular, if convex-concave function $\Phi(h; \mu_1, \mu_2)$ possesses a saddle point $h^*_*, (\mu^*_1, \mu^*_2)$ on $\mathcal{H} \times (\mathcal{M}_1 \times \mathcal{M}_2)$, the affine detector

$$\phi^*(\omega) = h^* T \omega + \frac{1}{2} \left[ \Phi_1(-h^*_*; \mu^*_1) - \Phi_2(h^*_*; \mu^*_2) \right]$$

admits risk bound

$$\text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2] \leq \exp\{\Phi(h^*_*; \mu^*_1, \mu^*_2)\}$$

4.9
Indeed, let \( h \in \mathcal{H} \). Setting \( \mu_1^* \in \text{Argmax}_{\mu_1 \in \mathcal{M}_1} \Phi_1(-h; \mu_1) \), \( \mu_2^* \in \text{Argmax}_{\mu_2 \in \mathcal{M}_2} \Phi_2(h; \mu_2) \), we have

\[
P \in \mathcal{P}_1 \coloneqq S[\mathcal{H}, \mathcal{M}_1, \Phi_1] \Rightarrow \exists \mu_1 \in \mathcal{M}_1 : \mathbb{E}_{\omega \sim \mathcal{P}} \left\{ e^{-h^T \omega} \mathcal{P}(d\omega) \right\} \leq e^{\Phi_1(-h; \mu_1)} \Rightarrow \mathbb{E}_{\omega \sim \mathcal{P}} \left\{ e^{-\phi(\omega)} \mathcal{P}(d\omega) \right\} \leq e^{\Phi_1(-h; \mu_1^*)} = e^\psi(h) \Rightarrow \text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2] \leq e^\psi(h).
\]

Similarly,

\[
P \in \mathcal{P}_2 \coloneqq S[\mathcal{H}, \mathcal{M}_2, \Phi_2] \Rightarrow \exists \mu_2 \in \mathcal{M}_2 : \mathbb{E}_{\omega \sim \mathcal{P}} \left\{ e^{h^T \omega} \mathcal{P}(d\omega) \right\} \leq e^{\Phi_2(h; \mu_2)} \Rightarrow \mathbb{E}_{\omega \sim \mathcal{P}} \left\{ e^{\phi(\omega)} \mathcal{P}(d\omega) \right\} \leq e^{\Phi_2(h; \mu_2^*)} + \kappa = e^\psi(h) \Rightarrow \text{Risk}_2[\phi|\mathcal{P}_1, \mathcal{P}_2] \leq e^\psi(h).
\]
Numerical Illustration. Given observation

\[ \omega = Ax + \sigma A \text{Diag} \{ \sqrt{x_1}, \ldots, \sqrt{x_n} \} \xi \quad [\xi \sim \mathcal{N}(0, I_n)] \]

of an unknown signal $x$ known to belong to a given convex compact set $M \subset \mathbb{R}^n_{++}$, we want to decide on two hypotheses $H_\chi : x \in X_\chi$, $\chi = 1, 2$, with risk 0.01.

$X_\chi$: convex compact subsets of $X$.

Novelty: Noise intensity depends on the signal!

- Introducing regular data $\mathcal{H}_\chi = \mathbb{R}^n$, $\mathcal{M}_\chi = X_\chi$,

\[ \Phi_\chi(h, \mu) = h^T A \mu + \frac{\sigma^2}{2} h^T [A \text{Diag} \{ \mu \} A^T] h \quad [\chi = 1, 2] \]

distribution of observations under $H_\chi$ belongs to $S[\mathcal{H}, \mathcal{M}_\chi, \Phi_\chi]$. 

4.11
• An affine detector for families $\mathcal{P}_{\chi}$ of distributions obeying $H_{\chi}$, $\chi = 1, 2$, is given by the saddle point of the function

$$\Phi(h; \mu_1, \mu_2) := \frac{1}{2} \left[ h^T [\mu_2 - \mu_1] + \frac{\sigma^2}{2} h^T A \text{Diag}\{\mu_1 + \mu_2\} A^T h \right]$$

♥ Data: $n = 16$, $\sigma = 0.1$, target risk 0.01,

• $A = U \text{Diag}\{0.01^{(i-1)/15}, i \leq 16\} V$ with random orthogonal $U$, $V$,

• $X_1 = \left\{ x \in \mathbb{R}^{16} : \begin{array}{l} 0.001 \leq x_1 \leq \delta \\ 0.001 \leq x_i \leq 1, i \geq 2 \end{array} \right\}$

• $X_2 = \left\{ x \in \mathbb{R}^{16} : \begin{array}{l} 2\delta \leq x_1 \leq 1 \\ 0.001 \leq x_i \leq 1, i \geq 2 \end{array} \right\}$

♥ Results:

$\delta = 0.1 \Rightarrow \text{Risk}[\phi_*|\mathcal{P}_1, \mathcal{P}_2] = 0.4346 \Rightarrow$ 6-repeated observation

$\delta = 0.01 \Rightarrow \text{Risk}[\phi_*|\mathcal{P}_1, \mathcal{P}_2] = 0.9201 \Rightarrow$ 56-repeated observation

• Safe “Gaussian o.s. approximation” of the above observation scheme requires 37-repeated observations to handle $\delta = 0.1$ and 3685-repeated observation to handle $\delta = 0.01$.

4.12
Sub-Gaussian case. For $\chi = 1, 2$, let $U_\chi \subset \Omega = \mathbb{R}^d$ and $\mathcal{V}_\chi \subset \text{int} S^d_+$ be convex compact sets. Setting

$$\mathcal{M}_\chi = U_\chi \times \mathcal{V}_\chi, \quad \Phi(h; u, \Theta) = h^T u + \frac{1}{2} h^T \Theta h : \mathcal{H} \times \mathcal{M}_\chi \to \mathbb{R},$$

the regular data $\mathcal{H} = \mathbb{R}^d, \mathcal{M}_\chi, \Phi$ specify the families

$$\mathcal{P}_\chi = S[\mathbb{R}^d, U_\chi \times \mathcal{V}_\chi, \Phi]$$

of sub-Gaussian distributions with parameters from $U_\chi \times \mathcal{V}_\chi$.

Saddle point problem responsible for design of affine detector for $\mathcal{P}_1, \mathcal{P}_2$ reads

$$\text{SadVal} = \min_{h \in \mathbb{R}^d} \max_{u_1 \in U_1, u_2 \in U_2, \Theta_1 \in \mathcal{V}_1, \Theta_2 \in \mathcal{V}_2} \frac{1}{2} \left[ h^T (u_2 - u_1) + \frac{1}{2} h^T [\Theta_1 + \Theta_2] h \right]$$

- Saddle point $(h^*_\chi; (u_1^*, u_2^*, \Theta_1^*, \Theta_2^*))$ does exist and satisfies

$$h^*_\chi = [\Theta_1^* + \Theta_2^*]^{-1} [u_1^* - u_2^*],$$

$$\text{SadVal} = -\frac{1}{4} [u_1^* - u_2^*][\Theta_1^* + \Theta_2^*]^{-1} [u_1^* - u_2^*] = -\frac{1}{4} h^*_\chi [u_1^* - u_2^*]$$

- The associated affine detector and its risk are

$$\phi^*_\chi (\omega) = h^*_\chi^T [\omega - \frac{1}{2} [u_1^* + u_2^*]] = [u_1^* - u_2^*]^T [\Theta_1^* + \Theta_2^*]^{-1} [\omega - \frac{1}{2} [u_1^* + u_2^*]]$$

$$\text{Risk}(\phi^*_\chi | \mathcal{P}_1, \mathcal{P}_2) \leq \exp\{\text{SadVal}\} = \exp\{-\frac{1}{4} [u_1^* - u_2^*][\Theta_1^* + \Theta_2^*]^{-1} [u_1^* - u_2^*]\}$$

4.13
Note: In the symmetric case $\mathcal{N}_1 = \mathcal{N}_2$ ($h^*; (u_{1*}, u_{2*}, \Theta_{1*}, \Theta_{2*})$) can be selected to have $\Theta_{1*} = \Theta_{2*} =: \Theta_*$. In this case, the affine detector we end up with is the minimum risk detector for $\mathcal{P}_1, \mathcal{P}_2$. 
What is “affine?” Quadratic Lifting

♣ We have developed a technique for building “presumably good” affine detectors for simple families of distributions.

**But:** Given observation \( \zeta \sim P \), we can subject it to nonlinear transformation \( \zeta \mapsto \omega = \psi(\zeta) \), e.g., to quadratic lifting

\[
\zeta \mapsto \omega = (\zeta, \zeta \zeta^T)
\]

and treat as our observation \( \omega \) rather than the “true” observation \( \zeta \).

**Note:** **Affine in \( \omega \) detectors are nonlinear in \( \zeta \).**

**Example:** Detectors affine in the quadratic lifting \( \omega = (\zeta, \zeta \zeta^T) \) of \( \zeta \) are exactly the quadratic functions of \( \zeta \).

♠ We can try to apply our machinery for building affine detectors to nonlinear transformations of true observations, thus arriving at nonlinear detectors.

**Bottleneck:** To apply the outlined strategy to a pair \( \mathcal{P}_1, \mathcal{P}_2 \) of families of distributions of interest, we need to cover the families \( \mathcal{P}_\chi^+ \) of distributions of \( \omega = \psi(\zeta) \) induced by distributions \( P \in \mathcal{P}_\chi \) of \( \zeta, \chi = 1, 2 \), by simple families of distributions.

**What is ahead:** Simple “coverings” of quadratic lifts of (sub)Gaussian distributions.
Situation: Given are:

- a compact nonempty set $U \subset \mathbb{R}^n$
- an affine mapping $u \mapsto A(u) = A[u; 1]: \mathbb{R}^n \to \mathbb{R}^d$
- a convex compact set $V \subset \text{int} \mathbb{S}^d_+$.

The above data specify families of probability distributions of random observations

$$\omega = (\zeta, \zeta^T), \quad \zeta = A(u) + \xi \in \mathbb{R}^d,$$

specifically,

- the family $G$ of all distributions of $\omega$ induced by deterministic $u \in U$ and *Gaussian* noise $\xi \sim \mathcal{N}(0, \Theta \in V)$
- the family $SG$ of all distributions of $\omega$ induced by deterministic $u \in U$ and *sub-Gaussian*, with parameters $(0, \Theta \in V)$ noise $\xi$

Goal: To cover $G$ ($SG$) by a simple family of distributions.
**Proposition.** Given the above data $U$, $A(u) = A[u; 1]$, $V$, let us select
- $\gamma \in (0, 1)$
- a computationally tractable convex compact set $\mathcal{Z} = \{Z \in S^{n+1} : Z \succeq 0, Z_{n+1,n+1} = 1\}$
- a matrix $\Theta_* \in S^d$ and $\delta \in [0, 2]$ such that
  $$\forall (\Theta \in \mathcal{V}) : \Theta \preceq \Theta_* \quad \text{and} \quad \|\Theta^{1/2}\Theta_*^{-1/2} - I_d\| \leq \delta$$
  (where $\|\cdot\|$ is the spectral norm)

Let us set

$$B = \begin{bmatrix} A \\ 0, \ldots, 0, 1 \end{bmatrix} \in \mathbb{R}^{(d+1) \times (n+1)}$$

$$\mathcal{M} = \mathcal{V} \times \mathcal{Z}$$

$$\mathcal{H} = \{(h, H) \in \mathbb{R}^d \times S^d : -\gamma \Theta_*^{-1} \leq H \leq \gamma \Theta_*^{-1}\}$$

$$\Phi_{A,Z}(h, H; \Theta, Z) = -\frac{1}{2} \ln \det(I - \Theta_*^{1/2} H \Theta_*^{1/2}) + \frac{1}{2} \text{Tr}([\Theta - \Theta_*]H) + \frac{\delta(2+\delta)\|\Theta_*^{1/2} H \Theta_*^{1/2}\|^2}{2(1 - \|\Theta_*^{1/2} H \Theta_*^{1/2}\|)}$$

$$+ \frac{1}{2} \text{Tr} \left( Z B^T \begin{bmatrix} H \\ h^T \end{bmatrix} h \right) + [H, h]^T \left( \Theta_*^{-1} - H \right)^{-1} [H, h] B \right) : \mathcal{H} \times \mathcal{M} \rightarrow \mathbb{R}$$

Then $\mathcal{H}$, $\mathcal{M}$, $\Phi_{A,Z}$ is efficiently computable regular data, and $\mathcal{G} \subset S[\mathcal{H}, \mathcal{M}, \Phi_{A,Z}]$. 

4.17
Proposition. Given the above data $U, A(u) = A[u; 1], \mathcal{V}$, let us select

- $\gamma, \gamma^+ \in (0, 1)$ with $\gamma < \gamma^+$
- a computationally tractable convex compact set

$$\mathcal{Z} \subset \mathcal{Z}^+ = \{Z \in S^{n+1} : Z \succeq 0, Z_{n+1,n+1} = 1\}$$

such that $[u; 1][u; 1]^T \in \mathcal{Z} \ \forall u \in U$

- A matrix $\Theta_* \in S^d$ and $\delta \in [0, 2]$ such that

$$\forall (\Theta \in \mathcal{V}) : \Theta \preceq \Theta_* \ \& \ \|\Theta^{1/2}\Theta_*^{-1/2} - I_d\| \leq \delta$$

Let us set

$$B = \begin{bmatrix} A \\ 0, ..., 0, 1 \end{bmatrix} \in \mathbb{R}^{(d+1) \times (n+1)}, \mathcal{H} = \{(h, H) \in \mathbb{R}^d \times S^d : -\gamma\Theta_*^{-1} \preceq H \preceq \gamma\Theta_*^{-1}\}$$

$$\mathcal{H}^+ = \{(h, H, G) \in \mathbb{R}^d \times S^d \times S^d : -\gamma^+\Theta_*^{-1} \preceq H \preceq G \preceq \gamma^+\Theta_*^{-1}, 0 \preceq G\}, \mathcal{M} = \mathcal{Z}$$

$$\Phi_{A,Z}(h, H; Z) = \min_{G : (h, H, G) \in \mathcal{H}^+} \left\{ -\frac{1}{2} \ln \det(I - \Theta_*^{1/2}G\Theta_*^{1/2}) + \frac{1}{2} \text{Tr} \left( ZB^T \begin{bmatrix} H \\ h \end{bmatrix} + [H, h]^T [\Theta_*^{-1} - G]^{-1} [H, h] ] B \right) \right\} : \mathcal{H} \times \mathcal{M} \to \mathbb{R}$$

Then $\mathcal{H}, \mathcal{M}, \Phi_{A,Z}$ is efficiently computable regular data, and $SG \subset S[\mathcal{H}, \mathcal{M}, \Phi_{A,Z}]$. 4.18
How to specify $\mathcal{Z}$. To apply the above construction, one should specify a computationally tractable convex compact set

$$\mathcal{Z} \subset \mathcal{Z}^+ = \{ Z \in S^{n+1} : Z \succeq 0, Z_{n+1,n+1} = 1 \}$$

the smaller the better, such that $u \in U \rightarrow [u; 1][u; 1]^T \in \mathcal{Z}$

- The ideal selection is

$$\mathcal{Z} = \mathcal{Z}[U] = \text{Conv}\{[u; 1][u; 1]^T : u \in U \}$$

However: $\mathcal{Z}[U]$ usually is computationally intractable.

Important exception:

$$Q \succ 0, U = \{ u : u^T Qu \leq 1 \} \Rightarrow \mathcal{Z}[U] = \{ Z \in \mathcal{Z}^+ : \sum_{i,j=1}^{n} Z_{ij} Q_{ij} \leq 1 \}$$
“Simple” case: When $U$ is given by quadratic inequalities:

$$U = \{ u \in \mathbb{R}^n : [u; 1]^T Q_s [u; 1] \leq q_s, 1 \leq s \leq S \}$$

we can set

$$Z = \{ Z \in S^{n+1} : Z \succeq 0, Z_{n+1,n+1} = 1, \text{Tr}(Q_s Z) \leq q_s, 1 \leq s \leq S \}.$$ 

• **Warning:** (*) can yield very conservative outer approximation of $Z[U]$. This conservatism with luck can be reduced by passing from the original description of $U$ to an equivalent one, with emphasis on eliminating/updating linear constraints. For example,

• a constraint of the form $|a^T x - c| \leq r$ should be replaced with $(a^T x - c)^2 \leq r^2$

**Note:** every linear constraint in the description of $U$ can be written as $\alpha - a^T x \geq 0$ and augmented by redundant constraint $a^T x \geq \beta$, with appropriately selected $\beta$. The resulting pair of constraints is equivalent to $|a^T x - c| \leq r$ with $c = \frac{1}{2}[\alpha + \beta]$ and $r = \frac{1}{2}[\alpha - \beta]$.

• It could make sense to write the linear constraints in the description of $U$ in the form $\alpha - a^T x \geq 0$ and add to these constraints their pairwise products.
Quadratic Lifting – Does it Pay?

♦ Situation: Let for $\chi = 2, 1$ be given
  - convex compact sets $U_\chi \subset \mathbb{R}^{n_\chi}$
  - affine mappings $u_\chi \mapsto A_\chi(u_\chi) : \mathbb{R}^{n_\chi} \to \mathbb{R}^d$
  - convex compact sets $V_\chi \subset \text{int } S^d_+$.

These data define families $G_\chi$ of Gaussian distributions:

$$G_\chi = \{ \mathcal{N}(A_\chi(u_\chi), \Theta_\chi) : u_\chi \in U_\chi, \Theta_\chi \in V_\chi \}$$

♣ Our machinery offers two types of detectors for $G_1, G_2$:
  - Affine detector $\phi_{\text{aff}}$ yielded by the solution to the saddle point problem

$$\text{SadVal}_{\text{aff}} = \min_{h \in \mathbb{R}^d} \max_{u_1 \in U_1, u_2 \in U_2, \Theta_1 \in V_1, \Theta_2 \in V_2} \frac{1}{2} \left[ h^T [A_2(u_2) - A_1(u_1)] + \frac{1}{2} h^T [\Theta_1 + \Theta_2] h \right]$$

with $\text{Risk}(\phi_{\text{aff}}|G_1, G_2) \leq \exp\{\text{SadVal}_{\text{aff}}\}$

  - Quadratic detector $\phi_{\text{lift}}$ yielded by the solution to the saddle point problem

$$\text{SadVal}_{\text{lift}} = \min_{(h,H) \in \mathcal{H}} \max_{\Theta_1 \in V_1, \Theta_2 \in V_2} \frac{1}{2} \left[ \Phi_{A_1, Z_1}(-h, -H; \Theta_1) + \Phi_{A_2, Z_2}(h, H; \Theta_2) \right]$$

with $\text{Risk}(\phi_{\text{lift}}|G_1, G_2) \leq \exp\{\text{SadVal}_{\text{lift}}\}$

4.21
♠ Fact: Assume that the sets $\mathcal{V}_\chi$ contain $\succeq$-largest elements. Then with proper selection of the “design parameters” $Z_\chi, \Theta(\chi)$ participating in the construction of $\Phi A_\chi Z_\chi, \chi = 1, 2$, passing from affine to quadratic detectors helps:

$$\text{SadVal}_{\text{lift}} \leq \text{SadVal}_{\text{aff}}$$

♥ Numerical illustration:

- $U_1 = U_1^\rho = \{u \in \mathbb{R}^{12} : u_i \geq \rho, 1 \leq i \leq 12\}$, $U_2 = U_2^\rho = -U_1^\rho$, $A_1 = A_2 \in \mathbb{R}^{8 \times 13}$;
- $\mathcal{V}_\chi = \{\Theta(\chi) = \sigma_\chi^2 I_8\}$

<table>
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<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>unrestricted $H$ and $h$</th>
<th>$H = 0$</th>
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<td>4</td>
<td>0.41</td>
<td>1.00</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Risk of quadratic detector $\phi(\zeta) = h^T \zeta + \frac{1}{2} \zeta^T H \zeta + \chi$

♣ We see that • when deciding on families of Gaussian distributions with common covariance matrix and expectations varying in associated with the families convex sets, passing from affine to quadratic detectors does not help.

• in general, both affine and purely quadratic components in a quadratic detector are useful.

• when deciding on families of Gaussian distributions in the case where distributions from different families can have close expectations, affine detectors are useless, while the quadratic ones are not.

4.22
Illustration: Simple Change Point Detection

Frames from a noisy “movie”

When the picture starts to change?
Model: We observe one by one vectors (“vectorized” 2D images)

\[ \omega_t = x_t + \xi_t, \]

- \( x_t \): deterministic image
- \( \xi_t \sim \mathcal{N}(0, \sigma^2 I_d) \): independent across observation noises.

Note: We know a range \([\sigma, \overline{\sigma}]\) of \(\sigma\), but perhaps do not know \(\sigma\) exactly.

- We know that \( x_1 = x_2 \) and want to check whether \( x_1 = \ldots = x_K \) (“no change”) or there is a change.

Goal: Given an upper bound \( \epsilon > 0 \) on the probability of false alarm, we want to design a sequential change detection routine capable to detect change, if any.
Approach:

- Pass from observations $\omega_t$, $1 \leq t \leq K$, to observations

$$\zeta_t = \omega_t - \omega_1 = \underbrace{x_t - x_1}_{y_t} + \underbrace{\xi_t - \xi_1}_{\eta_t}, \quad 2 \leq t \leq K$$

- Test hypothesis $H_0 : y_2 = \ldots = y_K = 0$ vs. alternative

$$\bigcup_{k=2}^{K} H^\rho_k, \quad H^\rho_k : y_2 = \ldots = y_{k-1} = 0, \|y_k\|_2 \geq \rho$$

via our machinery for testing

magenta hypothesis $H_0$

vs.

brown hypotheses $H^\rho_2, \ldots, H^\rho_K$

via quadratic liftings $\zeta_t \zeta_t^T$ of observations $\zeta_t$ up to closeness

$C$: all brown hypotheses are close to each other and are not close to the magenta hypothesis

- We intend to find the smallest $\rho$ for which the $C$-risk of the resulting inference is $\leq \epsilon$, and utilize this inference in change point detection.
How It Works

♠ Setup: \( \dim y = 256^2 = 65536, \sigma = 10, \sigma^2/\sigma^2 = 2, K = 9, \varepsilon = 0.01 \)

♠ Inference: At time \( t = 2, \ldots, K \), compute

\[
\phi_*(\zeta_t) = -2.7138 \frac{\|\zeta_t\|_2^2}{10^5} + 366.9548.
\]

\[\phi_*(\zeta_t) < 0 \implies \text{conclude that the change took place and terminate}
\]

\[\phi_*(\zeta_t) \geq 0 \implies \text{conclude that there was no change so far and proceed to the next image, if any}
\]

♠ Note:

● When magenta hypothesis \( H_0 \) holds true, the probability not to claim change on time horizon 2, \ldots, \( K \) is at least 0.99.

● When a brown hypothesis \( H_{\rho_k}^\rho \) holds true, the change at time \( \leq K \) is detected with probability at least 0.99, provided \( \rho \geq \rho_* = 2716.6 \) (average per pixel energy in \( y_k \) at least by 12% larger than \( \sigma^2 \))

● No test can 0.99-reliably decide via \( \zeta_1, \ldots, \zeta_k \) on \( H_{\rho_k}^\rho \) vs. \( H_0 \) when \( \rho/\rho_* < 0.965 \).

● In the movie, the change takes place at time 3 and is detected at time 4.
ESTIMATING SIGNALS IN GAUSSIAN O.S. AND BEYOND

- Problem of interest
- Developing tools
  - Conic Programming
  - Conic Duality
- Optimizing linear estimates
  - Ellitopic case
  - Spectratopic case
- Near-optimality of linear estimates
- Beyond linearity: polyhedral estimates
Situation: “In the nature” there exists a signal $x$ known to belong to a given convex compact set $\mathcal{X} \subset \mathbb{R}^n$. We observe corrupted by noise affine image of the signal (“indirect observations”):

$$\omega = Ax + \xi \in \mathbb{R}^m$$

- $A$: given $m \times n$ sensing matrix
- $\xi$: $\mathcal{N}(0, \sigma^2 I)$ observation noise

Goal: To recover the image $Bx$ of $x$ under a given linear mapping
- $B$: given $\nu \times n$ matrix.

Risk of a candidate estimate $\hat{x}(\cdot) : \Omega \to \mathbb{R}^\nu$ is defined as

$$\text{Risk}[\hat{x} | \mathcal{X}] = \sup_{x \in \mathcal{X}} \sqrt{\mathbb{E}_\xi \left\{ \|Bx - \hat{x}(Ax + \xi)\|_2^2 \right\}}$$

$\Rightarrow$ Risk$^2$ is the worst-case, over $x \in \mathcal{X}$, expected $\| \cdot \|_2^2$ recovery error.

With this worst-case quantification of risk, the “golden standard” is the minimax risk

$$\text{RiskOpt}[\mathcal{X}] = \inf_{\hat{x}} \text{Risk}[\hat{x} | \mathcal{X}],$$

inf being taken over all estimates – all (measurable) functions $\hat{x}(\cdot) : \mathbb{R}^m \to \mathbb{R}^\nu$. .

5.1
**Agenda:** Under appropriate assumptions on $\mathcal{X}$, we shall show that

A. One can build, in a computationally efficient fashion, (nearly) the best, in terms of risk, estimate in the family of linear estimates

$$\hat{x}(\omega) = \hat{x}_H(\omega) = H^T \omega$$

$[H \in \mathbb{R}^{m \times \nu}]$

B. The resulting linear estimate is nearly optimal among all estimates, linear and nonlinear alike.

C. Under appropriate assumptions on a norm $\| \cdot \|$ and a family $\mathcal{P}$ of distributions of observation noise, the results of A, B can be extended to the situation where

— the recovery error is measured in norm $\| \cdot \|$,  
— distribution $P$ of observation noise is known to belong to $\mathcal{P}$,  
— the $\| \cdot \|_2$-risk

$$\text{Risk}[\hat{x}|\mathcal{X}] = \sup_{x \in \mathcal{X}} \sqrt{\mathbb{E}_\xi \left\{ \|Bx - \hat{x}(Ax + \sigma \xi)\|_2 \right\}}$$

is replaced with $$(\| \cdot \|, \mathcal{P})$$-risk

$$\text{Risk}_{\| \cdot \|, \mathcal{P}}[\hat{x}|\mathcal{X}] = \sup_{x \in \mathcal{X}} \sup_{P \in \mathcal{P}} \mathbb{E}_{\xi \sim P} \left\{ \|Bx - \hat{x}(Ax + \xi)\| \right\}$$

5.2
Why linear estimates?

♠ As it was announced, a “nearly optimal” linear estimate can be built in a computationally efficient fashion.

♠ In contrast,

- Exactly minimax optimal estimate is unknown even in the simplest case when the observation is \( \omega = x + \xi \) with \( \xi \sim \mathcal{N}(0, \sigma^2) \) and \( x \in \mathcal{X} = [-1, 1] \)
- The “magic wand” of Statistics – the Maximum Likelihood estimate — is known to be optimal in the “noise goes to 0” asymptotics and can be disastrously bad before this asymptotics starts.
When $\sigma \ll 1$, $\sigma^2 n \geq O(1)$, and $\epsilon \leq O(\sigma)$, the risk of MLE is $O(1)$, while the risk of the linear estimate $\hat{x}(\omega) = \omega_n$ is $O(\sigma) \ll O(1)$.

**Note:** As $\sigma \to 0$, the ML estimate regains optimality, but this happens the later the larger is $n$. 
When passing from a Linear Programming program
\[
\min_x \left\{ c^T x : Ax - b \geq 0 \right\}
\]
to a convex one, the traditional wisdom is to replace linear inequality constraints
\[
a_i^T x - b_i \geq 0
\]
with nonlinear ones:
\[
g_i(x) \geq 0 \quad [g_i \text{ are concave}]
\]
There exists, however, another way to introduce nonlinearity, namely, to replace the coordinate-wise \textit{vector} inequality
\[
y \geq z \iff y - z \in \mathbb{R}^m_+ = \{ u \in \mathbb{R}^m : u_i \geq 0 \ \forall i \} \quad [y, z \in \mathbb{R}^m]
\]
with another \textit{vector} inequality
\[
y \geq_K z \iff y - z \in K \quad [y, z \in \mathbb{R}^m]
\]
where \( K \) is a \textit{regular cone} (i.e., closed, pointed and convex cone with a nonempty interior) in \( \mathbb{R}^m \).
\[ y \geq_K z \iff y - z \in K \]  

[y, z \in \mathbb{R}^m]

K: closed, pointed and convex cone in \( \mathbb{R}^m \) with a nonempty interior.

Requirements on K ensure that \( \geq_K \) obeys the usual rules for inequalities:

- \( \geq_K \) is a partial order:
  
  \[
  x \geq_K x \forall x \quad \text{[reflexivity]}
  \]
  
  \[
  (x \geq_K y \& y \geq_K x) \Rightarrow x = y \quad \text{[antisymmetry]}
  \]
  
  \[
  (x \geq_K y, y \geq_K z) \Rightarrow x \geq_K z \quad \text{[transitivity]}
  \]

- \( \geq_K \) is compatible with linear operations: the validity of \( \geq_K \) inequality is preserved when we multiply both sides by the same nonnegative real and add to it another valid \( \geq_K \)-inequality;

- in a sequence of \( \geq_K \)-inequalities, one can pass to limits:
  
  \[
  \{a_i \geq_K b_i, i = 1, 2, \ldots \& a_i \to a \& b_i \to b\} \Rightarrow a \geq_K b
  \]

- one can define the strict version \( >_K \) of \( \geq_K \):
  
  \[
  a >_K b \iff a - b \in \text{int } K.
  \]

Arithmetics of \( >_K \) and \( \geq_K \) inequalities is completely similar to the arithmetics of the usual coordinate-wise \( \geq \) and \( > \).
(LP problem:)
\[
\min_x \{ c^T x : Ax - b \geq 0 \} \iff \min_x \{ c^T x : Ax - b \in \mathbb{R}^m_+ \}
\]

(General Conic problem:)
\[
\min_x \{ c^T x : Ax - b \geq_K 0 \} \iff \min_x \{ c^T x : Ax - b \in K \}
\]

- \((A, b)\) – data of conic problem
- \(K\) - structure of conic problem

(\(\spadesuit\) Note: Every convex problem admits equivalent conic reformulation)  
(\(\spadesuit\) Note: With conic formulation, convexity is “built in”; with the standard MP formulation convexity should be kept in mind as an additional property.)

(??) A general convex cone has no more structure than a general convex function. Why conic reformulation is “structure-revealing”?

(!!) As a matter of fact, just 3 types of cones allow to represent an extremely wide spectrum (“essentially all”) of convex problems!

5.7
\[
\min_{x} \left\{ c^T x : Ax - b \geq_{K} 0 \right\} \iff \min_{x} \left\{ c^T x : Ax - b \in K \right\}
\]

♠ Three Magic Families of cones:

• **LP**: Nonnegative orthants $\mathbb{R}^m_+$ – direct products of $m$ nonnegative rays $\mathbb{R}_+ = \{ s \in \mathbb{R} : s \geq 0 \}$ giving rise to Linear Programming programs $\min_{s} \left\{ c^T x : a^T_{\ell} x - b_{\ell} \geq 0, \ 1 \leq \ell \leq q \right\}$.

• **CQP**: Direct products of Lorentz cones $L^p_+ = \{ u \in \mathbb{R}^p : u_p \geq \left( \sum_{i=1}^{p-1} u_i^2 \right)^{1/2} \}$ giving rise to Conic Quadratic programs $\min_{x} \left\{ c^T x : \| A_{\ell} x - b_{\ell} \|_2 \leq c^T_{\ell} x - d_{\ell}, \ 1 \leq \ell \leq q \right\}$.

• **SDP**: Direct products of Semidefinite cones $S^p_+ = \{ M \in S^p : M \succeq 0 \}$ giving rise to Semidefinite programs $\min_{x} \left\{ c^T x : \lambda_{\min}(A_{\ell}(x)) \geq 0, \ 1 \leq \ell \leq q \right\}$.

where $S^p$ is the space of $p \times p$ real symmetric matrices, $A_{\ell}(x) \in S^p$ are affine in $x$ and $\lambda_{\min}(S)$ is the minimal eigenvalue of $S \in S^p$. 
What can be reduced to $\mathcal{LP}/\mathcal{CQP}/\mathcal{SDP}$?

Calculus of Conic programs

♣ Let $\mathcal{K}$ be a family of regular cones closed w.r.t. taking direct products.

♠ Definition: • A $\mathcal{K}$-representation of a set $X \subset \mathbb{R}^n$ is a representation

$$X = \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m : Ax + Bu - b \in K\}$$

where $K \in \mathcal{K}$.

• $X$ is called $\mathcal{K}$-representable, if $X$ admits a $\mathcal{K}$-r.

♥ Note: Minimizing a linear objective $c^T x$ over a $\mathcal{K}$-representable set $X$ reduces to a conic program on a cone from $\mathcal{K}$.

Indeed, given ($\ast$), problem $\min_{x \in X} c^T x$ is equivalent to

$$\text{Opt} = \min_{x,u} \{c^T x : Ax + Bu - b \in K\}$$

♠ Definition: • A $\mathcal{K}$-representation of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a $\mathcal{K}$-representation of the epigraph of $f$:

$$\text{Epi}\{f\} := \{(x,t) : t \geq f(x)\}$$

$$= \{x, t : \exists v : Px + pt + Qv - q \in K\}, \quad K \in \mathcal{K}$$

• $f$ is called $\mathcal{K}$-representable, if $f$ admits a $\mathcal{K}$-r.
Note:

- A level set of a $\mathcal{K}$-r. function is $\mathcal{K}$-r.:

  $$\text{Epi}\{f\} := \{(x, t) : t \geq f(x)\}$$
  $$= \{x, t : \exists v : Px + pt + Qu - q \in \mathcal{K}\}$$
  $$\Rightarrow \{x : f(x) \leq c\} = \{x : \exists v : Px + Qu - [q - cp] \in \mathcal{K}\}$$

- Minimization of a $\mathcal{K}$-r. function $f$ over a $\mathcal{K}$-r. set $X$ reduces to a conic program on a cone from $\mathcal{K}$:

  $$x \in X \iff \exists u : Ax + Bu - b \in \mathcal{K}_X$$
  $$t \geq f(x) \iff \exists v : Px + pt + Qv - q \in \mathcal{K}_f$$

  $$\min_{x \in X} f(x) \Leftrightarrow \min_{t, x, u, v} \left\{ t : [Ax + Bu - b; Px + pt + Qv - q] \in \mathcal{K}_X \times \mathcal{K}_f \right\} \in \mathcal{K}$$
♣ Investigating “expressive abilities” of generic Magic conic problems reduces to answering the question

What are $\mathcal{LP}/\mathcal{CQP}/\mathcal{SDP}$-r. functions/sets?

♠ “Built-in” restriction is Convexity: A $\mathcal{K}$-representable set/function must be convex.

♦ Good news: Convexity, essentially, is the only restriction: for all practical purposes, all convex sets/functions arising in applications are $\mathcal{SDP}$-r. Quite rich families of convex functions/sets are $\mathcal{LP}/\mathcal{CQP}$-r.

♥ Note: Nonnegative orthants are direct products of (1-dimensional) Lorentz cones, and Lorentz cones are intersections of semidefinite cones and properly selected linear subspaces $\Rightarrow \mathcal{LP} \subset \mathcal{CQP} \subset \mathcal{SDP}$.
Let \( \mathcal{K} \) be a family of regular cones closed w.r.t. taking direct products and passing from a cone \( K \) to its dual cone

\[
K_* = \{ \lambda : \langle \lambda, \xi \rangle \geq 0 \ \forall \xi \in K \}
\]

Note: \( K_* \) is regular cone provided \( K \) is so, and

\[
(K_*)_* = K
\]

Fact: \( \mathcal{K} \)-representable sets/functions admit fully algorithmic calculus: all basic convexity-preserving operations with functions/sets, as applied to \( \mathcal{K} \)-r. operands, produce \( \mathcal{K} \)-r. results, and the resulting \( \mathcal{K} \)-r.’s are readily given by \( \mathcal{K} \)-r.’s of the operands. “Calculus rules” are independent of what \( \mathcal{K} \) is.

⇒ Starting with “raw materials” (characteristic for \( \mathcal{K} \) elementary \( \mathcal{K} \)-r. sets/functions) and applying calculus rules, we can recognize \( \mathcal{K} \)-representability and get explicit \( \mathcal{K} \)-r.’s of sets/functions of interest.
Basics of “calculus of $\mathcal{K}$-representability”:

**[Sets:]** If $X_1, \ldots, X_k$ are $\mathcal{K}$-r. sets, so are their
- intersections,
- direct products,
- images under affine mappings,
- inverse images under affine mappings.

**[Functions:]** If $f_1, \ldots, f_k$ are $\mathcal{K}$-r. functions, so are their
- linear combinations with nonnegative coefficients,
- superpositions with affine mappings.

Moreover, if $F, f_1, \ldots, f_k$ are $\mathcal{K}$-r. functions, so is the superposition $F(f_1(x), \ldots, f_k(x))$ provided that $F$ is monotonically nondecreasing in its arguments.

More advanced convexity-preserving operations preserve $\mathcal{K}$-representability under (pretty mild!) regularity conditions. This includes
- for sets: taking conic hulls and convex hulls of (finite) unions and passing from a set to its recessive cone, or polar, or support function
- for functions: partial minimization, projective transformation, and taking Fenchel dual.

**Note:** Calculus rules are simple and algorithmic

⇒ Calculus can be run on a compiler [used in cvx].
\[
\begin{align*}
\text{min } c^T x + d^T y \\
y \geq 0, \ Ax + By \leq b
\end{align*}
\]

\[
\begin{bmatrix}
2y_1^2 y_2^3 y_3^3 + 3y_2^2 y_3^2 y_4^2 & \leq e^T x + 4y_1^2 y_2^2 y_3^2 + 5y_3^2 y_4^2 \\
x_1 - x_2 & x_3 + x_2 \\
x_3 + x_2 & x_2 - x_4 & x_5 - 6 \\
x_5 - 6 & x_6 + x_7 & -x_8 \\
-x_8 & x_5
\end{bmatrix} \preceq 0
\]

\[
\text{Det} \left( \begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 \\
x_2 & x_6 & x_7 & x_8 & x_9 \\
x_3 & x_7 & x_{10} & x_{11} & x_{12} \\
x_4 & x_8 & x_{11} & x_{13} & x_{14} \\
x_5 & x_9 & x_{12} & x_{14} & x_{15}
\end{bmatrix} \right) \geq 1
\]

\[
\text{Sum of 2 largest singular values of} \quad \begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 \\
x_4 & x_5 & x_7 & x_8 & x_9 \\
x_7 & x_8 & x_{10} & x_{11} & x_{12} \\
x_{10} & x_{11} & x_{13} & x_{14} & x_{15}
\end{bmatrix} \quad \text{is} \leq 6
\]

\[
1 - \sum_{i=1}^6 [x_i - x_{i+1}] s^i \leq 0, \quad \frac{3}{2} \leq s \leq 6 \\
\sum_{i=1}^4 x_{2i} \cos(i \phi) - \sum_{i=1}^4 x_i \sin(i \phi) \leq 1, \quad \frac{\pi}{3} \leq \phi \leq \frac{\pi}{2}
\]

- the blue part of the problem is in \(\mathcal{LP}\)
- the blue-magenta part of the problem is in \(\mathcal{CQP}\) and can be approximated, in a polynomial time fashion, by \(\mathcal{LP}\)
- the entire problem is in \(\mathcal{SDP}\)

and the reductions to \(\mathcal{LP}/\mathcal{CQP}/\mathcal{SDP}\) are “fully algorithmic.”

5.14
Conic Duality

♣ Conic Programming admits nice Duality Theory completely similar to LP Duality.

Primal problem:

\[
\begin{align*}
\min_x & \left\{ c^T x : \begin{array}{ll}
Ax - b & \geq_K 0 \\
Rx & = r
\end{array} \right\} \\
\Leftrightarrow & \min_\xi \left\{ e^T \xi : \xi \in [\mathcal{L} - b] \cap K \right\} \quad \text{(P)}
\end{align*}
\]

[passing to primal slack \( \xi = Ax - b \)]

\[
\begin{align*}
e : A^T e + R^T f &= c \text{ for some } f \\
\mathcal{L} &= \{ Au : Ru = 0 \}
\end{align*}
\]

Dual problem:

\[
\begin{align*}
\max_{y,z} & \left\{ b^T y : A^T y + R^T z = c, \ y \geq_{K^*} 0 \right\} \\
\Leftrightarrow & \max_y \left\{ b^T y : y \in K^*, \ \exists z : A^T y + R^T z = c \right\} \\
& \max_y \left\{ b^T y : y \in [\mathcal{L}^\perp + e] \cap K^* \right\} \quad \text{(D)}
\end{align*}
\]

\([K^* : \text{cone dual to } K]\)
Note:

- the dual problem is conic along with primal
- the duality is completely symmetric

Note: Cones from Magic Families are self-dual, so that the dual of a Linear/Conic Quadratic/Semidefinite program is of exactly the same type.
Derivation of the Dual Problem

Primal problem:
\[
\text{Opt}(P) = \min_x \left\{ c^T x : A_i x - b_i \in K_i^i, i \leq m \right\} \quad (P)
\]

Goal: find a systematic way to bound \(\text{Opt}(P)\) from below.

Simple observation: When \(y_i \in K_i^i\), the scalar inequality \(y_i^T A_i x \geq y_i^T b_i\) is a consequence of the constraint \(A_i x - b_i \in K_i^i\). If \(z\) is a vector of the same dimension as \(r\), the scalar inequality \(z^T R x \geq z^T r\) is a consequence of the constraint \(R x = r\).

\(\Rightarrow\) Whenever \(y_i \in K_i^i\) for all \(i\) and \(z\) is a vector of the same dimension as \(r\), the scalar linear inequality
\[
[\sum_i A_i^T y_i + R^T z]^T x \geq \sum_i b_i^T y_i + r^T z
\]
is a consequence of the constraints in \((P)\).

\(\Rightarrow\) Whenever \(y_i \in K_i^i\) for all \(i\) and \(z\) is a vector of the same dimension as \(r\) such that
\[
\sum_i A_i^T y_i + R^T z = c,
\]
the quantity \(\sum_i b_i^T y_i + r^T z\) is a lower bound on \(\text{Opt}(P)\).

The Dual problem
\[
\text{Opt}(D) = \max_{y_i,z} \left\{ \sum_i b_i^T y_i + r^T z : y_i \in K_i^i, i \leq m, \sum_i A_i^T y_i + R^T z = c \right\} \quad (D)
\]
is just the problem of maximizing this lower bound on \(\text{Opt}(P)\).
Definition: A conic problem

\[
\min_x \left\{ c^T x : \begin{array}{l}
A_i x - b_i \in K^i, \ i \leq m \\
A x \leq b \\
R x = r
\end{array} \right\} \tag{C}
\]

is called strictly feasible, if there exists a feasible solution \( \bar{x} \) where all conic and \( \leq \) constraints are satisfied strictly:

\[
A_i \bar{x} - b_i \in \text{int} \ K^i \ \forall i \ & \ A \bar{x} < b,
\]

and is called essentially strictly feasible, if there exists a feasible solution \( \bar{x} \) where all non-polyhedral constraints are satisfied strictly:

\[
A_i \bar{x} - b_i \in \text{int} \ K^i \ \forall i.
\]
Conic Programming Duality Theorem. Consider a conic problem
\[ \text{Opt}(P) = \min_x \left\{ c^T x : A_i x - b_i \in K^i, i \leq m \right\} \] (P)
along with its dual
\[ \text{Opt}(D) = \max_{y_i, z} \left\{ \sum_i b_i^T y_i + r^T z : y_i \in K^i_*, i \leq m \right\} \] (D)

Then:
♠ [Symmetry] Duality is symmetric: the dual problem is conic, and its dual is (equivalent to) the primal problem;
♠ [Weak duality] One has \( \text{Opt}(D) \leq \text{Opt}(P) \);
♠ [Strong duality] Let one of the problems be essentially strictly feasible and bounded. Then the other problem is solvable, and
\[ \text{Opt}(D) = \text{Opt}(P) \]

In particular, if both problems are essentially strictly feasible, both are solvable with equal optimal values.
Conic Programming Optimality Conditions:

Let both \((P)\) and \((D)\) be essentially strictly feasible. Then a pair \((x, \{y_i\}, z)\) of primal and dual feasible solutions is comprised of optimal solutions to the respective problems if and only if

- **Zero Duality Gap**
  \[
  \text{DualityGap}(x, \{y_i\}, z) := c^T x - \left[ \sum_i b_i^T y_i + r^T z \right] = 0
  \]
  Indeed, \(\text{DualityGap}(x, \{y_i\}, z) = [c^T x - \text{Opt}(P)] + [\text{Opt}(D) - \sum_i b_i^T y_i + r^T z] \geq 0\)

  and if and only if

- **Complementary Slackness**
  \[
  [A_i x - b_i]^T y_i = 0, \quad i \leq m
  \]
  Indeed, \(\sum_i [A_i x - b_i]^T y_i = [\sum_i A_i^T y_i] x - \sum_i b_i^T y_i = [c - R^T z]^T x - \sum_i b_i^T y_i = c^T x - \sum_i b_i^T y_i + r^T z = c^T x - [\sum_i b_i^T y_i + r^T z] \geq 0\)
  \(= \text{DualityGap}(x, \{y_i\}, z)\)
Conic Duality, same as the LP one, is

- fully algorithmic: to write down the dual, given the primal, is a purely mechanical process
- fully symmetric: the dual problem “remembers” the primal one

Cf. Lagrange Duality:

\[
\begin{align*}
\min_x \{ f(x) : g_i(x) \leq 0, \ i = 1, ..., m \} \quad (P) \\
\downarrow \\
\max_{y \geq 0} L(y) \quad (D)
\end{align*}
\]

\[
L(y) = \min_x \left\{ f(x) + \sum_i y_i g_i(x) \right\}
\]

- Dual “exists in the nature”, but is given implicitly; its objective, typically, is not available in a closed form
- Duality is asymmetric: given \(L(\cdot)\), we, typically, cannot recover \(f\) and \(g_i\)...
Schur Complement Lemma

Lemma: Symmetric block matrix \[
\begin{bmatrix}
P & S^T \\
S & R \\
\end{bmatrix}
\] with \(R \succ 0\) is positive semidefinite if and only if the matrix \(P - S^T R^{-1} S\) is so.

Proof: \[
\begin{bmatrix}
P & S^T \\
S & R \\
\end{bmatrix} \succeq 0 \text{ iff }
\]
\[
0 \leq \min_{u,v} [u^T Pu + 2u^T S^T v + v^T R v] = \min_u \left[ \min_v [u^T Pu + 2u^T S^T v + v^T R v] \right] \text{ achieved when } v = -R^{-1} Su
\]
\[
= \min_u u^T \left[ P - S^T R^{-1} S \right] u.
\]
Optimizing Linear Estimates

Situation: “In the nature” there exists a signal $x$ known to belong to a given convex compact set $\mathcal{X} \subset \mathbb{R}^n$. We observe corrupted by noise affine image of the signal:

$$\omega = Ax + \sigma \xi \in \Omega = \mathbb{R}^m$$

- $A$: given $m \times n$ sensing matrix
- $\xi$: random noise

Goal: To recover the image $Bx$ of $x$

- $B$: given $\nu \times n$ matrix.

Risk of a candidate estimate $\hat{x}(\cdot): \Omega \to \mathbb{R}^\nu$ is

$$\text{Risk}[\hat{x}|\mathcal{X}] = \sup_{x \in \mathcal{X}} \sqrt{\mathbb{E}_\xi \left\{ \| Bx - \hat{x}(Ax + \sigma \xi) \|_2^2 \right\}}$$

Assumption on noise: $\xi$ is zero mean with unit covariance matrix.

⇒ The risk of a linear estimate $\hat{x}_H(\omega) = H^T \omega$ ($H$: contrast matrix) is given by

$$\text{Risk}^2[\hat{x}_H|\mathcal{X}] = \max_{x \in \mathcal{X}} \mathbb{E}_\xi \left\{ \| [B - H^T A]x - \sigma H^T \xi \|_2^2 \right\}$$

$$= \max_{x \in \mathcal{X}} \left\{ \| [B - H^T A]x \|_2^2 + \sigma^2 \mathbb{E}_\xi \{ \text{Tr}(H^T \xi^T H) \} \right\}$$

$$= \sigma^2 \text{Tr}(H^T H) + \max_{x \in \mathcal{X}} \text{Tr}([B - H^T A]xx^T[B^T - A^T H]).$$

$\psi(H)$
Risk^2[\hat{x}_H|\mathcal{X}] = \sigma^2 \text{Tr}(H^T H) + \Psi(H), \quad \Psi(H) = \max_{x \in \mathcal{X}} \text{Tr}([B - H^T A] xx^T [B^T - A^T H]).

☆ Note: \(\Psi\) is convex ⇒ building the minimum risk linear estimate reduces to solving convex minimization problem

\[ \text{Opt} = \min_H \left[ \Psi(H) + \sigma^2 \text{Tr}(H^T H) \right]. \tag{*} \]

But: Convex function \(\Psi\) is given implicitly and can be difficult to compute, making (\(*)\) difficult as well.
\[
\begin{align*}
\text{Opt} &= \min_H \left[ \sigma^2 \text{Tr}(H^T H) + \Psi(H) \right] \\
\Psi(H) &= \max_{x \in \mathcal{X}} \text{Tr} \left( [B - H^T A] xx^T [B^T - A^T H] \right)
\end{align*}
\]

\(\heartsuit\) **Fact:** Basically, the only cases when \((*)\) is known to be easy are those when

- \(\mathcal{X}\) is given as a convex hull of finite set of moderate cardinality
- \(\mathcal{X}\) is an ellipsoid.

\(\mathcal{X}\) is a box \(\Rightarrow\) computing \(\Psi\) is NP-hard...

\(\spadesuit\) When \(\Psi\) is difficult to compute, we can to replace \(\Psi\) in the design problem \((*)\) with an efficiently computable convex upper bound \(\Psi^+(H)\).

We are about to consider a family of sets \(\mathcal{X}\) – *ellitopes* – for which reasonably tight bounds \(\Psi^+\) of desired type are available.
A basic ellitope is a set \( \mathcal{Y} \subset \mathbb{R}^N \) given as

\[
\mathcal{Y} = \{ y \in \mathbb{R}^N : \exists t \in \mathcal{T} : y^T S_k y \leq t_k, \; k \leq K \}
\]

where

- \( S_k \succeq 0 \) are positive semidefinite matrices with \( \sum_k S_k \succ 0 \)
- \( \mathcal{T} \) is a convex compact subset of \( K \)-dimensional nonnegative orthant \( \mathbb{R}^K_+ \) such that
  - \( \mathcal{T} \) contains some positive vectors
  - \( \mathcal{T} \) is monotone: if \( 0 \leq t' \leq t \) and \( t \in \mathcal{T} \), then \( t' \in \mathcal{T} \) as well.

An ellitope \( \mathcal{X} \) is linear image of a basic ellitope:

\[
\mathcal{X} = \{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^N, t \in \mathcal{T} : x = F y, \; y^T S_k y \leq t_k, \; k \leq K \}
\]

- \( F \) is a given \( n \times N \) matrix,

Note: Every ellitope is a symmetric w.r.t. the origin convex compact set.
Examples of basic ellitopes:

A. Ellipsoid centered at the origin

\((K = 1, \mathcal{T} = [0; 1])\)

B. (Bounded) intersection of \(K\) ellipsoids/elliptic cylinders centered at the origin

\((\mathcal{T} = \{t \in \mathbb{R}^K : 0 \leq t_k \leq 1, k \leq K\})\)

C. Box \(\{x \in \mathbb{R}^n : -1 \leq x_i \leq 1\}\)

\((\mathcal{T} = \{t \in \mathbb{R}^n : 0 \leq t_k \leq 1, k \leq K = n\}, x^T S_k x = x_k^2\})

D. \(\ell_p\)-ball \(\mathcal{X} = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}\) with \(p \geq 2\)

\((\mathcal{T} = \{t \in \mathbb{R}_+^n : \|t\|_{p/2} \leq 1\}, x^T S_k x = x_k^2, k \leq K = n\})

♠ Ellitopes admit fully algorithmic calculus: if \(\mathcal{X}_i, 1 \leq i \leq I\), are ellitopes, so are their

- intersection \(\bigcap_i \mathcal{X}_i\)
- direct product \(\mathcal{X}_1 \times ... \times \mathcal{X}_I\)
- arithmetic sum \(\mathcal{X}_1 + ... + \mathcal{X}_I\)
- linear images \(\{Ax : x \in \mathcal{X}_i\}\)
- inverse linear images \(\{y : Ay \in \mathcal{X}_i\}\) under linear embedding \(A\)
Observation: Let

\[ \mathcal{X} = \{ x : \exists (t \in T, y) : x = Fy, y^T S_k y \leq t_k, k \leq K \} \] (\#)

be an ellitope. Given a quadratic form \( x^T W x \), \( W \in S^n \), we have

\[ \max_{x \in \mathcal{X}} x^T W x \leq \min_{\lambda} \left\{ \phi_T(\lambda) : \lambda \geq 0, \sum_{k=1}^{K} \lambda_k S_k \succeq F^T W F \right\} \]

\[ \phi_T(\lambda) = \max_{t \in T} t^T \lambda : \text{ support function of } T \]

Indeed, we have

\[ \lambda \geq 0, F^T W F \preceq \sum_k \lambda_k S_k, x \in \mathcal{X} \Rightarrow \exists (t \in T, y) : y^T S_k y \leq t_k \ \forall k \leq K, x = Fy \Rightarrow \exists (t \in T, y) : x^T W x = y^T F^T W F y \leq \sum_k \lambda_k y^T S_k y \leq \sum_k \lambda_k t_k \leq \phi_T(\lambda) \Rightarrow x^T W x \leq \phi_T(\lambda). \]
\[ \mathcal{X} = \{ x : \exists (t \in \mathcal{T}, y) : x = F y, y^T S_k y \leq t_k, k \leq K \} \]  

\[ \text{Corollary: Let } \mathcal{X} \text{ be the ellitope } (\ast). \text{ Then the function} \]
\[ \psi(H) = \max_{x \in \mathcal{X}} \text{Tr}((B - H^T A)x x^T (B^T - A^T H)) \]
\[ = \max_{x \in \mathcal{X}} x^T [(B^T - A^T H)(B - H^T A)]x \]
\[ \text{can be upper-bounded as} \]
\[ \psi(H) \leq \bar{\psi}(H) := \min_{\lambda} \left\{ \phi_T(\lambda) : \lambda \geq 0, \right. \]
\[ F^T [B^T - A^T H][B - H^T A] F \preceq \sum_k \lambda_k S_k \}
\[ \text{[Schur Complement Lemma]} \]
\[ = \min_{\lambda} \left\{ \phi_T(\lambda) : \lambda \geq 0, \left[ \sum_k \lambda_k S_k \right] F^T [B^T - A^T H] \right. \]
\[ \left. \left[ \frac{F^T B^T - A^T H}{I_{\nu}} \right] \preceq 0 \right\} \]

The function \( \bar{\psi}(H) \) is real-valued and convex, and is efficiently computable whenever \( \phi_T \) is so, that is, whenever \( \mathcal{T} \) is computationally tractable.
**Bottom line:** Given matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{\nu \times n}$ and an ellitope

$$\mathcal{X} = \{x : \exists (t \in \mathcal{T}, y) : x = Fy, y^T S_k y \leq t_k, k \leq K\}$$

contained in $\mathbb{R}^n$, consider the convex optimization problem

$$\operatorname{Opt} = \min_{H, \lambda} \left\{ \phi_T(\lambda) + \sigma^2 \text{Tr}(H^T H) : \lambda \geq 0, \left[ \sum_k \lambda_k S_k \right] F^T [B^T - A^T H] I_k \succeq 0 \right\}.$$ 

Assuming the noise $\xi$ in observation $\omega = Ax + \sigma \xi$ zero mean with unit covariance matrix, the risk of the linear estimate $\hat{x}_{H*}(\cdot)$ induced by the optimal solution $H_*$ to the problem (this solution clearly exists provided $\sigma > 0$) satisfies the risk bound

$$\operatorname{Risk}[\hat{x}_{H*}|\mathcal{X}] \leq \sqrt{\operatorname{Opt}}.$$ 

**Note:** We shall see eventually that in the case of $\xi \sim \mathcal{N}(0, I_m)$, $\operatorname{Opt}$ is "nearly" the same as the ideal minimax risk

$$\operatorname{RiskOpt} = \inf_{\hat{x}(\cdot)} \operatorname{Risk}[\hat{x}|\mathcal{X}],$$

where $\inf$ is taken w.r.t. all, not necessarily linear, estimates $\hat{x}(\cdot)$. 

5.30
How It Works: Inverse Heat Equation

♣ Situation: Square plate is heated at time 0 and is rest to cool; the temperature at the plate’s boundary is all the time is kept 0. Given given noisy measurements, taken along $m$ points, of plate’s temperature at time $t_1$, we want to recover distribution of temperature at a given time $t_0$, $0 < t_0 < t_1$.

♠ The model: The temperature field $u(t; p, q)$ evolves according to Heat Equation

$$
\frac{\partial}{\partial t} u(t; p, q) = \left[ \frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2} \right] u(t; p, q), \quad t \geq 0, \quad (p, q) \in S
$$

- $t$: time
- $S = \{(p, q), -1 \leq p, q \leq 1\}$: the plate

with boundary conditions $u(t; p, q)\big|_{(p,q)\in\partial S} \equiv 0$.

♥ It is convenient to represent $u(t; p, q)$ by its expansion

$$
u(t; p, q) = \sum_{k, \ell} x_{k\ell}(t) \phi_k(p) \phi_{\ell}(q), \quad (*)
$$

$$
\phi_k(s) = \begin{cases}
\cos(\omega_{2i-1}s), & \omega_{2i-1} = (i - 1/2)\pi \quad k = 2i - 1 \\
\sin(\omega_2i s), & \omega_2i = i\pi \quad k = 2i
\end{cases}
$$

Note: $\phi_k(s)$ are harmonic oscillations vanishing at $s = \pm 1$. 5.31
\[
    u(t; p, q) = \sum_{k, \ell} x_{k\ell}(t) \phi_k(p) \phi_\ell(q), \quad (*)
\]
\[
    \phi_k(s) = \begin{cases} 
        \cos(\omega_{2i-1}s), & \omega_{2i-1} = (i - 1/2)\pi \quad k = 2i - 1 \\
        \sin(\omega_{2i}s), & \omega_{2i} = i\pi \quad k = 2i
    \end{cases}
\]

**Note:**

- \{\phi_{k\ell}(p, q) = \phi_k(p)\phi_\ell(q)\}_{k, \ell} form an orthonormal basis in \(L_2(S)\)
- \(\phi_{k\ell}(\cdot)\) meet the boundary conditions
  \[
  \phi_{k\ell}(p, q)\bigr|_{(p, q) \in \partial S} = 0
  \]
- in terms of the coefficients \(x_{k\ell}(t)\), the Heat Equation becomes
  \[
  \frac{d}{dt} x_{k\ell}(t) = -[\omega_k^2 + \omega_\ell^2] x_{k\ell}(t) \Rightarrow x_{k\ell}(t) = e^{-[\omega_k^2+\omega_\ell^2]t} x_{k\ell}(0).
  \]

♥ We select integer discretization parameter \(N\) and
- restrict (*) to terms with \(1 \leq k, \ell \leq 2N - 1\)
- discretize the spatial variable \((p, q)\) to reside in the grid
  \[
  G_N = \{P_{ij} = (p_i, p_j) = \left(\frac{i}{N} - 1, \frac{j}{N} - 1\right), \ 1 \leq i, j \leq 2N - 1\}
  \]

**Note:** Restricting functions \(\phi_{k\ell}(\cdot, \cdot), 1 \leq k, \ell \leq 2N - 1\) on grid \(G_N\), we get orthogonal basis in \(\mathbb{R}^{(2N-1) \times (2N-1)}\).
We arrive at the model as follows:

- The signal \( x \) underlying observation is

\[
x = \{ x_{k\ell} := x_{k\ell}(t_0), 1 \leq k, \ell \leq 2N - 1 \} \in \mathbb{R}^{(2N-1) \times (2N-1)}
\]

- The observation is

\[
\omega = A(x) + \sigma \xi \in \mathbb{R}^m, \xi \sim \mathcal{N}(0, I_m)
\]

\[
[A(x)]_\nu = \sum_{k, \ell=1}^{2N-1} x_{k\ell} e^{-[\omega_k^2 + \omega_\ell^2][t_1-t_0]} \phi_k(p_i(\nu)) \phi_\ell(p_j(\nu)) x_{k\ell}
\]

- \((p_i(\nu), p_j(\nu)) \in S, 1 \leq \nu \leq m\): measurement points

- We want to recover the restriction \( B(x) \) of \( u(t_0; p, q) \) to some grid, say, square grid

\[
G_K = \{(r_i = \frac{i}{K} - 1, r_j = \frac{j}{K} - 1), 1 \leq i, j \leq 2K - 1 \} \subset S,
\]

resulting in

\[
[B(x)]_{ij} = \sum_{k, \ell=1}^{2N-1} \phi_k(r_i) \phi_\ell(r_j) x_{k\ell}
\]
• We assume that the initial distribution of temperatures \([u(0; p_i, p_j)]_{i,j=1}^{2N-1}\) satisfies 
\[\|u\|_2 \leq R,\] 
for some given \(R\), implying that \(x\) resides in the ellitope, namely, the ellipsoid

\[
\mathcal{X} = \left\{ \{x_{k\ell}\} \in \mathbb{R}^{(2N-1) \times (2N-1)} : \sum_{k,\ell} \left[ e^{[\omega_k^2 + \omega_\ell^2]t_0}x_{k\ell} \right]^2 \leq R^2 \right\}
\]
$$u(t; p_i, p_j) = \sum_{k, \ell} e^{-[\omega_k^2 + \omega_\ell^2][t-t_0]} \phi_k(p_i) \phi_\ell(p_j) x_{k\ell}$$

$$[A(x)]_\nu = \sum_{k, \ell=1}^{2N-1} x_{k\ell} e^{-[\omega_k^2 + \omega_\ell^2][t_1-t_0]} \phi_k(p_{i(\nu)}) \phi_\ell(p_{j(\nu)}) x_{k\ell}$$

♣ **Bad news:** Contributions of high frequency (with large $\omega_k^2 + \omega_\ell^2$) signal components $x_{k\ell}$ to $A(x)$ decrease exponentially fast with high decay rate as $t_1 - t_0$ grows

⇒ High frequency components $x_{k\ell}$ are impossible to recover from observations at time $t_1$, unless $t_1$ is very small.

$$\mathcal{X} = \left\{ \{x_{k\ell}\} : \sum_{k, \ell} \left[ e^{[\omega_k^2 + \omega_\ell^2]t_0} x_{k\ell} \right]^2 \leq R^2 \right\}$$

$$[B(x)]_{ij} = \sum_{k, \ell=1}^{2N-1} \phi_k(r_i) \phi_\ell(r_j) x_{k\ell}$$

♣ **Good news:** High frequency components $x_{k\ell}$ of $x \in \mathcal{X}$ are very small, provided $t_0$ is not too small

⇒ There is no necessity to recover well high frequency components of signal from observations!
**Numerical results** $N = 32, m = 125, K = 6, t_0 = 0.01, t_1 = 0.02, \sigma = 0.001, R = 15$

Minimax risk of optimal linear estimate: 0.1707

$63 \times 63$ grid $G_{63}$ and $m = 125$ measurement points

Sample results

- left: $b = B(x)$
- center: sample optimal linear recovery $\hat{b} = H^T \omega$ of $b = B(x)$
- right: naive recovery $\tilde{b} = B(\tilde{x})$, $\tilde{x}$: Least Squares solution to $A(x) = \omega$
How It Works: Denosing & Deblurring Images

- A grayscale image can be thought of as a 2D $m \times n$ array $[x_{ij}]_{0 \leq i < m, \ 0 \leq j < n}$ with entries (pixels’ intensities) in $[0, 255]$
- Taking picture can be modeled as observing noisy convolution

$$
\omega_{ij} = \sum_{0 \leq p \leq \mu, \ 0 \leq q \leq \nu} \kappa_{pq} x_{i-p,j-q} + \xi_{ij}, \ 0 \leq i \leq m + \mu, 0 \leq j < n + nu
$$

(\ast)

of the image and a given blurring kernel $[\kappa_{ij}]_{0 \leq p < \mu, \ 0 \leq q < \nu}$.

**Note:** In (\ast), $x_{ij} = 0$ outside of the actual range $\{0 \leq i < m, 0 \leq j < n\}$ of $i, j$.

**Note:** “Centering” image – subtracting from $x_{ij}$ entries in $x$ the midpoint $S$ of the range $[0, 255]$ of pixels’ intensities and updating $\omega_{ij}$ accordingly, the images become 2D arrays from the box

$$
\mathcal{X}_\infty == \{ x \in \mathbb{R}^{m \times n} : |x_{ij}| \leq S \},
$$

and the recovery problem falls into our framework.

5.37
\[ x \mapsto \kappa \ast x + \xi \quad \Rightarrow \quad \hat{x} \approx x \]

**Bad news:** Linear dimensions \( mn \) of typical images are in the range of \( 10^5 \text{–} 10^6 \), making straightforward design of linear estimates \( \omega \mapsto \hat{x} = H^T \omega \) intractable—linear dimensions of contrast matrices should be in the range of \( 10^{10} \text{–} 10^{12} \).

**Good news:** Extending \( x \) and \( \kappa \) to \( M := [m + \mu] \times N := [n + \nu] \) arrays \( x^+, \kappa^+ \) by adding zero entries to \( x, \kappa \), and passing to 2D Discrete Fourier Transforms \( \chi = \mathcal{F} x^+, \theta = \mathcal{F} \kappa^+ \) of these arrays, observation scheme becomes extremely simple:

\[
\zeta := \mathcal{F} \omega = \theta \bullet \chi + \sqrt{MN} \sigma \eta
\]

[\( \bullet \text{ : entrywise product, } \eta \text{ : (complex-valued) white Gaussian noise with unit covariance matrix} \)]

**Note:** DFT multiplies \( \| \cdot \|_2 \) by \( \sqrt{MN} \Rightarrow \text{when the recovery error is measured in } \| \cdot \|_2 \), recovering \( x \) from \( \omega \) is equivalent to recovering \( \chi \) from \( \zeta \).

**Besides this,** when a priori information on \( x \) translates into simple constraints on \( \chi \), like

\[
\sum_{r,s} \alpha_{rs} |\chi_{rs}| \leq \alpha \quad \text{and/or} \quad \sum_{r,s} \beta_{rs} |\chi_{rs}|^2 \leq \beta \quad \text{and/or} \quad |\chi_{rs}| \leq \gamma_{rs} \quad \forall r, s, 0 \leq r < M, 0 \leq s < N \quad (!)
\]

frequency representations \( \chi \) of signals of interest become points of a simple (complexified) ellitope, and sensing matrix \( A \) becomes (complex-valued) diagonal

\( \Rightarrow \) Working in frequency domain, we lose nothing when looking for linear estimates with diagonal (complex-valued) contrast matrices.

**Moreover,** when the number of constraints (!) is small, designing the best linear estimate with diagonal contrast matrix reduces to solving a low-dimensional convex problem and takes few seconds even when \( MN \) is in the range of millions.
... in frequency domain recovery problem becomes \( \zeta = \theta \cdot \chi + \sqrt{MN} \sigma \eta \Rightarrow \hat{\chi} \approx \chi \), and easy-to-utilize a priori information on \( \chi \) are constraints of the form

\[ \sum_{r,s} \alpha_{rs} |\chi_{rs}| \leq \alpha \quad \text{and/or} \quad \sum_{r,s} \beta_{rs} |\chi_{rs}|^2 \leq \beta \quad \text{and/or} \quad |\chi_{rs}| \leq \gamma_{rs} \quad \forall r, s, 0 \leq r < M, 0 \leq s < N \quad (!) \]

**Note:** Our “built in” box constraint \( \|x\|_\infty \leq L \) does not translate into a simple constraint on \( \chi \); the best simple (conservative!) frequency domain version of this constraint is the bound

\[ \|\chi\|_2 \leq \sqrt{MN} \cdot \sqrt{mnL} \quad (E) \]

on the \( \| \cdot \|_2 \)-norm of \( \chi \).

**Similarly,** the standard in Image Reconstruction bounds

\[ \text{TV}(x) := \sum_{i,j} |x_{i+1,j} - x_{i,j}| + \sum_{i,j} |x_{i,j+1} - x_{i,j}| \leq U \]

on Total Variation of \( x \) do not translate into simple constraints on \( \chi \).

- **However:** we can impose on \( \chi \), in addition to \((E)\), empirical upper bounds on \( \|\chi\|_\infty \) and \( \|\chi\|_1 \) by inspecting a “representative library” of images.

- **Warning:** When the blur is present (i.e., \( \kappa \) is not a \( \delta \)-function), the recovery problem can easily become ill-posed, since convolution can “kill” some frequencies (formally: some of the entries in \( \theta \) can be very small).
Blurred noisy observations (top) and recoveries (bottom) of $1200 \times 1600$ image, ill-posed case [with bound on signal’s energy]

$\sigma = 1.992$

$\sigma = 0.498$

$\sigma = 0.031$
Blurred noisy observations (top) and recoveries (bottom) of $1200 \times 1600$ image, ill-posed case
[with rudimentary form of Total Variation constraints]
Blurred noisy observations (top) and recoveries (bottom) of $1200 \times 1600$ image, well-posed case

$\sigma = 31.88$

$\sigma = 7.969$

$\sigma = 0.498$
Theorem Let $C$ be a symmetric $n \times n$ matrix and $\mathcal{X}$ be an ellitope:

$$\mathcal{X} = \{ x \in \mathbb{R}^n : \exists (t \in \mathcal{T}, y) : x = F y, y^T S_k y \leq t_k \forall k \leq K \}.$$ 

Then the efficiently computable quantity

$$\text{Opt} = \min_{\lambda} \left\{ \phi_\mathcal{T}(\lambda) : \lambda \geq 0, F^T C F \preceq \sum_k \lambda_k S_k \right\}$$

$$\left[ \phi_\mathcal{T}(\lambda) = \max_{t \in \mathcal{T}} \lambda^T t \right]$$

is a tight upper bound on

$$\text{Opt}_* = \max_{x \in \mathcal{X}} x^T C x :$$

namely,

$$\text{Opt}_* \leq \text{Opt} \leq 3 \ln(\sqrt{3}K) \text{Opt}_*.$$ 

Note: $\text{Opt}_*$ is difficult to compute within 4% accuracy when $\mathcal{X}$ is as simple as the unit box in $\mathbb{R}^n$. 

5.43
Let $\mathcal{X}$ be given by quadratic inequalities:
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t, k \leq K \} \neq \emptyset \]

[\mathcal{T} : \text{nonempty convex compact set}]

We have
\[
\text{Opt}^* := \max_{x \in \mathcal{X}} x^T C x \leq \text{Opt} := \min_\lambda \{ \phi_\mathcal{T}(\lambda) : \lambda \geq 0, C \leq \sum_k \lambda_k S_k \} \leq \Theta \cdot \text{Opt}^* 
\]

What can be said about tightness factor $\Theta$?

**Facts:**

A. Assuming $K = 1$ and Slater condition: $\bar{x}^T S_1 \bar{x} < t$ for some $\bar{x}$ and some $t \in \mathcal{T}$, one can set $\Theta = 1$.

[famous $S$-Lemma]

B. Assuming that $x^T S_k x = x_k^2, k \leq K = \dim x, \mathcal{T} = [0; 1]^K$, and $C$ is Laplacian of a graph (i.e., off-diagonal entries in $C$ are nonpositive and all row sums are zero), one can set $\Theta = 1.1382...$

[MAXCUT Theorem of Goemans and Williamson, 1995]

Note: Laplacian of a graph always is $\preceq 0$
\( \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k, k \leq K \} \neq \emptyset \)

[\mathcal{T} : \text{nonempty convex compact set}]

\[ \Rightarrow \text{Opt}^* := \max_{x \in \mathcal{X}} x^T C x \leq \text{Opt} := \min_\lambda \{ \phi_T(\lambda) : \lambda \geq 0, C \preceq \sum_k \lambda_k S_k \} \leq \Theta \cdot \text{Opt}^* \]

C. Assuming that \( C \succeq 0 \) and all matrices \( S_k \) are diagonal, one can set \( \Theta = \frac{\pi}{2} = 1.5708... \)

[\( \frac{\pi}{2} \) Theorem, Nesterov, 1998]

D. Assuming \( \mathcal{X} \) is an ellitope (i.e., \( S_k \succeq 0, \sum_k S_k \succ 0 \) and \( \mathcal{T} \) contains a positive vector), one can set \( \Theta = 3 \ln(\sqrt{3}K) \)

Note: In the case of D, \( \Theta \) indeed can be as large as \( O(\ln(K)) \)
A byproduct of Theorem is the following useful fact:

**Theorem [upper-bounding of operator norms]** Let \( \| \cdot \|_x \) be a norm on \( \mathbb{R}^N \) such that the unit ball \( \mathcal{X} \) of the norm is an ellitope:

\[
\mathcal{X} := \{ x : \|x\|_x \leq 1 \} = \{ x : \exists (t \in \mathcal{T}, y) : x = Py, y^T S_k y \leq t, k \leq K \}
\]

Let, further, \( \| \cdot \| \) be a norm on \( \mathbb{R}^M \) such that the unit ball \( \mathcal{B}_* \) of the norm \( \| \cdot \|_* \) conjugate to \( \| \cdot \| \) is an ellitope:

\[
\mathcal{B}_* := \{ u \in \mathbb{R}^m : u^Tv \leq 1 \forall (v, \|v\| \leq 1) \} = \{ u : \exists (r \in \mathcal{R}, z) : u = Qz, z^T R_\ell z \leq r, \ell \leq L \}
\]

Then the efficiently computable quantity

\[
\text{Opt}(C) = \min_{\lambda, \mu} \left\{ \phi_\mathcal{T}(\lambda) + \phi_\mathcal{R}(\mu) : \begin{array}{c}
\lambda \geq 0, \mu \geq 0 \\
\sum_\ell \mu_\ell R_\ell \left[ \frac{1}{2} P^T C P \right] \sum_k \lambda_k S_k \geq 0
\end{array} \right\}
\]

is a convex in \( C \) upper bound on the operator norm

\[
\|C\|_{\| \cdot \|_x \to \| \cdot \|} = \max_x \{ \|Cx\| : \|x\|_x \leq 1 \}
\]

of the mapping \( x \mapsto Cx \), and this bound is reasonably tight:

\[
\|C\|_{\| \cdot \|_x \to \| \cdot \|} \leq \text{Opt}(C) \leq 3 \ln(\sqrt{3}(K + L)) \|C\|_{\| \cdot \|_x \to \| \cdot \|}.
\]
Indeed, the operator norm in question is the maximum of a quadratic form over an ellitope:

\[
\|C\|_{\|\cdot\|_x \rightarrow \|\cdot\|} = \max_{x \in \mathcal{X}, u \in B^*} [x; u]^T \begin{bmatrix} C^T & C \\ Q^T CP \\ P^T C^T Q \\ \end{bmatrix} [x; u] \\
= \max_{[y; z] \in \mathcal{W}} [y; z]^T \begin{bmatrix} C^T & C \\ Q^T CP \\ P^T C^T Q \\ \end{bmatrix} [y; z]
\]

where \( \mathcal{W} \) is the basic ellitope given by

\[
\mathcal{W} = \left\{ [y; z] : \exists [t; r] \in \mathcal{T} \times \mathcal{R} : \begin{array}{l}
y^T S_k y \leq t_k, k \leq K \\
z^T R_\ell z \leq r_\ell, \ell \leq L
\end{array} \right\}.
\]
What is inside

♠ The above results on tightness of semidefinite relaxation, we speak about tightness of the Semidefinite Relaxation upper bound on the maximum of a quadratic form over an ellitope:

\[
\begin{align*}
\operatorname{Opt}_* &= \max_{x,t} \left\{ x^T C x : x^T S_k x \leq t_k, k \leq K, t \in T \right\} \\
\end{align*}
\]

(\*)

♠ **Fact:** Semidefinite relaxation admits an alternative description as follows: Let us associate with (\*) another optimization problem where instead of deterministic candidate solutions \((x, t)\) we are looking for random solutions \((\xi, \tau)\) satisfying the constraints at average:

\[
\begin{align*}
\operatorname{Opt}^+ &= \max_{\xi, \tau} \left\{ \mathbb{E}\{\xi^T C \xi\} : \right. \\
&\left. \frac{\mathbb{E}\{\xi^T S_k \xi\}}{\mathbb{E}\{\tau\}} \leq \frac{\mathbb{E}\{\tau_k\}}{\mathbb{E}\{\tau\}} \right\} \\
\end{align*}
\]

(!)

• **Immediate observation:** Property of a random solution \((\xi, \tau)\) to be feasible for (!) depends solely on the matrix \(Q = \mathbb{E}\{\xi \xi^T\}\) and the vector \(t = \mathbb{E}\{\tau\}\), so that

\[
\begin{align*}
\operatorname{Opt}^+ &= \max_{Q, t} \left\{ \operatorname{Tr}(CQ) : \right. \\
&\left. \operatorname{Tr}(S_k Q) \leq t_k, Q \succeq 0, t \in T \right\} \\
= \max_{Q, t} \left\{ \operatorname{Tr}(CQ) : \right. \\
&\left. \operatorname{Tr}(S_k Q) \leq t_k, Q \succeq 0, [t; 1] \in T \right\} \\
\end{align*}
\]

(\#)
\[ \text{Opt}_* = \max_{x,t} \left\{ x^T C x : \exists (t \in T) : x^T S_k x \leq t_k \right\} \quad (\ast) \]

\[ \text{Opt}^+ = \max_{Q,t} \left\{ \text{Tr}(CQ) : \text{Tr}(S_k Q) \leq t_k, Q \succeq 0, [t; 1] \in T \right\} \quad (#) \]

\[ \blacklozenge \textbf{Note:} \quad (\#) \text{ is strictly feasible and bounded, and the problem} \]

\[ \text{Opt} = \min_{\lambda} \left\{ \phi_T(\lambda) : \lambda \geq 0, C \preceq \sum_k \lambda_k S_k \right\} \]

specifying Semidefinite relaxation upper bound on Opt is is nothing but the conic dual to (\#) \( \Rightarrow \text{Opt}^+ = \text{Opt} \).

\[ \bullet \text{ (\#) suggests the following recipe for quantifying the conservatism of the upper bound Opt on Opt}_*: \]

\[ \begin{align*}
\text{— Find an optimal solution } Q_*, t_* \text{ to (\#) and treat } Q_* \succeq 0 \text{ as the covariance matrix of random vector } \xi \text{ (many options!)} \\
\text{— Random solutions } (\xi, t_*) \text{ satisfy (\ast) “at average.” Try to “correct” them to get feasible solutions to (\ast) and look how “costly” this correction is in terms of the objective. For example, in Goemans-Williamson MAXCUT and in Nesterov’s } \pi/2 \text{ Theorems, where } x^T C x \text{ is maximized over the unit box } \{ \| x \|_\infty \leq 1 \}, \text{ one selects } \xi \sim \mathcal{N}(0, Q_*) \text{ and “corrects” } \xi \text{ according to } \xi \mapsto \text{sign}[\xi].
\end{align*} \]
\[ \text{Opt}_* = \max_{x,t} \left\{ x^T C x : \exists (t \in T) : x^T S_k x \leq t_k \right\} \quad (\ast) \]

\[ \text{Opt}^+ = \max_{Q,t} \left\{ \text{Tr}(CQ) : \text{Tr}(S_k Q) \leq t_k, Q \succeq 0, [t; 1] \in T \right\} \quad (\#) \]

\[\star\] This is how the above recipe works in the general ellitopic case:

**A.** Let \((Q_*, t^*)\) be an optimal solution to \((\#).\) Set

\[ \bar{C} := Q_*^{1/2} C Q_*^{1/2} = U D U^T \]

(U is orthogonal, \(D\) is diagonal).

**B.** Let \(\xi = Q_*^{1/2} U \zeta\) with Rademacher random \(\zeta\) (\(\zeta_i\) take values \(\pm 1\) with probability \(1/2\) and are independent across \(i\)), so that \(E\{\xi \xi^T = Q_*\). Note that

\[
\xi^T C \xi = \zeta^T U^T [Q_*^{1/2} C Q_*^{1/2}] U \zeta = \zeta^T D \zeta \\
\equiv \text{Tr}(D) = \text{Tr}(Q_*^{1/2} C Q_*^{1/2}) = \text{Tr}(CQ_*) \\
= \text{Opt},
\]

\[
E\{\xi^T S_k \xi\} = E\{\zeta^T U^T Q_*^{1/2} S_k Q_*^{1/2} U \zeta\} \\
= \text{Tr}(U^T Q_*^{1/2} S_k Q_*^{1/2} U) \\
= \text{Tr}(Q_*^{1/2} S_k Q_*^{1/2}) = \text{Tr}(S_k Q_*) \\
\leq t_k^*, k \leq K
\]
\[ \xi^T C \xi \equiv \text{Opt} \quad (a) \]
\[ \mathbb{E}\{\xi^T S_k \xi\} \leq t_k^*, \ k \leq K \quad (b) \]

C. Since \( S_k \succeq 0 \) and \( \xi \) is “light-tail” (it comes from Rademacher random vector), simple bounds on probabilities of large deviations combine with (b) to imply that

\[ \forall (\gamma \geq 0, k \leq K) : \]
\[ \text{Prob}\{\xi : \xi^T S_k \xi > \gamma t_k^*\} \leq O(1) \exp\{-O(1)\gamma\} \]

⇒ with \( \gamma^* = O(1) \ln(K + 1) \), there exists a realization \( \hat{\xi} \) of \( \xi \) such that \( \hat{\xi}^T S_k \hat{\xi} \leq \gamma^* t_k^*, \ k \leq K \)

⇒ \( (\xi^* = \hat{\xi} / \sqrt{\gamma^*}, t^*) \) is feasible for

\[ \text{Opt}_* = \max_{x, t} \left\{ x^T C x : \exists (t \in T) : x^T S_k x \leq t_k \right\} \quad (*) \]

⇒ \( \text{Opt}_* \geq \hat{\xi}^T C \hat{\xi} / \gamma^* = \text{Opt} / \gamma^* \) (look at (a)!)
“Simple bounds on probabilities of large deviations” stem from the following Mini-Lemma: Let $P$ be positive semidefinite $N \times N$ matrix with trace $\leq 1$ and $\zeta$ be $N$-dimensional Rademacher random vector. Then

$$E\left\{ \exp\left\{ \zeta^T P \zeta / 3 \right\} \right\} \leq \sqrt{3}.$$ 

Mini-Lemma $\Rightarrow$ bounds: We have

$$\xi^T S_k \xi = \zeta^T U^T Q_{*}^{1/2} S_k Q_{*}^{1/2} U \zeta / t_k^* P_k$$

and $\text{Tr}(P_k) = \text{Tr}(Q_{*}^{1/2} S_k Q_{*}^{1/2}) / t_k^* = \text{Tr}(S_k Q_{*}) / t_k^* \leq 1$

$\Rightarrow$ [Mini-Lemma] $E\left\{ \exp\left\{ \zeta^T P_k \zeta / 3 \right\} \right\} \leq \sqrt{3}$

$\Rightarrow$ $\text{Prob}\{\zeta^T P_k \zeta > 3\rho\} \leq \sqrt{3} e^{-\rho}$

$\Rightarrow$ $\text{Prob}\{\xi^T S_k \xi > \gamma t_k^*\} = \text{Prob}\{\zeta^T P_k \zeta > \gamma\} \leq \sqrt{3} e^{-\gamma / 3}.$
Proof of Mini-Lemma: Let $P = \sum_i \sigma_i f_i f_i^T$ be the eigenvalue decomposition of $P$, so that $f_i^T f_i = 1$, $\sigma_i \geq 0$, and $\sum_i \sigma_i \leq 1$. The function

$$f(\sigma_1, \ldots, \sigma_N) = \mathbb{E}\left\{ e^{\frac{1}{3} \sum_i \sigma_i \zeta f_i f_i^T \zeta} \right\}$$

is convex on the simplex $\{\sigma \geq 0, \sum_i \sigma_i \leq 1\}$ and thus attains it maximum over the simplex at a vertex, implying that for some $h = f_i$, $h^T h = 1$, it holds

$$\mathbb{E}\{e^{\frac{1}{3} \zeta^T P \zeta}\} \leq \mathbb{E}\{e^{\frac{1}{3} (h^T \zeta)^2}\}.$$ 

Let $\xi \sim \mathcal{N}(0, 1)$ be independent of $\zeta$. We have

$$\mathbb{E}_\zeta \left\{ \exp\left\{ \frac{1}{3} (h^T \zeta)^2 \right\} \right\} = \mathbb{E}_\zeta \left\{ \mathbb{E}_\xi \left\{ \exp\left\{ \left[ \sqrt{2/3} h^T \zeta \right] \xi \right\} \right\} \right\}$$

$$= \mathbb{E}_\xi \left\{ \mathbb{E}_\zeta \left\{ \exp\left\{ \left[ \sqrt{2/3} h^T \zeta \right] \xi \right\} \right\} \right\}$$

$$= \mathbb{E}_\xi \left\{ \prod_{s=1}^N \mathbb{E}_\zeta \left\{ \exp\left\{ \sqrt{2/3} \xi h_s \zeta_s \right\} \right\} \right\}$$

$$= \mathbb{E}_\xi \left\{ \prod_{s=1}^N \cosh\left( \sqrt{2/3} \xi h_s \right) \right\} \leq \mathbb{E}_\xi \left\{ \prod_{s=1}^N \exp\left\{ \xi^2 h_s^2 / 3 \right\} \right\}$$

$$= \mathbb{E}_\xi \left\{ \exp\left\{ \xi^2 / 3 \right\} \right\} = \sqrt{3}$$

5.53
Extensions

♣ So far, we have considered a problem of recovering $Bx$ from observation

$$\omega = Ax + \xi \in \mathbb{R}^m$$

where

- $x$ is unknown signal known to belong to a given basic ellitope

$$\mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k, k \leq K \}$$

**Note:** Assuming signal set $\mathcal{X}$ basic ellitope rather than ellitope is w.l.o.g.: when $\mathcal{X} = F\mathcal{Y}$ with basic ellitope $\mathcal{Y}$, we lose nothing when assuming that the signal is $y$ rather than $x = Fy$ and replacing $A$, $B$ with $AF$, $BF$.

- $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{\nu \times n}$ are given matrices
- $\xi \sim \mathcal{N}(0, \sigma^2 I_m)$ is observation noise
- (squared) risk of a candidate estimate is the worst-case, over $x \in \mathcal{X}$, expected squared $\| \cdot \|_2$-norm of recovery error:

$$\text{Risk}^2[\hat{x}|\mathcal{X}] = \sup_{x \in \mathcal{X}} \mathbb{E}\left\{\|Bx - \hat{x}(Ax + \xi)\|_2^2\right\}.$$
We are about to extend our results to the situation where

• Noise $\xi$ not necessary is zero mean Gaussian; we allow the distribution $P$ of noise to be unknown in advance and to depend on signal $x$.

Assumption: We are given a convex compact set $\Pi \subset \text{int } S^m_+$ such that the variance matrix of $P$ admits an upper bound from $\Pi$:

$$P \in \mathcal{P}[\Pi] := \left\{ P : \exists Q \in \Pi : \text{Var}[P] := E_{\xi \sim P} \{ \xi \xi^T \} \preceq Q \right\}$$

• We measure recovering error in a given norm $\| \cdot \|$, not necessarily the Euclidean one, and define risk of a candidate estimate $\hat{x}(\cdot)$ as

$$\text{Risk}_{Q,\| \cdot \|}[\hat{x} | \mathcal{X}] = \sup_{x \in \mathcal{X}} \sup_{P \in \mathcal{P}[\Pi]} E_{\xi \sim P} \{ \| Bx - \hat{x}(Ax + \xi) \| \}$$

Assumption: The unit ball $B_*$ of the norm conjugate to $\| \cdot \|$ is an ellitope:

$$\| u \| = \max_{h \in B_*} h^T u,$$

$$B_* = \{ h : \exists (y \in \mathbb{R}^M, r \in \mathcal{R}) : h = Fy, y^T R_{\ell} y \leq r_{\ell} \forall \ell \leq L \}$$
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in T : x^T S_k x \leq t_k \ \forall k \leq K \} \]

\[ \omega = Ax + \xi \Rightarrow \hat{x}_H(\omega) = H^T \omega \approx Bx \]

**Building Presumably Good Linear Estimate**

♣ We have

\[
\text{Risk}_{\|\cdot\|, \mathcal{P}[\Pi]}[\hat{x}_H | \mathcal{X}] = \sup_{x \in \mathcal{X}} \sup_{P \in \mathcal{P}[\Pi]} \mathbb{E}_{\xi \sim P} \left\{ \| Bx - H^T (Ax + \xi) \| \right\}
\]

\[
\leq \sup_{x \in \mathcal{X}} \sup_{P \in \mathcal{P}[\Pi]} \mathbb{E}_{\xi \sim P} \left\{ \|[B - H^T Ax]\| + \|H^T \xi\| \right\}
\]

\[
\leq \Phi(H) + \Psi_{\Pi}[H],
\]

\[
\Phi(H) = \max_{x \in \mathcal{X}} \|[B - H^T Ax]\|,
\]

\[
\Psi_{\Pi}(H) = \sup_{P \in \mathcal{P}[\Pi]} \mathbb{E}_{\xi \sim P} \left\{ \|H^T \xi\| \right\}
\]
Next,

\[ \mathcal{B}_* = \{ u = My : y \in \mathcal{Y} \}, \]

\[ \mathcal{Y} = \{ y : \exists r \in \mathcal{R} : y^T R \ell y \leq r \ell \ \forall \ell \leq L \} \]

whence

\[ \Phi(H) := \max_{x \in \mathcal{X}} \| [B - H^T A] x \| = \max_{[u; x] \in \mathcal{B} \times \mathcal{X}} [u; x]^T \left[ \frac{1}{2} [B^T - A^T H] \right] \left[ \frac{1}{2} [B - H^T A] \right] [u; x] \]

\[ = \max_{[y; x] \in \mathcal{Y} \times \mathcal{X}} [y; x]^T \left[ \frac{1}{2} [B^T - A^T H] F y \right] \left[ \frac{1}{2} F^T [B - H^T A] \right] [y; x] \]  

[semidefinite relaxation; note that \( \mathcal{Y} \times \mathcal{X} \) is an ellitope]  

\[ \leq \Phi(H) := \min_{\lambda, \mu} \left\{ \phi_T(\lambda) + \phi_R(\mu) : \lambda \geq 0, \mu \geq 0, \sum_\ell \mu_\ell R_\ell \geq \frac{1}{2} [A^T H - B^T] F \right\} \]

\[ \leq 0 \}

\[ \phi_T(\lambda) = \max_{t \in T} \lambda^T t, \phi_R(\mu) = \max_{r \in \mathcal{R}} \mu^T r \]
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k, k \leq K \} \]
\[ \mathcal{B}_* = \{ u = My : y \in \mathcal{Y} \}, \mathcal{Y} = \{ y : \exists r \in \mathcal{R} : y^T R \ell y \leq r_\ell \ \forall \ell \leq L \} \]
\[ \omega = Ax + \xi \Rightarrow \hat{x}_H(\omega) = H^T \omega \approx Bx \]
\[ \downarrow \]
\[ \text{Risk}_{\|\|, \mathcal{P}[\mathcal{N}]}[\hat{x}_H|\mathcal{X}] \leq \Phi(H) + \Psi_\Pi(H), \]

\[ \Phi(H) = \min_{\lambda, \mu} \left\{ \phi_T(\lambda) + \phi_R(\mu) : \begin{bmatrix} \sum_{\ell} \mu_\ell R_\ell & \frac{1}{2} M^T [H^T A - B] \\ \frac{1}{2} [A^T H - B^T] M & \sum_k \lambda_k S_k \end{bmatrix} \succeq 0 \right\} \]

\[ \Psi_\Pi(H) = \sup_{P \in \mathcal{P}[\mathcal{N}]} \mathbb{E}_{\xi \sim P} \{ \| H^T \xi \| \} \]

\[ \dag \quad \text{Lemma: One has} \]

\[ \Psi_\Pi(H) \leq \overline{\Psi}_\Pi(H) := \min_{\Theta, \kappa} \left\{ \Gamma_\Pi(\Theta) + \phi_\mathcal{R}(\kappa) : \begin{bmatrix} \kappa \geq 0 \\ \sum_{\ell} \kappa_\ell R_\ell & \frac{1}{2} M^T H^T \Theta \end{bmatrix} \succeq 0 \right\} \]

\[ \Gamma_\Pi(\Theta) = \max_{Q \in \Pi} \text{Tr}(Q \Theta). \]

5.58
Lemma:

\[ \|z\| = \max_y \left\{ z^T M y : \exists r \in R : y^T R_\ell y \leq r_\ell, \ell \leq L \right\} \]
\[ \Gamma_\Pi(\Theta) = \max_{Q \in \Pi} \text{Tr}(Q \Theta). \]

\[ \Downarrow \]

\[ \Psi_\Pi(H) \leq \bar{\Psi}_\Pi(H) := \min_{\Theta, \nu} \left\{ \Gamma_\Pi(\Theta) + \phi_R(\nu) : \nu \geq 0, \begin{bmatrix} \sum_\ell \nu_\ell R_\ell & \frac{1}{2} M^T H^T \\ \frac{1}{2} H M & \Theta \end{bmatrix} \succeq 0 \right\} \]

Indeed, let \((\nu, \Theta)\) be feasible for the problem specifying \(\bar{\Psi}_\Pi\), and let \(\text{Var}[P] \preceq Q \in \Pi\). We have

\[ \|H^T \xi\| = \max_{u \in B} \left\{ -u^T H^T \xi \right\} = \max_{y \in Y} \left\{ -y^T M^T H^T \xi \right\} \leq \max_{y \in Y} \left\{ y^T \left[ \sum_\ell \nu_\ell R_\ell \right] y + \xi^T \Theta \xi \right\} \]
\[ = \max_{r \in R, y} \left\{ y^T \left[ \sum_\ell \nu_\ell R_\ell \right] y + \xi^T \Theta \xi : y^T R_\ell y \leq r_\ell, \ell \leq L \right\} \leq \max_{r \in R} \left\{ \sum_\ell \nu_\ell r_\ell + \xi^T \Theta \xi \right\} \]
\[ \leq \phi_R(\nu) + \xi^T \Theta \xi = \phi_R(\nu) + \text{Tr}(\Theta[\xi \xi^T]). \]

Taking expectation in \(\xi\), we get

\[ E_{\xi \sim P} \left\{ \|H^T \xi\| \right\} \leq \phi_R(\nu) + \text{Tr}(\Theta \text{Var}[P]) \leq \phi_R(\nu) + \Gamma_\Pi(\Theta). \]

and the conclusion of Lemma follows.
Illustration: When \( \| \cdot \| = \| \cdot \|_p, \ p \in [1, 2] \), Lemma implies that whenever \( \text{Var}[P] \preceq Q \), one has

\[
E_{\xi \sim P} \left\{ \| H^T \xi \|_p \right\} \leq \left\| \left[ \| \text{Col}_1 [Q^{1/2} H] \|_2; \ldots; \| \text{Col}_k [Q^{1/2} H] \|_2 \right] \right\|_p
\]
Consider convex optimization problem

\[
\text{Opt} = \min_{H, \lambda, \mu, \kappa, \Theta} \left\{ \phi_T(\lambda) + \phi_R(\mu) + \phi_R(\kappa) + \Gamma_\Pi(\Theta) : \lambda \geq 0, \mu \geq 0, \kappa \geq 0, \right. \\
\left. \sum_{\ell} \mu_\ell R_\ell \left[ \frac{1}{2}[A^T H - B^T]M \right] \frac{1}{2} M^T [H^T A - B] \frac{1}{2} M^T \right. \\
\left. \sum_k \lambda_k S_k \right) \succeq 0, \left[ \sum_{\ell} \kappa_\ell R_\ell \left[ \frac{1}{2} HM \right] \frac{1}{2} M^T H^T \right] \frac{1}{2} \Theta \succeq 0 \right\}
\]

\[
\Gamma_\Pi(\Theta) = \max_{Q \in \Pi} \text{Tr}(\Theta Q)
\]

The problem is solvable, and the \(H\)-component \(H^*\) of its optimal solution yields linear estimate

\[
\hat{x}_{H^*}(\omega) = H^T \omega
\]

such that

\[
\text{Risk}_{\|\cdot\|, P[\Pi]}[\hat{x}_{H^*}|\mathcal{X}] \leq \text{Opt}.
\]
\[
\text{Opt} = \min_{H, \lambda, \mu, \kappa, \Theta} \left\{ \phi_T(\lambda) + \phi_R(\mu) + \phi_R(\kappa) + \Gamma_\Pi(\Theta) : \lambda \geq 0, \mu \geq 0, \kappa \geq 0, \right. \\
\left. \frac{1}{2} \left[ A^T H - B^T \right] M \right\} \geq 0, \left[ \frac{1}{2} M^T H A - B \right] \geq 0, \left[ \frac{1}{2} M^T H M \right] \geq 0 \\
[\Gamma_\Pi(\Theta) = \max_{Q \in \Pi} \text{Tr}(\Theta \cdot Q)]
\]

**Example I:** \( \Pi = \{Q \in S^m_+ : Q \preceq Q_*\} \Rightarrow \Gamma_\Pi(\Theta) = \text{Tr}(Q_* \Theta) 

**Example II: Poisson o.s.** Assume our observations are as follows:

- \( \mathcal{X} \) is compact convex subset of \( \mathbb{R}^n_+ \)
- \( A \) is an \( m \times n \) entrywise nonnegative matrix
- observation \( \omega \) stemming from \( x \in \mathcal{X} \) is random vector with independent of each other entries \( \omega_i \sim \text{Poisson}([Ax]_i) \)

\[
\Rightarrow \omega = Ax + \xi_x \\
\xi_x := \omega - Ax : \text{zero mean random vector with covariance matrix } \text{Diag}\{Ax\}
\]

\( \Rightarrow \text{we can take } \Pi = \{\text{Diag}\{Ax\} : x \in \mathcal{X}\} \)
\( \Rightarrow \Gamma_\Pi(\Theta) = \max_{x \in \mathcal{X}} \sum_i [Ax]_i \Theta_{ii} \)
**Example III: Discrete o.s.** Assume our observations are as follows:

- \( \mathcal{X} \) is closed convex subset of probabilistic simplex \( \Delta_n = \{ x \in \mathbb{R}^n_+ : \sum_i x_i = 1 \} \)
- \( A \) is an \( m \times n \) column-stochastic matrix
- observation stemming from \( x \in \mathcal{X} \) is a sample of \( K \) i.i.d. realizations \( \omega_k \) of discrete random variable with distribution \( Ax \)

Encoding the values of \( \omega_k \) by standard basic orths in \( \mathbb{R}^m \), on a close inspection we lose nothing when assuming that our observation is

\[
\omega = \frac{1}{K} \sum_{k=1}^K \omega_k = Ax + \xi_x
\]

\( \xi_x := \omega - Ax \): zero mean random variable with covariance matrix \( \frac{1}{K} \left[ \text{Diag}\{Ax\} - Axx^T A^T \right] \preceq \frac{1}{K} \text{Diag}\{Ax\} \)

\[\Rightarrow\] we can take \( \Pi = \left\{ \frac{1}{K} \text{Diag}\{Ax\} : x \in \mathcal{X} \right\} \)

\[\Rightarrow\] \( \Gamma_\Pi(\Theta) = \frac{1}{K} \max_{x \in \mathcal{X}} \sum_i [Ax]_i \Theta_{ii} \leq \frac{1}{K} \max_i \Theta_{ii} \)
Fact: In the case of zero mean Gaussian observation noise, the estimate $\hat{x}_{H^*}$ is near-optimal:

♠ Theorem: We have

$$\text{Risk}_{\|\cdot\|,\mathcal{P}[\Pi]}[\hat{x}_{H^*}|\mathcal{X}] \leq \text{Opt} \leq O(1)\sqrt{\ln(2K)\ln(2L)}\text{RiskOpt}_{\|\cdot\|,\mathcal{P}[\Pi]}[\mathcal{X}],$$

where

- $O(1)$ is a positive absolute constant,
- $K$ and $L$ are “sizes” of the ellitopes
- $\mathcal{X} = \{x : \exists t \in \mathcal{T} : x^TS_kx \leq t_k, k \leq K\}$,
- $\mathcal{B}^* = M\mathcal{Y}$, $\mathcal{Y} = \{y : \exists r \in \mathcal{R} : y^TR_\ell y \leq r_\ell, \ell \leq L\}$,
- $\text{RiskOpt}_{\|\cdot\|,\mathcal{P}[\Pi]}[\mathcal{X}] = \int\sup_{\hat{x}(\cdot)}\sup_{Q \in \Pi}\mathbb{E}_{\xi \sim \mathcal{N}(0,Q)}\{\|x - \hat{x}(Ax + \xi)\}\}$ is the minimax optimal risk taken w.r.t. Gaussian zero mean observation noises with co-variance matrices from $\Pi$. 

5.64
Variation: Recovery of partially stochastic signals

So far, we have considered the problem of recovering the image $Bx$ of unknown deterministic signal $x$ known to belong to a given signal set $X$ from noisy observations

$$\omega = Ax + \xi$$

of linear image of the signal.

In some applications, it makes sense to consider similar problem when the signal has a stochastic component.
Example: Kalman’s Filter. Consider linear dynamical system

\[
\begin{align*}
    y_1 &= \zeta_0 \\
    y_{t+1} &= P_t y_t + u_t + \zeta_t, \; t = 1, 2, \ldots, T \\
    \omega_t &= C_t y_t + \xi_t
\end{align*}
\]

- \( y_t \in \mathbb{R}^n \): states
- \( u_t \): controls
- \( \omega_t \in \mathbb{R}^m \): observations
- \( \zeta_t \): random “process noise”
- \( \xi_t \): random observation noise
- \( P_t, C_t \): known matrices.

What we want is to recover from observations \( \omega_1, \ldots, \omega_T \) linear image

\[
z := R[y_1; \ldots; y_{T+1}]
\]

of the state trajectory, e.g., \( y_T \) (“filtering”) or \( y_{T+1} \) (“forecast”).

Note: In the classical Kalman Filter,
- \( \zeta_0, \ldots, \zeta_T \) are independent of each other zero mean Gaussian
- \( \xi_1, \ldots, \xi_T \) are independent of each other and of \( \zeta_t \)'s zero mean Gaussian
- \( u_1, \ldots, u_T \) are deterministic and known (reduces to the case when \( u_t \equiv 0 \))
\[ y_1 = \zeta_0, \ y_{t+1} = P_t y_t + u_t + \zeta_t, \ \omega_t = C_t y_t + \xi_t \]

\[
(\omega_1, ..., \omega_T) \Rightarrow ?? z = R[y_1; ...; y_{T+1}] 
\]

- When modeling the situation as an estimation problem, we can use state equation to express the states \( y_t \) as known linear functions of controls \( u_t \) and process noises \( \zeta_t \), thus arriving at the model

\[
\omega = A[u; \zeta] + \xi \Rightarrow ?? z = B[u; \zeta] \\
[\omega = [\omega_1; ...; \omega_T], \ u = [u_1; ...; u_T], \ \zeta = [\zeta_0; ...; \zeta_T], \ \xi = [\xi_1; ...; \xi_T]]
\]

- When quantifying the performance of a candidate estimate \( \hat{x}(\omega) \), it makes sense to look at risk of the form

\[
\text{Risk}[\hat{x}] = \sup_u E_{\xi,\zeta} \{ \| B[u; \zeta] - \hat{x}(A[u; \zeta] + \xi) \| \}.
\]
Situation: We observe noisy linear image

$$\omega = A[u; \zeta] + \xi = A_d u + A_s \zeta + \xi$$

of “signal” $x = [u; \zeta]$ with deterministic component $u$ and stochastic component $\zeta$. We assume that

- $u$ is “uncertain-but-bounded” – is known to belong to a given set $U$
- $\zeta$ and $\xi$ have partially known distributions, specifically, for given $Q_\zeta \succ 0, Q_\xi \succ 0$

it holds

$$\text{Var}[\xi] = E\{\xi \xi^T\} \preceq Q_\xi, \quad \text{Var}[\zeta] = E\{\zeta \zeta^T\} \preceq Q_\zeta$$

Given matrix $B = [B_d, B_s]$ and a norm $\| \cdot \|$ on the image space of $B$, we want to recover $B[u; \zeta] = B_d u + B_s \zeta$, quantifying the recovery error in $\| \cdot \|$. The performance of a candidate estimate $\hat{x}(\cdot)$ is quantified by

$$\text{Risk}[\hat{x}] = \sup_{u \in U} \sup_{P \in \mathcal{P}} E_{[\xi; \zeta] \sim P} \left\{ \| B[u; \zeta] - \hat{x}(A[u; \zeta] + \xi) \| \right\}$$

$$\mathcal{P} : \text{probability distributions such that } E_{[\xi; \zeta] \sim P} \{\xi \xi^T\} \preceq Q_\xi, E_{[\xi; \zeta] \sim P} \{\zeta \zeta^T\} \preceq Q_\zeta$$

Goal: To build “presumably good” linear estimate $\hat{x}_H(\omega) = H^T \omega$.  

5.68
\[ \omega = A_d u + A_s \zeta + \xi \& u \in U \& \text{Var}[\xi] \preceq Q_\xi \& \text{Var}[\zeta] \preceq Q_\zeta \]

\[ \hat{x}_H(\omega) := H^T \omega \approx B_d u + B_s \zeta \]

**Assumption:** \( U \) is a basic ellitope, and the unit ball of the norm \( \| \cdot \|_* \) dual to \( \| \cdot \| \) is an ellitope:

\[ U = \{ u : \exists t \in T : u^T S_k u \leq t_k, k \leq K \} \]
\[ \{ v : \| v \|_* \leq 1 \} = \{ v : \exists r \in R, w : v = Mw, w^T R_\ell w \leq r_\ell, \ell \leq L \} \]

\[ \cdot \] For a candidate linear estimate \( \hat{x}_H(\omega) = H^T \omega, u \in U \), and a distribution \( P \) of \([\xi; \zeta]\) satisfying the bounds on the matrices of second moments of \( \xi \) and \( \zeta \) we have

\[ \mathbf{E}_{[\xi;\zeta] \sim P} \left\{ \| B_d u + B_s \zeta - H^T [A_d u + A_s \zeta + \xi] \| \right\} \]
\[ \leq \| B_d - H^T A_d \| u \| + \mathbf{E}_{[\xi;\zeta] \sim P} \left\{ \| H^T \xi \| \right\} + \mathbf{E}_{[\xi;\zeta] \sim P} \left\{ \| [B_s - H^T A_s] \zeta \| \right\} \]

As we know,

\[ u \in U \Rightarrow \| [B_d - H^T A_d] u \| \leq \min_{\lambda \geq 0, \nu \geq 0} \left\{ \phi_T(\lambda) + \phi_R(\nu) : \begin{bmatrix} \frac{1}{2} [B_d^T - A_d^T H] \sum_k \lambda_k S_k & \frac{1}{2} M^T [B_d - H^T A_d^T] \end{bmatrix} \succeq 0 \right\} \]

\[ \text{Var}[\xi] \preceq Q_\xi \Rightarrow \mathbf{E}_\xi \left\{ \| H^T \xi \| \right\} \leq \min_{\mu \geq 0, G} \left\{ \text{Tr}(G Q \xi) + \phi_R(\mu) : \begin{bmatrix} \frac{1}{2} M^T H^T & \frac{1}{2} H M \end{bmatrix} \sum_\ell \mu_\ell R_\ell \right\} \succeq 0 \}

\[ \text{Var}[\zeta] \preceq Q_\zeta \Rightarrow \mathbf{E}_\zeta \left\{ \|[B_s - H^T A_s] \zeta \| \right\} \leq \min_{\mu \geq 0, G} \left\{ \text{Tr}(G Q \zeta) + \phi_R(\mu) : \begin{bmatrix} \frac{1}{2} M^T [B_s - H^T A_s] & \frac{1}{2} [B_s^T - A_s^T H] M \end{bmatrix} \sum_\ell \mu_\ell R_\ell \right\} \succeq 0 \}

5.69
\[ \omega = A_d u + A_s \zeta + \xi \& u \in \{ u : \exists t \in T : u^T S_k u \leq t_k, k \leq K \} \& \text{Var}[\xi] \leq Q_\xi \& \text{Var}[\zeta] \leq Q_\zeta \]

\[ \hat{x}_H(\omega) := H^T \omega \approx B_d u + B_s \zeta \]

\[ \{ v : ||v||_* \leq 1 \} = \{ v : \exists r \in \mathbb{R}, w : v = M w, w^T R_\ell w \leq r_\ell, \ell \leq L \} \]

**Bottom line:** In the situation at hand, consider the convex optimization problem

\[
\text{Opt} = \min_{H, \lambda, \nu, \mu, \mu', G, G'} \left\{ \phi_T(\lambda) + \phi_R(\nu) + \phi_R(\mu) + \phi_R(\mu') + \text{Tr}(Q_\xi G) + \text{Tr}(Q_\zeta G') : \right. \\
\lambda \geq 0, \nu \geq 0, \mu \geq 0, \mu' \geq 0, \left. \begin{bmatrix}
\frac{1}{2} M^T H^T & \frac{1}{2} H M \\
\sum_{\ell} \mu_\ell R_\ell & \sum_{\ell} \mu_\ell R_\ell
\end{bmatrix} \succeq 0, \begin{bmatrix}
\frac{1}{2} M^T [B_d - H^T A_d^T] & \frac{1}{2} M^T [B_d - H^T A_d^T] \\
\sum_{k} \lambda_k S_k & \sum_{k} \lambda_k S_k
\end{bmatrix} \succeq 0, \\
\frac{1}{2} M^T [B_s - H^T A_s^T] & \frac{1}{2} M^T [B_s - H^T A_s^T] \\
\sum_{\ell} \mu'_\ell R_\ell & \sum_{\ell} \mu'_\ell R_\ell
\end{bmatrix} \succeq 0 \right\} 
\]

The problem is efficiently solvable, and the \( H \)-component \( H_* \) of its optimal solution gives rise to linear estimate \( \hat{x}_{H_*}(\Omega) = H^T_* \omega \) such that

\[ \text{Risk}[\hat{x}_{H_*}] \leq \text{Opt}. \]
How it works

**System:** Discretized pendulum $\ddot{x} = -\dot{x} - \kappa x$:

$$
\begin{bmatrix}
  x_{t+1} \\
  v_{t+1}
\end{bmatrix} =
\begin{bmatrix}
  0.9990 & 0.0951 \\
  -0.0190 & 0.9039
\end{bmatrix}
\begin{bmatrix}
  x_t \\
  v_t
\end{bmatrix} + (u_t + \zeta_t)
\begin{bmatrix}
  0.0048 \\
  0.0951
\end{bmatrix},
1 \leq t \leq 128
$$

$$
\omega_t = x_t
$$

$$
\begin{bmatrix}
  x_1 \\
  v_1
\end{bmatrix} \sim \mathcal{N}(0, I),
\zeta_t \sim \mathcal{N}(0, 0.05^2),
\xi \sim \mathcal{N}(0, 0.05^2),
|u_t| \leq 0.1
$$

Sample trajectory and forecasts

Errors/mean errors/error bounds

5.71
Recovery under uncertain-but-bounded noise

So far, we have considered recovering $Bx, x \in \mathcal{X}$, from observation

$$\omega = Ax + \xi$$

affected by *random* noise $\xi$. We are about to consider the case when $\xi$ is “uncertain-but-bounded:” all we know is that

$$\xi \in \mathcal{H}$$

with a given convex and compact set $\mathcal{H}$.

In the case in question, natural definition of risk of a candidate estimate $\hat{x}(\cdot)$ is

$$\text{Risk}_{\mathcal{H},\|\|}[\hat{x}(\cdot)|\mathcal{X}] = \sup_{x \in \mathcal{X}, \xi \in \mathcal{H}} \|Bx - \hat{x}(Ax + \xi)\|.$$

**Observation:** *Signal recovery under uncertain-but-bounded noise reduces to the situation where there is no observation noise at all.*

Indeed, let us treat as the signal underlying observation the pair $z = [x; \xi] \in Z := X \times \mathcal{H}$ and replace $A$ with $\bar{A} = [A, 0]$ and $B$ with $\bar{B} = [B, 0]$, so that

$$\omega = \bar{A}[x; \xi] \quad \& \quad Bx = \bar{B}[x; \xi],$$

thus reducing signal recovery to recovering $\bar{B}z, z \in Z$, from noiseless observation $\bar{A}z$.  

5.72
Let us focus on the problem of recovering the image $Bx \in \mathbb{R}^\nu$ of unknown signal $x \in \mathbb{R}^n$ known to belong to signal set $\mathcal{X} \subset \mathbb{R}^n$ via observation

$$\omega = Ax \in \mathbb{R}^m.$$ Given norm $\| \cdot \|$ on $\mathbb{R}^\nu$, we quantify the performance of an estimate $\hat{x}(\cdot) : \mathbb{R}^m \to \mathbb{R}^\nu$ by its $\| \cdot \|$-risk

$$\text{Risk}_{\| \cdot \|}[\hat{x} | \mathcal{X}] = \sup_{x \in \mathcal{X}} \| Bx - \hat{x}(Ax) \|.$$ 

Observation: Assuming that $\mathcal{X}$ is computationally tractable convex compact set and $\| \cdot \|$ is computationally tractable, it is easy to build an efficiently computable optimal within factor 2 nonlinear estimate:

Given $\omega$, let us solve the convex feasibility problem

Find $y \in \mathcal{Y}[\omega] := \{ y \in \mathcal{X} : Ay = \omega \}$.

and take, as $\hat{x}(\omega)$, the vector $By$, where $y$ is (any) solution to the feasibility problem.

Note: When $\omega$ stems from a signal $x \in \mathcal{X}$, the set $\mathcal{Y}[\omega]$ contains $x$

$\Rightarrow \hat{x}(\cdot)$ is well defined.
\[ x \in \mathcal{X}, \omega = Ax \Rightarrow \hat{x}(\omega) = By \]
\[ y \in \mathcal{Y}[\omega] = \{y \in \mathcal{X} : Ay = \omega\} \]

\[ ♠ \text{ Performance analysis: Let} \]
\[ R = \max_{y,z} \left\{ \frac{1}{2} \|B[y - z]\| : y, z \in \mathcal{X}, A[y - z] = 0 \right\} \]
\[ = \frac{1}{2} \|B[y^* - z^*]\| \quad [y^*, z^* \in \mathcal{X}, A[y^* - z^*] = 0] \]

**Claim A:** For every estimate \( \tilde{x}(\cdot) \) it holds \( \text{Risk}_{\|\cdot\|}[\tilde{x}|\mathcal{X}] \geq R \).
Indeed, the observation \( \omega = Ay^* = Az^* \) stems from both \( y^* \) and \( z^* \), whence the \( \|\cdot\| \)-risk of every estimate is at least \( \frac{1}{2} \|y^* - z^*\| = R \).

**Claim B:** One has \( \text{Risk}_{\|\cdot\|}[\hat{x}|\mathcal{X}] \leq 2R \).
Indeed, let \( \omega = Ax \) with \( x \in \mathcal{X} \), and let \( \hat{x}(\omega) = B\hat{y} \) with \( \hat{y} \in \mathcal{Y}[\omega] \). Then both \( x, \hat{y} \) belong to \( \mathcal{Y}[\omega] \)
\[ \Rightarrow \frac{1}{2} \|B[x - \hat{y}]\| \leq R. \]
We have built optimal, within factor 2, estimate. How to upper-bound its $\| \cdot \|$-risk?

**Observation:** Let $\mathcal{X}$ and the unit ball $\mathcal{B}^*$ of the norm $\| \cdot \|_*$ conjugate to $\| \cdot \|$ be ellitopes:

$$
\mathcal{X} = \{ x = Py : y \in \mathcal{Y} := \{ y : \exists t \in \mathcal{T} : y^T S_k y \leq t_k, k \leq K \} \}
$$

$$
\mathcal{B}^* = \{ u = Qv : v \in \mathcal{V} := \{ v : \exists r \in \mathcal{R} : v^T R_\ell v \leq r_\ell, \ell \leq L \} \}
$$

Then the $\| \cdot \|$-risk of the optimal, within factor 2, efficiently computable nonlinear estimate $\hat{x}(\cdot)$ can be tightly lower- and upper-bounded as follows.

- Assuming $\ker A \cap \mathcal{X} \neq \{0\}$ (otherwise the risk is zero), the set $\mathcal{X}_A = \{ x \in \mathcal{X} : Ax = 0 \}$ is an ellitope:

$$
\mathcal{X}_A = \{ x = Fw, w \in \mathcal{W} := \{ w : \exists t \in \mathcal{T} : w^T T_k w \leq t_k, k \leq K \} \}
$$

Indeed, setting $E = \{ y : APy = 0 \}$, the set $\mathcal{Y}_A = \{ y \in E : \exists t \in \mathcal{T} : y^T S_k y \leq t_k, k \leq K \}$ is a basic ellitope in some $\mathbb{R}^{N'} \Rightarrow \mathcal{X}_A = \{ Py : y \in \mathcal{Y}_A \}$ is an ellitope.

$$
\Rightarrow R := \max_{x', x'' \in \mathcal{X}} \left\{ \frac{1}{2} \| B [x' - x''] \| : A [x' - x''] = 0 \right\} = \max_{x \in \mathcal{X}_A} \| Bx \| = \max_{w \in \mathcal{W}} \| BFw \|
$$

$$
= \| BF \|_{\| \cdot \| \rightarrow \| \cdot \|} [\| \cdot \| : \text{norm with the unit ball } \mathcal{W}] 
$$

$$
\Rightarrow R \leq \text{Opt} \leq 3 \ln(\sqrt{3}[K + L]) R, \text{ with } \text{Opt} \text{ given by}
$$

$$
\text{Opt} = \min_{\lambda, \mu} \left\{ \phi_T(\lambda) + \phi_R(\mu) : \begin{aligned}
\lambda & \geq 0, \mu \geq 0 \\
\frac{1}{2} Q^T BF & \geq 0 \\
\sum_{\ell} \mu_\ell R_\ell & \geq 0 \\
\frac{1}{2} F^T B^T Q & \geq 0 \\
\sum_k \lambda_k T_k & \geq 0
\end{aligned} \right\}
$$

$$
\Rightarrow \text{The optimal } \| \cdot \|\text{-risk is } \geq R \geq \frac{\text{Opt}}{3 \ln(\sqrt{3}[K + L])}, \text{ and } \text{Risk}_{\| \cdot \|}[\hat{x}] \leq 2R \leq 2\text{Opt}.
$$

5.75
In fact, under mild assumptions a linear estimate is near-optimal:

**Theorem.** Consider the problem of recovering $Bx$ in $\| \cdot \|$, $x \in \mathcal{X}$, via observation $\omega = Ax$. Let the signal set $\mathcal{X}$ and the unit ball $B_*$ of the norm conjugate to $\| \cdot \|$ be ellitopes:

$$
\mathcal{X} = \{ x = Py : y \in \mathcal{Y} := \{ y : \exists t \in T : y^T S_k y \leq t_k, \ k \leq K \} \}
$$

$$
B_* = \{ u = Qz : z \in \mathcal{Z} = \{ \exists r \in \mathcal{R} : z^T R_\ell z \leq r_\ell, \ \ell \leq L \} \}
$$

Then the linear estimate $\hat{x}(\omega) = H^T \omega$ yielded by the $H$-component of optimal solution to the efficiently solvable optimization problem

$$
\text{Opt} = \min_{\lambda, \mu, H} \left\{ \phi_T(\lambda) + \phi_R(\mu) : \lambda \geq 0, \mu \geq 0 \left[ \sum_\ell \mu_\ell R_\ell \left\| \frac{1}{2} [B - H^T A] P \right\| \sum_k \lambda_k S_k \right] \geq 0 \right\}
$$

is near-optimal:

$$
\text{Risk}_{\| \cdot \|}[\hat{x}_H_* | \mathcal{X}] \leq \text{Opt} \leq O(1) \ln(K + L) \text{Risk}^*_{\| \cdot \|}[\mathcal{X}],
$$

where

$$
\text{Risk}^*_{\| \cdot \|}[\mathcal{X}] = \inf_{\hat{x}()} \text{Risk}_{\| \cdot \|}[\hat{x} | \mathcal{X}],
$$

$\inf$ being taken over all estimates, linear and nonlinear alike, is the minimax optimal risk.

5.76
From Ellitopes to Spectratopes

♠ Fact: All our results extend from ellitopes – sets of the form

\[
\{ y \in \mathbb{R}^N : \exists t \in \mathcal{T}, z : y = Pz, z^T S_k z \leq t_k, k \leq K \} \]

\[
\left[ S_k \succeq 0, \sum_k S_k \succ 0, \mathcal{T} \subset \mathbb{R}^K_+ : \text{monotone convex compact}, \mathcal{T} \cap \text{int} \mathbb{R}^K_+ \neq \emptyset \right] \tag{E}
\]

which played the roles of signal sets, ranges of bounded noise, and unit balls of the norms conjugate to the norms \( \| \cdot \| \) in which the recovering error is measured, to a wider family – spectratopes

basic spectratope: \( \mathcal{Y} = \{ y \in \mathbb{R}^N : \exists t \in \mathcal{T}, S_k^2[y] \leq t_k I_{d_k}, k \leq K \} \)

spectratope:

\[
S_k[y] = \sum_j y_j S_k^{kj}, S_k^{kj} \in S^d_{d_k} : \text{linear mappings with values in } S^d_{d_k} \]

\[
y \neq 0 \Rightarrow \sum_k S_k^2[y] \neq 0 \text{ [equivalent to } \mathcal{Y} \text{ being bounded]} \]

\[
\mathcal{T} \text{ as in (E)} \tag{S}
\]
Note:

- Every ellitope is a spectratope.

It suffices to verify that basic ellitope $\mathcal{X} = \{x : \exists t \in \mathcal{T} : x^T S_k x \leq t_k, k \leq K\}$ is a basic spectratope. Indeed, representing $S_k = \sum_{i=1}^{r_k} f_{ki} f_{ki}^T$, we have

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k, k \leq K \right\} = \left\{ x \in \mathbb{R}^n : \exists \{t_{ki} \geq 0, 1 \leq k \leq K, 1 \leq i \leq r_i\} : \left[ \sum_i t_{1i}; \ldots; \sum_i t_{Ki} \right] \in \mathcal{T} : [f_{ki}^T x]^2 \leq t_{ki} I_1 \forall (k \leq K, i \leq r_k) \right\}$$

- Denoting by $\| \cdot \|_{2,2}$ the spectral norm, matrix box

$$\mathcal{X} = \left\{ x \in \mathbb{R}^{p \times q} : \| x \|_{2,2} \leq 1 \right\} = \left\{ x \in \mathbb{R}^{p \times q} : \left[ \begin{array}{c} x \end{array} \right]_2 \leq I_{p+q} \right\}$$

and its symmetric version

$$\mathcal{X} = \left\{ x \in S^n : -I_n \preceq x \preceq I_n \right\} = \left\{ x \in S^n : x^2 \preceq I_n \right\}$$

are spectratopes $\Rightarrow$ access to matrix boxes as signal sets and nuclear norm as the recovery norm

- Spectratopes admit the same fully algorithmic calculus as ellitopes
Modifications of the results when passing from ellitopes to spectratopes are as follows:

A. The “size” $K$ of an ellitope ($E$) (logs of these sizes participate in our tightness factors) in the case of spectratope ($S$) becomes $D = \sum_k d_k$. 
B. Semidefinite relaxation bound for the quantity

\[ \text{Opt}_* = \max_y \{ y^T B y : \exists t \in T, z : y = P z, S_k^2[z] \preceq t_k I_{d_k}, k \leq K \} \]

\[ = \max_{z,t} \left\{ z^T \hat{B} z : t \in T, S_k^2[z] \preceq t_k I_{d_k}, k \leq K \right\}, \quad \hat{B} = P^T B P \]

is as follows. We associate with \( S_k[z] = \sum_j z_j S^k j \), \( S^k j \in S^{d_k} \), two linear mappings:

\[ Q \mapsto S_k[Q] : S^{\text{dim} z} \rightarrow S_{d_k} : \]
\[ \Lambda \mapsto S^*_k[\Lambda] : S_{d_k} \rightarrow S^{\text{dim} z} : \]

\[ \left[ S^*_k[\Lambda] \right]_{ij} = \frac{1}{2} \text{Tr}(\Lambda S^k_i S^k j + S^k j S^k i) = \frac{1}{2} \text{Tr}(\Lambda S^k i S^k j) \]

Note:

- \( S^2_k[z] = S_k[zz^T] \)
- the mappings \( S_k \) and \( S^*_k \) are conjugates of each other w.r.t. to the Frobenius inner product:

\[ \text{Tr}(S_k[Q]) = \text{Tr}(Q S^*_k[\Lambda]) \quad \forall (Q \in S^{\text{dim} z}, \Lambda \in S_{d_k}) \]

Selecting \( \Lambda_k \succeq 0, k \leq K \), such that \( \sum_k S^*_k[\Lambda_k] \succeq \hat{B} \), for

\[ z \in \mathcal{Z} = \{ z : \exists t \in T : S_k^2[z] \preceq t_k I_{d_k}, k \leq K \} \]

we have

\[ \exists t \in T : S_k^2[z] \preceq t_k I_{d_k}, \forall k \Rightarrow z^T \hat{B} z \preceq z^T \left[ \sum_k S^*_k[\Lambda_k] \right] z = \sum_k z^T S_k^2[z] \leq \sum_k \text{Tr}(S^*_k[\Lambda_k][zz^T]) \]

\[ = \sum_k \text{Tr}(\Lambda_k S_k[zz^T]) = \sum_k \text{Tr}(\Lambda_k S_k^2[z]) \leq \sum_k t_k \text{Tr}(\Lambda_k) \leq \phi_T(\lambda[\Lambda]), \]

\[ \phi_T(\lambda) = \max_{t \in T} t^T \lambda, \lambda[\Lambda] = [\text{Tr}(\Lambda_1); \ldots; \text{Tr}(\Lambda_K)] \]

\[ \Rightarrow \text{Opt}_* \leq \text{Opt} := \min_{\Lambda = \{ \Lambda_k, k \leq K \}} \left\{ \phi_T(\lambda[\Lambda]) : \Lambda_k \succeq 0, k \leq K, \hat{B} \preceq \sum_k S^*_k[\Lambda_k] \right\} \]

5.80
Theorem. Semidefinite relaxation bound

\[ \text{Opt} := \min_{\Lambda = \{\Lambda_k, k \leq K\}} \left\{ \phi_T(\lambda[\Lambda]) : \Lambda_k \succeq 0, k \leq K, \hat{B} \preceq \sum_k S_k^*[\Lambda_k] \right\} \]

on the quantity

\[ \text{Opt}_* = \max_y \left\{ y^T B y : \exists t \in T, z : y = Pz, S_k^2[z] \preceq t_k I_{d_k}, k \leq K \right\} \]

\[ = \max_{z,t} \left\{ z^T \hat{B} z : t \in T, S_k^2[z] \preceq t_k I_{d_k}, k \leq K \right\} \]

is tight:

\[ \text{Opt}_* \leq \text{Opt} \leq 2 \ln(2 \sum_k d_k) \text{Opt}_*. \]

Note: Proof follows the one for the ellitopic case.

But: The role of elementary Mini-Lemma in the spectratopic case is played by the following fundamental matrix concentration result:

Noncommutative Khintchine Inequality [Lust-Picard 1986, Pisier 1998, Buchholz 2001] Let \( A_i \in S^d, 1 \leq i \leq N, \) be deterministic matrices such that

\[ \sum_i A_i^2 \preceq I_d, \]

and let \( \zeta \) be \( N \)-dimensional \( \mathcal{N}(0, I_N) \) or Rademacher random vector. Then for all \( s \geq 0 \) it holds

\[ \text{Prob} \left\{ \| \sum_i \zeta_i A_i \|_{2,2} \geq s \right\} \leq 2d \exp\{-s^2/2\}. \]
C. Assuming that the signal set $\mathcal{X}$ and the unit ball $B_*$ of the norm conjugate to $\| \cdot \|$ spectratopes:

\[
\mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : S_k^2[x] \preceq t_k I_{d_k}, k \leq K \}
\]
\[
B_* = M \mathcal{Y}, \quad \mathcal{Y} = \{ y \in \mathbb{R}^N : \exists r \in \mathcal{R} : R^2_\ell[y] \preceq r_\ell I_{f_\ell}, \ell \leq L \}
\]

and that the distribution of noise in observation $\omega = Ax + \xi$ belongs to $\mathcal{P}[\Pi]$, the problem responsible for building presumably good linear estimate of $Bx$ via $\omega$ reads

\[
\begin{align*}
\text{Opt} = \min_{H, \Lambda, \gamma, \gamma', \Theta} & \left\{ \phi_T(\lambda[\Lambda]) + \phi_R(\lambda[\gamma]) + \phi_R(\lambda[\gamma']) + \Gamma_{\Pi}(\Theta) : \\
& \Lambda = \{ \Lambda_k \succeq 0 \}_{k \leq K}, \quad \gamma = \{ \gamma_\ell \succeq 0 \}_{\ell \leq L}, \quad \gamma' = \{ \gamma'_\ell \succeq 0 \}_{\ell \leq L} \\
& \left[ \begin{array}{c}
\sum_\ell R^*_\ell[\gamma_\ell] \\
\frac{1}{2}[A^TH - B^T]M
\end{array} \right] \left[ \begin{array}{c}
\frac{1}{2}M^TH^TA - B \\
\sum_k S^*_k[\Lambda_k]
\end{array} \right] \begin{bmatrix} 1 \\ \frac{1}{2}M^TH^T \end{bmatrix} \begin{bmatrix} \Gamma_{\Pi}(\Theta) \end{bmatrix} \succeq 0 \\
\left[ \begin{array}{c}
\sum_\ell R^*_\ell[\gamma'_\ell] \\
\frac{1}{2}HM
\end{array} \right] \begin{bmatrix} 1 \\ \Theta \end{bmatrix} \succeq 0
\right\}
\end{align*}
\]

\[
\begin{align*}
\Gamma_{\Pi}(\Theta) &= \max_{Q \in \Pi} \text{Tr}(\Theta Q), \quad \phi_G(h) = \max_{g \in G} g^T h \\
S_k[x] &= \sum_i x_k S_k^i[x], \quad S^*_k[\Lambda_k] = [\text{Tr}(S^{kp}\Lambda_k S^{kq})]_{p,q \leq n} \\
R_\ell[y] &= \sum_i y_i R^*_\ell, \quad R^*_\ell[\gamma_\ell] = [\text{Tr}(R^{\ell p}\gamma_\ell R^{\ell q})]_{p,q \leq N} \\
\lambda[\{ U_1, \ldots, U_s \}] &= [\text{Tr}(U_1); \ldots; \text{Tr}(U_s)]
\end{align*}
\]

The risk $\text{Risk}_{\| \cdot \|, \mathcal{P}[\Pi]}[\hat{x}_{H_*}|\mathcal{X}]$ of the linear estimate $\hat{x}_{H_*}(\omega) = H^T_* \omega$ yielded by the $H$-component of optimal solution to the problem does not exceed $\text{Opt}$. 5.82
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : S_k^2[x] \preceq t_k I_{d_k}, k \leq K \} \]
\[ \mathcal{B}_* = MY, \mathcal{Y} = \{ y \in \mathbb{R}^N : \exists r \in \mathcal{R} : R_\ell^2[y] \preceq r_\ell I_{f_\ell}, \ell \leq L \} \]

D. Near-optimality statement reads as follows:

The \( \| \cdot \| \)-risk of the just defined presumably good linear estimate \( \hat{x}_{H*} \) is within moderate factor of minimax optimal Gaussian risk:

\[
\text{Risk}_{\| \cdot \|, P[\Pi]}[\hat{x}_{H*}|\mathcal{X}] \leq \text{Opt} \leq O(1) \sqrt{\ln(2D) \ln(2F)} \text{Risk}_{\| \cdot \|, P[\Pi]}[\mathcal{X}]
\]

where

\[
D = \sum_k d_k, \quad F = \sum \ell f_\ell
\]

are the spectratopic sizes of \( \mathcal{X} \) and \( \mathcal{B}_* \), and

\[
\text{Risk}_{\| \cdot \|, P[\Pi]}[\mathcal{X}] = \inf_{\hat{x}} \sup_{Q \in \Pi} \max_{x \in \mathcal{X}} \mathbb{E}_{\xi \sim \mathcal{N}(0, Q)} \{ \| Bx - \hat{x}(Ax + \xi) \| \}
\]

is the Gaussian minimax optimal risk, i.e., the minimax risk associated with zero mean Gaussian noises with covariance matrices from \( \Pi \).
Proof of Near-Optimality: Executive Sketch

**Preliminaries, 1.** We shall use the following important

**Anderson’s Lemma.** Let \( f : \mathbb{R}^N \to \mathbb{R} \) be an even nonnegative summable function such that the sets \( \{ u : f(u) \geq t \} \) are convex for all \( t \geq 0 \), and let \( X \) be a symmetric w.r.t. the origin closed convex subset of \( \mathbb{R}^n \). Then the function

\[
\int_{X + te} f(u) du \quad [e \in \mathbb{R}^N]
\]

is nonincreasing in \( t \geq 0 \). As a result, if \( W \in \mathbb{S}^N_+ \), \( \| \cdot \| \) is a norm on \( \mathbb{R}^\nu \) and \( Y \) is an \( \nu \times N \) matrix, one has

\[
\text{Prob}_{\eta \sim \mathcal{N}(0, W)} \{ \| Y\eta + e \| \geq r \} \geq \text{Prob} \{ \| E\eta \| \geq r \} \quad \forall (e \in \mathbb{R}^N, r \geq 0).
\]
Preliminaries, 2. The upper risk bound $\text{Opt}$ on the risk of our presumably good linear estimate is in fact the saddle point value in the convex-concave saddle point problem:

\[
\text{Opt} = \min_{H, \Lambda, \gamma, \gamma', \Theta} \left\{ \phi_T(\lambda[\Lambda]) + \phi_R(\lambda[\gamma]) + \phi_R(\lambda[\gamma']) + \Gamma_n(\Theta) : \right.
\]

\[
\begin{align*}
\Lambda &= \{\Lambda_k \succeq 0\}_{k \leq K}, \\
\gamma &= \{\gamma_\ell \succeq 0\}_{\ell \leq L}, \\
\gamma' &= \{\gamma'_\ell \succeq 0\}_{\ell \leq L}, \\
&\left[ \begin{bmatrix} \sum_\ell R_\ell^*[\gamma_\ell] & \frac{1}{2}MT[H^T A - B] \\ \frac{1}{2}[A^T H - B^T]M & \sum_k S_k^*[\Lambda_k] \end{bmatrix} \right] \succeq 0
\end{align*}
\]

\[
\Xi = \left\{ \begin{bmatrix} \sum_\ell R_\ell^*[\gamma'_\ell] & \frac{1}{2}MT[H^T A - B] \\ \frac{1}{2}[A^T H - B^T]M & \sum_k S_k^*[\Lambda_k] \end{bmatrix} \succeq 0 \\
\frac{1}{2}HM & \frac{1}{2}HM \end{bmatrix} \succeq 0 \right\}
\]

It is easily seen that the problem has a saddle point $(Q_*, \ldots)$. From now on we restrict the observation noise to be $\mathcal{N}(0, Q_*)$. 

\[5.85\]
Preliminaries, 3. The crucial role in the proof is played by the following Main Lemma. Let the unit ball $B_*$ of the norm conjugate to norm $\| \cdot \|$ on $\mathbb{R}^\nu$ be a spectratope:

$$B_* = MY, \ Y = \{ y \in \mathbb{R}^N : \exists r \in \mathcal{R} : R^2_\ell[y] \preceq r_\ell I_\ell, \ell \leq L \},$$

let $Y$ be an $S \times \nu$ matrix, and $\eta \sim \mathcal{N}(0, \Sigma)$ with $0 \prec \Sigma \in S^S$. Then the upper bound

$$\Psi_\Sigma(Y) = \min_{\Upsilon, \Theta} \left\{ \phi_\mathcal{R}(\lambda[\Upsilon]) + \text{Tr}(\Sigma \Theta) : \begin{bmatrix} \sum_{\ell} \mathcal{R}_\ell^*[\Upsilon_\ell] & \frac{1}{2} MY^T \Theta \\ \frac{1}{2} YM & \Theta \end{bmatrix} \succeq 0 \right\}$$

on the quantity $\mathbb{E}_{\xi} \left\{ \| Y^T \xi \| \right\}$ is tight:

$$\mathbb{E}_{\eta \sim \mathcal{N}(0, \Sigma)} \left\{ \| Y^T \eta \| \right\} \leq \Psi_\Sigma(Y) \leq O(1)\sqrt{\ln(2F)} \mathbb{E}_{\eta \sim \mathcal{N}(0, \Sigma)} \left\{ \| Y^T \eta \| \right\} F = \sum_{\ell} f_\ell$$

Besides this, for every $\delta \in (0, 1)$ it holds

$$\text{Prob}_{\eta \sim \mathcal{N}(0, \Sigma)} \left\{ \| Y^T \eta \| > \frac{\delta}{\sqrt{\ln(2F/\delta)}} \text{Opt} \right\} \geq 1 - O(1)\delta.$$

Proof of Main Lemma heavily utilizes Conic Duality.

5.86
\[ \omega = Ax + \xi : x \in \mathcal{X}, \xi \sim \mathcal{N}(0, Q_*) , Q_* > 0 \]

**Step 1:** All we need is to upper-bound \( \text{Opt} \) in terms of the minimax optimal risk

\[ \text{Risk}_{\text{Opt}}\|\cdot\|[\mathcal{X}] = \inf \sup \mathbb{E}_{\xi \sim \mathcal{N}(0, Q_*)} \{ \| Bx - \hat{x}(Ax + \xi) \| \} . \]

Technically it is easier to upper-bound \( \text{Opt} \) in terms of the minimax \( \epsilon \)-risk \( \text{Risk}_{\epsilon} \), with some fixed \( \epsilon \). We use \( \epsilon = \frac{1}{8} \), resulting in

\[ \text{Risk}_{\frac{1}{8}} := \inf \min \left\{ \rho : \text{Prob}_{\xi \sim \mathcal{N}(0, Q_*)} \{ \| Bx - \hat{x}(Ax + \xi) \| > \rho \} \leq \frac{1}{8} \forall x \in \mathcal{X} \right\} \]

**Note:** \( \text{Risk}_{\frac{1}{8}} \leq 8 \text{Risk}_{\|\cdot\|}[\mathcal{X}] \Rightarrow \) upper-bounding \( \text{Opt} \) in terms of \( \text{Risk}_{\frac{1}{8}} \) automatically implies upper-bounding of \( \text{Opt} \) in terms of \( \text{Risk}_{\|\cdot\|} \).
\( \omega = Ax + \xi : x \in X, \xi \sim \mathcal{N}(0, Q_*), Q_* > 0 \)

**Step 2:** Let \( W \in \mathbb{S}^n_+ \). Consider the Bayesian version of our estimation problem, where the observation is

\[ \omega = A\eta + \xi \]

\( \xi \sim \mathcal{N}(0, Q_*), \eta \sim \mathcal{N}(0, W) \) are independent of each other

**Fact [well known]:** Since \( [\omega; \xi] \) is zero mean Gaussian, the conditional, given \( \omega \), expectation \( E_{|\omega} \{B\eta\} \) of \( B\eta \) is a linear function \( \bar{H}^T \omega \) of \( \omega \), implying that

\[ \text{For any estimate } \hat{x}(\cdot) \text{ and every } \rho > 0 \text{ one has, denoting by } P_{|w} \text{ the conditional, } \omega \text{ given, expectation,} \]

\[ \text{Prob}_{\xi,\eta}\{\|B\eta - \hat{x}(A\eta + \xi)\| > \rho\} = E_{\omega}\{P_{|w}\{\|B\eta - \hat{x}(\omega)\| \geq \rho\}\} \]

\[ \geq E_{\omega}\{P_{|w}\{Bx - E_{|w}\{B\eta\}\| \geq \rho\}\}\| = E_{\omega}\{P_{|w}\{Bx - \bar{H}^T \omega\}\}\} \]

\[ = \text{Prob}_{\xi,\eta}\{\|[B - \bar{h}^T A]\omega - \bar{H}^T \xi\| \geq \rho\}, \]

where \( \geq \) is due to Anderson’s Lemma as applied to the shift of the conditional, \( \omega \) given, distribution of \( B\eta \) by the mean of this distribution. By the same Anderson’s Lemma,

\[ \text{Prob}_{\xi,\eta}\{\|[B - \bar{h}^T A]\omega - \bar{H}^T \xi\| \geq \rho\} = E_{\xi}\{\text{Prob}_{\eta}\{\|[B - \bar{H}^T A]\eta - \bar{H}^T \xi\| \geq \rho\}\} \]

\[ \geq \text{Prob}_{\eta \sim \mathcal{N}(0, W)}\{\|[B - \bar{H}^T A]\eta\| \geq \rho\} \]

\[ \text{Prob}_{\xi,\eta}\{\|[B - \bar{h}^T A]\omega - \bar{H}^T \xi\| \geq \rho\} = E_{\eta}\{\text{Prob}_{\xi}\{\|[B - \bar{H}^T A]\eta - \bar{H}^T \xi\| \geq \rho\}\} \]

\[ \geq \text{Prob}_{\xi \sim \mathcal{N}(0, Q_*)}\{\|\bar{H}^T \xi\| \geq \rho\} \]

\[ \geq \max\left[\text{Prob}_{\eta \sim \mathcal{N}(0, W)}\{\|[B - \bar{H}^T A]\eta\| \geq \rho\}, \right. \]

\[ \left. \text{Prob}_{\xi \sim \mathcal{N}(0, Q_*)}\{\|\bar{H}^T \xi\| \geq \rho\}\right] \]

5.88
For every $W \in S^n_+$, every estimate $\hat{x}$ and every $\rho > 0$ one has

$$\text{Prob}_{[\xi;\eta]\sim N(0,Q) \times N(0,W)} \{ \| B \eta - \hat{x}(A \eta + \xi) \| \geq \rho \} \geq \max \left\{ \begin{array}{c} \text{Prob}_{\eta\sim N(0,W)} \{ \| (B - \bar{H}^T A) \eta \| \geq \rho \}, \\ \text{Prob}_{\xi\sim N(0,Q)} \{ \| \bar{H}^T \xi \| \geq \rho \} \end{array} \right\}, \quad (\ast)$$

**Step 3:** By Main Lemma, setting

$$\bar{\Psi}(H) = \min_{\gamma,\Theta} \left\{ \phi_R(\lambda[\gamma]) + \text{Tr}(Q_\ast \Theta) : \begin{array}{l} \gamma = \{ \gamma_\ell \geq 0 \}_{\ell \leq L} \\ \frac{1}{2} H M \begin{bmatrix} \frac{1}{2} M^T H^T & \Theta \end{bmatrix} \geq 0 \end{array} \right\}$$

$$\bar{\Phi}(W, H) = \min_{\gamma,\Theta} \left\{ \phi_R(\lambda[\gamma]) + \text{Tr}(W \Theta) : \begin{array}{l} \gamma = \{ \gamma_\ell \geq 0 \}_{\ell \leq L} \\ \frac{1}{2} [B^T - A^T H] M \begin{bmatrix} \frac{1}{2} M^T [B - H^T A] \end{bmatrix} \geq 0 \end{array} \right\}$$

for an appropriate absolute constant $O(1) > 0$ we have

$$\rho_* = \frac{O(1)}{\sqrt{\ln(2F)}} \Rightarrow \left\{ \begin{array}{l} \text{Prob}_{\eta\sim N(0,W)} \{ \| (B - \bar{H}^T A) \eta \| \geq \rho_* \bar{\Phi}(W, \bar{H}) \} \geq 1 - \frac{3}{8} \\ \text{Prob}_{\xi\sim N(0,Q_\ast)} \{ \| \bar{H}^T \xi \| \geq \rho_* \bar{\Psi}(\bar{H}) \} \geq 1 - \frac{3}{8} \end{array} \right\}$$

which combines with $(\ast)$ to imply that

$$\text{Prob}_{[\xi;\eta]\sim N(0,Q) \times N(0,W)} \{ \| B \eta - \hat{x}(A \eta + \xi) \| \geq \frac{1}{2} \rho_* \left[ \bar{\Phi}(W, \bar{H}) + \bar{\Psi}(\bar{H}) \right] \} \geq 1 - \frac{6}{8} = \frac{1}{4}$$

$$\Rightarrow \text{Prob}_{[\xi;\eta]\sim N(0,Q) \times N(0,W)} \{ \| B \eta - \hat{x}(A \eta + \xi) \| \geq \frac{1}{2} \rho_* \inf_{\bar{H}} \left[ \bar{\Phi}(W, H) + \bar{\Psi}(H) \right] \} \geq \frac{1}{4}$$

5.89
\[ X = \{ x \in \mathbb{R}^n : \exists t \in T : S_k^2[x] \leq t_k I_{d_k}, k \leq K \} \]

For appropriate positive absolute constant \( O(1) \), for every \( W \in S^n_+ \) and every estimate \( \hat{x}(\cdot) \) one has

\[
\text{Prob}_{[\xi; \eta] \sim N(0, Q_x) \times N(0, W)} \left\{ \| B\eta - \hat{x}(A\eta + \xi) \| \geq \frac{O(1)}{\sqrt{\ln(2F)}} \inf_H [\overline{\Phi}(W, H) + \overline{\Psi}(H)] \right\} \geq \frac{1}{4}. \tag{!}
\]

**Concluding steps:** Consider the parametric family of convex sets

\[ \mathcal{W}_\kappa = \{ W \in S^n_+ : \exists t \in T : S_k[W] \leq \kappa t_k I_{d_k}, k \leq K \} \]

where \( \kappa \in (0, 1] \), and the parametric family of convex-concave saddle point problems

\[
\text{Opt}(\kappa) = \sup_{W \in \mathcal{W}_\kappa} \inf_H [\overline{\Phi}(W, H) + \overline{\Psi}(H)]. \tag{\ast_\kappa}
\]

**Note:** When \( W \in \mathcal{W}_\kappa \) and \( \eta \sim N(0, W) \), the vector \( \eta/\sqrt{\kappa} \) “belongs to \( X \) at average:”

\[
\exists t \in T : \forall k \leq K : \kappa t_k I_{d_k} \geq S_k[W] = \sum_{i,j} E_{\eta \sim N(0, W)} \{ \eta_i \eta_j \} S^k_i S^k_j = E_{\eta \sim N(0, W)} \{ \sum_{i,j} \eta_i \eta_j S^k_i S^k_j \} = E_{\eta \sim N(0, W)} \{ S^2_k[\eta] \}.
\]

- It is not difficult to verify that for every \( \kappa \in (0, 1] \):
  - **a.** The convex-concave saddle point problem \( (\ast_\kappa) \) has a solution \( (W_\kappa, H_\kappa) \)
  - **b.** \( \text{Opt}(\kappa) \geq \sqrt{\kappa} \text{Opt}(1) \)
  - **c.** \( \text{Opt}(1) = \text{Opt} \) (miracle stemming from Conic Duality)
  - **d.** As \( \kappa \downarrow 0 \), \( \text{Prob}_{\eta \sim N(0, W_\kappa)} \{ \eta \notin X \} \) rapidly goes to 0:
    \[ \text{Prob}_{\eta \sim N(0, W_\kappa)} \{ \eta \notin X \} \leq 2 \exp\{-\frac{1}{2\kappa}\} \sum_k d_k. \]
    (stems from Noncommutative Khintchine Inequality)

- By **b**, **c** and (!), for every estimate \( \hat{x} \) and every \( \kappa \in (0, 1] \) we have

\[
\text{Prob}_{[\xi; \eta] \sim N(0, Q_x) \times N(0, W_\kappa)} \left\{ \| B\eta - \hat{x}(A\eta + \xi) \| \geq \frac{O(1)}{\sqrt{\ln(2F)}} \sqrt{\kappa} \text{Opt} \right\} > \frac{1}{4},
\]

5.90
... for every estimate $\hat{x}$ and every $\kappa \in (0, 1]$ we have
\[
\text{Prob}_{[\xi; \eta] \sim \mathcal{N}(0, Q_\kappa) \times \mathcal{N}(0, W_\kappa)} \left\{ \| B\eta - \hat{x}(A\eta + \xi) \| \geq \frac{O(1)}{\sqrt{\ln(2F)}} \sqrt{\kappa} \text{Opt} \right\} > \frac{1}{4},
\]
This relation easily combines with $d$ to yield the target upper bound
\[
\text{Risk}_{\frac{1}{8}} \geq \frac{O(1)\text{Opt}}{\sqrt{\ln(2D) \ln(2F)}}
\]
on $\text{Opt}$ in terms of $\text{Risk}_{\frac{1}{8}}$. □
Beyond linearity: Polyhedral estimates

♣ As before, our problem of interest is: *given noisy observation*

\[ \omega = Ax + \xi \in \mathbb{R}^m, \quad \xi \sim P_x, \]

*of unknown signal* \( x \) *known to belong to a given convex compact signal set* \( \mathcal{X} \subset \mathbb{R}^n \),

*we want to recover* \( Bx \in \mathbb{R}^\nu \) *in a given norm* \( \| \cdot \| \).

We have seen that under reasonable assumptions on problem’s data, efficiently computable via convex Programming *linear in* \( \omega \) *estimates are near-optimal. However: There are meaningful situations which go beyond the scope of “reasonable assumptions,” moreover, situations where linear estimation is provably far from being near-optimal.*
Example: Let $\mathcal{X} = \{ x \in \mathbb{R}^n : \| x \|_1 \leq 1 \}$ be the unit $\ell_1$-ball, observations be direct:

$$\omega = x + \sigma \eta, \; \eta \sim \mathcal{N}(0, I_n),$$

and we want to recover $B x \equiv x$ in Euclidean norm. For a linear estimate $H^T \omega$, worst-case expected squared recovery error is

$$\max_{x \in \mathcal{X}} \mathbb{E}_{\eta \sim \mathcal{N}(0,I_n)} \left\{ \| H^T (x + \sigma \xi) - x \|_2^2 \right\} = \max_i \| \text{Row}_i[I - H] \|_2^2 + \sigma^2 \text{Tr}(H^T H)$$

Its minimum over $n \times n$ matrices $H$ is achieved at the scalar matrix $H = h I_n$ with $h = \frac{1}{\sigma^2 n + 1}$ and equals

$$\text{Risk}_{\text{lin}}^2 = \frac{\sigma^2 n}{\sigma^2 n + 1}.$$  

When $\sigma^2 n \geq 1$, this squared risk is at least $1/2$.  

5.93
Now consider the estimate as follows: *given* \( \omega \), we estimate \( x \) by the optimal solution \( \hat{x}(\omega) \) to the convex optimization problem

\[
\text{Opt}(\omega) = \min \left\{ \| \omega - y \|_\infty : y \in X \right\}.
\]

Observe that when \( \omega = x + \sigma \eta \) with \( x \in X \), setting \( \hat{x} = \hat{x}(\omega) \) we have

\[
\text{Opt}(\omega) \leq \| \omega - x \|_\infty = \sigma \| \eta \|_\infty
\]

\[
\Rightarrow \| x - \hat{x} \|_\infty \leq \| x - \omega \|_\infty + \| \omega - \hat{x} \|_\infty \leq 2\sigma \| \eta \|_\infty
\]

\[
\Rightarrow \| x - \hat{x} \|_2 \leq \| x - \hat{x} \|_\infty \| x - \hat{x} \|_1 \leq 4\sigma \| \eta \|_\infty
\]

\[
\Rightarrow \text{Risk}^2[\hat{x}] := \max_{x \in X} \mathbb{E} \left\{ \| x - \hat{x}(Ax + \sigma \xi) \|_2^2 \right\} \leq 4\sigma \mathbb{E}_{\eta \sim N(0,I_n)} \{ \| \eta \|_\infty \}
\]

It is easily seen that \( \mathbb{E}_{\eta \sim N(0,I_n)} \{ \| \eta \|_\infty \} \leq 2\sqrt{\ln(2n)} \), whence

\[
\text{Risk}^2[\hat{x}] \leq 8\sigma \sqrt{\ln(2n)} \quad \text{&} \quad \text{Risk}^2_{\text{lin}} = \frac{\sigma^2 n}{\sigma^2 n + 1}.
\]

\( \Rightarrow \) When \( \sigma \) is small and \( \sigma^2 n \) is of order of 1, an appropriate nonlinear estimate significantly outperforms the best linear one – for the former, squared risk is nearly \( O(\sigma) \), and for the latter it is \( O(1) \).

\( \blacklozenge \) **What is ahead:** nonlinear polyhedral estimates with the “scope of near-optimality” strictly wider than the one for linear estimates.

5.94
\( \mathcal{X} \subset \mathbb{R}^n \) & \( \omega = Ax + \xi \in \mathbb{R}^m, \xi \sim P_x \) with \( x \in \mathcal{X} \) \[ \Rightarrow \] \( \hat{x}(\omega) \approx Bx \in \mathbb{R}^\nu \)

**Polyhedral Estimate: Construction.** Generic polyhedral estimate generalizes the estimate \( \hat{x} \) from Example and is as follows:
The estimate is specified by \( N \times m \) contrast matrix \( H \) and is given by

\[
\omega \mapsto \bar{x}(\omega) \in \text{Argmin}_{y \in \mathcal{X}} \| H^T [\omega - Ay] \|_\infty \mapsto \hat{x}(\omega) = B\bar{x}(\omega)
\]

**Polyhedral estimate: Risk Analysis.** In what follows, it is convenient to quantify the performance of a candidate estimate \( \hat{x}(\cdot) \) by its \( \epsilon \)-risk rather the worst-case, over \( x \in \mathcal{X} \), expected error. Specifically, given “reliability tolerance” \( \epsilon \in (0, 1) \), we define \((\epsilon, \| \cdot \|)-risk\) of a candidate estimate \( \hat{x}(\cdot) : \mathbb{R}^m \to \mathbb{R}^\nu \) as the worst case, over \( x \in \mathcal{X} \), width of “\( \| \cdot \| - (1 - \epsilon)\)-confidence interval:"

\[
\text{Risk}_{\epsilon, \| \cdot \|}[\hat{x}(\cdot)|\mathcal{X}] = \min \left\{ \rho : \text{Prob}_{\xi \sim P_x} \{ \| \hat{x}(Ax + \xi) - Bx \| > \rho \} \leq \epsilon \forall x \in \mathcal{X} \right\}
\]
\( \mathcal{X} \subset \mathbb{R}^n \) & \( \omega = Ax + \xi \in \mathbb{R}^m \), \( \xi \sim P_x \) with \( x \in \mathcal{X} \) \( \Rightarrow \) \( \hat{x}(\omega) \approx Bx \in \mathbb{R}^\nu \)

**Immediate observation:** Given reliability tolerance \( \epsilon \in (0, 1) \), assume that contrast matrix \( H \) satisfies

\[
\text{Prob}_{\xi \sim P_x} \{ \| H^T \xi \|_{\infty} > 1 \} \leq \epsilon \ \forall x \in \mathcal{X}
\]

Let \( \mathcal{X}_s = \frac{1}{2} [\mathcal{X} - \mathcal{X}] = \left\{ \frac{1}{2} [x - x'] : x, x' \in \mathcal{X} \right\} \) and

\[
\mathcal{R} = \max_z \left\{ 2 \| Bz \| : z \in \mathcal{X}_s, \| H^T A z \|_{\infty} \leq 1 \right\}
\]

for the polyhedral estimate \( \hat{x}_H \) associated with the contrast matrix \( H \) we have

\[
\text{Risk}_{\epsilon, \| \cdot \|} [\hat{x}_H | \mathcal{X}] \leq \mathcal{R}.
\]

Indeed, let us fix \( x \in \mathcal{X} \), and let \( \mathcal{E} = \{ \xi : \| H^T \xi \|_{\infty} \leq 1 \} \), so that \( P_x \{ \mathcal{E} \} \geq 1 - \epsilon \). For \( \xi \in \mathcal{E} \), setting \( \hat{x} = \hat{x}(Ax + \xi) \), we have \( \hat{x} = B\bar{x} \) with \( \bar{x} \in \text{Argmin} \ F(y) := \| H^T [Ax + \xi - Ay] \|_{\infty} \)

We have \( x \in \mathcal{X} \) and \( F(x) \leq \| H^T \xi \|_{\infty} \leq 1 \) since \( \xi \in \mathcal{E} \)

\[ \Rightarrow \bar{x} \in \mathcal{X} \) and \( F(\bar{x}) \leq 1 \]

\[ \Rightarrow 2 \geq F(x) + F(\bar{x}) = \| H^T \xi \|_{\infty} + \| H^T A [x - \bar{x}] + H^T \xi \|_{\infty} \geq \| H^T A [x - \bar{x}] \|_{\infty} \]

\[ \Rightarrow \text{for } z = \frac{1}{2} [x - \bar{x}] \in \mathcal{X}_s \text{ it holds } \| H^T A z \|_{\infty} \leq 1 \Rightarrow \| Bx - \hat{x} \| = \| Bx - B\bar{x} \| \leq 2 \| z \| \leq \mathcal{R}. \]

\[ \Rightarrow \text{when } x \in \mathcal{X} \) and \( \xi \in \mathcal{E} \) (which happens with \( P_x \)-probability at least \( 1 - \epsilon \)) it holds \( \| x - \hat{x}(Ax + \xi) \| \leq \mathcal{R}. \)
\[
\text{Prob}_{\xi \sim P_x} \{ \| H^T \xi \|_\infty > 1 \} \leq \epsilon \ \forall x \in X
\]
\[
\Rightarrow \text{Risk}_{\epsilon, \| \cdot \|, [\hat{x}_H | \mathcal{X}]} \leq \mathcal{R} := \max_z \left\{ 2 \| Bz \| : z \in \mathcal{X}_s, \| H^T A z \|_\infty \leq 1 \right\}. \quad (*)
\]

**Questions to be addressed:**

A. How to define a set \( \mathcal{H}_\epsilon \), the wider the better, of contrast matrices \( H \) satisfying (!)

B. How to upper-bound \( \mathcal{R} \) efficiently

**Note:** Optimization problem in (*) is a difficult problem of *maximizing convex* function over a convex set.

C. How to optimize, to the largest extent possible, \( \mathcal{R} \) over \( H \in \mathcal{H}_\epsilon \)
A. How to define a set $\mathcal{H}_\epsilon$ of contrast matrices $H$ satisfying

$$\text{Prob}_{\xi \sim P_x}\{\|H^T \xi\|_\infty \leq 1\} \geq 1 - \epsilon?$$

**Answering Question A.** In the sequel, we restrict ourselves with 3 observation schemes:

**Sub-Gaussian case:** For every $x \in \mathcal{X}$, the distribution $P_x$ of observation noise is sub-Gaussian with parameters $(0, \sigma^2 I_m)$:

$$E_{\xi \sim P_x}\{\exp\{h^T \xi\}\} \leq \frac{1}{2}h^T h \ \forall h.$$

Given positive integer $N$ and setting

$$\pi_G(h) = \vartheta_G \|h\|_2 \text{ where } \vartheta_G = \sigma \sqrt{2 \ln(2N/\epsilon)},$$

$$\mathcal{H}_\epsilon = \mathcal{H}_\epsilon^G = \{H \in \mathbb{R}^{m \times N} : \pi_G(\text{Col}_j[H]) \leq 1, 1 \leq j \leq N\}$$

we ensure that for every $H \in \mathcal{H}_\epsilon$ and every $(0, \sigma^2 I_m)$-sub-Gaussian $\xi$ it holds

$$\text{Prob}\{\|H^T \xi\|_\infty \leq 1\} \geq 1 - \epsilon.$$
**Discrete case:** $\mathcal{X}$ is a convex compact subset of the probabilistic simplex $\Delta_n = \{x \in \mathbb{R}^n : x \geq 0, \sum_i x_i = 1\}$, $A$ is column-stochastic matrix, and observation $\omega$ stemming from signal $x \in \mathcal{X}$ is

$$\omega = \frac{1}{K} \sum_{k=1}^{K} \zeta_k$$

with independent across $k \leq K$ random vectors $\zeta_k$, each taking values $e_i$ with probabilities $[Ax]_i, i = 1, \ldots, m$, $e_i$ being the basic orths in $\mathbb{R}^m$.

Setting

$$\pi_D(h) = 2\sqrt{\vartheta_D \max_{x \in \mathcal{X}} \sum_i [Ax]_i h_i^2 + \frac{16}{9} \vartheta_D^2 \|h\|_\infty^2} \quad \text{with} \quad \vartheta_D = \frac{\ln(2N/\epsilon)}{K},$$

$$\mathcal{H}_\epsilon = \mathcal{H}_\epsilon^D := \{H \in \mathbb{R}^{m \times N} : \pi_D(\text{Col}_j[H]) \leq 1, j \leq m\},$$

we ensure that for every $H \in \mathcal{H}_\epsilon$ and every $x \in \mathcal{X}$, for the random noise $\xi_x = \omega - Ax$, with the above $\omega$, it holds

$$\text{Prob}\{\|H^T \xi_x\|_\infty \leq 1\} \geq 1 - \epsilon.$$
Note: The crucial role in the justification of the above bounds on probabilities of large deviations of histograms from true distributions is played by the fundamental Bernstein Inequality: Let $X_1, \ldots, X_N$ be independent zero mean random variables with variations $\sigma^2_1, \ldots, \sigma^2_N$ such that $|X_i| \leq M < \infty$ for all $i$ and some $M$. Then for every $t \geq 0$ one has

$$\text{Prob} \left\{ \sum_{i=1}^{N} X_i \geq t \right\} \leq \exp \left\{ -\frac{t^2}{2 \left[ \sum_{i=1}^{N} \sigma^2_i + \frac{1}{3} Mt \right]} \right\}.$$
**Poisson case:** \( \mathcal{X} \) is a convex compact subset of the nonnegative orthant \( \mathbb{R}^n_+ \), \( A \) is entrywise nonnegative, and the observation \( \omega \) stemming from \( x \in \mathcal{X} \) is random vector with independent across \( i \) entries \( \omega_i \sim \text{Poisson}(\langle Ax \rangle_i) \).

In the Poisson case we set

\[
\pi_P(h) = 2\sqrt{\vartheta_P} \max_{x \in \mathcal{X}} \sum_i [Ax]_i h_i^2 + \frac{4}{9} \vartheta_P^2 \|h\|_\infty^2 \quad \text{with} \quad \vartheta_P = \ln(2N/\epsilon),
\]

\( \mathcal{H}_\epsilon = \mathcal{H}_\epsilon^P := \{ H \in \mathbb{R}^{m \times N} : \pi_P(\text{Col}_j[H]) \leq 1, 1 \leq j \leq N \} \).

thus ensuring that for every \( H \in \mathcal{H}_\epsilon \) and every \( x \in \mathcal{X} \), for the random noise \( \xi_x = \omega - Ax \), with the above \( \omega \), it holds

\[
\text{Prob}\{\|H^T \xi_x\|_\infty \leq 1\} \geq 1 - \epsilon.
\]

**Note:** In all 3 cases, the set \( \mathcal{H}_\epsilon \) of “admissible” \( m \times N \) contrast matrices is of the form

\[
\mathcal{H}_\epsilon = \{ H \in \mathbb{R}^{m \times N} : \pi(\text{Col}_j(H)) \leq 1, j \leq M \}
\]

where \( \pi(\cdot) \) is norm of the form

\[
\pi(\cdot) = \sqrt{\alpha \max_{y \in Y} \sum_i y_i h_i^2 + \beta \|h\|_\infty^2}\}
\]

\([Y \subset \mathbb{R}^m_+: \text{convex compact set}]\)

with \( \alpha > 0, \beta \geq 0 \) logarithmically depending on \( N/\epsilon \).
\[ \text{Risk}_{\epsilon, \|\cdot\|} \left[ \hat{x}_H | \mathcal{X} \right] \leq \mathcal{R}[H] = \max_z \left\{ 2\|Bz\| : z \in \mathcal{X}, \|H^T Az\|_\infty \leq 1 \right\} \]

**B. How to upper-bound \( \mathcal{R}[H] \)? C. How to optimize \( \mathcal{R}[H] \) over \( H \)?

**Answering Questions B, C, Version I**

♣ The reference case for what follows is the one of \( \|\cdot\| = \|\cdot\|_\infty \). In this case \( \mathcal{R}[H] \) is easy to compute by solving \( \nu \) convex optimization problems

\[
\max \left\{ [Bz]_i : z \in \mathcal{X}, \|H^T Az\|_\infty \leq 1 \right\}, \ i = 1, \ldots, \nu;
\]

It turns out that in this case it is easy to optimize \( \mathcal{R}[H] \) over \( H \in \mathcal{H}_\epsilon \).

**Assumptions of Version I** are as follows:

**A.1.** \( \|\cdot\| = \|\cdot\|_r \) with some \( r \in [1, \infty] \)

**A.2.** We have at our disposal a sequence \( \gamma = \{\gamma_i > 0, 1 \leq i \leq \nu\} \) and \( \rho \in [1, \infty] \) such that the image of \( \mathcal{X}_s \) under the mapping \( x \mapsto Bx \) is contained in the “scaled \( \|\cdot\|_\rho \)-ball”

\[
\mathcal{Y} = \{y \in \mathbb{R}^\nu : \|\text{Diag}\{\gamma\} y\|_\rho \leq 1\}.
\]

5.102
Observation: Let $B^T \ell$ be $\ell$-th row in $B$, $1 \leq \ell \leq \nu$. Under assumptions A.1-2, let $\epsilon \in (0, 1)$ and a positive real $N \geq \nu$ be given, and let $\pi(\cdot)$ be a norm on $\mathbb{R}^m$ such that

$$\forall (h : \pi(h) \leq 1, x \in \mathcal{X}) : \text{Prob}\{|h^T \xi| > 1\} \leq \epsilon/N.$$ 

Let, next, a matrix $H = [H_1, \ldots, H_\nu]$ with $H_\ell \in \mathbb{R}^{m \times m_\ell}$, $m_\ell \geq 1$, $\sum_\ell m_\ell = N$, and positive reals $\varsigma_\ell$, $1 \leq \ell \leq \nu$, satisfy the relations

1. $\pi(\text{Col}_j[H]) \leq 1$, $1 \leq j \leq N$;
2. $\max_x \{B_\ell^T x : x \in \mathcal{X}_s, \|H_\ell^T Ax\|_{\infty} \leq 1\} \leq \varsigma_\ell$, $1 \leq \ell \leq \nu$.

Then the quantity $\mathcal{R}[H]$ can be upper-bounded as follows:

$$\mathcal{R}[H] \leq \Psi(\varsigma) := 2 \max_v \{\|(v_1/\gamma_1; \ldots; v_\nu/\gamma_\nu)\|_r : \|v\|_\rho \leq 1, 0 \leq v_\ell \leq \gamma_\ell \varsigma_\ell, 1 \leq \ell \leq \nu\}.$$ 

implying that

$$\text{Risk}_{\epsilon,\|\cdot\|}[\hat{w}_H|\mathcal{X}] \leq \Psi(\varsigma).$$

Function $\Psi$ is nondecreasing on the nonnegative orthant and is easy to compute.
\textbf{A.1: } \| \cdot \| = \| \cdot \| _r \quad \textbf{A.2: } B \mathcal{X}_s \subset \mathcal{Y} = \{ y : \| \text{Diag} \{ \gamma \} y \| _\rho \leq 1 \}

Indeed, let $\bar{z} \in \mathcal{X}_s$ and $\| H^T A \bar{z} \| _\infty \leq 1$. Setting $y = B \bar{z}$, we have $y \in \mathcal{Y}$ due to $\bar{z} \in \mathcal{X}_s$ and \textbf{A.2}. Thus, $\| \text{Diag} \{ \gamma \} y \| _\rho \leq 1$. Besides this, by (b) relations $\bar{z} \in \mathcal{X}_s$ and $\| H^T A \bar{z} \| _\infty \leq 1$ combine with the symmetry of $\mathcal{X}_s$ to imply that $|y_\ell| = |B^T_\ell \bar{z}| \leq \varsigma_\ell$, $\ell \leq \nu$. Taking into account that $\| \cdot \| = \| \cdot \| _r$ by \textbf{A.1}, we see that

$$\begin{align*}
\Re[H] &= \max_z \{ 2\| Bz \|_r : z \in \mathcal{X}_s, \| H^T Az \|_\infty \leq 1 \} \\
&\leq 2 \max_y \{ \| y \|_r : |y_\ell| \leq \varsigma_\ell, \ell \leq \nu \& \| \text{Diag} \{ \gamma \} y \|_\rho \leq 1 \} \\
&= 2 \max_v \{ \| [v_1/\gamma_1; \ldots; v_\nu/\gamma_\nu] \|_r : \| v \|_\rho \leq 1, 0 \leq v_\ell \leq \gamma_\ell \varsigma_\ell, \ell \leq \nu \} = \Psi(\varsigma),
\end{align*}$$

as claimed. It is evident that $\Psi$ is nondecreasing on the nonnegative orthant.
Designing contrast. Using Observation, it is easy to minimize $\varsigma$'s (and thus upper risk bound $\Psi(\varsigma)$) over contrast matrices $H$ with $\pi$-norms of the columns not exceeding 1.

Preliminaries. Given a vector $b \in \mathbb{R}^n$ and a norm $\pi(\cdot)$ on $\mathbb{R}^m$, consider convex-concave saddle point problem

$$\text{Opt} = \inf_{g \in \mathbb{R}^m} \max_{x \in \mathcal{X}_S} \{ \phi(g, x) := [b - A^T g]^T x + \pi(g) \} \quad (SP)$$

along with the induced primal and dual problems

$$\text{Opt}(P) = \inf_{g \in \mathbb{R}^m} \left[ \phi(g) := \max_{x \in \mathcal{X}_S} \phi(g, x) \right] \quad (P)$$

$$\text{Opt}(D) = \max_{x \in \mathcal{X}_S} \left[ \phi(g) := \inf_{g \in \mathbb{R}^m} \phi(g, x) \right] \quad (D)$$

where $\theta(\cdot)$ is the norm conjugate to $\pi(\cdot)$ (we have used the evident fact that $\inf_{g \in \mathbb{R}^m} [f^T g + \pi(g)]$ is either $-\infty$ or 0 depending on whether $\theta(f) > 1$ or $\theta(f) \leq 1$). Since $\mathcal{X}_S$ is compact, we have $\text{Opt}(P) = \text{Opt}(D) = \text{Opt}$ by the Sion-Kakutani theorem. Besides this, $(D)$ is solvable (this is evident) and $(P)$ is solvable as well, since $\phi(g)$ is continuous due to the compactness of $\mathcal{X}_S$, and $\bar{\phi}(g) \geq \pi(g)$, so that $\bar{\phi}(\cdot)$ has bounded level sets. Let $\bar{g}$ be an optimal solution to $(P)$, let $\bar{x}$ be an optimal solution to $(D)$, and let $\bar{h} = \bar{g} / s(\bar{g})$, so that $s(\bar{h}) = 1$ and $\bar{g} = s(\bar{g}) \bar{h}$.

Claim: One has $\max_x \{ |b^T x| : x \in \mathcal{X}_S, |\bar{h}^T Ax| \leq 1 \} \leq \text{Opt}$. In addition, for any matrix $G = [g^1, ..., g^M] \in \mathbb{R}^{m \times M}$ with $\pi(g^j) \leq 1, 1 \leq j \leq M$, one has

$$\max_x \{ |b^T x| : x \in \mathcal{X}_S, \|G^T Ax\|_\infty \leq 1 \} = \max_x \{ b^T x : x \in \mathcal{X}_S, \|G^T Ax\|_\infty \leq 1 \} \geq \text{Opt}.$$
\[ \text{Opt} = \inf_{g \in \mathbb{R}^m} \left[ \pi(g) + \max_{x \in X_S} [b - A^T g]^T x \right] \quad (\$) \]

\[ \Rightarrow \text{optimal solution } \tilde{g} = \pi(\tilde{g}) \pi(\tilde{h}), \pi(\tilde{h}) = 1 \]

**Claim:** One has \( \max_x \{ |b^T x| : x \in X_S, |\tilde{h}^T Ax| \leq 1 \} \leq \text{Opt} \), and for any matrix \( G = [g^1, ..., g^M] \in \mathbb{R}^{m \times M} \) with \( \pi(g^j) \leq 1 \), \( 1 \leq j \leq M \), one has

\[ \max_x \{ |b^T x| : x \in X_S, \|G^T Ax\|_\infty \leq 1 \} = \max_x \{ b^T x : x \in X_S, \|G^T Ax\|_\infty \leq 1 \} \geq \text{Opt}. \]

Taking Claim for granted, we see that when \( H = [H_1, ..., H_\nu] \) is \( m \times N \) matrix with columns of \( \pi \)-norm not exceeding 1, then

\[ \varsigma_\ell = \max_x \{ B^T_\ell x : x \in X_S, \|H^T_\ell Ax\|_\infty \leq 1 \} \geq \text{Opt}_\ell = \inf_{g \in \mathbb{R}^m} \left[ \pi(g) + \max_{x \in X_S} [B_\ell - A^T g]^T x \right] \]

\[ \geq \max_x \{ |b^T x| : x \in X_S, |\tilde{h}^T_\ell Ax| \leq 1 \} \]

where \( \tilde{h}_\ell \) with \( \pi(\tilde{h}_\ell) = 1 \) is readily given by an optimal solution to the problem (\$) with \( b = B_\ell \).

\[ \Rightarrow \text{The best – resulting in the smallest possible values } \text{Opt}_\ell \text{ of the quantities } \varsigma_\ell, \ 1 \leq \ell \leq \nu, \text{ and thus in the smallest upper risk bound } \Psi(\varsigma) - m \times N, \ N \geq \nu, \text{ contrast matrix with } \pi \text{-norms of columns not exceeding } 1 \text{ is the } m \times \nu \text{ matrix } [\tilde{h}_1, \tilde{h}_2, ..., \tilde{h}_\nu]. \]
\begin{align*}
\text{Opt} &= \inf_{g \in \mathbb{R}^m} \left[ \phi(g) := \max_{x \in \mathcal{X}} \phi(g, x) := [b - A^T g]^T x + \pi(g) \right] \quad (P) \\
&= \inf_{g \in \mathbb{R}^m} \left[ \pi(g) + \max_{x \in \mathcal{X}} [b - A^T g]^T x \right], \\
&= \max_{x \in \mathcal{X}} \left[ \phi(g) := \inf_{g \in \mathbb{R}^m} \phi(g, x) \right] \\
&= \max_{x \in \mathcal{X}} \left[ \inf_{g \in \mathbb{R}^m} \left[ b^T x - [Ax]^T g + \pi(g) \right] \right] \\
&= \max_{x} \left[ b^T x : x \in \mathcal{X}, \theta(Ax) \leq 1 \right] \quad (D)
\end{align*}

where \( \theta(\cdot) \) is the norm conjugate to \( \pi(\cdot) \).

**Claim:** When \( \bar{g} = \pi(\bar{g}) \bar{h}, \pi(\bar{h}) = 1 \), is an optimal solution (which does exist) to \((P)\), one has
\[
\max_{x} \{ |b^T x| : x \in \mathcal{X}, |\bar{h}^T Ax| \leq 1 \} \leq \text{Opt},
\]
and for any matrix \( G = [g^1, ..., g^M] \in \mathbb{R}^{m \times M} \) with \( \pi(g^j) \leq 1, 1 \leq j \leq M \), one has
\[
\max_{x} \{ |b^T x| : x \in \mathcal{X}, \|G^T Ax\|_\infty \leq 1 \} = \max_{x} \{ b^T x : x \in \mathcal{X}, \|G^T Ax\|_\infty \leq 1 \} \geq \text{Opt}.
\]

**Justifying Claim:** Let \( x \) be a feasible solution to the optimization problem in \((1)\). Replacing, if necessary, \( x \) with \( -x \), we can assume that \( |b^T x| = b^T x \). We now have
\[
|b^T x| = b^T x = \left[ g^T Ax - \pi(\bar{g}) \right] + \left[ b - A^T \bar{g} \right]^T x + \pi(\bar{g}) \leq \text{Opt} + [\pi(\bar{g}) \bar{h}^T Ax - \pi(\bar{g})] \leq \phi(\bar{g}) = \text{Opt}
\]
\[
\leq \text{Opt} + \pi(\bar{g}) |\bar{h}^T Ax| - \pi(\bar{g}) \leq \text{Opt},
\]
as claimed in \((1)\). The equality in \((2)\) is due to the symmetry of \( \mathcal{X} \) w.r.t. the origin. To verify the inequality in \((2)\), let \( \bar{x} \) be an optimal solution to \((D)\), so that \( \bar{x} \in \mathcal{X} \) and \( \theta(A \bar{x}) \leq 1 \), implying, due to the fact that the columns of \( G \) are of \( \pi(\cdot) \)-norm \( \leq 1 \), that \( \bar{x} \) is a feasible solution to the optimization problem in \((2)\). As a result, the second quantity in \((2)\) is at least \( b^T \bar{x} = \text{Opt} \), and \((2)\) follows. \( \square \)
Note: An attractive feature of the contrast design we have just developed is that it is completely independent of the entities participating in Assumptions A.1-2 – these entities affect theoretical risk bounds of the resulting polyhedral estimate, but not the estimate itself.

Near-optimality. Unfortunately, for the proposed polyhedral estimate no really general results on near-optimality are known.
However: There are important special cases where near-optimality can be justified, most notably, the **Simple diagonal case** (one of the typical cases considered in the traditional Nonparametric Statistics), where

- $\mathcal{X} = \{ x \in \mathbb{R}^n : \| Dx \|_\rho \leq 1 \}$, where $D = \text{Diag}\{ \ell^\delta, \ell = 1, 2, ..., n \}$,
- $\| \cdot \| = \| \cdot \|_r$ with $1 \leq \rho \leq r < \infty$,
- $m = n = \nu$, $A = \text{Diag}\{ \ell^{-\alpha}, \ell = 1, ..., n \}$, $B = \text{Diag}\{ \ell^{-\beta}, \ell = 1, ..., n \}$,
  with
  \[
  \beta \geq \alpha \geq 0, \; \delta \geq 0 \; & (\beta - \alpha) r < 1
  \]
- We are in Sub-Gaussian case: $\xi_x$ is $(0, \sigma^2 I_n)$-sub-Gaussian, $x \in \mathcal{X}$.

Assuming that $\sigma, \epsilon, n$ are in the range $0 < \sqrt{\ln(2n/\epsilon)} \sigma \leq 1$ and $n$ is large enough:

\[
n \geq c \vartheta_G^{-1/\alpha + \delta + 1/\rho} \quad [\vartheta_G = \sigma \sqrt{2 \ln(2n/\epsilon)}]
\]

(here and what follows $c$ and $C$ depend solely on $\alpha, \beta, \delta, r, \rho$) our design results in

\[
H = [\sigma \kappa]^{-1} I_n \text{ with } \kappa = \sqrt{2 \ln(2n/\epsilon)}
\]

\[
\text{Risk}_{\epsilon, \| \cdot \|_r}([\hat{x}_H \mid \mathcal{X}] \leq C [\sigma \kappa]^{\varphi}, \; \varphi = \frac{\beta + \delta + 1/\rho - 1/r}{\alpha + \delta + 1/\rho},
\]

while the minimax optimal $(\epsilon, \| \cdot \|_r)$-risk is $\geq c \sigma^{\varphi}$.

⇒ the risk of our polyhedral estimate is within logarithmic in $n/\epsilon$ factor of the minimax optimal risk.

5.109
\begin{align*}
\text{Risk}_{\epsilon, \|\cdot\|}[\hat{x}_H|X] & \leq \mathcal{R}[H] = \max_z \left\{ 2\|Bz\| : z \in \mathcal{X}_s, \|H^T A z\|_{\infty} \leq 1 \right\} \\
\text{B. How to upper-bound } \mathcal{R}[H] ? \quad \text{C. How to optimize } \mathcal{R}[H] \text{ over } H ?
\end{align*}

\textbf{Answering Questions B, C, Version II}

\begin{itemize}
\item Our second approach to B, C resembles what we did when building linear estimates – it is based on a kind of semidefinite relaxation
\end{itemize}

\textbf{Definition.} Given a nonempty convex compact set $\mathcal{Y} \in \mathbb{R}^N$, we say that $\mathcal{Y}$ is compatible with $\mathcal{Y}$, if $\mathcal{Y}$ is a closed convex cone contained in $S_+^N \times \mathbb{R}^+$ and such that

\begin{itemize}
\item $\forall (V, \tau) \in \mathcal{Y}$ : $\max_{y \in \mathcal{Y}} y^T V y \leq \tau$
\item relations $(V, \tau) \in \mathcal{Y}$ and $\tau' \geq \tau$ imply that $(V, \tau') \in \mathcal{Y}$
\item $\mathcal{Y}$ contains a pair $(V, \tau)$ with $V \succ 0$.
\end{itemize}

\item We say that a cone $\mathcal{Y}$ compatible with $\mathcal{Y}$ is \textit{sharp}, if the only pair $(V, 0) \in \mathcal{Y}$ is with $V = 0$.

\textbf{Fact:} When $\text{Lin}(\mathcal{Y}) = \mathbb{R}^N$, every cone compatible with $\mathcal{Y}$ is sharp

\textbf{Fact:} When $\mathcal{Y}$ is compatible with a shift of $\mathcal{Y}$, $\mathcal{Y}$ is compatible with $\mathcal{Y}_s = \frac{1}{2}[\mathcal{Y} - \mathcal{Y}]$

Indeed, $\mathcal{Y}_s$ remains intact when shifting $\mathcal{Y}$, so that we can assume that $\mathcal{Y}$ is compatible with $\mathcal{Y}$. When $(V \tau) \in \mathcal{Y}$ and $y, y' \in \mathcal{Y}$, we have

\begin{align*}
\frac{1}{4} (y - y')^T V (y - y') + \frac{1}{4} (y + y')^T V (y - y') &= \frac{1}{2} [y^T V y + (y')^T V y'] \leq \tau \Rightarrow \frac{1}{4} (y - y') V (y - y') \leq \tau, y, y' \in Y.
\end{align*}
The role of compatibility in our context stems from the following

**Observation:** Assume that we have at our disposal cones $X$ and $V$ compatible, respectively, with $X_s$ and with the unit ball $B_\ast = \{ v \in \mathbb{R}^\nu : \| u \|_\ast \leq 1 \}$ of the norm $\| \cdot \|_\ast$ conjugate to the norm $\| \cdot \|$ in which we measure the recovery error. Given contrast matrix $H$ satisfying

$$\text{Prob}_{\xi \sim P_x} \{ \| H^T \xi \|_\infty > 1 \} \leq \epsilon \ \forall x \in X$$

let

$$\text{Opt}(H) = \min_{\lambda, (U, \mu), (V, \tau)} \left\{ 4 \sum_j \lambda_j + 4 \mu + \tau : \lambda \in \mathbb{R}^J_+, (U, \mu) \in X, (V, \tau) \in V, \begin{bmatrix} V \\ \frac{1}{2} B^T \end{bmatrix} \begin{bmatrix} A^T H \text{Diag} \{ \lambda \} H^T A + U \end{bmatrix} \succeq 0 \right\}$$

The quantity $\text{Opt}(H)$ is an efficiently computable upper bound on the quantity

$$\mathcal{R} = \max_z \left\{ 2 \| Bz \| : z \in X_s, \| H^T Az \|_\infty \leq 1 \right\}$$

and thus – an upper bound on the $(\epsilon, \| \cdot \|)$-risk of the polyhedral estimate $\hat{x}_H(\cdot)$ on $X$.

When $X$ and $V$ are sharp, the optimization problem specifying $\text{Opt}$ is solvable.

5.111
Opt\((H) = \min_{\lambda, (U, \mu), (V, \tau)} \left\{ 4 \sum_j \lambda_j + 4 \mu + \tau : \begin{bmatrix} V & \frac{1}{2} B \\
\frac{1}{2} B^T & A^T H \text{Diag}\{\lambda\} H^T A + U \end{bmatrix} \succeq 0 \right\} \right. \\
\succeq 0 \right\} \right. \\
\succeq 0 \right\} \right. \\
\succeq 0 \right\} \right. \\
\succeq 0 \right\} \right.}

\[\mathcal{R} = \max_z \{2\|Bz\| : z \in \mathcal{X}_S, \|H^T A z\|_\infty \leq 1\}\]

**Claim:** \(\mathcal{R} \leq \text{Opt}(H)\)

**Proof.** Let \(\lambda, (U, \mu), (V, \tau)\) be a feasible solution to (*) and \(z\) be a feasible solution for the problem specifying \(\mathcal{R}\), so that \(z \in 2\mathcal{X}_S\) and \(\|H^T A z\|_\infty \leq 2\), and let \(u \in B_*\). By the semidefinite constraint in (*) we have

\[u^T B z \leq u^T V u + z^T A^T H \text{Diag}(\lambda) H^T A z + z^T U z \leq \tau + \sum_j \lambda_j ([h^j]^T A z)^2 + z^T U z \leq \tau + \sum_j \lambda_j + 4 \mu.\]

Taking supremum over \(u \in B_*\), we get \(\|Bz\| \leq \tau + 4 \sum_j \lambda_j + 4 \mu\) for every feasible solution \(z\) to the problem specifying \(\mathcal{R} \Rightarrow \mathcal{R} \leq \tau + 4 \sum_j \lambda_j + 4 \mu\). Since \(\lambda, (U, \mu), (V, \tau)\) is an arbitrary feasible solution to (*), we get \(\mathcal{R} \leq \text{Opt}(H)\). \(\square\)
\[ H \in \mathbb{R}^{m \times N} : \forall x \in \mathcal{X} : \text{Prob}_{\xi \sim P_x} \{ \| H^T \xi \|_\infty \leq 1 \} \geq 1 - \epsilon \quad (a) \]

\[ \begin{align*}
\text{Risk}_{\epsilon,\|\cdot\|} [\hat{x}|\mathcal{X}] & \leq \text{Opt}(H) \\
\hat{x}_H(\omega) &= B \text{ Argmin} \| H^T[Ax - \omega] \|_\infty \\
\text{Opt}(H) &= \min_{\lambda,(U,\mu),(V,\tau)} \left\{ 4 \sum_j \lambda_j + 4\mu + \tau : \left[ \begin{array}{c|c} V & \frac{1}{2}B \\ \frac{1}{2}B^T & A^T H \text{Diag}\{\lambda\} H^T A + U \end{array} \right] \succeq 0 \right\} \quad (*)
\end{align*} \]

[\mathcal{X} \text{ is compatible with } \mathcal{X}, \mathcal{V} \text{ is compatible with } \mathcal{B}_*]

What is ahead:

In sub-Gaussian/Discrete/Poisson o.s., to enforce (a) we impose on the columns \( h_j \) of \( H \) the restriction \( p(h_j) \leq 1 \), with adjusted to \( N, \epsilon \), and the o.s. norm \( p \), thus defining the set

\[ \mathcal{H} = \{ H = [h_1, \ldots, h_N] \in \mathbb{R}^{m \times n} : p(h_j) \leq \epsilon, j \leq N \} \]

of admissible \( m \times N \) contrast matrices. What matters for us is the conic set

\[ \mathcal{H}_* = \{(G, \mu) : \exists \lambda \geq 0, h_1, \ldots, h_N : G = \sum_j \lambda_j h_j h_j^T, p(h_j) \leq 1 \forall j, \sum_j \lambda_j \leq \mu \} \]

Even for simple norm \( p \), \( \mathcal{H}_* \) can be difficult to handle computationally.

I. How to build a tight inner approximation of this set by something appropriate for subsequent optimization in (*)?

II. How to build cones \( \mathcal{X}, \mathcal{V} \), the larger the better, compatible with \( \mathcal{X}, \mathcal{B}_* \)?
**Question:** Given a norm $p$ on $\mathbb{R}^m$ and positive integer $N$, how to build a tight inner approximation of the conic set
\[ H_* = \{(G, \mu) : \exists \lambda \geq 0, h_1, \ldots, h_N : G = \sum_j \lambda_j h_j h_j^T, p(h_j) \leq 1 \forall j, \sum_j \lambda_j \leq \mu\} \]
by something appropriate for subsequent optimization over this something?

**Fact:** The norm $p(\cdot)$ associated with sub-Gaussian/Siscrete/Polyhedral case is of special form:
\[ p^2(h) = \theta([h]^2), \quad \theta(u) = \max_{z \in Z} z^T u, \quad [[h_1; \ldots; h_m]]^2 = [h_1^2; h_2^2; \ldots; h_m^2], \quad (1) \]
where $Z$ is a convex compact subset of $\mathbb{R}^m_+$ with a nonempty interior.

**Assumption:** From now on we assume that $p(\cdot)$ is given by (1), and that $N \geq m$.

**Observation:** When the columns $h_j$ of an $m \times N$ matrix $H$ satisfy $p(h_j) \leq 1$, and $\lambda \geq 0$, $\mu$ satisfy $\sum_j \lambda_j \leq \mu$, we have
\[ \theta \left( \text{Dg}\{\sum_j \lambda_j h_j h_j^T\} \right) \leq \mu \quad (*) \]
where $\text{Dg}\{G\} \in \mathbb{R}^m$ is the diagonal of a matrix $G \in S^m$. 
Indeed, $\theta(\cdot)$ clearly is convex and homogeneous of degree 1, whence under the premise of Observation one has
\[ \theta(\text{Dg}\{\sum_i \lambda_j h_j h_j^T\}) = \theta(\sum_{j=1}^N \lambda_j [h_j]^2) \leq \sum_j \lambda_j \theta([h_j]^2) \leq \left[ \sum_j \lambda_j \right] \left[ \max_j p^2(h_j) \right] \leq \mu \]
Observation:

\[ H_* = \{(G, \mu) : \exists \lambda \geq 0, h_1, ..., h_N : G = \sum_j \lambda_j h_j h_j^T, p(h_j) \leq 1 \forall j, \sum_j \lambda_j \leq \mu \} \]

\[ \lambda \geq 0, \sum_j \lambda_j \leq \mu, p(h_j) \leq 1, G = \sum_j \lambda_j h_j h_j^T \Rightarrow G \succeq 0 & \theta(\mathcal{D}G\{g\}) \leq \mu \]

\[ \left[ p^2(h) = \theta([h]^2), \theta(u) = \max_{z \in \mathcal{Z} \subseteq \mathbb{R}_+^m} z^T u \right] \]

Fact: Observation can be “nearly inverted:” one has

\[ H := \{(G, \mu) : G \succeq 0, \kappa \theta(\mathcal{D}g\{G\}) \leq \mu \} \subset H_* \subset \{(G, \mu) : G \succeq 0, \theta(\mathcal{D}g\{G\}) \leq \mu \}, \]

where

— \( \kappa = 1 \) when \( p \) is proportional to \( \| \cdot \|_2 \), and

— \( \kappa = 4 \ln(4m^2) \) for a general norm \( p \) of the form

\[ p^2(u) = \theta(u) := \max_{z \in \mathcal{Z} \subseteq \mathbb{R}_+^m} z^T u. \]

Thus, \( H \) is a reasonably tight computationally tractable (provided \( \mathcal{Z} \) is so) inner approximation of \( H_* \).

5.115
**Claim:** Relations

\[ H_* = \{(G, \mu) : \exists \lambda \geq 0, h_1, \ldots, h_N : G = \sum_j \lambda_j h_j h_j^T, \ p(h_j) \leq 1 \ \forall j, \ \sum_j \lambda_j \leq \mu\} \]

\[ p^2(u) = \theta(u) := \max_{z \in Z \subset \mathbb{R}^m} z^T u \]

imply that

\[ H := \{(G, \mu) : G \succeq 0, \kappa \theta(\text{Diag}\{G\}) \leq \mu\} \subset H_* \subset \{(G, \mu) : G \succeq 0, \theta(\text{Diag}\{G\}) \leq \mu\} \quad (\ast) \]

**Proof.** The right inclusion in \((\ast)\) has been proved. Let us prove the left inclusion. By homogeneity it suffices to prove that when \(G \succeq 0\) satisfies \(\theta(\text{Diag}\{G\}) \leq 1\), we can represent \(G\) as

\[ G = \sum_j \lambda_j h_j h_j^T \]

with \(\lambda \geq 0\) satisfying

\[ \sum_j \lambda_j \leq \kappa. \]

**Case of** \(p(\cdot) = \alpha \|\cdot\|_2\): Here \(Z = \{[\alpha^2; \ldots; \alpha^2]\}\), \(\theta(u) = \alpha^2 \sum_j u_j\), and on the close inspection we should prove that when \(G \succeq 0\) and \(\text{Tr}(G) \leq 1\), we have \(G = \sum_j \lambda_j h_j h_j^T\), with \(\lambda \geq 0, \sum_j \lambda_j = 1\), and \(\|h_j\|_2 \leq 1\) for all \(j\) – the fact readily given by eigenvalue decomposition of \(G\).

**General case:** Since \(G \succeq 0\), we have \(G = Q^2\) with some \(Q \in S^m\). Setting \(\sigma_i = G_{ii}\), we have

\[ 1 \geq \theta(\sigma) \& \sum_j Q_{ij}^2 = \sigma_i. \]

5.116
\[ G, Q \in S^m \& G = Q^2 \& \sum_j Q_{ij}^2 = \sigma_i \text{ with } \theta(\sigma) \leq 1 \]

• Let \( U \) be \( m \times m \) orthonormal matrix with magnitudes of entries not exceeding \( \gamma = \sqrt{2/m} \) (matrices of this type do exist). For a random Rademacher vector \( \xi \), setting \( Q\xi = Q\text{Diag}\{\xi\}U \), we get

\[ Q\xi Q^T \equiv G. \]

On the other hand, \( [Q\xi]_{ij} = \sum_{\ell=1}^m Q_{i\ell} \xi_{\ell} U_{\ell j} \), whence

\[ \mathbb{E}_\xi \{ [Q\xi]_{ij}^2 \} = \sum_{\ell=1}^m Q_{i\ell}^2 U_{\ell j}^2 \leq (2/m) \sum_{\ell=1}^m Q_{i\ell}^2 = 2\sigma_i/m. \]

It is easily seen that when \( \gamma \geq 1 \), we have for every \( i, j \):

\[ \text{Prob} \{ [Q\xi]_{ij}^2 > 2\gamma\sigma_i/m \} \leq 2 \exp\{-\gamma/2\}. \]

⇒ Setting \( \gamma = 2 \ln(4m^2) = \kappa/2 \), the probability for \( \xi \) to ensure \( [Q\xi]_{ij}^2 \leq 2\gamma\sigma_i/m \) for all \( i, j \) is at least 1/2

⇒ \( \exists Q\tilde{\xi} = [q_1, \ldots, q_m] \): \( G = Q\tilde{\xi} Q^T = \sum_j q_j q_j^T \) and \( [q_j]^2 \leq 2\gamma/m \sigma \Rightarrow p^2(q_j) \leq 2\gamma m \theta(\sigma) \leq 2\gamma \)

⇒ \( G = \sum_j \lambda_j h_j h_j^T \) with \( h_j = \sqrt{\frac{m}{2\gamma}} q_j \) and \( \lambda_j = \frac{2\gamma}{m} \)

⇒ \( G = \sum_j \lambda_j h_j h_j^T \) with \( p(h_j) \leq 1 \) and \( \lambda \geq 0, \sum_j \lambda_j \leq 2\gamma = \kappa. \) □
Compatibility of closed convex cone $\mathcal{Y} = \{(U, \tau)\} \subset S^N_{++} \times \mathbb{R}^+$ with convex compact set $\mathcal{Y} \subset \mathbb{R}^N$ means that

- $y^T U y \leq \tau \quad \forall (y \in \mathcal{Y}, (U, \tau) \in \mathcal{Y})$
- $\exists (\bar{U}, \bar{\tau}) \in \mathcal{Y}: \bar{U} \succ 0$
- $(U, \tau) \in \mathcal{Y}, \tau' \geq \tau \Rightarrow (U, \tau') \in \mathcal{Y}$.

How to build cone $U$, the wider the better, compatible with a given convex compact set $\mathcal{Y}$?

♠ We know two sources of cones compatible with $\mathcal{Y}$:
- cones coming from semidefinite relaxation on ellitopes/spectratopes
- cones coming from absolute norms.
Compatibility via ellitopes/spectratopes

Fact: Let $\mathcal{Y}$ be a convex compact subset of an ellitope:

$$\mathcal{Y} \subset \mathcal{Z} = \{z \in \mathbb{R}^N : \exists (t \in \mathcal{T}, x) : z = Pz, x^T S_k \leq q, k \leq K\}$$

$$[S_k \succeq 0, \sum_k S_k > 0]$$

Then the cone

$$\mathcal{Y} = \{(U, \tau) \in S^N_+ \times \mathbb{R}_+ : \exists \lambda \geq 0 : P^T U P \leq \sum_k \lambda_k S_k, \phi_T(\lambda) := \max_{t \in \mathcal{T}} t^T \lambda \leq \tau\}$$

is compatible with $\mathcal{Y}$.

When $\mathcal{Y}$ is a subset of spectratope:

$$\mathcal{Y} \subset \mathcal{Z} = \{z \in \mathbb{R}^N : \exists (t \in \mathcal{T}, x) : S_k^2[x] \leq t_k I_{d_k}, k \leq K\},$$

$$[S_k[x] = \sum_{j=1}^n x_j S^{kj}, S^{kj} \in S^{d_k}]$$

the cone

$$\mathcal{Y} = \{(U, \tau) \in S^N_+ \times \mathbb{R}_+ : \exists \Lambda_k \succeq 0 : P^T U P \leq \sum_k S_k^*[\Lambda_k], \phi_T(\lambda[\Lambda]) \leq \tau\}$$

$$[[S_k[\Lambda_k]]_{ij} = \text{Tr}(S^{ki} \Lambda_k S^{kj}), [\Lambda[\Lambda]]_k = \text{Tr}(\Lambda_k)]$$

is compatible with $\mathcal{Y}$.

This is readily given by what we know on semidefinite relaxation on ellitopes/spectratopes.

5.119
Compatibility via absolute norms

♠ Preliminaries: absolute norms. A norm \( \| \cdot \| \) on \( \mathbb{R}^N \) is called absolute, if it depends solely on the magnitudes of entries of a vector:

\[
\|z\| = \|\text{abs}[z]\|, \quad \text{abs}[[z_1; \ldots; z_N]] = [|z_1|; \ldots; |z_N|].
\]

Examples: The \( \ell_s \) norms \( \| \cdot \|_s \), are absolute; similarly, the block \( \ell_s \)-norm

\[
\|[z^1; \ldots; z^K]\| = \|[\|z^1\|_s^1; \|z^2\|_s^2; \ldots; \|z^K\|_s^K]\|_s \quad [s, s_1, \ldots, s_K \in [1.\infty]]
\]
is absolute.

Facts:

• An absolute norm \( \| \cdot \| \) is monotone in the magnitudes of entries: if \( \text{abs}[z] \leq \text{abs}[z'] \), then \( \|z\| \leq \|z'\| \).
• The norm \( \|y\|_* = \max_{x: \|x\| \leq 1} y^T x \) conjugate to an absolute norm \( \| \cdot \| \) is absolute as well.
**Observation:** An absolute norm $p(\cdot)$ on $\mathbb{R}^N$ can be “lifted” to an absolute norm $p^+(\cdot)$ on $S^N$ by setting

$$p^+(X) = p([\text{Col}_1[X]; \text{Col}_2[X]; \ldots; \text{Col}_N[X]]) , \ X \in S^N.$$

$p^+$ indeed is an absolute norm, and

$$p^+(xx^T) = p^2(x) \ \forall x \in \mathbb{R}^N.$$

**Example:** When $p(\cdot)$ is $\ell_\pi$-norm on $\mathbb{R}^N$, $p^+(\cdot)$ is $\ell_\pi$ norm on $S^N$. 

5.121
We say that an absolute norm $r(\cdot)$ fits an absolute norm $p(\cdot)$ on $\mathbb{R}^N$,

$$p(x) \leq 1 \Rightarrow r([x]^2) \leq 1.$$  

**Example:** When $p(\cdot) = \| \cdot \|_s$, $s \in [1,\infty]$, the norm

$$r(\cdot) = \begin{cases} \| \cdot \|_1, & 1 \leq s \leq 2 \\ \| \cdot \|_{s/2}, & s \geq 2 \end{cases}$$

fits $p(\cdot)$.

**Fact:** Let $p$ be an absolute norm on $\mathbb{R}^N$, let absolute norm $r(\cdot)$ fit $p(\cdot)$, and let

$$\mathcal{Y} \subset B_p := \{x \in \mathbb{R}^N : p(x) \leq 1\}.$$  

Then the set

$$\mathcal{Y} = \left\{(U, \tau) \in S^N_+ \times \mathbb{R}_+ : \exists W \in S^n, w \in \mathbb{R}^N_+ : U \preceq W + \text{Diag}\{w\} \quad \|W\|_{p^{**}} + r_*(w) \leq \tau \right\}$$

where $p^{**}$ is the norm on $S^N_+$ conjugate to $p^+$, and $r_*(\cdot)$ is the norm on $\mathbb{R}^N$ conjugate to $r(\cdot)$, is compatible with $\mathcal{Y}$.

Besides this, $p^{**}(\cdot) \leq q^+(\cdot)$, where $q(\cdot)$ is the norm conjugate to $p(\cdot)$.
Fact: Let $p$ be an absolute norm on $\mathbb{R}^N$, let absolute norm $r(\cdot)$ fit $p(\cdot)$, and let 

$$
\mathcal{Y} \subset B_p := \{ x \in \mathbb{R}^N : p(x) \leq 1 \}.
$$

Then the set

$$
\mathcal{Y} = \left\{ (U, \tau) \in S^N_+ \times \mathbb{R}_+ : \exists W \in S^n, w \in \mathbb{R}^N_+ : U \leq W + \text{Diag}\{w\} \right\}
$$

where $p^{++}$ is the norm on $S^N$ conjugate to $p^+$, and $r^*(\cdot)$ is the norm on $\mathbb{R}^N$ conjugate to $r(\cdot)$, is compatible with $\mathcal{Y}$. Besides this, $p^{++}(\cdot) \leq q^+(\cdot)$, where $q(\cdot)$ is the norm conjugate to $p(\cdot)$.

Indeed, let $(U, \tau) \in \mathcal{Y}$, so that $U \leq W + \text{Diag}\{w\}$ with $w \geq 0$ and $\|W\|_{p^{++}} + r^*(w) \leq \tau$. For $y \in \mathcal{Y}$ we have $p(y) \leq 1$ due to $\mathcal{Y} \subset B_p$, whence

$$
y^T U y = \text{Tr}(U[yy^T]) \leq \text{Tr}(W[yy^T]) + \text{Tr}(\text{Diag}\{w\}yy^T) \leq p^{++}(W)p^+(yy^T) + w^T[y^2]^2
$$

$$
\leq p^{++}(W)p^2(y) + r^*(w) \sqrt{[y^2]} \leq p^{++}(W) + r^*(w) \leq \tau
$$

$$
\Rightarrow \max_{y \in \mathcal{Y}} y^T U y \leq \tau.
$$

Besides this, when $U, V \in S^n$, denoting $U_j$ and $V_j$ the columns of $U$ and $V$, we have

$$
\text{Tr}(UV) = \sum_j U_j^T V_j \leq \sum_j p(U_j)q(V_j) \leq [p(U_1); \ldots; p(U_N)]^T[q(V_1); \ldots; q(V_N)]
$$

$$
\leq p([p(U_1); \ldots; p(U_N)])q([q(V_1); \ldots; q(V_N)]) = p^+(U)q^+(V)
$$

$$
\Rightarrow \|V\|_{p^{++}} = \max_{U : p^+(U) \leq 1} \text{Tr}(UV) \leq q^+(V).
$$

5.123
Example: Let \( p(\cdot) = \| \cdot \|_s \) with \( s \in [1, \infty] \). In this case
— we can take \( r(x) = \|x\|_{\bar{s}}, \bar{s} = \max[s/2, 1] \), resulting in
\[
r_*(w) = \|w\|_{\bar{s}_*}, \bar{s}_* = \frac{s}{s-1} = \begin{cases} +\infty, & 1 \leq s \leq 2 \\ \frac{s}{s-2}, & s > 2 \end{cases}
\]
— \( p^{+*}(\cdot) \) is \( \| \cdot \|_{s_*} \) on \( S^N \), \( s_* = \frac{s}{s-1} \)
and we conclude that the cone
\[
Y_s = \{(U, \tau) \in S^N_+ \times \mathbb{R}_+ : \exists (w \geq 0, W) : U \preceq W + \text{Diag}\{w\}, \|W\|_{s_*} + \|w\|_{\bar{s}_*} \leq \tau \}
\]
is compatible with any subset of the unit \( \ell_s \) ball.
Note: It is easily seen that when \( s \in [2, \infty] \), the expression for \( Y \) provably simplifies to
\[
Y_s = \left\{(U, \tau) \in S^N_+ \times \mathbb{R}_+ : \exists (w \geq 0) : U \preceq \text{Diag}\{w\}, \|w\|_{\frac{s}{s-2}} \leq \tau \right\}
\]
In the case in question \( Y_s \) is an ellitope, and \( Y_s \) happens to be exactly the cone compatible with this ellitope, as given by our “ellitopic” construction.
Note: In our context, the larger is a cone compatible with the set \( \mathcal{Y} \) in question (for us, this is either \( \mathcal{X}_S \), or \( \mathcal{B}_* \)), the better. The “ideal” choice would be
\[
Y = Y_*[\mathcal{Y}] = \{(U, \tau) : U \succeq 0, \tau \geq \max_{y \in \mathcal{Y}} y^T U y \}.
\]
This ideal cone is typically intractable computationally, this is why we have developed techniques for building tractable approximations of this cone from inside.
However: When \( \mathcal{Y} = \{y \in \mathbb{R}^N : \|y\|_2 \leq 1\} \), the cone
\[
Y_2 = \{(U, \tau) : 0 \preceq U \preceq \tau I_N \}
\]
is exactly the same as the “ideal” cone \( Y_*[\mathcal{Y}] \).
Assume that
- the o.s. is Gaussian: $\omega = Ax + \sigma \xi$, $\xi \sim \mathcal{N}(0, I_m)$
- the signal set $\mathcal{X}$ and the unit ball $B_*$ of the norm conjugate to the one used to measure the recovery error are ellitopes:

\[
\mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k, k \leq K \}
\]
\[
B_* = \{ u \in \mathbb{R}^m : \exists (r \in \mathcal{R}, z) : u = Mz, z^T R_\ell z \leq r_\ell, \ell \leq L \}
\]

In this case, our compatibility-based recipe for building presumably good polyhedral estimate combines with the machinery for building cones compatible with ellitopes to result in the polyhedral estimate $\hat{x}_H$ yielded by the optimal solution to the convex optimization problem

\[
\text{Opt} = \min_{\Theta, U, \lambda, \mu} \left\{ 2 \left[ \phi_T(\lambda) + \phi_R(\mu) + \kappa^2 \sigma^2 \text{Tr}(\Theta) \right] : \begin{bmatrix} \Theta \succeq 0, U \succeq 0, \lambda \geq 0, \mu \geq 0, \\ \frac{1}{2} B^T A^T \Theta A + \sum_k \lambda_k S_k \\ M^T U M \preceq \sum_\ell \mu_\ell R_\ell \end{bmatrix} \succeq 0, \right\}
\]

\[
\kappa = \sqrt{2 \ln(2m/\epsilon)}, \quad \phi_Z(\nu) = \max_{z \in Z} \nu^T z.
\]

The $m \times m$ contrast matrix $H$ is given by the $\Theta$-component $\Theta_*$ of an optimal solution to the problem: the columns $h_j$ of $H$ are the eigenvectors of $\Theta_*$ normalized to satisfy $\|h_j\|_2 = (\kappa \sigma)^{-1}$, and

\[
\text{Risk}_{\epsilon, \| \cdot \|}[\hat{x}_H|\mathcal{X}] \leq \text{Opt}.
\]

**Proposition:** Assume that $\epsilon \leq 1/8$. Then the resulting estimate is near-optimal:

\[
\text{Opt} \leq O(1) \kappa \sqrt{\ln(2K) \ln(2L) \text{RiskOpt}_\epsilon \frac{1}{8}} \leq O(1) \kappa \sqrt{\ln(2K) \ln(2L) \text{RiskOpt}_\epsilon},
\]

where $\text{RiskOpt}_\epsilon$ is the infimum, over all possible estimates, of $(\epsilon, \| \cdot \|)$-risks of the estimates on $\mathcal{X}$.

**Note:** Similar result holds true in the case when $\mathcal{X}$ and $B_*$ are spectratopes.
How It Works

♠ Setup:
• Unknown signal $x$ is restriction of function $h(t)$ of continuous time on the $n$-element equidistant grid on $[0, 4]$, with the magnitude of $h$ known to be $\leq 1$
• We want to recover the result of “numerical double-integration” of $h$ – the vector $Bx$ with
$$B_{ij} = \begin{cases} \frac{16}{n^2}[i - j + 1] & , i \geq j \\ 0 & , i > j \end{cases}$$
• We observe in Gaussian noise $\mathcal{N}(0, \sigma^2 I_m)$ the restriction of $x$ onto $m$ randomly selected points of the grid; this selection specifies $A$.
• The recovery error is measured in $\| \cdot \|_2$.
♠ We are in the case when the signal set $\mathcal{X}$ is the unit box:
$$\mathcal{X} = \{ x \in \mathbb{R}^n : x_i^2 \leq 1, 1 \leq i \leq n \}$$

Note that our $\mathcal{X}$ and $B_*$ are ellitopes, so that we can build efficiently
— the provably near-optimal linear estimate $\text{Lin}$,
— the polyhedral estimate $\text{PolyI}$,
— the provably near-optimal polyhedral estimate $\text{PolyII}$,

with $\text{PolyI}, \text{PolyII}$ yielded by the first, resp. the second of our techniques for designing polyhedral estimates.
♠ In the experiments to be reported, $n = 64$, $m = 32$, and $\epsilon = 0.1$. 

5.126
Recovery errors for Lin (left column), PolyI (right column), and PolyII (middle column)
Horizontal lines: solid — upper bound on Risk_{0.1,\|\cdot\|_2} of PolyII
dotted — upper bound on Risk_{\|\cdot\|_2} of Lin.
Data over 20 simulations per each value of $\sigma$
How It Works (continued)
Denoising and Deblurring Images

Let us look how Polyhedral Estimate works when recovering images \( x \in \mathbb{R}^{m \times n} \) from their blurred noisy observations

\[
\omega = \kappa \star x + \xi
\]

Same as with linear estimates, we pass to frequency domain, where the observation become

\[
\zeta = \theta \cdot \chi + \eta
\]

[\( \chi \): DFT of \( x \); \( \theta \): DFT of \( \kappa \); \( \eta \): complex-valued white Gaussian noise]

and a priori information on \( \chi \) reduces to a small number of (empirically identified) simple constraints of the form

\[
\sum_{r,s} \alpha_{rs} |\chi_{rs}| \leq \alpha, \quad \sqrt{\sum_{r,s} \beta_{rs} |\chi_{rs}|^2} \leq \beta, \quad 0 \leq |\chi_{rs}| \leq \gamma_{rs} \forall r, s
\]

By both theoretical and computational reasons, we use the simplest possible – just the unit – contrast matrix, resulting in extremely simple (nothing more than Bisection!) recovery routine

\[
\hat{\chi} = \text{argmin}_{\chi} \left\{ \max_{r,s} |\zeta_{rs} - \theta_{rs} \chi_{rs}| : \sum_{r,s} \alpha_{rs}^k |\chi_{rs}| \leq \alpha^k, \quad \sqrt{\sum_{r,s} \beta_{rs}^k |\chi_{rs}|^2} \leq \beta^k, \quad k \leq K, \quad |\chi_{rs}| \leq \gamma_{rs} \forall r, s \right\}
\]
True image

Observation, $\sigma = 1.280$

Recovery

True image

Observation, $\sigma = 0.128$

Recovery
Observation, $\sigma = 1.280$

Observation, $\sigma = 0.128$
True image | Observation, $\sigma = 1.280$ | Recovery
---|---|---
True image | Observation, $\sigma = 0.128$ | Recovery

5.131
Observation, $\sigma = 1.280$

Observation, $\sigma = 0.128$
Observation, $\sigma = 1.280$

Observation, $\sigma = 0.128$
True image | Observation, $\sigma = 1.280$ | Recovery
---|---|---

True image | Observation, $\sigma = 0.128$ | Recovery
---|---|---

5.134
Observation, $\sigma = 1.280$

Recovery

Observation, $\sigma = 0.128$

Recovery
Observation, $\sigma = 1.280$

Observation, $\sigma = 0.128$

Recovery

Recovery
Observation, $\sigma = 1.280$

Observation, $\sigma = 0.128$

Recovery

Recovery
True image | Observation, $\sigma = 1.280$ | Recovery
---|---|---
True image | Observation, $\sigma = 0.128$ | Recovery

5.138
True image

Observation, $\sigma = 1.280$

Recovery

True image

Observation, $\sigma = 0.128$

Recovery

5.139
True image  |
| Observation, $\sigma = 1.280$  |
| Recovery  |

True image  |
| Observation, $\sigma = 0.128$  |
| Recovery  |
Observation, $\sigma = 1.280$

Observation, $\sigma = 0.128$

Recovery
<table>
<thead>
<tr>
<th>True Image</th>
<th>Observation, $\sigma = 1.280$</th>
<th>Recovery</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="True Image" /></td>
<td><img src="image2" alt="Observation" /></td>
<td><img src="image3" alt="Recovery" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>True Image</th>
<th>Observation, $\sigma = 0.128$</th>
<th>Recovery</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image4" alt="True Image" /></td>
<td><img src="image5" alt="Observation" /></td>
<td><img src="image6" alt="Recovery" /></td>
</tr>
</tbody>
</table>

5.142
True image

Observation, $\sigma = 1.280$

Recovery

True image

Observation, $\sigma = 0.128$

Recovery

5.143
Observation, $\sigma = 1.280$

Observation, $\sigma = 0.128$

Recovery

Recovery
Observation, $\sigma = 1.280$

Observation, $\sigma = 0.128$
ESTIMATING SIGNALS IN MONOTONE GENERALIZED LINEAR MODELS

- Generalized Linear Model
- Developing tools
  - Variational inequalities with monotone operators
- Sample Average Approximation estimate
- Stochastic Approximation estimate
What the story is about

♣ Ultimate Goal: To recover unknown signal $x \in \mathbb{R}^n$ from observations

$$\omega^K = (\omega_1, ..., \omega_K)$$

given by

**Generalized Linear Model:** $\omega_k = (y_k, \eta_k)$, where

— $\omega_k$, $k = 1, ..., K$, are i.i.d.

— the common distribution $P$ of regressors $\eta_k$ is independent of signal $x$

— the joint distribution of label $y_k \in \mathbb{R}^m$ and regressor $\eta_k \in \mathbb{R}^{n \times m}$ depends solely on signal $x$, and

$$E_{\mid \eta_k} \{ y_k \} = \psi(\eta_k^T x)$$

$\psi(\cdot): \mathbb{R}^m \rightarrow \mathbb{R}^m$: known link function

$E_{\mid \eta_k} \{ \cdot \}$: conditional, given $\eta_k$, expectation over $y_k$

• We assume that a priori information on signal $x$ reduces to $x \in \mathcal{X}$, for a given convex compact set $\mathcal{X} \subset \mathbb{R}^n$.  

6.1
\{(y_k, \eta_k)\}_{k \leq K} \text{ i.i.d., } E_{\eta_k}\{y_k\} = \psi(\eta_k^T x), x \in \mathcal{X} \Rightarrow ?? \ x

**Examples of GLM’s:**

**Linear model:** \(\psi(s) \equiv s\). *Assuming additive signal- and regressor-independent noise*, the problem becomes to recover signal \(x\) from observations \((y_k, \eta_k), k \leq K\), where regressors \(\eta_k\) are i.i.d. with independent of \(x\) distribution,

\[y_k = \eta_k^T x + \xi_k,\]

and \(\xi_k, k \leq K\), are independent of \(\eta_k\) i.i.d. zero mean observation noises.

♣ *Linear model admits “special treatment” which was our previous subject.*
Logit model (Logistic regression): $m = 1$, $\psi(s) = \exp\{s\}/(1 + \exp\{s\})$, $\eta_k \in \mathbb{R}^n$, $1 \leq k \leq K$, are i.i.d. Given $\eta_k$, $y_k$ takes value 1 with probability $\psi(\eta_k^T x)$ and value 0 with complementary probability.

Probit model: Exactly as Logistic Regression, but with the cdf of the standard Gaussian distribution in the role of link: $\psi(s) = \Phi(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} \exp\{-t^2/2\} dt$.

Both Logit and Probit models are widely used in Regression Analysis with binary dependent variables.
Signal Recovery in GLM

The standard signal recovery in GLM model is given by Maximum Likelihood (ML).

Assuming the conditional, signal $x$ and regressor $\eta$ given, distribution of the label $y$ to have density $p(y, \eta^T x)$ w.r.t. some reference measure, the conditional by the sequence of regressors log-likelihood of the sequence of labels as a function of candidate signal $z$ is $\sum_{k=1}^{K} \ln(p(y_k, \eta_k^T z))$. The ML estimate $\hat{x}$ of the signal underlying observations is obtained by maximizing log-likelihood in $z \in \mathcal{X}$.

- In **Linear model** with Gaussian noise the ML estimate is given by Least Squares:

  $$\hat{x} \in \text{Argmin}_{z \in \mathcal{X}} \sum_{k=1}^{K} \|y_k - \eta_k^T z\|_2^2$$

- In **Logit model** the ML estimate is

  $$\hat{x} \in \text{Argmin}_{z \in \mathcal{X}} \sum_{k=1}^{K} \left[ \ln \left( 1 + \exp\{\eta_k^T z\} \right) - y_k \eta_k^T z \right]$$

- In **Probit model** the ML estimate is

  $$\hat{x} \in \text{Argmin}_{z \in \mathcal{X}} \sum_{k=1}^{K} \left[ -\ln \left( 1 - \Phi(\eta_k^T z) \right) - y_k \ln \left( \frac{\Phi(\eta_k^T z)}{1 - \Phi(\eta_k^T z)} \right) \right]$$

In all these cases likelihood maximization (which we convert to minimizing minus log-likelihood) happens to be convex, and thus efficiently solvable, problem.
**However:** Minimizing minus log-likelihood in GLM can be a *nonconvex* problem. For example, this happens when the link function $\psi(s) = \exp\{s\}/(1 + \exp\{s\})$ in Logit model is replaced with $\psi(s) = \frac{1}{2} + \frac{1}{\pi}\tan(s)$:
Common wisdom is to recover $x$ by minimizing minus log-likelihood by Newton method and to hope for the better.

*With non-concave log-likelihood, this approach can fail...*

**Question:** *Can we do better?*

**Answer:** Yes! Under monotonicity assumption on the link function, there exists an alternative to Maximum Likelihood computationally efficient signal recovery with provably reasonably good performance.

♠ **Monotonicity assumption**, in nutshell, requires from $\psi(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ to be monotone:

$$\langle \psi(s) - \psi(s'), s - s' \rangle \geq 0 \ \forall s, s' \in \mathbb{R}^m$$

**Motivation:** Recovering signal $x$ from noisy observations hardly can be easier than recovering $w = \eta^T x$ from noiseless observation

$$y = \psi(w). \quad (\ast)$$

*Monotonicity of $\psi$ is, basically, the weakest general-type structural assumption which ensures computational tractability of the square system of nonlinear equations $(\ast)$.  

6.6
Executive Summary on Variational Inequalities with Monotone Operators

**Definition:** Let $X \subset \mathbb{R}^N$ be a closed convex set and $G : X \to \mathbb{R}^N$ be a vector field. $G$ is called monotone on $X$, if

$$\langle G(y) - G(y'), y - y' \rangle \geq 0 \ \forall y, y' \in X. \quad (\star)$$

If $(\star)$ can be strengthened to

$$\langle G(y) - G(y'), y - y' \rangle \geq \alpha \| y - y' \|^2 \ \forall y, y' \in X, \quad [\alpha > 0]$$

$G$ is called strongly monotone, with modulus $\alpha$, on $X$.

**Examples:**

A. Univariate ($N = 1$) monotone vector fields on closed convex subset $X$ of $\mathbb{R}$ are exactly non-decreasing real-valued functions on $X$.

B. If $f : X \to \mathbb{R}$ is convex differentiable on $X$, the gradient field $G(x) = \nabla f(x)$ of $f$ is monotone on $X$. The same holds true for (any) subgradient field of convex function $f : X \to \mathbb{R}$, provided that subdifferential of $f$ at every point $x \in X$ is nonempty.

C. Let $X = U \times V$, and $f(u, v)$ be differentiable on $X$ convex in $u \in U$ and concave in $v \in V$ function. Then the vector field

$$G(u, v) = [\nabla_u f(u, v); -\nabla_v f(u, v)]$$

is monotone on $X$. The same holds true when smoothness of $f$ is weakened to Lipschitz continuity, and $\nabla_u, \nabla_v$ are replaced with respective partial sub- and supergradients.

6.7
**Fact:** Let $G : X \to \mathbb{R}^N$ be continuously differentiable vector field on a closed convex subset $X$, int $X \neq \emptyset$, of $\mathbb{R}^N$. $G$ is monotone on $X$ iff the symmeterized Jacobian

$$J_s[G](x) := \frac{1}{2} \left[ \frac{\partial G(x)}{\partial x} + \left[ \frac{\partial G(x)}{\partial x} \right]^T \right]$$

is positive semidefinite for all $x \in X$. $G$ is strongly monotone with modulus $\alpha > 0$ on $X$ iff $J_s[G](x) \succeq \alpha I_N$, $x \in X$.

**Variational Inequality** $\text{VI}(G, X)$ associated with closed convex set $X$ and a monotone on $X$ vector field $G$ reads

$$\text{find } z_* \in X : \langle G(z), z - z_* \rangle \geq 0 \ \forall z \in X \quad \text{(*)}$$

Vectors $z_* \in X$ satisfying (\text{*}) are called weak solutions to $\text{VI}(G, X)$. A strong solution to $\text{VI}(G, X)$ is a point $z_* \in X$ such that

$$\langle G(z_*), z - z_* \rangle \geq 0 \ \forall z \in X.$$  

- A strong solution is a weak one, since by monotonicity $\langle G(z), z - z_* \rangle \geq \langle G(z_*), z - z_* \rangle$, $z, z_* \in X$. The inverse is true provided that $G$ is continuous on $X$. 

6.8
\[ X \subset \mathbb{R}^m: \text{closed and convex} \quad G : X \to \mathbb{R}^m: \text{monotone on } X \]

Weak solution to \(\text{VI}(G, X)\): \(z_\ast \in X\) such that \(\langle G(z), z - z_\ast \rangle \geq 0 \ \forall z \in X\)

Strong solution to \(\text{VI}(G, X)\): \(z_\ast \in X\) such that \(\langle G(z_\ast), z - z_\ast \rangle \geq 0 \ \forall z \in X\)

**Facts:**

- Weak solutions to \(\text{VI}(G, X)\) form a closed convex subset of \(X\); this set is nonempty, *provided \(X\) is bounded*.
- When \(G\) is a subgradient field of continuous convex function \(f : X \to \mathbb{R}\), weak solutions to \(\text{VI}(G, X)\) are exactly the minimizers of \(f\) on \(X\). More generally, when \(G\) is the monotone vector field associated with continuous convex-concave \(f(u, v) : X = U \times V \to \mathbb{R}\), the weak solutions to \(\text{VI}(G, X)\) are exactly the saddle points of \(f\) on \(U \times V\).
Fact: Approximating weak solutions to Monotone Variational Inequalities is computationally tractable task – all basic algorithms of convex minimization admit “VI versions.”

Let us define inaccuracy $\text{Res}(x|G, X)$ of a candidate solution $z \in X$ to the VI

$$\text{find } z_* \in X : \langle G(z), z - z_* \rangle \geq 0 \ \forall z \in X$$

as

$$\text{Res}(z|G, X) = \sup_{y \in X} \langle G(y), z - y \rangle,$$

so that $\text{Res}(z|G, X) \geq 0$ and $\text{Res}(z|G, X) = 0$ iff $z$ is a weak solution to $\text{VI}(G, X)$.
Fact: Approximating weak solutions to Monotone Variational Inequalities is computationally tractable task – all basic algorithms of convex minimization admit “VI versions”

For example, assuming that

— $X$ is closed convex set contained in a given $\| \cdot \|_2$-ball of radius $R$ and containing ball of a given radius $r > 0$,

— $G$ is monotone on $X$ and $\|G(x)\|_2 \leq V$, $x \in X$, for some known $V$,

for every $\epsilon \in (0, VR)$, a solution $z \in X$ with $\text{Res}(z | G, X \leq \epsilon)$ can be found

• by Ellipsoid method – in $O(1)N^2 \ln \left( \frac{NVR}{\epsilon} \cdot \frac{R}{r} + 1 \right)$ iterations, with the computational effort per iteration dominated by the necessity
  (a) to check whether a point belongs to $X$, and if not - to separate the point from $X$ by a linear form,
  (b) to compute the value of $G$ at a point of $X$, and
  (c) to perform, on the top of (a), (b), $O(N^2)$ additional arithmetic operations

• by Subgradient Descent – in $O(1)\frac{V^2R^2}{\epsilon^2}$ iterations, with computational effort per iteration dominated by the necessity to compute metric projection of a point onto $X$ and the value of $G$ at a point;

• by Mirror Prox – in $O(1)\frac{LR^2}{\epsilon}$ iterations, provided $G$ is Lipschits continuous, with constant $L$, on $X$, with the same iteration complexity as for Subgradient Descent.
Strongly Monotone Variational Inequalities

**Fact:** Let $G$ be strongly monotone, with modulus $\alpha > 0$, on convex compact set $X$. Then the weak solution $z_\ast$ to $\text{VI}(G, X)$ is unique, and for every $z \in X$ it holds

\[
(a) \quad \alpha \|z - z_\ast\|^2 \leq \langle G(z), z - z_\ast \rangle \\
(b) \quad \alpha \|z - z_\ast\|^2 \leq 4 \text{Res}(z|G, X)
\]

Indeed, setting $z_t = z_\ast + t(z - z_\ast)$, for $0 < t < 1$ we have

\[
\langle G(z), z - z_t \rangle \geq \alpha \|z - z_t\|^2_t + \langle G(z_t), z - z_t \rangle
\]

by strong monotonicity, and $\langle G(z_t), z - z_t \rangle = \frac{1-t}{t} \langle G(z_t), z_t - z_\ast \rangle \geq 0$.

$\Rightarrow$ $\langle G(z), z - z_t \rangle \geq \alpha \|z - z_t\|^2_t \forall t \in (0, 1)$

$\Rightarrow [t \to +0]$ $(a)$.

Next, by $(a)$ applied to $z_\frac{1}{2}$ in the role of $z$, $\langle G(z_\frac{1}{2}), z_\frac{1}{2} - z_\ast \rangle \geq \frac{\alpha}{4} \|z - z_\ast\|^2_2$

$\Rightarrow$ $\text{Res}(z|G, X) \geq \langle G(z_\frac{1}{2}), z - z_\frac{1}{2} \rangle = \langle G(z_\frac{1}{2}), z_\frac{1}{2} - z_\ast \rangle \geq \frac{\alpha}{4} \|z - z_\ast\|^2_2$

$\Rightarrow (b)$. 

6.12
\{(y_k, \eta_k)\}_{k \leq K} \text{ i.i.d., } E_{\eta_k}\{y_k\} = \psi(\eta_k^T x), x \in \mathcal{X} \Rightarrow x \Rightarrow \Rightarrow x

**Main Observation:** Under (slightly refined) Monotonicity Assumption "\(\psi\) is continuous and monotone on \(\mathbb{R}^m\)"

the signal \(x\) underlying observations in GLM is the unique weak solution to a Variational Inequality \(\text{VI}(G, \mathcal{X})\) with strongly monotone on \(\mathcal{X}\) vector field \(G\).

Indeed, given GLM, let \(P\) be the distribution of regressors \(\eta_k\), and let

\[
F(z) = E_{\eta \sim P}\{\eta \psi(\eta^T z)\}
\]

Observe that for fixed \(\eta \in \mathbb{R}^{n \times m}\), \(z \mapsto F_{\eta}(z) := \eta \psi(\eta^T z)\) is a vector field on \(\mathbb{R}^n\) and this field is monotone and continuous along with \(\psi\):

\[
z, z' \in \mathbb{R}^N \Rightarrow \langle \eta \psi(\eta^T z) - \eta \psi(\eta^T z'), z - z' \rangle = \langle \psi(\eta^T z) - \psi(\eta^T z'), \eta^T z - \eta^T z' \rangle \geq 0.
\]

Under mild regularity assumptions, monotonicity and continuity are preserved when taking expectation w.r.t. \(\eta\). Assuming from now on that

— the distribution \(P\) of \(\eta\) has finite moments of all orders, and

— \(\psi(\cdot) : \mathbb{R}^m \to \mathbb{R}^m\) is monotone, continuous, and with polynomial growth at infinity,

*the vector field \(F\) is well defined, continuous, and monotone.*
\[
\{(y_k, \eta_k)\}_{k \leq K} \text{ i.i.d., } \mathbb{E}_{\eta_k \{y_k\}} = \psi(\eta_k^T x), \; x \in \mathcal{X} \quad \Rightarrow \quad \Rightarrow \quad F(z) = \mathbb{E}_{\eta \sim P \{\eta \psi(\eta^T z)\}}
\]

Let us make

**Assumption A:** *The monotone vector field* \( F \) *is strongly monotone, with modulus* \( \alpha > 0 \), *on* \( \mathcal{X} \).

It is immediately seen that a simple *sufficient* condition for Assumption A is strong monotonicity of \( \psi \) on bounded subsets of \( \mathbb{R}^m \) *plus* positive definiteness of the second order moment matrix \( \mathbb{E}_{\eta \sim P \{\eta \eta^T\}} \) *plus* compactness of \( \mathcal{X} \).

Observe that

**A:** *Underlying observations signal* \( x \) *is zero of continuous and monotone vector field*

\[
G(z) = F(z) - F(x) : \mathcal{X} \to \mathbb{R}^n;
\]

under Assumption A, *\( G \) is strongly monotone, with modulus* \( \alpha > 0 \), *on* \( \mathcal{X} \).

**B.** *For every fixed* \( z \in \mathcal{X} \) *and every* \( k \), *observation* \( (y_k, \eta_k) \) *induces unbiased estimate*

\[
G_{y_k, \eta_k}(z) = \eta_k \psi(\eta_k^T z) - \eta_k y_k.
\]

of \( G(z) \).

Indeed,

\[
\mathbb{E}_{y, \eta} \{\eta \psi(\eta^T z) - \eta y\} = \mathbb{E}_{\eta \sim P \{\eta \psi(\eta^T z) - \eta \mathbb{E}_{\eta} \{y\}\}} = \mathbb{E}_{\eta \sim P \{\eta \psi(\eta^T z) - \eta \psi(\eta^T x)\}} = F(z) - F(x)
\]

6.14
\{(y_k, \eta_k)\}_{k \leq K} \text{ i.i.d., } E_{\eta_k} \{y_k\} = \psi(\eta_k^T x), \ x \in \mathcal{X} \ \Rightarrow \ ? \ x

F(z) = E_{\eta \sim P} \{\eta \psi(\eta^T z)\} : \text{strongly monotone with modulus } \alpha > 0 \text{ on } \mathcal{X}, \ G(z) = F(z) - F(x)

A : \ x \text{ is the unique weak solution to VI}(G, \mathcal{X})

B : \text{Observable vector fields } G_{y_k, \eta_k}(z) = \eta_k \psi(\eta_k^T z) - \eta_k y_k

\text{are unbiased estimates of vector field } G(z)

\textbf{Conclusion:} We can recover } x \text{ via solving } \text{VI}(G, \mathcal{X}) \text{ by an algorithm capable to work with unbiased stochastic estimates of } G(\cdot) \text{ instead of the actual values of } G.
\{(y_k, \eta_k)\}_{k \leq K} \text{i.i.d., } E_{\eta_k} \{y_k\} = \psi(\eta_k^T x), \ x \in \mathcal{X} \ \Rightarrow \ ?\ ? \ x

F(z) = E_{\eta \sim P} \{\eta \psi(\eta^T z)\} : \text{strongly monotone with modulus } \alpha > 0 \text{ on } \mathcal{X}, \ G(z) = F(z) - F(x)

A: \ x \text{ is the unique weak solution to } \text{VI}(G, \mathcal{X})
B: \text{Observable vector fields } G_{y_k, \eta_k}(z) = \eta_k \psi(\eta_k^T z) - \eta_k y_k \\
\text{are unbiased estimates of vector field } G(z)

\♠ There are two basic approaches to solving "stochastic" monotone VI:

**Sample Average Approximation:** Approximate the "vector field of interest" $G(x)$ by its empirical approximation

$$G_{\omega K}(z) = \frac{1}{K} \sum_{k=1}^{K} \left[ \eta_k \psi(\eta_k^T z) - \eta_k y_k \right]$$

which is monotone along with $\psi$, find a weak solution $\hat{x}(\omega^K)$ to $\text{VI}(G_{\omega K}, \mathcal{X})$ and take $\hat{x}$ as the SAA estimate of $x$.

**Stochastic Approximation:** Run stochastic analogy of the simplest First Order algorithm for solving deterministic monotone VI’s – the Stochastic Approximation

$$z_k = \text{Proj}_\mathcal{X} [z_{k-1} - \gamma_k G_{y_k, \eta_k}(z_{k-1})], \ k = 1, 2, \ldots, K$$

- $\text{Proj}_\mathcal{X}[z] = \text{argmin}_{y \in \mathcal{X}} \|y - z\|_2$: metric projection onto $\mathcal{X}$
- $z_0 \in \mathcal{X}$ (arbitrary) deterministic starting point
- $\gamma_k > 0$: deterministic stepsizes
Sample Average Approximation Estimate

\[ \omega^K = \{ \omega_k = (y_k, \eta_k) \}_{k \leq K} \text{ i.i.d., } E[\eta_k y_k] = \psi(\eta_k^T x), \ x \in \mathcal{X} \Rightarrow \hat{x}_{SAA}^{\omega^K} \in \mathcal{X} : \langle G^{\omega^K}(z), z - \hat{x}_{SAA}^{\omega^K} \rangle \geq 0 \forall z \in \mathcal{X} \]

\[ F(z) = E_{\eta \sim P}\{\eta \psi(\eta^T z)\} : \text{strongly monotone with modulus } \alpha > 0 \text{ on } \mathcal{X}, \ G(z) = F(z) - F(x) \]

\[ \Rightarrow G^{\omega^K}(z) = \frac{1}{K} \sum_{k=1}^{K} [\eta_k \psi(\eta_k^T z) - \eta_k y_k] : E_{\omega^K \sim P^K}\{G^{\omega^K}(z)\} = G(z) \]

\[ \Rightarrow \hat{x}_{SAA}^{\omega^K} \text{ minimizes } G^{\omega^K}(z) \text{ over } \mathcal{X}. \]

There exists rather sophisticated theoretical performance analysis of SAA recovery, resulting, under mild assumptions, in tight non-asymptotic upper bounds on the recovery error \( E\{\|\hat{x}(\omega^K) - x\|_2^2\} \).

Assume that the link function \( \psi \) (which we have assumed to be a continuous monotone vector field on \( \mathbb{R}^m \)) is the gradient field of a (automatically convex) continuously differentiable function \( \Psi \):

\[ \psi(s) = \nabla \psi(s). \]

**Note:** The assumption definitely holds true when \( \psi \) is univariate, as in Logit and Probit models.

**Observation:** When \( \psi = \nabla \psi \), the SAA \( G^{\omega^K}(z) \) is the gradient field of a continuously differentiable convex function as well:

\[ G^{\omega^K}(z) = \nabla z [G^{\omega^K}(z) := \frac{1}{K} \sum_{k=1}^{K} [\psi(\eta_k^T z) - z^T \eta_k y_k]] \]

\[ \Rightarrow \text{The SAA estimate } \hat{x}_{SAA}^{\omega^K} \text{ minimizes } G^{\omega^K}(z) \text{ over } \mathcal{X}. \]

6.17
Examples:
Linear model $\psi(s) \equiv s = \nabla_s s^T s / 2$. In this case, the SAA estimate reduces to Least Squares:

$$\hat{x}_{\text{SAA}}(\omega^K) \in \text{Argmin}_{z \in \mathcal{X}} \frac{1}{2K} \sum_{k=1}^{K} \|y_k - \eta_k^T z\|_2^2$$

Note: For linear model with regressor- and signal-independent Gaussian noise:

$$y_k = \eta_k^T x + \xi_k, \ 1 \leq k \leq K$$

[noises $\xi_k \sim N(0, \sigma^2 I)$ are independent of regressors and of each other]

the SAA estimate is the same as the ML one.
Logit model \( \psi(s) = \exp\{s\}/(1 + \exp\{s\}) \). The SAA estimate is

\[
\hat{x}_{SAA}(\omega^K) \in \text{Argmin} \frac{1}{K} \sum_{k=1}^{K} \left[ \ln(1 + \exp\{\eta_k^T z\}) - y_k \eta_k^T z \right]
\]

and happens to be the same as the ML estimate.

Probit model \( \psi(s) = \Phi(s) = \text{Prob}_{\xi \sim \mathcal{N}(0,1)}\{\xi \leq s\} \). Here

\[
\begin{align*}
\hat{x}_{SAA}(\omega^K) & \in \text{Argmin} \frac{1}{K} \sum_{k=1}^{K} \left[ (\eta_k^T z) \Phi(\eta_k^T z) + (2\pi)^{-1/2} \exp\{-(\eta_k^T z)^2/2\} - y_k \eta_k^T z \right] \\
\hat{x}_{ML}(\omega^K) & \in \text{Argmin} \frac{1}{K} \sum_{k=1}^{K} \left[ y_k \ln \left( \frac{(1 - \Phi(\eta_k^T z))}{\Phi(\eta_k^T z)} \right) - \ln(1 - \Phi(\eta_k^T z)) \right]
\end{align*}
\]
Note: In the above GLM’s, finding ML estimates happened to be efficiently solvable convex problems. It is not so in general.

Example: \( y_k = \tan(\eta x) + 3\xi_k \) with i.i.d. regressors \( \eta_k \sim N(0, 1) \) and independent of regressors i.i.d. noises \( \xi_k \sim N(0, 1) \). With \( \mathcal{X} = [-20, 20] \), \( K = 20 \), this is what can happen:

- Magenta curve: graph of the objective \( \Psi_{SAA} \) to be minimized on \( \mathcal{X} \) to get the SAA estimate
- Blue curve: graph of the objective \( \Psi_{ML} \) to be minimized on \( \mathcal{X} \) to get the ML estimate
- Abscissae of vertical segments:
  - green: true signal \( \approx 1.4047 \)
  - magenta: \( \hat{x}_{SAA} \approx 0.8910 \) – minimizer of \( \Psi_{SAA} \)
  - blue: local minimizer \( \approx 0.4300 \) of \( \Psi_{ML} \); the global minimizer of \( \Psi_{ML} \) on \( \mathcal{X} \) is \( \hat{x}_{ML} = -20 \)

Note: With one-dimensional signal, the ML estimate can be computed by "brute force." With multidimensional signal, potential nonconvexity of minus log-likelihood can result in severe computational difficulties. For the SAA estimate, computational tractability is "built in."

6.20
Stochastic Approximation Estimate

\[ z_k = \text{Proj}_{\mathcal{X}} [z_{k-1} - \gamma_k G_{y_k, \eta_k}(z_{k-1})], \quad k = 1, 2, \ldots, K \]

- \( \text{Proj}_{\mathcal{X}}[z] = \arg\min_{y \in \mathcal{X}} \|y - z\|_2 \): metric projection onto \( \mathcal{X} \)
- \( z_0 \in \mathcal{X} \) (arbitrary) deterministic starting point
- \( \gamma_k > 0 \): deterministic stepsizes

The \textit{basic} performance analysis for the SA estimate is as follows. Let us augment Assumption A with

**Assumption B:** For some \( M < \infty \) and for every signal \( x \in \mathcal{X} \), denoting by \( P_x \) the common distribution of observations \( \omega_k = (y_k, \eta_k) \), \( k \leq K \), stemming from signal \( x \), one has

\[
E_{(y, \eta) \sim P_x} \left\{ \|\eta y\|_2^2 \right\} \leq M^2 \quad \forall x \in \mathcal{X}.
\]
\{(y_k, \eta_k) \sim P_x \} \_{k \leq K} \text{ i.i.d., } E_{\eta_k} \{y_k\} = \psi(\eta_k^T x), \text{ } x \in \mathcal{X} \implies \text{ ?? } \rightarrow \text{ ?? } x

**A:** \(F(z) = E_{\eta \sim P}\{\eta \psi(\eta^T z)\} : \text{ strongly monotone with modulus } \alpha > 0 \text{ on } \mathcal{X}, \ G(z) = F(z) - F(x)\)

**B:** \(E_{(y, \eta) \sim P_x}\{\|\eta y\|_2^2\} \leq M^2 \forall x \in \mathcal{X}\)

\[G_{y,\eta}(z) = \eta \psi(\eta^T z) - \eta y\]

\[z_k = \text{Proj}_{\mathcal{X}} \left[ z_{k-1} - \gamma_k G_{y_k,\eta_k}(z_{k-1}) \right], \text{ } 1 \leq k \leq K\]

**Simple standard fact:** **Under Assumptions A, B** and with stepsizes

\[\gamma_k = \frac{1}{(k + 1)\alpha}, \text{ } 1 \leq k \leq K, \quad (\star)\]

whatever be signal \(x \in \mathcal{X}\) underlying observations \(\omega_k = (y_k, \eta_k)\), for the SA iterates \(z_k\) it holds

\[E_{\omega^k \sim P_x^k}\left\{\|z_k - x\|_2^2\right\} \leq \frac{4M^2}{(k + 1)^2\alpha^2}, \text{ } 1 \leq k \leq K \quad (\dagger)\]

[\(P_x^k\): distribution of observation \(\omega^k = (\omega_1, ..., \omega_k)\), the signal being \(x\)]

**Good news:** **Typically, the \(O(1/k)\)-rate of convergence established in (\dagger) is the best rate allowed by Statistics.**

**Another good news:** Error bound (\dagger) is non-asymptotic and is governed by the true modulus of strong monotonocity \(\alpha\) of \(F\) and the true ”magnitude of uncertainty” \(M\).

**Not so good news:** To ensure (\dagger), we need to use stepsizes (\star) with \(\alpha\) lower-bounding the true modulus of strong monotonocity of \(F\) on \(\mathcal{X}\). Overestimating this modulus could completely destroy (\dagger).

6.22
\[ F(z) = \mathbf{E}_{\eta \sim P} \left\{ \eta \psi(\eta^T z) \right\} : \mathcal{X} \to \mathbb{R}^n \& \langle F(z) - F(z'), z - z' \rangle \geq \alpha \|z - z'\|_2^2 \& G(z) = F(z) - F(x) \Rightarrow \]
\[ \langle G(z), z - x \rangle \geq \alpha \|z - x\|^2, z \in \mathcal{X} \quad (a) \]
\[ G_{y,\eta}(z) = \eta \psi(\eta^T y) - \eta y \& \mathbf{E}_{(y,\eta) \sim P_x} \{ G_{y,\eta}(z) \} = G(z), z \in \mathcal{X} \quad (b) \]
\[ \omega_k = (y_k, \eta_k) \sim P_x \text{ i.i.d. } \mathbf{E}_{\eta_k \sim \{ y_k \}} = \psi(\eta_k^T x) \quad (c) \]
\[ \mathbf{E}_{(y,\eta) \sim P_x} \left\{ \|\eta y\|_2^2 \right\} \leq M^2, \quad w \in \mathcal{X} \quad (d) \]

**Proof of Standard Fact:**

- Claim: from (b)-(d) it follows that

\[ \forall (x, z \in \mathcal{X}) : \| F(z) \| \leq M \& \mathbf{E}_{(y,\eta) \sim P_x} \{ \| G_{y,\eta}(z) \|_2 \} \leq 4M^2 \quad (e) \]

Indeed, denoting by \( P \) the distribution of regressors (it is independent of the signal), we have

\[ \forall (x \in \mathcal{X}) : M^2 \geq \mathbf{E}_{(y,\eta) \sim P_x} \left\{ \|\eta y\|_2^2 \right\} = \mathbf{E}_{\eta \sim P} \left\{ \mathbf{E}_{\eta \sim \{ y \}} \{ \|\eta y\|_2^2 \} \right\} \geq \mathbf{E}_{\eta \sim P} \{ \|\eta \mathbf{E}_{\eta \sim \{ y \}} \{ y \} \|_2^2 \} = \mathbf{E}_{\eta \sim P} \left\{ \|\eta \psi(\eta^T x)\|_2^2 \right\} \]

\[ \Rightarrow \left\{ \begin{array}{l}
\| F(z) \|_2 \leq \mathbf{E}_{\eta \sim P} \{ \| \psi(\eta^T z) \|_2 \} \leq \mathbf{E}_{\eta \sim P} \{ \| \psi(\eta^T z) \|_2^2 \} \leq \sqrt{\mathbf{E}_{\eta \sim P} \{ \| \psi(\eta^T x) \|_2^2 \}} \leq M \\
\mathbf{E}_{(y,\eta) \sim P_x} \{ \| G_{y,\eta}(z) \|_2^2 \} = \mathbf{E}_{(y,\eta) \sim P_x} \{ \| \psi(\eta^T z) - \eta y \|_2^2 \} \leq 2 \left[ \mathbf{E}_{\eta \sim P} \{ \| \psi(\eta^T x) \|_2^2 \} + \mathbf{E}_{(y,\eta) \sim P_x} \{ \| \eta y \|_2^2 \} \right] \leq 4M^2
\end{array} \right. 
\]

- Let us fix signal \( x \in \mathcal{X} \) underlying observations \( \omega_k = (y_k, z_k) \). Observe that by construction \( z_k \) is a deterministic function of \( \omega_k = (\omega_1, ..., \omega_k) : z_k = Z_k(\omega^k) \). Setting \( D_k(\omega^k) = \frac{1}{2} \| Z_k(\omega^k) - x \|_2^2 \), we have

\[ D_k(\omega^k) \leq \frac{1}{2} \| Z_k-1(\omega^{k-1}) - x \|_2^2 - \gamma_k G_{y_k,\eta_k}(Z_k-1(\omega^{k-1})) \|_2^2 \\
= D_{k-1}(\omega^{k-1}) - \gamma_k \langle G_{y_k,\eta_k}(Z_k-1(\omega^{k-1})), Z_k-1(\omega^{k-1}) - x \rangle + \frac{1}{2} \gamma_k^2 \| G_{y_k,\eta_k}(Z_k-1(\omega^{k-1})) \|_2^2 
\]

Taking expectation and invoking (b), (a), (e) and the fact that \( (y_k, \eta_k) \sim P_x \) are independent across \( k \), we get

\[ d_k := \mathbf{E}_{\omega^k \sim P_x} \{ D_k(\omega^k) \} \leq d_{k-1} - \gamma_k \mathbf{E}_{\omega^{k-1} \sim P_x} \{ \langle G(Z_k-1(\omega^{k-1})), Z_k-1(\omega^{k-1}) - x \rangle \} + 2\gamma_k^2 M^2 \\
\leq (1 - 2\alpha \gamma_k) d_{k-1} + 2\gamma_k^2 M^2. \]
\[ D_k(\omega^k) = \frac{1}{2} \| Z_k(\omega^k) - x \|_2^2, \quad d_k := \mathbb{E}_{\omega^k \sim P^k_x} \left\{ D_k(\omega^k) \right\} \leq (1 - 2\alpha \gamma_k) d_{k-1} + 2\gamma_k^2 M^2, \quad 1 \leq k \leq K \]

\[ \gamma_k = \frac{1}{(k+1)\alpha} \]

- Let us prove by induction in \( k \) that with \( S = \frac{2M^2}{\alpha^2} \) for \( k = 0, 1, \ldots, K \) it holds

\[ d_k \leq \frac{S}{k+1} \]  \hspace{1cm} (\star_k)

**Base \( k = 0 \):** Let \( D \) be the \( \| \cdot \|_2 \)-diameter of \( X \) and \( z_\pm \in X \) be such that \( \| z_+ - z_- \|_2 = D \). Invoking (e) and strong monotonicity, with modulus \( \alpha \), of \( F \) on \( X \), we have

\[ \alpha D^2 \leq \langle F(z_+) - F(z_-), z_+ - z_- \rangle \leq 2M D \Rightarrow D \leq \frac{2M}{\alpha} \Rightarrow d_0 \leq \frac{D^2}{2} \leq \frac{2M^2}{\alpha^2}, \]

implying (\( \star_0 \)).

**Step \( k - 1 \Rightarrow k \):** Assuming \( k \geq 1 \) and (\( \star_{k-1} \)) true, note that \( 2\alpha \gamma_k = \frac{2}{k+1} \leq 1 \). Invoking (!) and (\( \star_{k-1} \)), we get

\[ d_k \leq \left[ 1 - \frac{2}{k+1} \right] \frac{S}{k} + \frac{2M^2}{(k+1)^2 \alpha^2} = S \left[ \frac{1}{k} \left( 1 - \frac{2}{k+1} \right) + \frac{1}{(k+1)^2} \right] = \frac{S}{k+1} \left[ 1 - \frac{1}{k} + \frac{1}{k+1} \right] \leq \frac{S}{k+1}. \]

Induction is complete.

- Since \( d_k = \frac{1}{2} \mathbb{E}_{\omega^k \sim P^k_x} \left\{ \| Z_k(\omega^k) - x \|_2^2 \right\}, \) (\( \star_k \)) reads

\[ \mathbb{E}_{\omega^k \sim P^k_x} \left\{ \| Z_k(\omega^k) - x \|_2^2 \right\} \leq \frac{4M^2}{(k+1)\alpha^2}. \]

\[ \square \]

6.24
How It Works

**Experiment:** We consider four univariate link functions:

- **Logit**, \( y \in \{0, 1\} \)
  \[ \psi(s) = \frac{\exp\{s\}}{1 + \exp\{s\}} \]
  \[ \text{Prob}_{y \in \{y = 1\}} = \psi(\eta^T x) \]

- **Linear**, \( y \in \mathbb{R} \)
  \[ \psi(s) = s \]
  \[ y \sim N(\psi(\eta^T x), 1) \]

- **Hinge**, \( y \in \mathbb{R} \)
  \[ \psi(s) = \max\{s, 0\} \]
  \[ y \sim N(\psi(\eta^T x), 1) \]

- **Ramp**, \( y \in \mathbb{R} \)
  \[ \psi(s) = \min\{1, \max[0, s]\} \]
  \[ y \sim N(\psi(\eta^T x), 1) \]

- In all four cases, \( \mathcal{X} = \{x \in \mathbb{R}^{100} : \|x\|_2 \leq 1\} \), \( \eta_k \sim N(0, I_{100}) \)

**Note:** When we know in advance the common distribution \( P \) of regressors \( \eta_k \), the vector field

\[ F(z) = \mathbf{E}_{\eta \sim P} \{\eta \psi(\eta^T z)\} \]

becomes known. In addition, when \( P = \mathcal{N}(0, I_n) \), \( F \) becomes extremely simple:

\[ F(x) = \psi(\|x\|_2) \frac{x}{\|x\|_2}, \quad \psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s\psi(ts)e^{-s^2/2} ds \]

6.25
\[ \eta \sim \mathcal{N}(0, I_n) \Rightarrow F(x) = \Psi(\|x\|_2) \frac{x}{\|x\|_2} \]

Functions \( \Psi \) for our four cases:

- **Logit**
- **Linear**
- **Hinge**
- **Ramp**

Modulae of strong monotonicity of vector fields \( F(\cdot) \) on \( \{ z : \|z\|_2 \leq R \} \) vs. \( R \)

6.26
Average $\| \cdot \|_2$-recovery errors for SA (o) and SAA (+) estimates vs $K = 500, 2000, 8000, 32000$
“Single-Observation” Case

♦ Situation: We observe deterministic sequence of regressors \( \{\eta_k \in \mathbb{R}^{n \times m}\}_{k \leq K} \) and sequence of random labels \( y^K = \{y_k \in \mathbb{R}^m\}_{k \leq K} \). The labels \( y_1, \ldots, y_K \) are independent of each other with distributions \( P_{x,k} \) parameterized by unknown signal \( x \in \mathcal{X} \subset \mathbb{R}^n \), and

\[
E_{y_k \sim P_{x,k}} \{y_k\} = \psi(\eta_k^T x), \quad x \in \mathcal{X}.
\]

Our goal is to recover \( x \) given \( \eta^K \) and \( y^K \).

Note: In fact we have a single-observation GLM with deterministic regressor \( \eta^K \), random label \( y^K \), and link function \( \psi^K \) given by

\[
\eta^K = [\eta_1, \ldots, \eta_K] \in \mathbb{R}^{n \times mK}, \quad y^K = \begin{bmatrix} y_1 \\ \vdots \\ y_K \end{bmatrix} \in \mathbb{R}^{mK}, \quad \psi^K([u_1; \ldots; u_K]) = \begin{bmatrix} \psi(u_1) \\ \vdots \\ \psi(u_K) \end{bmatrix} : \mathbb{R}^{mK} \to \mathbb{R}^{mK}.
\]

Indeed, we clearly have

\[
E_{y^K \sim P_{x,1} \times \cdots \times P_{x,K}} \{y^K\} = \psi^K([\eta^K]^T x), \quad x \in \mathcal{X}
\]

\( \Rightarrow \) We can apply our machinery!

6.28
Situation (reworded): We are given
- a deterministic regressor matrix \( \eta \in \mathbb{R}^{n \times M} \)
- a convex compact signal set \( \mathcal{X} \subset \mathbb{R}^n \)
- a random observation ("label") \( y \in \mathbb{R}^M \) with distribution \( P_x \) parameterized by signal \( x \in \mathcal{X} \) in such a way that

\[
E_{y \sim P_x} \{y\} = \phi(\eta^T x)
\]

for a given link function \( \phi(\cdot) : \mathbb{R}^M \rightarrow \mathbb{R}^M \)

Given \( y \) and \( \eta \), we want to recover \( x \).

Note: Under the circumstances the vector field

\[
F(z) = \eta \phi(\eta^T z) : \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

becomes fully observable!

Assumptions:

A': The vector field \( \phi(\cdot) : \mathbb{R}^M \rightarrow \mathbb{R}^M \) is continuous and monotone, so that \( F(\cdot) \) is continuous and monotone on \( \mathbb{R}^n \); in addition, \( F \) is strongly monotone, with modulus \( \alpha > 0 \), on \( \mathcal{X} \).

B': For some \( \sigma < \infty \) it holds

\[
E_{y \sim P_z} \left\{ \| \eta [y - \phi(\eta^T z)] \|_2^2 \right\} \leq \sigma^2 \forall z \in \mathcal{X}
\]
Under the circumstances, the SAA estimate $\hat{x}_{\text{SAA}}(y)$ of signal $x$ underlying observation $y$ is the weak solution of $\text{VI}(G_y, \mathcal{X})$ with

$$G_y(z) = \eta \phi(\eta^T z) - \eta y$$

**Proposition** Under Assumptions $A'$, $B'$ one has

$$\mathbb{E}_{y \sim P_x} \{ \| \hat{x}_{\text{SAA}}(y) - x \|_2^2 \} \leq \frac{\sigma^2}{\alpha^2} \forall x \in \mathcal{X}.$$
\[ G_y(z) = \eta \phi(\eta^T z) - \eta y : \alpha\text{-strongly monotone on } \mathcal{X} \]

**Proof of Proposition:** Let \( x \) be the signal underlying observation, \( y \) be a realization of the observation, and let \( \hat{x} = \hat{x}_{SAA}(y) \), so that \( \hat{x} \) is a weak and therefore a strong, by \( A' \), solution to \( \operatorname{VI}(G_y, \mathcal{X}) \). It suffices to verify that

\[ \| \hat{x} - x \| \leq \alpha^{-1} \| \eta [y - \phi(\eta^T x)] \Delta \|_2 \]  \hspace{1cm} (!)

Setting \( G(z) = F(z) - F(x) \), we have

\[
G_y(z) = F(z) - \eta y = F(z) - F(x) + [F(x) - \eta y] = G(z) - \eta [y - \phi(\eta^T x)] = G(z) - \Delta; \\
\hat{x} \text{ solves } \operatorname{VI}(G_y, \mathcal{X}) \Rightarrow 0 \leq \langle G_y(\hat{x}), x - \hat{x} \rangle = \langle G(\hat{x}), x - \hat{x} \rangle - \langle \Delta, x - \hat{x} \rangle \Rightarrow \\
-\langle G(\hat{x}), x - \hat{x} \rangle \leq -\langle \Delta, x - \hat{x} \rangle \\
G(x) = 0 \Rightarrow \langle G(x), x - \hat{x} \rangle = 0 \hspace{1cm} (a)
\]

so that

\[
\alpha \| x - \hat{x} \|_2^2 \leq \langle G(x) - G(\hat{x}), x - \hat{x} \rangle \leq -\langle \Delta, x - \hat{x} \rangle \leq \| \Delta \|_2 \| x - \hat{x} \|_2 \Rightarrow (!)
\]

\[ \square \]

6.31
Example: Assume that

- $\phi$ is continuous and strongly monotone, with modulus $\kappa > 0$, on the entire $\mathbb{R}^M$,
- $n \times M$ regressor $\eta$ is a realization of random matrix $\mathbf{H}$ with independent of each other $\mathcal{N}(0, 1)$ entries,
- $y = \phi(\eta^T x) + \xi$, where $\xi \sim \mathcal{N}(0, \lambda^2 I_M)$ is independent of $\eta$,
- $M \gg n$.

In this case, with probability rapidly approaching 1 as $M \to \infty$,

- $F(z) = \eta \phi(\eta^T z)$ is strongly monotone, with modulus $\alpha = O(1) \kappa M$, on $\mathbb{R}^n$,
- $\mathbb{E}_{y \sim \mathcal{P}_x} \left\{ \| \eta[y - \phi(\eta^T x)] \|_2^2 \right\} = \mathbb{E}_{\xi \sim \mathcal{N}(0, \lambda^2 I_M)} \left\{ \| \eta \xi \|_2^2 \right\} \leq \sigma^2 : = O(1) \lambda^2 M n$

$\Rightarrow$ Modulo rapidly going to 0 as $M \geq O(1) n$ grows probability of getting "pathological" $\eta$, we have

$$\mathbb{E} \left\{ \| \hat{x}_{\text{SAA}}(y) - x \|_2^2 \right\} \leq \frac{\sigma^2}{\alpha^2} \leq O(1) \frac{\lambda^2 n}{\kappa^2 M}.$$
Illustration: Image reconstruction from blurred noisy observation

\[ y = [\kappa \ast x]^{1/2} + \sigma \xi \]

\( \kappa \): nonnegative 2D kernel, \( \| \kappa \|_1 = 1 \)
\( x \): 2D image to be recovered
\( \xi \): white Gaussian noise
\( \sigma \): 1.2 \( \approx 0.075 \sqrt{\|x\|_{\infty}} \)

True image

Observation \( y \)

SAA recovery \( \mathcal{X} \): nonnegative part of TV ball

SAA recovery \( \mathcal{X} \): nonnegative orthant