Preface
Fact: Many inference procedures in Statistics reduce to optimization.

Example: MLE – Maximum Likelihood Estimation

Problem: Given a parametric family \( \{ p_{\theta}(\cdot) : \theta \in \Theta \} \) of probability densities on \( \mathbb{R}^d \) and a random observation \( \omega \) drawn from some density \( p_{\theta^*}(\cdot) \) from the family, estimate the parameter \( \theta^* \).

Maximum Likelihood Estimate: Given \( \omega \), maximize \( p_{\theta}(\omega) \) over \( \theta \in \Theta \) and use the maximizer \( \hat{\theta} = \hat{\theta}(\omega) \) as an estimate of \( \theta^* \).

Note: In MLE, optimization is used for number crunching only and has nothing to do with motivation and performance analysis of MLE.

Fact: Most of traditional applications of Optimization in Statistics are of “number crunching” nature. While often vitally important, “number crunching” applications are beyond our scope.
What is in our scope, are inference routines motivated and justified by Optimization Theory – Convex Analysis, Optimality Conditions, Duality...

As a matter of fact, our "working horse" will be Convex Optimization. This choice is motivated by

- nice geometry of convex sets, functions, and optimization problems

- computational tractability of convex optimization implying computational efficiency of statistical inferences stemming from Convex Optimization.
SPARSITY-ORIENTED SIGNAL PROCESSING

- Signal Recovery from Indirect Observations
- Sparse $\ell_1$ Recovery: Motivation
- Validating $\ell_1$ Recovery
  - $s$-Goodness and Nullspace Property
  - Quantifying Nullspace Property
  - Regular and Penalized $\ell_1$ Recoveries
  - Restricted Isometry Property
  - Tractability Issues
Basic Signal Processing problem is to recover unknown signal $x_\ast$ (which is an $n$-dimensional vector) from its observation

$$y = A(x_\ast) + \xi$$

- $A(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$: known “signal-to-observation” transformation
- $\xi$: observation noise.
- In many applications, the signal-to-observation transformation is just linear:
  $$A(x) = Ax$$ for some known $m \times n$ matrix $A$.
- Assume from now on that $A(\cdot)$ is linear
  $\Rightarrow$ the recovery problem is just to solve a system of linear equations
  $$Ax = b : = A x_\ast$$

given $m \times n$ matrix $A$ and a noisy observation $y$ of the “true” right hand side $b$. 

1.1
Problem of interest: to solve a linear system

\[ Ax = b : = Ax^* \]

given \( m \times n \) matrix \( A \) and a noisy observation \( y \) of the “true” right hand side \( b \).

As of now, there are two typical settings of the problem:

- \( m \geq n \) (typically, \( m \gg n \)) — we have (much) more observations than unknowns. This is the classical case studied in numerical Linear Algebra (where noise is non-random) and Statistics (where noise is random).

  Unless \( A \) is “pathological,” the only difficulty here is the presence of noise. The challenge is to reproduce well the true signal while suppressing as much as possible the influence of noise.

- \( m < n \) (and even \( m \ll n \)) – we have (much) less observations than unknowns.

Till early 2000’s, this case was thought of as completely meaningless. Indeed, as Linear Algebra says, an under-determined (with more unknowns than equations) system of linear equations either has no solutions at all, or has infinitely many solutions which can be arbitrarily far away from each other.

\[ \Rightarrow \text{When } m < n, \text{ the true signal cannot be recovered from observations even in the noiseless case!} \]

Remedy: Add some information on the true signal.
♠ **Problem of interest:** to solve a linear system
\[ Ax = b := Ax_\ast \]
given \( m \times n \) matrix \( A \) and a noisy observation \( y \) of the “true” right hand side \( b \) in the case of \( m \ll n \)

♠ **Sparsity-oriented remedy:** Reduce the problem to the one where the signal is sparse – has \( s \ll n \) nonzero entries, and utilize sparsity in your recovery routine.

♠ **Fact:** Many real-life signals \( x \) when presented by their coefficients in properly selected basis (“dictionary”) \( B \):
\[ x = Bu \]
- columns of \( B \): vectors of basis \( B \)
- \( u \): coefficients of \( x \) in basis \( B \)
become sparse (or nearly so): \( u \) has just \( s \ll n \) nonzero entries (or can be well approximated by vector with \( s \ll n \) nonzero entries).
Illustration: 25 sec fragment of audio signal “Mail must go through” (dimension 1,058,400) and its Discrete Fourier Transform:

![Time Domain Graph]

![Frequency Domain Graph]

<table>
<thead>
<tr>
<th>% of leading Fourier coefficients kept</th>
<th>energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>25%</td>
<td>99.8%</td>
</tr>
<tr>
<td>15%</td>
<td>99.6%</td>
</tr>
<tr>
<td>5%</td>
<td>98.2%</td>
</tr>
<tr>
<td>1%</td>
<td>79.0%</td>
</tr>
</tbody>
</table>
Illustration: The $256 \times 256$ image

can be thought of as $256^2 = 65536$-dimensional vector (write down the intensities of pixels column by column). “As is,” this vector is not sparse and cannot be approximated well by highly sparse vectors. This is what happens when we keep several leading (i.e., largest in magnitude) entries and zero out all other entries:

1% of leading entries kept

10% of leading entries kept

25% of leading entries kept

50% of leading entries kept
However, the image (same as other “non-pathological” images) is nearly sparse when represented in wavelet basis:

1% of leading wavelet coeff. (99.70% of energy) kept
5% of leading wavelet coeff. (99.93% of energy) kept
10% of leading wavelet coeff. (99.96% of energy) kept
25% of leading wavelet coeff. (99.99% of energy) kept
Similar, albeit less intense, phenomenon takes place when representing typical images in frequency domain:

1% of leading wavelet coeff. (99.70% of energy) kept

5% of leading wavelet coeff. (99.93% of energy) kept

1% of leading Fourier coeff. (96.41% of energy) kept

5% of leading Fourier coeff. (99.46% of energy) kept

10% of leading wavelet coeff. (99.96% of energy) kept

25% of leading wavelet coeff. (99.99% of energy) kept

10% of leading Fourier coeff. (99.76% of energy) kept

25% of leading Fourier coeff. (99.95% of energy) kept
When recovering a signal $x_*$ admitting a sparse (or nearly so) representation $Bu_*$ in a known basis $B$ from observations
\[ y = Ax_* + \xi, \]
the situation reduces to the one when the signal to be recovered is just sparse.

Indeed, we can first recover sparse $u_*$ from observations
\[ y = Ax_* + \xi = [AB]u_* + \xi. \]

After an estimate $\hat{u}$ of $u_*$ is built, we can estimate $x_*$ by $B\hat{u}$.

$\Rightarrow$ In fact, sparse recovery is about how to recover a sparse $n$-dimensional signal $x$ from $m \ll n$ observations
\[ y = Ax_* + \xi. \]
(?) How to recover a \textit{sparse} (or nearly so) \(n\)-dimensional signal \(x\) from \(m \ll n\) observations

\[ y = Ax^* + \xi \]

\(\clubsuit\) To get an idea, consider the case when \(x^*\) is exactly sparse – has \(s \ll n\) nonzero entries – and there is no observation noise:

\[ y = Ax^* \]

- \textit{If we knew the positions} \(i_1, \ldots, i_s\) \textit{of the nonzero entries in} \(x^*\), \textit{we could recover} \(x^*\) \textit{by solving the system with just} \(s\) \textit{unknowns}:

\[
y = \begin{bmatrix} A_{i_1}, \ldots, A_{i_s} \end{bmatrix} \cdot \begin{bmatrix} x_{i_1}; \ldots; x_{i_s} \end{bmatrix}. \quad (!)\]

When \(s \leq m\) (which, with \(s \ll n\), still allows for \(m \ll n\)), we would get an \textit{over-determined} system of linear equations on the nonzero entries in \(x\). Assuming \(A\) “non-pathologic,” so that every \(s \leq m\) columns of \(A\) are linearly independent, (!) has a unique solution which can be easily found.

\textbf{But:} \textit{We never know in advance where the nonzeros in} \(x\) \textit{are located!}
(?) How to recover a *sparse* (or nearly so) $n$-dimensional signal $x$ from $m \ll n$ observations

$$y = Ax_\ast + \xi$$

♠ A straightforward way to account for the fact that we *never know where the nonzeros in $x_\ast$ stand*, is to look for the sparest solution to the system $y = Ax$. This amounts to solving the optimization problem

$$\min_x \text{nnz}(x) \text{ s.t. } y = Ax \quad (!)$$

- nnz($x$): # of nonzero entries in $x$.
- It is easily seen that *if $x_\ast$ is $s$-sparse and every $2s$ columns in $A$ are linearly independent* (which is so when $2s \leq m$, unless $A$ is pathological), *then $x_\ast$ is the unique optimal solution to (!), and thus our procedure recovers $x_\ast$ exactly.*

**But:** nnz($z$) is a bad (nonconvex and discontinuous) function, so that (!) is a disastrously complicated combinatorial problem. Seemingly, the only way to solve (!) is to use brute force search where we test one by one all collections of potential locations of nonzero entries in a solution. Brute force is completely unrealistic: to recover $s$-sparse signal, it would require looking through **at least**

$$N = \binom{n}{s-1} = \frac{n!}{(s-1)!(n-s+1)!}$$

candidate solutions.

- with $s = 17, n = 128$, $N$ is as large as $1.49 \cdot 10^{21}$
- with $s = 49, n = 1024$, $N$ is as large as $3.94 \cdot 10^{84}$
How to recover a sparse (or nearly so) $n$-dimensional signal $x$ from $m \ll n$ observations

$$y = Ax_\star + \xi$$

- Solving problem

$$\min_x \text{nnz}(x) \text{ s.t. } y = Ax \quad (!)$$

would yield the desired recovery, but (!) is heavily computationally intractable...

- **Partial remedy:** Replace the difficult to minimize objective $\text{nnz}(\theta)$ with an “easy-to-minimize” objective, specifically, with $\|\theta\|_1 = \sum_i |\theta_i|$, thus arriving at $\ell_1$-recovery

$$\hat{x} = \text{argmin}_x \{ \sum_i |x_i| : Ax = y := Ax_\star \} \quad (!!)$$

- **Observation:** (!!) is just an LO program!

Indeed,

- the constraints in (!!) are linear equalities.
- $|x_i| = \max[x_i, -x_i]$, so that the terms in the objective can be “linearized.”

- The LO reformulation of (!!) is

$$\min_{x,z} \left\{ \sum_j z_j : Ax = y, z_j \geq x_j, z_j \geq -x_j \forall j \leq n \right\}.$$
• **In the noiseless case**, $\ell_1$ recovery is given by
  \[
  \hat{x} = \text{argmin}_x \left\{ \sum_i |x_i| : Ax = y := Ax_\ast \right\}
  \]

  ♠ When the observation $y$ is noisy:
  \[
  y = Ax_\ast + \xi
  \]
  the constraint $Ax = y$ on a candidate recovery should be relaxed.

• **When we know an upper bound** $\delta$ on some norm $\|\xi\|$ of the noise $\xi$, a natural version of $\ell_1$ recovery is
  \[
  \hat{x} \in \text{Argmin}_x \left\{ \sum_i |x_i| : \|Ax - y\| \leq \delta \right\}
  \]
  
  **Note:** When $\|\xi\| = \|\xi\|_\infty := \max_i |\xi_i|$ (“uniform norm”), (*) reduces to the LO program
  \[
  \min_{x,z} \left\{ \sum_j z_j : -z_j \leq x_j \leq z_j, \ 1 \leq j \leq n \right\}
  \]
  \[
  y_i - \delta \leq [Ax]_i \leq y_i + \delta, \ 1 \leq i \leq m
  \]

• **When the noise $\xi$ is random with zero mean**, there are reasons to define $\ell_1$ recovery by **Dantzig Selector**:
  \[
  \hat{x} \in \text{Argmin}_x \left\{ \sum_i |x_i| : \|Q(Ax - y)\|_\infty \leq \delta \right\}
  \]
  with $M \times m$ contrast matrix $Q$ and $\delta > 0$ chosen according to noise’s structure and intensity. This again is reducible to LO program, specifically,
  \[
  \min_{x,z} \left\{ \sum_j z_j : -z_j \leq x_j \leq z_j, \ 1 \leq j \leq n \right\}
  \]
  \[
  -\delta \leq [QAx - Qy]_i \leq \delta, \ 1 \leq i \leq M
  \]

• **Note:** In Dantzig Selector proper, $Q = A^T$.  

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How to recover a *sparse* (or nearly so) $n$-dimensional signal $x$ from $m \ll n$ observations

$$y = Ax^* + \xi$$

(!) Use $\ell_1$ minimization

$$\hat{x} \in \text{Argmin}_x \{\sum_i |x_i| : \|Ax - y\| \leq \delta\}$$

♣ Compressed Sensing theory shows that under appropriate assumptions on $A$, in a meaningful range of sizes $m, n$ and sparsities $s$, $\ell_1$-minimization recovers the unknown signal $x^*$ — *exactly*, when $x^*$ is $s$-sparse and there is no observation noise,

— *within inaccuracy* $\leq C(A)[\delta_n + \delta_s]$ in the general case

- $\delta_n$: magnitude of noise
- $\delta_s$: deviation of $x^*$ from its best $s$-sparse approximation
♣ **Bad news:** “Appropriate assumptions on $A$” are difficult to verify

Partial remedy: there are conservative verifiable sufficient conditions for “appropriate assumptions.”

♣ **Good news:** For $A$ drawn at random from natural distributions, “appropriate assumptions” are satisfied with overwhelming probability.

- E.g., when entries in $m \times n$ matrix $A$ are, independently of each other, sampled from Gaussian distribution, the resulting matrix, with probability approaching 1 as $m, n$ grow, ensures the validity of $\ell_1$ recovery of sparse signals with as many as

$$s = O(1) \frac{m}{\ln(n/m)}$$

nonzero entries.

♣ **More good news:** In many applications (Imaging, Radars, Magnetic Resonance Tomography,...), signal acquisition via randomly generated matrices $A$ makes perfect sense and results in significant acceleration of the acquisition process.

In these applications, signals of interest are sparse in properly selected bases

$\Rightarrow$ *With accelerated acquisition, no information is lost!*
Example: Single-Pixel Camera:
How it works:
Sparse recovery via Dantzig Selector

32 nonzero entries
No noise
Recovery error: $7.8 \cdot 10^{-9}$

32 nonzero entries
Noise's StD 0.01
Recovery error: 0.064

32 nonzero entries
Noise's StD 0.10
Recovery error: 0.66

64 nonzero entries
No noise
Recovery error: 1.47

$:\text{signal}$ $\rightarrow\text{: recovery}$

$256 \times 512$ Gaussian sensing matrix $A$
Validity of sparse signal recovery via $\ell_1$ minimization

♠ Notational convention: From now on, for a vector $x \in \mathbb{R}^n$
- $I_x = \{ j : x_j \neq 0 \}$ is the support of $x$.
- for a subset $I$ of the index set $\{1, \ldots, n\}$, $x_I$ is the vector obtained from $x$ by zeroing out entries with indexes not in $I$, and $I^o$ is the complement of $I$:

$$I^o = \{ i \in \{1, \ldots, n\} : i \notin I \}.$$  
- for $s \leq n$, $x^s$ is the vector obtained from $x$ by zeroing our all but the $s$ largest in magnitude entries.
- $x^s$ is the best $s$-sparse approximation of $x$ in any one of the $\ell_p$ norms, $1 \leq p \leq \infty$.
- for $s \leq n$ and $p \in [1, \infty]$, we set

$$\|x\|_{s,p} = \|x^s\|_p.$$
Validity of $\ell_1$ minimization in the noiseless case

♠ The minimal requirement on sensing matrix $A$ which makes $\ell_1$-minimization valid is to guarantee the correct recovery of exactly $s$-sparse signals in the noiseless case, and we start with investigating this property.

♠ $s$-Goodness: An $m \times n$ sensing matrix $A$ is called $s$-good, if whenever the true signal $x$ underlying noiseless observations is $s$-sparse, this signal will be recovered exactly by $\ell_1$-minimization.

Equivalently: $A$ is $s$-good, if

$$\text{nnz}(x_*) \leq s$$
$$\Rightarrow x_* \text{ is the unique optimal solution to } \min_x \{\|x\|_1 : Ax = Ax_* \}$$

♠ Necessary and sufficient condition for $s$-goodness is Nullspace Property:

\[
\text{For every } 0 \neq z \in \text{Ker} A := \{z : Az = 0\} \text{ it holds } \|z\|_{s,1} < \frac{1}{2}\|z\|_1.
\]

• Nullspace Property can be derived from LO Optimality Conditions, same as can be verified directly.
• **s-goodness ⇒ Nullspace Property:**
Nullspace Property does \textit{not} take place
⇒ \( \exists 0 \neq z \in \Ker A : \|z^s\|_1 \geq \frac{1}{2}\|z\|_1 \)
⇒ \( Az^s = A[z^s - z], \|z^s\|_1 \geq \|z^s - z\|_1 \)
⇒ \( s\)-sparse signal \( x_* = z^s \) is not the unique optimal solution
to \( \min_x \{\|x\|_1 : Ax = Ax_*\} \) – contradiction

• **Nullspace Property ⇒ s-goodness:** Let Nullspace Property take place and \( x_* \) be \( s\)-sparse, and let \( u \) be an optimal solution to \( \min_x \{\|x\|_1 : Ax = Ax_*\} \).
Denoting by \( I \) the support of \( x_* \), for \( z = u - x_* \) we have \( z \in \Ker A \) and

\[
  z_I = u_I - [x_*]_I = u_I - x_* \quad \& \quad z_{I^0} = u_{I^0}
\]
⇒ \( \|z_I\|_1 \geq \|x_*\|_1 - \|u_I\|_1 \quad \& \quad \|z_{I^0}\|_1 = \|u_{I^0}\|_1 \)
⇒ \( \|z_I\|_1 - \|z_{I^0}\|_1 \geq \|x_*\|_1 - \|u_I\|_1 - \|u_{I^0}\|_1 \)
\[
= \|x_*\|_1 - \|u\|_1 \geq 0
\]
⇒ \( \|z_I\|_1 - \|z_{I^0}\|_1 \geq 0 \)
⇒ \( \|z\|_{s,1} \geq \|z_I\|_1 \geq \frac{1}{2}[\|z_I\|_1 + \|z_{I^0}\|_1] = \frac{1}{2}\|z\|_1 \)
⇒ \( z = 0 \)
Questions to be addressed:

What happens when $A$ is $s$-good, but $\ell_1$ recovery is “imperfect,” e.g.

- $x$ is not exactly $s$-sparse, and/or

- there is observation noise

How to verify, given $A$ and $s$, that $A$ is $s$-good
Quantifying Nullspace Property and Imperfect $\ell_1$ Recovery

In order to address the above questions, we need to “quantify” Nullspace Property.

Nullspace Property states that

$$\{z \in \text{Ker} A \& \|z\|_1 = 1 \} \Rightarrow \|z\|_{s,1} < 1/2,$$

or, which is the same,

$$\exists \kappa < 1/2 : \|z\|_{s,1} \leq \kappa \|z\|_1 \forall z \in \text{Ker} A \quad (!)$$

Equivalent form of necessary and sufficient condition (!) for $s$-goodness of $m \times n$ sensing matrix $A$ reads:

$A$ is $s$-good if and only if for some constants $\kappa < 1/2$, $C < \infty$, and some norm $\| \cdot \|$ it holds

$$\|x\|_{s,1} \leq C\|Ax\| + \kappa\|x\|_1 \forall x \in \mathbb{R}^n \quad (!!)$$

Indeed, (!!) clearly implies (!). Assume (!), and let $\bar{x}$ be $\| \cdot \|_1$-closest to $x$ element of Ker$A$, so that $\|x - \bar{x}\|_1 \leq c\|Ax\|$ with $c$ independent of $x$. We have

$$\|x\|_{s,1} \leq \|\bar{x}\|_{s,1} + \|x - \bar{x}\|_1 \leq \kappa \|\bar{x}\|_1 + \|x - \bar{x}\|_1$$

$$\leq \kappa \|x\|_1 + [1 + \kappa]\|x - \bar{x}\|_1 \leq [1 + \kappa]c\|Ax\| + \kappa \|x\|_1$$

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\[ \exists C : \|x\|_{s,1} \leq C \|Ax\| + \kappa \|x\|_1 \quad \forall x \in \mathbb{R}^n \tag{!!} \]

\[ \star \] It makes sense to rewrite the latter condition in a more flexible form linking

- \( m \times n \) sensing matrix \( A \),
- sparsity level \( s \),
- \( m \times N \) contrast matrix \( H \),
- \( \text{norm } \| \cdot \| \text{ on } \mathbb{R}^N \),
- \( \text{condition's parameter } q \in [1, \infty], \) and
- \( \text{parameter } \kappa \in (0, 1/2) \)

**Condition** \( Q_q(s, \kappa) : \)

\[
\|x\|_{s,q} := \|x^s\|_q \leq s^{\frac{1}{q}} \|H^T Ax\| + \kappa s^{\frac{1}{q}-1} \|x\|_1 \quad \forall x \in \mathbb{R}^n
\]

\[ \star \] We treat condition \( Q_q(s, \kappa) \) as a condition on contrast matrix \( H \) and norm \( \| \cdot \| \).

\[ \star \] **Note:** \( A \) is \( s \)-good if and only if the Nullspace Property holds, or, which is the same, if and only if the condition \( Q_1(s, \kappa) \) with some \( \kappa < 1/2 \) is satisfiable (e.g., with \( N = n \), \( H = CAT^T \) with properly selected \( C \), and \( \| \cdot \| = \| \cdot \|_\infty \)).
**Condition** $Q_q(s, \kappa)$:

\[
\|x\|_{s,q} := \|x^s\|_q \leq s^q \|H^T A x\| + \kappa s^{q-1} \|x\|_1 \forall x \in \mathbb{R}^n
\]

♠ **Immediate observations:**

- **The larger is** $q$, **the stronger is** $Q_q(s, \kappa)$: If $H, \| \cdot \|$ satisfy $Q_q(s, \kappa)$ and $p \in [1, q]$, then $H, \| \cdot \|$ satisfy $Q_p(s, \kappa)$.

Indeed, if $H, \| \cdot \|$ satisfy $Q_q(s, \kappa)$, then

\[
\|x\|_{s,p} \leq \|x\|_{s,q} s^{p-1} \leq s^p \|H^T A x\| + \kappa s^{q-1} \|x\|_1
\]

\[
= s^p \|H^T A x\| + \kappa s^{p-1} \|x\|_1.
\]

- **Satisfiability of the weakest condition** $Q_1(s, \kappa)$ for some $\kappa < 1/2$ is necessary and sufficient for $s$-goodness of $A$.

♠ **Fact:** Conditions $Q_q(s, \kappa)$ underly instructive bounds on recovery error for imperfect $\ell_1$ recovery.
Example A: Regular $\ell_1$-Recovery

Regular $\ell_1$ recovery of signal $x$ from observations

$$y = Ax + \eta$$

is given by

$$\hat{x}_{\text{reg}}(y) \in \text{Argmin}_u \left\{ \|u\|_1 : \|H^T(Au - y)\| \leq \rho \right\}$$

where $H, \| \cdot \|, \rho \geq 0$ are construction’s parameters.

Theorem. Let $s$ be a positive integer, $q \in [1, \infty]$ and $\kappa \in (0, 1/2)$. Assume that $H, \| \cdot \|$ satisfy $Q_q(s, \kappa)$, and let

$$\Xi_\rho = \{ \eta : \|H^T \eta\| \leq \rho \}.$$ 

Then for all $x \in \mathbb{R}^n$ and $\eta \in \Xi_\rho$ one has

$$\|\hat{x}_{\text{reg}}(Ax + \eta) - x\|_p \leq \frac{4(2s)^{1/p}}{1 - 2\kappa} \left[ \rho + \frac{\|x - x^s\|_1}{2s} \right], \ 1 \leq p \leq q.$$ 

Note: Regular $\ell_1$ recovery requires a priori information on noise needed to select $\rho$ with “meaningful” $\Xi_\rho$ and does not require a priori information on sparsity $s$. 

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∀η ∈ Ξρ = \{η : \|H^Tη\| ≤ ρ\} ∀x:

\|\hat{x}_{reg}(Ax + η) - x\|_p ≤ \frac{4(2s)^{1/p}}{1 - 2κ}\left[ρ + \frac{\|x - x^s\|_1}{2s}\right], 1 ≤ p ≤ q.

♠ Comments:

A. ρ stems from observation errors:

• η ≡ 0 ⇒ we can set ρ = 0, resulting in zero recovering error for exactly s-sparse signals

• η is “uncertain but bounded”: η ∈ U for some known and bounded U

⇒ we can set ρ = \max_{u∈U} \|H^Tu\|

• η ∼ \mathcal{N}(0, σ^2I_m) ⇒ given tolerance β and setting

ρ = σ\sqrt{2 \ln(N/β)} \max_i \|Col_i[H]\|_2

we get

\text{Prob}\{η : \|H^Tη\|_∞ ≤ ρ\} ≥ 1 - β

When \|·\| = \|·\|_∞, this allows to build explicitly “confidence domains” for regular ℓ_1 recovery.

B. Pay attention to the factor s^{−1} at the “near-sparsity” term \|x - x^s\|_1.

C. Adjusting H and \|·\|, we can, to some extent, account for the nature of observation errors.
Example B: Penalized $\ell_1$ Recovery

Penalized $\ell_1$ recovery of signal $x$ from observations

$$y = Ax + \eta$$

is given by

$$\hat{x}_{\text{pen}}(y) \in \text{Argmin}_u \left\{ \|u\|_1 + \lambda \|H^T(Au - y)\| \right\}$$

where $H$, $\| \cdot \|$, $\lambda > 0$ are construction’s parameters.

♣ Theorem. Given $A$, positive integer $s$, and $q \in [1, \infty]$, assume that $H$, $\| \cdot \|$ satisfy $Q_q(s, \kappa)$ with $\kappa < 1/2$, and let $\lambda \geq 2s$. Then for all $\eta \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$, for $1 \leq p \leq q$ it holds

$$\|\hat{x}_{\text{pen}}(Ax + \eta) - x\|_p \leq \frac{4 \lambda^p \left[ \frac{1}{2} + \frac{\lambda}{4s} \right]}{1 - 2\kappa} \left[ \|H^T\eta\| + \frac{\|x - x^s\|_1}{2s} \right].$$

In particular, with $\lambda = 2s$, for $1 \leq p \leq q$ it holds

$$\|\hat{x}_{\text{pen}}(Ax + \eta) - x\|_p \leq \frac{4(2s)^p}{1 - 2\kappa} \left[ \|H^T\eta\| + \frac{\|x - x^s\|_1}{2s} \right].$$

Note: Penalized $\ell_1$ recovery requires a priori knowledge of sparsity level $s$ and does not require any information on noise.

Note: When $\lambda = 2s$, for all $x$ it holds

$$\forall (\rho \geq 0, \eta \in \Xi_\rho := \{\eta : \|H^T\eta\| \leq \rho\}) :$$

$$\|\hat{x}_{\text{pen}}(Ax + \eta) - x\|_p \leq \frac{4(2s)^p}{1 - 2\kappa} \left[ \rho + \frac{\|x - x^s\|_1}{2s} \right], 1 \leq p \leq q.$$
$H, \| \cdot \|$ satisfy $Q_q(s, \kappa)$

$y = Ax + \eta, \eta \sim \mathcal{N}(0, \sigma^2 I_N)$

$x \in \mathbb{R}^n$ is $s$-sparse

\[ \begin{align*}
\text{Prob} \left\{ \| \hat{x}_{\text{reg}}(Ax + \eta) - x \|_p \leq C(H, \kappa, \ln(1/\epsilon))\sigma_s^\frac{1}{p} \right\} & \geq 1 - \epsilon \\
\text{Prob} \left\{ \| \hat{x}_{\text{pen}}(Ax + \eta) - x \|_p \leq C(H, \kappa, \ln(1/\epsilon))\sigma_s^\frac{1}{p} \right\} & \geq 1 - \epsilon \\
1 \leq p \leq q
\end{align*} \]

**Note:** Given direct observations $y = x + \eta$ of $s$-dimensional signal $x$ with $\eta \sim \mathcal{N}(0, \sigma^2 I_s)$, the expected $\| \cdot \|_p$-norm of recovery error in optimal recovery is $O(1)\sigma_s^\frac{1}{p}$. 

1.27
How to Verify Validity Conditions for $\ell_1$-Recovery?

♣ **Bad news:** Given $A$ and $s$, the Nullspace Property is difficult to verify. Similarly, when $q < \infty$ and $\kappa < 1/2$, it is difficult to verify whether the condition $Q_q(s, \kappa)$ is satisfied by given $H$, $\| \cdot \|$, same as it is difficult to verify whether the condition is satisfiable at all.

♠ **Relatively good news:** There are natural ensembles of random sensing matrices for which properly selected $H$, $\| \cdot \|$ with overwhelming probability satisfy $Q_2(s, \kappa)$ and thus are $s$-good.

♣ **Definition.** An $m \times n$ sensing matrix $A$ satisfies Restricted Isometry Property RIP$(\delta, k)$, if

$$(1 - \delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta)\|x\|_2^2 \forall (x : \text{nnz}(x) \leq k).$$

♠ **Theorem** Let $m \times n$ sensing matrix $A$ satisfy RIP$(\delta, 2s)$ for some $\delta < 1/3$ and positive integer $s$. Then

- **The pair** $(H = \frac{s^{-1/2}}{\sqrt{1-\delta}}I_m, \| \cdot \|_2)$ **satisfies the condition**

$$Q_2 \left( s, \frac{\delta}{1-\delta} \right);$$

- **The pair** $(H = \frac{1}{1-\delta}A, \| \cdot \|_\infty)$ **satisfies the condition**

$$Q_2 \left( s, \frac{\delta}{1-\delta} \right).$$
Theorem Given $\delta \in (0, \frac{1}{5}]$, with properly selected positive $c = c(\delta)$, $d = d(\delta)$, $f = f(\delta)$ for all $m \leq n$ and all positive integers $k$ such that

$$k \leq \frac{m}{c \ln(n/m) + d}$$

the probability for a random $m \times n$ matrix $A$ with independent $\mathcal{N}(0, \frac{1}{m})$ entries to satisfy RIP$(\delta, k)$ is at least

$$1 - \exp\{-fm\}.$$ 

Similar result holds true for Rademacher matrices – those with i.i.d. entries taking values $\pm 1/\sqrt{m}$ with probabilities 0.5.

Note: $k$ can be “nearly” as large as $m$!
Sketch of the proof

Let $A$ be Gaussian random $m \times n$ matrix from Theorem, $I \subset \{1, \ldots, n\}$ be fixed $k$-element index set, and $A_I = [A_{ij} : i \leq m, j \in I]$. Let us fix $\alpha \in (0, 0.1]$.

Fact: For fixed $u \in \mathbb{R}^k$ with $\|u\|_2 = 1$ one has

$$\text{Prob}\{A : \|A_I u\|_2^2 \not\in [1 - \alpha, 1 + \alpha]\} \leq 2e^{-\frac{m}{5} \alpha^2}.$$ 

[observe that $A_I u \sim \mathcal{N}(0, \frac{1}{m} I_m)$ and use standard bounds on the tails of $\chi^2$-distribution]

⇒ Let $\Gamma$ be $\alpha$-net on the unit sphere $S_k$ in $\mathbb{R}^k$. Then

$$\text{Prob}\{A : \exists u \in S_k : \|A_I u\|_2^2 \not\in [1 - 4\alpha, 1 + 4\alpha]\} \leq \pi := 2|\Gamma|e^{-\frac{m}{5} \alpha^2}.$$ 

[By Fact, $\text{Prob}\{A : \|A_I u\|_2^2 \not\in [1 - \alpha, 1 + \alpha] \text{ } \forall u \in \Gamma\} \geq 1 - \pi$. Since the quadratic form $f(u) := u^T A_I^T A_i u$ is Lipschitz continuous on $S_k$ with constant $2M := 2 \max_{u \in S_k} \|A_I u\|_2^2$, we have]

$$A \in \mathcal{E} \Rightarrow \left\{ \begin{array}{l}
\min_{u \in S_k} f(u) \geq \min_{u \in \Gamma} -\alpha M \leq 1 - (1 + \alpha) M \\
M = \max_{u \in S_k} f(u) \leq \max_{u \in \Gamma} f(u) + \alpha M \leq (1 + \alpha) M + 1
\end{array} \right.,$$

and the conclusion follows.]

⇒ $\forall (I, |I| = k)$:

$$\text{Prob}\{A : (1 - 4\alpha) I_k \preceq A_I^T A_I \preceq (1 + 4\alpha) I_k\} \geq 1 - 2 \left[1 + 2/\alpha \right]^k e^{-\frac{m}{5} \alpha^2}$$

[Comparing volumes, the cardinality of a minimal $\alpha$-net on $S_k$ is $\leq F$]

⇒ $\text{Prob}\{A : A \text{ is not RIP}(4\alpha, k)\} \leq \binom{n}{k} \left[1 + 2/\alpha \right]^k e^{-\frac{m}{5} \alpha^2}$

⇒ Theorem.
♠ **Bad news:** No (series of) explicitly computable (even by a randomized computation) $\text{RIP}(0.1, k)$ “low” ($2m \leq n$) $m \times n$ matrices with “large” $k$ (namely, $k \gg \sqrt{m}$) are known.

♥ The natural idea – “generate at random a low $m \times n$ matrix and check whether it satisfies $\text{RIP}(0.1, k)$” with “large” $k$; if yes, output the matrix” – fails – fails: while typical random matrices do possess $\text{RIP}(0.1, k)$ with “large” $k$, we do not know how to verify this property in a computationally efficient fashion.
Designing/checking RIP matrices is similar to other situation where we do know that a typical randomly selected object possesses some property, but we neither can point out an individual object with this property, nor can check efficiently whether a given object possesses it. Some examples:

- **Complexity of Boolean functions** [Shannon, 1949]: For a Boolean function \( f \) of \( n \) Boolean variables, the minimal number of AND, OR, NOT switches in a circuit computing the function is upper-bounded by \( O(1) \frac{2^n}{n} \), and as \( n \) grows, this bound becomes sharp with overwhelming probability.

  However: No individual functions with nonlinear “Boolean complexity” are known...

- **Lindenstrauss-Johnson Theorem** For a Gaussian “low” \( m \times n \) matrix \( A \), the image \( \{Ax : x \in B_n\} \) of the unit \( n \)-dimensional box \( B_n = \{x \in \mathbb{R}^n : \|x\|_{\infty} \leq 1\} \) under the mapping \( x \mapsto Ax \) with overwhelming, as \( n \to \infty \), probability is in-between two similar ellipsoids with the ratio of linear sizes not exceeding \( 1 + O(1)(n/m)^2 \).

  However: No individual matrices \( A \) with \( AB_n \) reasonably close to an ellipsoid are known...

**Note:** For every \( \epsilon \in (0, 1) \) and every \( n \), one can explicitly point out a polytope \( P \) given by \( O(1)n \ln(1/\epsilon) \) linear inequalities on \( O(1)n \ln(1/\epsilon) \) variables such that the projection of \( P \) onto the plane of the first \( n \) variables is in-between \( \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\} \) and \( \{x \in \mathbb{R}^n : \|x\|_2 \leq 1 + \epsilon\} \). However, this “fast polyhedral approximation” of Euclidean ball deals with polytopes \( P \) quite different from boxes...
We have seen that RIP-matrices $A$ yield easy-to-satisfy condition $Q_2(s, \kappa)$.
Unfortunately, RIP is difficult to verify...

**Good news:** Condition $Q_\infty(s, \kappa)$ is fully computationally tractable.
**Theorem** Let $A$ be an $m \times n$ sensing matrix, $s$ be a sparsity level, and $\kappa \geq 0$. Whenever $\bar{H}, \| \cdot \|$ satisfy $Q_\infty(s, \kappa)$, there exists an $m \times n$ matrix $H$ such that

$$\| \text{Col}_j [I_n - H^T A] \|_\infty \leq s^{-1} \kappa, \ 1 \leq j \leq n.$$ 

As a result, $H, \| \cdot \|_\infty$ satisfy $Q_\infty(s, \kappa)$. Besides this,

$$\| H^T \eta \|_\infty \leq \| \bar{H}^T \eta \| \ \forall \eta \in \mathbb{R}^m.$$ 

In addition, $m \times n$ contrast matrix $H$ such that $H, \| \cdot \|_\infty$ satisfy $Q_\infty(s, \kappa)$ with as small $\kappa$ as possible can be found as follows: we consider $n$ LP programs

$$\text{Opt}_i = \min_{\nu, h} \\{ \nu : \| A^T h - e^i \|_\infty \leq \nu \}, \quad (#_i)$$

where $e^i$ is $i$-th basic orth in $\mathbb{R}^n$, find optimal solutions $\text{Opt}_i, h_i$ to these problems, and make $h_i, i = 1, \ldots, n$, the columns of $H$; the corresponding value of $\kappa$ is

$$\kappa_* = s \max_i \text{Opt}_i.$$ 

Finally, there exists a transparent alternative description of the quantities $\text{Opt}_i$ (and thus – of $\kappa_*$):

$$\text{Opt}_i = \max_x \{ x_i : \| x \|_1 \leq 1, Ax = 0 \}.$$
Let $A$ be an $m \times n$ sensing matrix, $s$ be a sparsity level, and $\kappa \geq 0$. Whenever $\bar{H}, \| \cdot \|$ satisfy $Q_\infty(s, \kappa)$, there exists an $m \times n$ matrix $H$ such that

$$\|\text{Col}_j[I_n - H^T A]\|_\infty \leq s^{-1}\kappa, \quad 1 \leq j \leq n,$$

As a result, $H, \| \cdot \|_\infty$ satisfy $Q_\infty(s, \kappa)$. Besides this,

$$\|H^T \eta\|_\infty \leq \|\bar{H}^T \eta\| \forall \eta \in \mathbb{R}^m.$$

**Proof** uses Basic fact of Convex Geometry: A norm $\| \cdot \|$ on $\mathbb{R}^N$ induces the conjugate norm

$$\|f\|_* = \max_{h : \|h\| \leq 1} f^T h.$$

One always has $|f^T h| \leq \|f\|_* \|h\| \& \|h\| = \max_{f : \|f\|_* \leq 1} f^T h$

Now,

$i \leq n$

$\Rightarrow x_i \leq \|x\|_{s, \infty} \leq \|\bar{H}^T A x\| + \kappa s^{-1}\|x\|_1 \forall x$ [by $Q_\infty(s, \kappa)$]

$\Rightarrow \max_x \left\{x_i - \|\bar{H}^T A x\| : \|x\|_1 \leq 1\right\} \leq s^{-1}\kappa$

$\Leftrightarrow \max \min_{x : \|x\|_1 \leq 1} \left\{[e^i]^T x - f^T \bar{H}^T A x\right\} \leq s^{-1}\kappa$

$\Leftrightarrow \min \max_{f : \|f\|_* \leq 1} \left\{[e^i - A^T \bar{H} f]^T x\right\} \leq s^{-1}\kappa$

$\Leftrightarrow \forall i \leq n \exists f_i \in \mathbb{R}^N : \|e^i - A^T \bar{H} f_i\|_\infty \leq s^{-1}\kappa \& \|f_i\|_* \leq 1.$
\[ \forall i \leq n \exists f_i \in \mathbb{R}^N : \| e^i - A^T \bar{H} f_i \|_\infty \leq \kappa \text{ and } \| f_i \|_* \leq 1. \]

Let \( h_i = \bar{H} f_i \) and \( H = [h_1, \ldots, h_n] \). Then

\[ [I_n - H^T A]_{ij} = [I_n - A^T H]_{ji} = [e^i - A^T h_i]_j = [e^i - A^T \bar{H} f_i]_j \]

\[ \Rightarrow \max_{i,j} |[I_n - H^T A]_{ij}| \leq \max_i \max_j |[e^i - A^T \bar{H} f_i]_j| \]

\[ \leq \max_i \| e^i - A^T \bar{H} f_i \|_\infty \leq s^{-1} \kappa \]

\[ \Rightarrow \| \text{Col}_i[I_n - H^T A] \|_\infty \leq s^{-1} \kappa \forall i \]

Further,

\[ \| \text{Col}_i[I_n - H^T A] \|_\infty \leq s^{-1} \kappa \forall i \]

\[ \Rightarrow \| [I_n - H^T A] x \|_\infty \leq s^{-1} \kappa \| x \|_1 \forall x \in \mathbb{R}^n \]

\[ \Rightarrow \| x \|_\infty - \| H^T A x \|_\infty \leq s^{-1} \kappa \| x \|_1 \forall x \in \mathbb{R}^n \]

\[ \Rightarrow H, \| \cdot \|_\infty \text{ satisfy } Q_\infty(s, \kappa) \]

In addition,

\[ \| H^T \eta \|_\infty = \max_i |h^T_i \eta| = \max_i |f^T_i \bar{H} \eta| \leq \max_i \| f_i \|_* \| \bar{H}^T \eta \| \]

\[ \leq \| H^T \eta \| \forall \eta. \]
... In addition, \( m \times n \) contrast matrix \( H \) such that \( H, \| \cdot \|_\infty \) satisfy \( Q_\infty(s, \kappa) \) with as small \( \kappa \) as possible can be found as follows: we consider \( n \) LP programs

\[
\text{Opt}_i = \min_{\nu, h} \left\{ \nu : \| A^T h - e^i \|_\infty \leq \nu \right\}, \quad (\#i)
\]

where \( e^i \) is \( i \)-th basic orth in \( \mathbb{R}^n \), find optimal solutions \( \text{Opt}_i, h_i \) to these problems, and make \( h_i, i = 1, \ldots, n \), the columns of \( H \); the corresponding value of \( \kappa \) is \( \kappa^* = s \max_i \text{Opt}_i \).

**Proof:** By the above reasoning, if \( H, \| \cdot \| \) satisfy \( Q_\infty(s, \kappa) \), then \( \forall (i \leq n) \exists h_i : \| e^i - A^T h_i \|_\infty \leq s^{-1} \kappa \), and if \( h_i, i \leq n \), satisfy \( \| e^i - A^T h_i \|_\infty \leq s^{-1} \kappa \) for some \( \kappa \), then \( H := [h_1, \ldots, h_n], \| \cdot \|_\infty \) satisfy \( Q_\infty(s, \kappa) \).

... Finally, there exists a transparent alternative description of the quantities \( \text{Opt}_i \) (and thus – of \( \kappa^* \));

\[
\text{Opt}_i = \max_x \left\{ x_i : \| x \|_1 \leq 1, Ax = 0 \right\}.
\]

**Proof:**

\[
\text{Opt}_i = \min \left\{ t : -t \leq e^i_j - [A^T h]_j \leq t, \forall j \right\} \\
= \max_{\lambda, \mu} \left\{ [\lambda - \mu]_i : \sum_i \lambda_i + \sum_i \mu_i = 1, \lambda \geq 0, \mu \geq 0 \right\} \quad \text{[LP duality]} \\
= \max_x \left\{ x_i : A^T x = 0, \| x \|_1 \leq 1 \right\}
\]
Illustration

$k$-th Hadamard matrix $\mathcal{H}_k$ is $n_k \times n_k$ matrix, $n_k = 2^k$, with entries $\pm 1$ given by the recurrence

$$\mathcal{H}_0 = [1]; \mathcal{H}_{k+1} = \begin{bmatrix} \mathcal{H}_k & \mathcal{H}_k \\ \mathcal{H}_k & -\mathcal{H}_k \end{bmatrix}$$

Note: $\mathcal{H}_k$ is symmetric and is proportional to orthogonal matrix: $\mathcal{H}_k^T \mathcal{H}_k = n_k I_{n_k} \Rightarrow$ When $k > 0$, the only eigenvalues of $\mathcal{H}_k$ are $\sqrt{n_k}$ and $-\sqrt{n_k}$ with multiplicities $n_k/2$ each.

- Let $k > 1$, $m_k = n_k/2 = 2^{k-1}$, and let $a_1, ..., a_{m_k}$ be an orthonormal system of eigenvectors of $\mathcal{H}_k$ with eigenvalue $\sqrt{n_k}$. Let $A_k$ be the $m_k \times n_k$ matrix with the rows $a_1^T, ..., a_{m_k}^T$.

Fact: Let $s < \frac{1}{2} \sqrt{n_k} = 2^{k/2 - 1}$. Then the matrix $A_k$ is $s$-good. Moreover, there exists (and can be efficiently computed) contrast matrix $H_k$ such that $(H_k, \| \cdot \|_\infty)$ satisfies the condition $Q_\infty(s, \kappa_s = s/\sqrt{n_k})$, and $\| \text{Col}_i[H_k] \|_2 \leq \sqrt{2 + 2/\sqrt{n_k}}$ for all $j$. 

1.38
Verifiable Sufficient condition for satisfiability of $Q_q(s, \kappa)$:
Let $m \times n$ matrix $H$ satisfy the condition

$$\| \text{Col}_j [I_n - H^T A] \|_{s,q} \leq s^{\frac{1}{q} - 1} \kappa, \ 1 \leq j \leq n \quad (!)$$

Then $H, \| \cdot \|$ satisfy $Q_q(s, \kappa)$.

Proof:

$(!) \Rightarrow \| [I_n - H^T A] x \|_{s,q} \leq s^{\frac{1}{q} - 1} \kappa \| x \|_1 \ \forall x$

$\Rightarrow \| x \|_{s,q} - \| H^T A x \|_{s,q} \leq s^{\frac{1}{q} - 1} \kappa \| x \|_1 \ \forall x$

$\Rightarrow \| x \|_{s,q} \leq \| H^T A x \|_{s,q} + s^{\frac{1}{q} - 1} \kappa \| x \|_1$

$\Rightarrow \| x \|_{s,q} \leq s^{\frac{1}{q}} \| H^T A x \|_{\infty} + s^{\frac{1}{q} - 1} \kappa \| x \|_1 \ \forall x$

Note: (!) is an explicit system of convex constraints on $H$

$\Rightarrow$ The sufficient condition (!) for $H, \| \cdot \|_{\infty}$ to satisfy $Q_q(s, \kappa)$ is computationally tractable.

Note: When $q = \infty$, feasibility of (!) is necessary and sufficient for satisfiability of $Q_\infty(s, \kappa)$: ($H \in \mathbb{R}^{m \times n}, \| \cdot \|_{\infty}$) satisfies $Q_\infty(s, \kappa)$ if and only if

$$\| \text{Col}_j [I_n - H^T A] \|_{\infty} \leq s^{-1} \kappa \ \forall j.$$
Let $m \times n$ matrix $H$ satisfy the condition

$$\| \text{Col}_j[I_n - H^T A]\|_{s,q} \leq s^{\frac{1}{q}} \kappa, \quad 1 \leq j \leq n$$

(1)

Then $H, \| \cdot \|$ satisfy $Q_q(s, \kappa)$.

The above statement, whatever simple, has an instructive origin. Consider the following problem:

(?) **Given a convex function** $\phi(x) : \mathbb{R}^n \to \mathbb{R}$ and a convex set

$$X = \{ x \in \text{Conv}\{f_1, \ldots, f_N\} : Ax = 0 \}$$

$[A \in \mathbb{R}^{m \times n}]$

we want to compute/upper-bound efficiently the quantity

$$\phi_* = \max_{x \in X} \phi(x).$$

**Example:** Verifying the Nullspace Property of matrix $A$ reduces to checking whether the quantity

$$\phi_* := \max_{x \in X} \left[ \phi(x) := \|x\|_{s,1} \right],$$

$X = \{ x \in \text{Conv}\{\pm e_1, \pm e_2, \ldots, \pm e_n\} : Ax = 0 \}$

$[e_i : \text{basic orths}]$

is or is not $< 1/2$. 

1.40
\[ \phi_* = \max_{x \in X} \phi(x), \quad X = \{x \in \text{Conv}\{f_1, ..., f_N\} : Ax = 0\} \]

- \( \phi_* \) is the maximum of a convex function over a bounded polyhedral set and as such is in general NP-hard to compute. However, we can point out a simple scheme for efficient upper-bounding \( \phi_* \):

\[
\forall H \in \mathbb{R}^{m \times n} : \\
\phi_* = \max_x \{\phi(x) : x \in \text{Conv}\{f_1, ..., f_N\}, Ax = 0\} \\
= \max_x \{\phi([I - H^T A]x) : x \in \text{Conv}\{f_1, ..., f_N\}, Ax = 0\} \\
\leq \max_x \{\phi([I - H^T A]x) : x \in \text{Conv}\{f_1, ..., f_N\}\}
\]

\[
\Rightarrow \phi_* \leq \bar{\phi} := \min_H \left[ \max_{j \leq N} \phi([I - H^T A]f_j) \right],
\]

and \( \bar{\phi} \) is efficiently computable (as an optimal value in a convex problem).

- **Note:** As applied to

\[
\phi(x) = \|x\|_{s,1}, \quad X = \{x \in \text{Conv}\{\pm e_1, ..., \pm e_n\} : Ax = 0\},
\]

the above bounding scheme results in the verifiable sufficient condition

\[
\exists (\kappa < 1/2, H) : \|\text{Col}_j[I - H^T A]\|_{s,1} \leq \kappa, \quad 1 \leq j \leq n
\]

for \( s \)-goodness of \( A \). This is as a hint for developing the above verifiable sufficient conditions for \( Q_q(s, \kappa) \).
Bad news: When $m \times n$ sensing matrix $A$ is “essentially non-square”, namely, $n \geq 2m$, the above verifiable sufficient conditions for the validity of $Q_q(s, \kappa)$ can be satisfiable only in the range

$$s \leq \sqrt{2m}$$

which is much less than the range

$$s \leq O(1) \frac{m}{\ln(n/m)}$$

where random Gaussian/Rademacher $m \times n$ sensing matrices satisfy RIP($\frac{1}{4}, 2s$) with overwhelming probability, thus implying satisfiability of $Q_2(s, \frac{1}{3})$.

Note:
A. No series of individual essentially non-square $m \times n$ sensing matrices $A$ with $m, n \to \infty$ which are provably $s$-good for $s \geq O(1)\sqrt{m}$ are known
B. For $k = 1, 2, \ldots$ one can easily point out individual $2^{k-1} \times 2^k$ sensing matrices for which condition $Q_\infty(s, \frac{1}{3})$ is satisfiable whenever $s \leq \frac{\sqrt{2m}}{3}$.
C. Whenever $A$ satisfies RIP($\delta, 2k$) and $s \leq \frac{1-\delta}{3\delta} \sqrt{k}$, the pair $(H = \sqrt{\frac{k}{1-\delta}} A, \| \cdot \|_\infty)$ satisfies $Q_\infty(s, \frac{1}{3})$
D. For properly selected $C > 0$ and every $m, n$, one can point out individual $m \times n$ sensing matrix which is $C\sqrt{m}$-good.
\* \* Mutual Incoherence. \* \*

Let $A$ be $m \times n$ sensing matrix without zero columns. Mutual Incoherence of $A$ is the quantity

$$\mu(A) = \max_{i \neq j} \frac{\text{Col}_i^T [A] \text{Col}_j [A]}{\text{Col}_i^T [A] \text{Col}_i [A]}$$

**Observation:** The $m \times n$ matrix $H$ with columns $\frac{\text{Col}_j [A]}{\text{Col}_j^T [A] \text{Col}_j [A]}$, $j = 1, \ldots, n$, satisfies

$$\forall j : \| \text{Col}_j [I_n - H^T A]\|_\infty \leq \frac{\mu(A)}{1 + \mu(A)}$$

$\Rightarrow$ $H, \| \cdot \|_\infty$ satisfy $Q_\infty \left(s, \frac{s \mu(A)}{1 + \mu(A)} \right)$ for every $s$. In particular, $A$ is $s$-good, provided that

$$\frac{2\mu(A)}{1 + \mu(A)} < \frac{1}{s}.$$
HYPOTHESIS TESTING, I

- Preliminaries
  - Tests & Risks
  - Repeated Observations
  - 2-Point Lower Risk Bound
- Pairwise Tests via Euclidean Separation
- From Pairwise to Multiple Hypothesis Testing
Hypothesis Testing Problem: Given

- observation space $\Omega$ where our observations take values,
- $L$ families $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_L$ of probability distributions on $\Omega$, and
- an observation $\omega$ — a realization of random variable with unknown probability distribution $P$ known to belong to one of the families $\mathcal{P}_\ell$: $P \in \bigcup_{\ell=1}^{L} \mathcal{P}_\ell$,

we want to decide to which one of the families $\mathcal{P}_\ell$ the distribution $P$ belongs.

Equivalent wording: Given the outlined data, we want to decide on $L$ hypotheses $H_1, \ldots, H_L$, with $\ell$-th hypothesis $H_\ell$ stating that $P \in \mathcal{P}_\ell$.

A test is a function $\mathcal{T}(\cdot)$ on $\Omega$. The value $\mathcal{T}(\omega)$ of this function at a point $\omega \in \Omega$ is a subset of the set $\{1, \ldots, L\}$.

- relation $\ell \in \mathcal{T}(\omega)$ is interpreted as “given observation $\omega$, the test accepts the hypothesis $H_\ell$”
- relation $\ell \notin \mathcal{T}(\omega)$ is interpreted as “given observation $\omega$, the test rejects the hypothesis $H_\ell$”

$\mathcal{T}$ is called simple, if $\mathcal{T}(\omega)$ is a singleton for every $\omega \in \Omega$. 
For a simple test $\mathcal{T}$, its risks are defined as follows:

- $\ell$-th partial risk of $\mathcal{T}$ is the (worst-case) probability to reject $\ell$-th hypothesis when it is true:

$$\text{Risk}_\ell(\mathcal{T}|H_1, \ldots, H_L) = \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{\omega \sim P} \{ \ell \notin \mathcal{T}(\omega) \}$$

- total risk of $\mathcal{T}$ is the sum of all partial risks:

$$\text{Risk}_{\text{tot}}(\mathcal{T}|H_1, \ldots, H_L) = \sum_{1 \leq \ell \leq L} \text{Risk}_\ell(\mathcal{T}|H_1, \ldots, H_L).$$

- risk of $\mathcal{T}$ is the maximum of all partial risks:

$$\text{Risk}(\mathcal{T}|H_1, \ldots, H_L) = \max_{1 \leq \ell \leq L} \text{Risk}_\ell(\mathcal{T}|H_1, \ldots, H_L).$$

Note: What was called test is in fact a deterministic test. A randomized test is a deterministic function $\mathcal{T}(\omega, \eta)$ of observation $\omega$ and independent of $\omega$ random variable $\eta \sim P_\eta$ with once for ever fixed distribution (say, $P_\eta = \text{Uniform}[0, 1]$). The values $\mathcal{T}(\omega, \eta)$ of $\mathcal{T}$ are subsets of $\{1, \ldots, L\}$ (singletons for a simple test).

- Given observation $\omega$, we “flip a coin” (draw a realization of $\eta$), accept hypotheses $H_\ell$, $\ell \in \mathcal{T}(\omega, \eta)$, and reject all other hypotheses.

- Partial risks of randomized test are

$$\text{Risk}_\ell(\mathcal{T}|H_1, \ldots, H_L) = \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{(\omega,\eta) \sim P \times P_\eta} \{ \ell \notin \mathcal{T}(\omega, \eta) \}.$$}

Exactly as above, these risks give rise to the total risk and risk of $\mathcal{T}$. 2.2
Testing from repeated observations. There are situations where an inference can be based on several observations $\omega_1, ..., \omega_K$ rather than on a single observation. Our related setup is as follows:

We are given $L$ families $\mathcal{P}_\ell$, $\ell = 1, ..., L$, of probability distributions on observation space $\Omega$ and a collection

$$\omega^K = (\omega_1, ..., \omega_K)$$

and want to make conclusions on how the distribution of $\omega^K$ "is positioned" w.r.t. the families $\mathcal{P}_\ell$, $1 \leq \ell \leq L$. Specifically, we are interested in three situations of type:

A. Stationary $K$-repeated observations: $\omega_1, ..., \omega_K$ are independently of each other drawn from a distribution $P$. Our goal is to decide, given $\omega^K$, on the hypotheses $P \in \mathcal{P}_\ell$, $\ell = 1, ..., L$.

Equivalently: Families $\mathcal{P}_\ell$ of probability distributions of $\omega \in \Omega$, $1 \leq \ell \leq L$, give rise to the families

$$\mathcal{P}^{\circ,K}_\ell = \{P^K = \underbrace{P \times ... \times P}_K : P \in \mathcal{P}_\ell\}$$

of probability distributions on $\Omega^K = \underbrace{\Omega \times ... \times \Omega}_K$. Given observation $\omega^K \in \Omega^K$, we want to decide on the hypotheses

$$H^{\circ,K}_\ell : \omega^K \sim P^K \in \mathcal{P}^{\circ,K}_\ell, \; 1 \leq \ell \leq L.$$
B. Semi-stationary $K$-repeated observations: “The nature” selects somehow a sequence $P_1, \ldots, P_K$ of distributions on $\Omega$, and then draws, independently across $k$, observations $\omega_k$ from these distributions:

$$\omega_k \sim P_k \text{ are independent across } k \leq K$$

Our goal is to decide, given $\omega^K = (\omega_1, \ldots, \omega_K)$, on the hypotheses $\{P_k \in \mathcal{P}_\ell, 1 \leq k \leq K\}$, $\ell = 1, \ldots, L$.

Equivalently: Families $\mathcal{P}_\ell$ of probability distributions of $\omega \in \Omega$, $1 \leq \ell \leq L$, give rise to the families

$$\mathcal{P}^{\oplus, K} = \bigoplus_{k=1}^K \mathcal{P}_\ell := \{P^K = P_1 \times \ldots \times P_K : P_k \in \mathcal{P}_\ell, 1 \leq k \leq K\}$$

of probability distributions on $\Omega^K = \underbrace{\Omega \times \ldots \times \Omega}_{K}$. Given observation $\omega^K \in \Omega^K$, we want to decide on the hypotheses

$$H_{\ell}^{\oplus, K} : \omega^K \sim P^K \in \mathcal{P}^{\oplus, K}\_\ell, \ 1 \leq \ell \leq L.$$
C. Quasi-stationary $K$-repeated observations: We observe random sequence $\omega^K = (\omega_1, ..., \omega_K)$ generated as follows:

There exists a random sequence $\zeta_1, ..., \zeta_K$ of driving factors such that for $1 \leq k \leq K$

- $\omega_k$ is a deterministic function of $\zeta^k = (\zeta_1, ..., \zeta_k)$
- conditional, $\zeta^{k-1}$ given, distribution of $\omega_k$ always belongs to $\mathcal{P}_\ell$.

Our goal is to decide, given $\omega^K$, on the underlying $\ell$.

Equivalently: Families $\mathcal{P}_\ell$ of probability distributions on $\Omega$, $1 \leq \ell \leq L$, give rise to the quasi-direct products $\mathcal{P}_\ell^{\otimes,K} = \bigotimes_{k=1}^{K} \mathcal{P}_\ell$ of families $\mathcal{P}_\ell$.

The family $\bigotimes_{k=1}^{K} \mathcal{P}_\ell$ is comprised of all probability distributions on $\Omega^K = \Omega \times ... \times \Omega$ which can be obtained from $\mathcal{P}_\ell$ via the above “driving factors” mechanism.

Given observation $\omega^K \in \Omega^K$, we want to decide on the hypotheses

$$H_{\ell}^{\otimes,K} : \omega^K \sim P^K \in \mathcal{P}_\ell^{\otimes,K}, \ 1 \leq \ell \leq L.$$
Important fact: 2-point lower risk bound. Consider simple pairwise test deciding on two simple hypotheses on the distribution $P$ of observation $\omega \in \Omega$:

$$H_1: P = P_1, \ H_2: P = P_2.$$ 

Let $P_1, P_2$ have densities $p_1, p_2$ w.r.t. some reference measure $\Pi$ on $\Omega$. Then the total risk of every test $\mathcal{T}$ deciding on $H_1, H_2$ admits lower bound as follows:

$$\text{Risk}_{\text{tot}}(\mathcal{T}|H_1, H_2) \geq \int_{\Omega} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega).$$

As a result,

$$\text{Risk}(\mathcal{T}|H_1, H_2) \geq \frac{1}{2} \int_{\Omega} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega). \quad (\ast)$$

Note: The bound does not depend on the choice of $\Pi$ (for example, we can always take $\Pi = P_1 + P_2$).
\[
\text{Risk}(\mathcal{T}|H_1, H_2) \geq \frac{1}{2} \int_{\Omega} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega). \quad (?)
\]

**Proof** (for deterministic test). Simple test deciding on \(H_1, H_2\) must accept \(H_1\) and reject \(H_2\) on some subset \(\Omega_1\) of \(\Omega\) and must reject \(H_1\) and accept \(H_2\) on the complement \(\Omega_2 = \Omega \setminus \Omega_1\) of this set. We have

\[
\text{Risk}_1(\mathcal{T}|H_1, H_2) = \int_{\Omega_2} p_1(\omega) \Pi(d\omega) \geq \int_{\Omega_2} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega)
\]

\[
\text{Risk}_2(\mathcal{T}|H_1, H_2) = \int_{\Omega_1} p_2(\omega) \Pi(d\omega) \geq \int_{\Omega_1} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega)
\]

\[
\Rightarrow \quad \text{Risk}_{\text{tot}}(\mathcal{T}|H_1, H_2) \geq \int_{\Omega_2} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega) + \int_{\Omega_1} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega)
\]

\[
= \int_{\Omega} \min[p_1(\omega), p_2(\omega)] \Pi(d\omega) \quad \Box
\]
Corollary. Consider \( L \) hypotheses \( H_\ell : P \in \mathcal{P}_\ell, \ell = 1, 2, \ldots, L \), on the distribution \( P \) of observation \( \omega \in \Omega \), let \( \ell \neq \ell' \) and let \( P_\ell \in \mathcal{P}_\ell, P_{\ell'} \in \mathcal{P}_{\ell'} \). The risk of any simple test \( T \) deciding on \( H_1, \ldots, H_L \) can be lower-bounded as

\[
\text{Risk}(T|H_1, \ldots, H_L) \geq \frac{1}{2} \int_{\Omega} \min [P_\ell(d\omega), P_{\ell'}(d\omega)],
\]

where, by convention, the integral in the right hand side is

\[
\int_{\Omega} \min[p_\ell(\omega), p_{\ell'}(\omega)] \Pi(d\omega),
\]

with \( p_\ell, p_{\ell'} \) being the densities of \( P_\ell, P_{\ell'} \) w.r.t. \( \Pi = P_\ell + P_{\ell'} \). Indeed, risk of \( T \) cannot be less than the risk of the naturally induced by \( T \) simple test deciding on two simple hypotheses \( P = P_\ell, P = P_{\ell'} \), specifically, the simple test which, given observation \( \omega \) accepts the hypothesis \( P = P_1 \) whenever \( \ell \in T(\omega) \) and accepts the hypothesis \( P = P_{\ell'} \) otherwise.
Pairwise Hypothesis Testing via Euclidean Separation

Situation: Let $\Omega = \mathbb{R}^d$, and let our observation be

$$\omega = x + \xi$$

where the deterministic vector $x$ is the signal of interest, and $\xi$ is random observation noise with probability density $p(\cdot)$ of the form

$$p(u) = f(\|u\|_2)$$

where $f(\cdot)$ is a strictly monotonically decreasing function on the nonnegative ray.

Simple example: standard (zero mean, unit covariance) Gaussian noise: $p(u) = (2\pi)^{-d/2}e^{-u^Tu/2}$.

Our goal is to decide on two simple hypotheses on the signal underlying observation, the first stating that $x = x^1$, and the second stating that $x = x^2$, where $x^1$, $x^2$ are two given points.

Equivalent wording: We are given two probability distributions, $P_1$ and $P_2$, on $\mathbb{R}^d$, with densities $p_1(u) = p(u - x^1)$ and $p_2(u) = p(u - x^2)$, and want to decide on two simple hypotheses $H_1 : P = P_1$, $H_2 : P = P_2$ on the distribution $P$ of our observation.
Assuming \( x^1 \neq x^2 \), let \( 2\delta = \|x^1 - x^2\|_2 \), \( e = \frac{x^1 - x^2}{\|x^1 - x^2\|_2} \),

\[ \Pi = \{ \omega : \|\omega - x^1\|_2 = \|\omega - x^2\|_2 \} = \{ \omega : \phi(\omega) = 0 \}, \]

\[ \phi(\omega) = e^T \omega - \frac{1}{2} e^T [x^1 + x^2] \]

Consider test \( T \) which, given observation \( \omega = x + \xi \), accepts the hypothesis \( H_1 : P = P_1 \) (i.e., \( x = x^1 \)) when \( \phi(\omega) \geq 0 \), and accepts the hypothesis \( H_2 : P = P_2 \) (i.e., \( x = x^2 \)) otherwise. We have

\[
\text{Risk}_1(T|H_1, H_2) = \int_{\omega: \phi(\omega) < 0} p_1(\omega) d\omega = \int_{u:e^T u \geq \delta} f(\|u\|_2) du \\
= \int_{\omega: \phi(\omega) \geq 0} p_2(\omega) d\omega = \text{Risk}_2(T|H_1, H_2)
\]

Since \( p(u) \) is strictly decreasing function of \( \|u\|_2 \), we have also

\[
\min[p_1(u), p_2(u)] = \begin{cases} 
  p_1(u), & \phi(u) \geq 0 \\
  p_2(u), & \phi(u) \leq 0
\end{cases}
\]

whence

\[
\text{Risk}_1(T|H_1, H_2) + \text{Risk}_2(T|H_1, H_2) = \int_{\omega: \phi(\omega) < 0} p_1(\omega) d\omega + \int_{\omega: \phi(\omega) \geq 0} p_2(\omega) d\omega = \int_{\mathbb{R}^d} \min[p_1(u), p_2(u)] du
\]

\( \Rightarrow \) Test \( T \) is the minimum risk simple test deciding on \( H_1, H_2 \).
Extension: Given observation $\omega = x + \xi$ with observation noise $\xi$ possessing probability density

$$p(u) = f(\|u\|_2),$$

where $f(\cdot)$ is a strictly decreasing function on the nonnegative ray, we want do decide on two composite hypotheses $H_1, H_2$:

$$H_1 : x \in X_1, \quad H_2 : x \in X_2,$$

where $X_1, X_2$ are nonempty nonintersecting, closed and convex sets, and one of the sets is bounded.

Elementary fact: With $X_1, X_2$ as above, consider the convex minimization problem

$$\text{Opt} = \min_{x^1 \in X_1, x^2 \in X_2} \frac{1}{2} \|x^1 - x^2\|_2.$$

The problem is solvable. Let $(x^1_*, x^2_*)$ be an optimal solution, and let

$$\phi(\omega) = e^T \omega - c, \quad e = -\frac{x^1_* - x^2_*}{\|x^1_* - x^2_*\|_2}, \quad c = \frac{1}{2} e^T [x^1_* + x^2_*]$$

Then the stripe $\{\omega : -\text{Opt} \leq \phi(\omega) \leq \text{Opt}\}$ separates $X_1$ and $X_2$:

$$\phi(x^1) \geq \phi(x^1_*) = \text{Opt} \ \forall x^1 \in X_1,$$

$$\phi(x^2) \geq \phi(x^2_*) = -\text{Opt} \ \forall x^2 \in X_2$$
\[ \phi(\omega) = \text{Opt} \]

\[ \phi(\omega) = -\text{Opt} \]

\[ \star \text{ We have associated with two non-intersecting closed convex } X_1, X_2, \text{ one of the sets being bounded,} \]

\[ \text{— convex optimization problem} \]

\[ \text{Opt} = \min_{x^1 \in X_1, x^2 \in X_2} \frac{1}{2} \| x^1 - x^2 \|_2 \]

\[ \text{— linear function} \]

\[ \phi(\omega) = e^T \omega - \frac{1}{2} e^T [x^1_* + x^2_*], \]

\[ e = \frac{1}{2 \text{Opt}} [x^1_* - x^2_*] \]

where \([x^1_*, x^2_*]\) is an optimal solution to the above problem. While this solution not necessarily is uniquely defined by \(X_1, X_2\), \(\phi(\cdot)\) is uniquely defined by \(X_1, X_2\).
Given $\delta_1 \geq 0, \delta_2 \geq 0$ with $\delta_1 + \delta_2 = 2\text{Opt}$, $\phi(\cdot)$ specifies simple \textit{Euclidean Separation Test} $\mathcal{T}$ induced by $X_1, X_2, \delta_1, \delta_2$:

$$
\mathcal{T}(\omega) = \begin{cases} 
\{1\}, & \phi(\omega) \geq \frac{1}{2}[\delta_2 - \delta_1] \\
\{2\}, & \text{otherwise}
\end{cases}
$$

\textbf{Fact:} Let $\xi \sim p(\cdot)$, where $p(u) = f(\|u\|_2)$ with strictly decreasing $f(t), t \geq 0$. Given observation $\omega = x + \xi$ the \textit{Euclidean Separation Test} $\mathcal{T}$ decides on the hypotheses

$$H_1 : x \in X_1, \ H_2 : x \in X_2$$

with risks satisfying

$$\text{Risk}_1(\mathcal{T}|H_1, H_2) \leq \int_{\delta_1}^{\infty} \gamma(s)ds, \quad \text{Risk}_2(\mathcal{T}|H_1, H_2) \leq \int_{\delta_2}^{\infty} \gamma(s)ds$$

where $\gamma(\cdot)$ is the univariate marginal density of $\xi$, that is, probability density of the scalar random variable $h^T \xi$, where $\|h\|_2 = 1$.  

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In addition, when $\delta_1 = \delta_2 = \text{Opt}$, $T$ is the minimum risk test deciding on $H_1, H_2$. The risk of this test is

$$\text{Risk}(T \mid H_1, H_2) = \int_{\text{Opt}}^{\infty} \gamma(s)ds.$$
Extension: Under the premise of Fact, i.e., when

- the observation is $\omega = x + \xi$ with $\xi \sim p(\cdot) = f(\| \cdot \|_2)$, where
  - $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly decreasing, and
  - the hypotheses to be decided upon are $H_1 : x \in X_1$, $H_2 : x \in X_2$ with closed convex nonintersecting and nonempty $X_1, X_2$, one of the sets being bounded, the risk bounds $\text{Risk}_\ell(T|H_1, H_2) \leq \int_\delta^\infty \gamma(s)ds$, $\ell = 1, 2$ for the Euclidean Separation Test stem from the fact that under the circumstances,

For every half-space $E = \{ u \in \mathbb{R}^d : e^T u \geq \delta \}$, where $\| e \|_2 = 1$ and $\delta \geq 0$, one has

$$\text{Prob}_{\xi \sim p(\cdot)} \{ \xi \in E \} \leq \int_\delta^\infty \gamma(s)ds.$$
Given an even probability density $\gamma(\cdot)$ on the axis such that
$$\int_{\delta}^{\infty} \gamma(s) ds < \frac{1}{2}$$
whenever $\delta > 0$, let us associate with it the family $P_{\gamma}^{d}$ of all probability distributions $P$ on $\mathbb{R}^d$ such that

A: distribution $P$ possesses even density, and

B: whenever $e \in \mathbb{R}^d$, $\|e\|_2 = 1$, and $\delta \geq 0$, we have
$$\text{Prob}_{\xi \sim P}\{\xi : e^T\xi \geq \delta\} \leq \Gamma(\delta) := \int_{\delta}^{\infty} \gamma(s) ds$$

By the same reasons as in Fact, we have the following

Proposition. Whenever the distribution $P$ of noise $\xi$ in observation $\omega = x + \xi$ belongs to $P_{\gamma}^{d}$ and $X_1, X_2$ are non-intersecting closed convex sets, one of the sets being bounded, the risks of the Euclidean Separation Test $T$ induced by $X_1, X_2$ and $\delta_1, \delta_2$ can be upper-bounded as

$$\text{Risk}_\ell(T|H_1, H_2) \leq \Gamma(\delta_\ell) := \int_{\delta_\ell}^{\infty} \gamma(s) ds, \ \ell = 1, 2.$$
Example: Gaussian mixtures. Let \( \eta \) be an \( d \)-dimensional Gaussian random vector with zero mean and covariance matrix \( \Theta \) (notation: \( \eta \sim \mathcal{N}(0, \Theta) \)). Let, further, \( Z \) be independent of \( \eta \) positive random variable. Gaussian mixture is the probability distribution of the random vector \( \xi = \sqrt{Z} \eta \). Examples of Gaussian mixtures are:

- Gaussian distribution \( \mathcal{N}(0, \Theta) \) (take \( Z \) identically equal to 1),
- multidimensional Student’s \( t \)-distribution with \( \nu \) degrees of freedom (\( \nu/Z \) has \( \chi^2 \)-distribution with \( \nu \) degrees of freedom)

Immediate Observations:

- Let \( Z \) be a random variable taking values in \([0, 1]\), let \( \eta \sim \mathcal{N}(0, \Theta) \) with \( \Theta \preceq I_d \) (i.e., the matrix \( I_d - \Theta \) is positive semidefinite) be independent of \( Z \), and let

\[
\gamma_g(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}
\]

be the standard (zero mean, unit variance) Gaussian density on the axis. Then the distribution of the Gaussian mixture \( \xi = \sqrt{Z} \eta \) belongs to the family \( \mathcal{P}_d^{\gamma_g} \).

- With \( \gamma \) given by the distribution \( P_Z \) of \( Z \) according to

\[
\gamma_Z(s) = \int_{z>0} \frac{1}{\sqrt{2\pi z}} e^{-s^2/2z} P_Z(dz),
\]

the distribution of random variable \( \sqrt{Z} \eta \), with \( \eta \sim \mathcal{N}(0, \Theta) \), \( \Theta \preceq I_d \), independent of \( Z \), belongs to the family \( \mathcal{P}_d^{\gamma_Z} \).
Let $\gamma(\cdot), \mathcal{P}_\gamma^d, X_1, X_2$ be as in Proposition, and assume we have access to semi-stationary $K$-repeated observations

$$\omega^K = \{\omega_k = x_k + \xi_k : 1 \leq k \leq K\}$$

where

- $\{x_k : 1 \leq k \leq K\}$ is a deterministic sequence of signals,
- $\xi_k \sim P_k, 1 \leq k \leq K$, are independent across $k$ noises, and
- $\{P_k, 1 \leq k \leq K\}$ is a deterministic sequence of distributions from $\mathcal{P}_\gamma^d$.

Given $\omega^K$, we want to decide on the hypotheses $H^K_1 : x_k \in X_1, 1 \leq k \leq K$ and $H^K_2 : x_k \in X_2, 1 \leq k \leq K$.

Equivalently: The sets $X_\ell, \ell = 1, 2$, give rise to families $\mathcal{P}_\ell$ of probability distributions on $\Omega = \mathbb{R}^d$; $\mathcal{P}_\ell$ is comprised of distributions $P$ of random vectors of the form $x + \xi$, with deterministic $x \in X_\ell$ and with the distribution of noise $\xi$ belonging to $\mathcal{P}_\gamma^d$. The families $\mathcal{P}_\ell$, in turn, give rise to hypotheses

$$H^K_{\ell} = H^K_{\ell} : P^K \in \mathcal{P}_\ell^{\bigoplus,K}, \ell = 1, 2,$$

on the distribution $P^K$ of $K$-repeated observation $\omega^K = (\omega_1, ..., \omega_K)$. Given $\omega^K$, we want to decide on the hypotheses $H^K_1, H^K_2$. 2.18
\[ \omega^K = \{ \omega_k = x_k + \xi_k : 1 \leq k \leq K \} \]

\[ H^K \ell : x_k \in X_\ell, 1 \leq k \leq K, \xi_k \sim P_k \in \mathcal{P}_d : \text{ independent across } k \]

Let us use the majority test \( T^{\text{maj}}_K \) defined as follows:

- we build the Euclidean separator of \( X_1, X_2 \), thus arriving at the affine function

\[ \phi(\omega) = e^T \omega - c \quad [\|e\|_2 = 1] \]

such that the stripe

\[ \{\omega : -\text{Opt} \leq \phi(\omega) \leq \text{Opt} \} \]

with

\[ \text{Opt} = \min_{x^1 \in X_1, x^2 \in X_2} \frac{1}{2} \|x^1 - x^2\|_2, \]

separates \( X_1, X_2 \);

- given \((\omega_1, ..., \omega_K)\), we compute reals \( v_k = \phi(\omega_k), 1 \leq k \leq K \), and accept \( H^K_1 \) when the number of nonnegative \( v_k \)'s is at least \( K/2 \), otherwise we accept \( H^K_2 \).
★ Risk analysis. Assume that $H_1^K$ takes place, so that $\{x_k\}$ form some deterministic sequence of points from $X_1$, and $\xi_k$ are drawn, independently across $k$, from some distributions $P_k \in \mathcal{P}^d$. With $\{x_k\}$ and $\{P_k\}$ fixed, $v_k$ are independent across $k$, and probability for $v_k$ to be negative is, by our previous results, $\leq \epsilon_* := \Gamma(\text{Opt}) := \int_{\text{Opt}}^{\infty} \gamma(s) ds$, where

$$\text{Opt} = \min_{x^1 \in X_1, x^2 \in X_2} \frac{1}{2} \|x^1 - x^2\|_2.$$  

Consequently, the probability to reject $H_1^K$ under the circumstances is $\leq \epsilon_K := \sum_{K/2 \leq k \leq K} \binom{K}{k} \epsilon_*^k (1 - \epsilon_*)^{K-k}$. By “symmetric” reasoning, the probability to reject $H_2^K$ when the hypothesis is true is $\leq \epsilon_K$ as well.

- We arrive at

★ Proposition. The risk of $T_{maj}^K$ can be upper-bounded as

$$\text{Risk}(T_{maj}^K | H_1^K, H_2^K) \leq \sum_{K/2 \leq k \leq K} \binom{K}{k} \epsilon_*^k (1 - \epsilon_*)^{K-k} \left[ \epsilon_* = \int_{\text{Opt}}^{\infty} \gamma(s) ds, \text{Opt} = \min_{x^1 \in X_1, x^2 \in X_2} \frac{1}{2} \|x^1 - x^2\|_2 \right]$$

Fact: Conclusion remains true in the case of quasi-stationary observations.
Quiz: We have used “evident” observation as follows:

Let $w_1, \ldots, w_K$ be independent random variables taking values 0 and 1, and let the probability for $w_i$ to take value 1 be some $p_i \in [0, 1]$. Then for every fixed $M$ the probability of the event “at least $M$ of $w_1, \ldots, w_K$ are equal to 1” as a function of $p_1, \ldots, p_K$ is nondecreasing in every one of $p_i$’s. (In our context, $w_i$ were the signs of $v_i$).

Why this observation is true?
From Pairwise to Multiple Hypotheses Testing

♣ **Situation:** We are given $L$ families of probability distributions $\mathcal{P}_\ell$, $1 \leq \ell \leq L$, on observation space $\Omega$, and observe a realization of random variable $\omega \sim P$ taking values in $\Omega$. Given $\omega$, we want to decide on the $L$ hypotheses

$$H_\ell : P \in \mathcal{P}_\ell, \ 1 \leq \ell \leq L.$$ 

**Our ideal goal** would be to find a low-risk simple test deciding on the hypotheses.

**However:** It may happen that the “ideal goal” is not achievable, for example, when some pairs of families $\mathcal{P}_\ell$ have nonempty intersections. When $\mathcal{P}_\ell \cap \mathcal{P}_{\ell'} \neq \emptyset$ for some $\ell \neq \ell'$, there is no way to decide on the hypotheses with risk $< 1/2$.

**But:** *Impossibility to decide reliably on all $L$ hypotheses “individually” does not mean that no meaningful inferences can be done.*
Example: Consider the 3 colored rectangles on the plane:

and 3 hypotheses, with $H_\ell$, $1 \leq \ell \leq 3$, stating that our observation is $\omega = x + \xi$ with deterministic “signal” $x$ belonging to $\ell$-th rectangle and $\xi \sim \mathcal{N}(0, \sigma^2 I_2)$.

Whatever small $\sigma$ be, no test can decide on the 3 hypotheses with risk $< 1/2$; e.g., there is no way to decide reliably $H_1$ vs. $H_2$. However, we may hope that when $\sigma$ is small, an observation allows us to discard reliably some of the hypotheses. For example, if $H_1$ is true, we hopefully can discard $H_3$.

When handling multiple hypotheses which cannot be reliably decided upon “as they are,” it makes sense to speak about testing the hypotheses “up to closeness.”
\[ \omega \sim P, \quad H_\ell : P \in \mathcal{P}_\ell, \quad 1 \leq \ell \leq L \]

♣ **Closeness relation** \( C \) on \( L \) hypotheses \( H_1, \ldots, H_L \) is defined as some set of pairs \((\ell, \ell')\) with \( 1 \leq \ell, \ell' \leq L \); we interpret the relation \((\ell, \ell') \in C\) as the fact that the hypotheses \( H_\ell \) and \( H'_\ell \) are close to each other.

We always assume that

- \( C \) contains all “diagonal pairs” \((\ell, \ell), 1 \leq \ell \leq L\) (“every hypothesis is close to itself”)
- \((\ell, \ell') \in C\) if and only if \((\ell', \ell) \in C\) (“closeness is symmetric relation”)

**Note:** By symmetry of \( C \), the relation \((\ell, \ell') \in \mathcal{T}\) is in fact a property of **unordered** pair \(\{\ell, \ell'\}\).
“Up to closeness” risks. Let $\mathcal{T}$ be a test deciding on $H_1, \ldots, H_L$; given observation $\omega$, $\mathcal{T}$ accepts all hypotheses $H_\ell$ with indexes $\ell \in \mathcal{T}(\omega)$ and rejects all other hypotheses. We say that $\ell$-th partial $C$-risk of test $\mathcal{T}$ is $\leq \epsilon$, if whenever $H_\ell$ is true: $\omega \sim P \in \mathcal{P}_\ell$, the $P$-probability of the event

$\mathcal{T}$ accepts $H_\ell: \ell \in \mathcal{T}(\omega)$

and

all hypotheses $H_{\ell'}$ accepted by $\mathcal{T}$ are $C$-close to $H_\ell$: $(\ell, \ell') \in C \forall \ell' \in \mathcal{T}(\omega)$

is at least $1 - \epsilon$.

$\ell$-th partial $C$-risk of $\mathcal{T}$ is the smallest $\epsilon$ with the outlined property:

\[
\operatorname{Risk}_C^\ell (\mathcal{T}|H_1, \ldots, H_L) = \sup_{P \in \mathcal{P}_\ell} \operatorname{Prob}_{\omega \sim P} \{[\ell \notin \mathcal{T}(\omega)] \text{ or } [\exists \ell' \in \mathcal{T}(\omega): (\ell, \ell') \notin C] \}
\]

$C$-risk of $\mathcal{T}$ is the largest of the partial $C$-risks of the test:

\[
\operatorname{Risk}_C (\mathcal{T}|H_1, \ldots, H_L) = \max_{1 \leq \ell \leq L} \operatorname{Risk}_C^\ell (\mathcal{T}|H_1, \ldots, H_L).
\]
\[ \omega \sim P, \ \ H_\ell : \ P \in \mathcal{P}_\ell, \ 1 \leq \ell \leq L \]
\[ \mathcal{C} : \text{closeness relation} \]

\[ \text{♣ Multiple Hypothesis Testing via Pairwise Tests.} \] Assume that for every unordered pair \( \{\ell, \ell'\} \) with \((\ell, \ell') \notin \mathcal{C} \) we are given a simple test \( T_{\{\ell, \ell'\}} \) deciding on \( H_\ell \) vs. \( H_{\ell'} \) via observation \( \omega \). Our goal is to “assemble” the tests \( T_{\{\ell, \ell'\}}, (\ell, \ell') \notin \mathcal{C} \), into a test \( T \) deciding on \( H_1, \ldots, H_L \) up to closeness \( \mathcal{C} \).

\[ \text{♣ The construction:} \]
- For \((\ell, \ell') \notin \mathcal{C} \), so that \( \ell \neq \ell' \), we define function \( T_{\ell \ell'}(\omega) \) as follows:

\[ T_{\ell \ell'}(\omega) = \begin{cases} 1, & T_{\{\ell, \ell'\}}(\omega) = \{\ell\} \\ -1, & T_{\{\ell, \ell'\}}(\omega) = \{\ell'\} \end{cases} \]

**Note:** \( T_{\{\ell, \ell'\}} \) is a simple test \( \Rightarrow T_{\ell \ell'}(\cdot) \) is well defined and takes values \( \pm 1 \).

- For \((\ell, \ell') \in \mathcal{C} \), we set \( T_{\ell \ell'}(\cdot) \equiv 0 \).

**Note:** By construction, we have \( T_{\ell \ell'}(\omega) \equiv -T_{\ell' \ell}(\omega), \ 1 \leq \ell, \ell' \leq L \).

- The test \( T \) is as follows: given observation \( \omega \), we build the \( L \times L \) matrix \( T(\omega) = [T_{\ell \ell'}(\omega)] \) and accept exactly those of the hypotheses \( H_\ell \) for which \( \ell \)-th row in \( T(\omega) \) is nonnegative, that is, all tests \( T_{\{\ell, \ell'\}} \) with \((\ell, \ell') \notin \mathcal{C} \) accept \( H_\ell \), observation being \( \omega \).
Example: • \( L = 4 \)

• \( C = \{(1, 1), (2, 2), (3, 3), (4, 4), \{1, 2\}, \{2, 3\}, \{3, 4\}\} \)

Given tests \( T_{\{1,3\}}, T_{\{1,4\}}, T_{\{2,4\}} \) and observation \( \omega \)

♠ When \( T_{\{1,3\}} \) accepts \( H_1 \), \( T_{\{1,4\}} \) accepts \( H_1 \), \( T_{\{2,4\}} \) accepts \( H_4 \), we get

\[
T(\omega) = \begin{bmatrix}
0 & 0 & +1 & +1 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
-1 & +1 & 0 & 0
\end{bmatrix}
\]

\( \Rightarrow \) Aggregated test \( T \) accepts \( H_1 \)

♠ When \( T_{\{1,3\}} \) accepts \( H_1 \), \( T_{\{1,4\}} \) accepts \( H_1 \), \( T_{\{2,4\}} \) accepts \( H_2 \), we get

\[
T(\omega) = \begin{bmatrix}
0 & 0 & +1 & +1 \\
0 & 0 & 0 & +1 \\
-1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0
\end{bmatrix}
\]

\( \Rightarrow \) Aggregated test \( T \) accepts \( H_1 \) and \( H_2 \)
Observation: When $T$ accepts some hypothesis $H_\ell$, all hypotheses accepted by $T$ are $C$-close to $H_\ell$.

Indeed, if $\ell$-th row in $T(\omega)$ is nonnegative and $\ell'$ is not $C$-close to $\ell$, we have $T_{\ell\ell'}(\omega) \geq 0$ and $T_{\ell\ell'}(\omega) \in \{-1, 1\}$

$\Rightarrow T_{\ell\ell'}(\omega) = 1$

$\Rightarrow T_{\ell'\ell}(\omega) = -T_{\ell\ell'}(\omega) = -1$

$\Rightarrow \ell'$-th row in $T(\omega)$ is not nonnegative

$\Rightarrow \ell'$ is not accepted.
\textbf{Risk analysis.} For \((\ell, \ell') \notin C\), let

\[
\epsilon_{\ell \ell'} = \text{Risk}_1(T_{\{\ell, \ell'\}}|H_\ell, H_{\ell'}) = \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{\omega \sim P}\{\ell \notin T_{\{\ell, \ell'\}}(\omega)\}
\]

\[
= \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{\omega \sim P}\{T_{\ell \ell'}(\omega) = -1\} = \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{\omega \sim P}\{T_{\ell \ell'}(\omega) = 1\}
\]

\[
= \sup_{P \in \mathcal{P}_\ell} \text{Prob}_{\omega \sim P}\{\ell' \in T_{\{\ell, \ell'\}}(\omega)\}
\]

\[
= \text{Risk}_2(T_{\{\ell, \ell'\}}|H_{\ell'}, H_\ell).
\]

\textbf{Proposition.} One has

\[
\text{Risk}_C^C(T|H_1, \ldots, H_L) \leq \epsilon_\ell := \sum_{\ell': (\ell, \ell') \notin C} \epsilon_{\ell \ell'}.
\]

Indeed, let us fix \(\ell\), and let \(H_\ell\) be true. Let \(P \in \mathcal{P}_\ell\) be the distribution of observation \(\omega\), and let \(I = \{\ell' \leq L : (\ell, \ell') \notin C\}\). For \(\ell' \in I\), let \(E_{\ell'}\) be the event \(\{\omega : T_{\ell \ell'}(\omega) = -1\}\). We have \(\text{Prob}_{\omega \sim P}(E_{\ell'}) \leq \epsilon_{\ell \ell'}\) (by definition of \(\epsilon_{\ell \ell'}\))

\[
\Rightarrow \text{Prob}_{\omega \sim P}\left(\bigcup_{\ell' \in I} E_{\ell'}\right) \leq \epsilon_\ell.
\]

When the event \(E\) does not take place, we have \(T_{\ell \ell'}(\omega) = 1\) for all \(\ell' \in I\)

\[
\Rightarrow T_{\ell \ell'}(\omega) \geq 0 \text{ for all } \ell', 1 \leq \ell' \leq L
\]

\[
\Rightarrow \ell \in T(\omega)
\]

\[
\Rightarrow (\text{by Observation}) \{\ell \in T(\omega)\} \& \{(\ell, \ell') \in C \forall \ell' \in T(\omega)\}.
\]

By definition of partial \(C\)-risk, we get

\[
\text{Risk}_C^C(T|H_1, \ldots, H_L) \leq \text{Prob}_{\omega \sim P}(E) \leq \epsilon_\ell.
\]
Testing Multiple Hypotheses via Euclidean Separation

♠ Situation: We are given \( L \) nonempty, closed and bounded convex sets \( X_\ell \subset \mathbb{R}^d, 1 \leq \ell \leq L \), and a family \( \mathcal{P}^d_\gamma \) of noise distributions, a closeness \( C \), and semi-stationary \( K \)-repeated observation

\[
\omega^K = \{\omega_k = x_k + \xi_k, 1 \leq k \leq K\},
\]

so that

- \( \{x_k, 1 \leq k \leq K\} \), is a deterministic sequence of signals,
- \( \xi_k \sim P_k, 1 \leq k \leq K \), are independent across \( k \) noises, and
- \( \{P_k, 1 \leq k \leq K\} \), is a deterministic sequence of distributions from \( \mathcal{P}^d_\gamma \).

Given \( \omega^K \), we want to decide up to closeness \( C \) on \( L \) hypotheses

\[
H_\ell : \{x_k \in X_\ell, 1 \leq k \leq K\}.
\]
Given $\omega^K$, we want to decide up to closeness $C$ on $L$ hypotheses

$$H_\ell : \{ x_k \in X_\ell, 1 \leq k \leq K \}.$$  

**Equivalently:** The sets $X_\ell \subset \mathbb{R}^d$ along with $\mathcal{P}_d^\gamma$ specify $L$ families of distributions $\mathcal{P}_\ell$, $1 \leq \ell \leq L$; specifically, $\mathcal{P}_\ell$ is comprised of probability distributions of random variables $x + \xi$, where deterministic $x$ belongs to $X_\ell$, and the distribution of random noise $\xi$ belongs to $\mathcal{P}_d^\gamma$. Given $\omega^K$, we want to decide, up to closeness $C$, on $L$ hypotheses

$$H_\ell : P^K \in \mathcal{P}_\ell^\oplus^K, 1 \leq \ell \leq L$$

on the distribution $P^K$ of observation $\omega^K$.

♠ **Standing Assumption:** Whenever $\ell, \ell'$ are not $C$-close: $(\ell, \ell') \notin C$, the sets $X_\ell, X_{\ell'}$ do not intersect.

♠ **Strategy:** We intend to assemble pairwise Euclidean separation tests.
♠ **Building blocks.** For \((\ell, \ell') \notin C\), we solve convex optimization problems

\[
\text{Opt}_{\ell \ell'} = \min_{u \in X_\ell, v \in X_{\ell'}} \frac{1}{2} \|u - v\|_2. \quad (P_{\ell \ell'})
\]

**Note:** By Standing Assumption, \(\text{Opt}_{\ell \ell'} > 0\). Optimal solution \((u_*, v_*)\) to \((P_{\ell \ell'})\) defines affine functions

\[
\phi_{\ell \ell'}(\omega) = e^{T}_{\ell \ell'} \omega - c_{\ell \ell'}
\]

\[
e_{\ell \ell'} = \frac{u_* - v_*}{\|u_* - v_*\|_2}, \quad c_{\ell \ell'} = \frac{1}{2} e^{T}_{\ell \ell'} [u_* + v_*]
\]

**Note:** We have \(\phi_{\ell \ell'}(\cdot) \equiv -\phi_{\ell' \ell}(\cdot)\) for all \((\ell, \ell') \notin C\).

♥ As we know, whenever \(\delta_{\ell \ell'} \geq 0, \delta_{\ell' \ell} \geq 0\) satisfy

\[
2\text{Opt}_{\ell \ell'} = \delta_{\ell \ell'} + \delta_{\ell' \ell}
\]
it holds

\[
\forall (u \in X_\ell, P \in \mathcal{P}_d^\gamma) : \quad \text{Prob}_{\xi \sim P} \{ \phi(u + \xi) < \frac{1}{2} [\delta_{\ell' \ell} - \delta_{\ell \ell'}] \} \leq \Gamma(\delta_{\ell \ell'}) := \int_{\delta_{\ell \ell'}}^\infty \gamma(s) ds
\]

\[
\forall (v \in X_{\ell'}, P \in \mathcal{P}_d^\gamma) : \quad \text{Prob}_{\xi \sim P} \{ \phi(u + \xi) \geq \frac{1}{2} [\delta_{\ell' \ell} - \delta_{\ell \ell'}] \} \leq \Gamma(\delta_{\ell' \ell}) := \int_{\delta_{\ell' \ell}}^\infty \gamma(s) ds
\]
\( \ell, \ell' : (\ell, \ell') \not\in \mathcal{C} \)

\[ \text{Opt}_{\ell \ell'} = \min_{u \in X_\ell, v \in X_{\ell'}} \frac{1}{2} \|u - v\|_2 > 0 = \text{Opt}_{\ell' \ell} \]

\[ u_*, v_*, \phi_{\ell \ell'}(\omega) = e_{\ell \ell'}^T \omega - c_{\ell \ell'} \equiv -\phi_{\ell' \ell}(\omega) \]

\[ e_{\ell \ell'} = \frac{u_* - v_*}{\|u_* - v_*\|_2}, \quad c_{\ell \ell'} = \frac{1}{2} e_{\ell \ell'}^T [u_* + v_*] \]

\[ \delta_{\ell \ell'} \geq 0, \delta_{\ell' \ell} \geq 0, 2 \text{Opt}_{\ell \ell'} = \delta_{\ell \ell'} + \delta_{\ell' \ell} \quad \text{(*)} \]

\[ \forall (u \in X_\ell, P \in \mathcal{P}_\ell^d) : \quad \text{Prob}_{\xi \sim P} \left\{ \phi(u + \xi) < \frac{1}{2} [\delta_{\ell \ell} - \delta_{\ell' \ell}] \right\} \]

\[ \leq \Gamma(\delta_{\ell \ell}) := \int_{\delta_{\ell \ell}}^{1} \gamma(s) ds \]

\[ \Rightarrow \]

\[ \forall (v \in X_{\ell'}, P \in \mathcal{P}_{\ell'}^d) : \quad \text{Prob}_{\xi \sim P} \left\{ \phi(v + \xi) \geq \frac{1}{2} [\delta_{\ell' \ell} - \delta_{\ell \ell'}] \right\} \]

\[ \leq \Gamma(\delta_{\ell' \ell}) := \int_{\delta_{\ell' \ell}}^{1} \gamma(s) ds \quad \text{(!)} \]

\[ \begin{cases} \{ \ell \}, & \phi_{\ell \ell'}(\omega) \geq \frac{1}{2} [\delta_{\ell \ell} - \delta_{\ell' \ell}] \\ \{ \ell' \}, & \phi_{\ell' \ell}(\omega) < \frac{1}{2} [\delta_{\ell' \ell} - \delta_{\ell \ell'}] \end{cases} \]

\[ \mathcal{T}_{\{\ell, \ell'\}}(\omega) = \mathcal{T}_{\{\ell, \ell'\}} \]

\[ \text{Further, we use out general construction to assemble pairwise tests } \mathcal{T}_{\{\ell, \ell'\}} : (\ell, \ell') \not\in \mathcal{C} \text{ into single-observation test } \mathcal{T} \text{ deciding on } H_1, ..., H_L \]

**Note:** By (!), the associated with tests \( \mathcal{T}_{\{\ell, \ell'\}} \) quantities \( \epsilon_{\ell \ell'} \) satisfy the relations \( \epsilon_{\ell \ell'} \leq \Gamma(\delta_{\ell \ell'}) := \int_{\delta_{\ell \ell'}}^{1} \gamma(s) ds \), whence

\[ \text{Risk}_C^\mathcal{C}(\mathcal{T} | H_1, ..., H_L) \leq \sum_{\ell' : (\ell, \ell') \not\in \mathcal{C}} \Gamma(\delta_{\ell \ell'}). \]

2.33
\[ \ell, \ell' : (\ell, \ell') \not\in C \Rightarrow \text{Opt}_{\ell \ell'} = \min_{u \in X_{\ell}, v \in X'_{\ell}} \frac{1}{2} \| u - v \|_2 \]

\[ \Rightarrow \delta_{\ell \ell'} \geq 0, \delta_{\ell' \ell} \geq 0, 2 \text{Opt}_{\ell \ell'} = \delta_{\ell \ell'} + \delta_{\ell' \ell} \]

\[ T : \text{Risk}^C_\ell (T | H_1, ..., H_L) \leq \sum_{\ell' : (\ell, \ell') \not\in C} \Gamma(\delta_{\ell \ell'}) \]

\[ \Rightarrow \Gamma(\delta) = \int_\delta^\infty \gamma(s) \, ds \]

\[ \text{♠ Single-observation case } K = 1 : \text{ optimizing the construction.} \]

Let us optimize the (upper bounds) on partial \( C \)-risks of the assembled test \( T \) over the “free parameters” \( \delta_{\ell \ell'}, \ell, \ell' \not\in C \), of the construction.

\[ \text{♥ A natural model here is as follows: given nonnegative weight matrix } W \text{ and nonnegative vectors } \alpha, \beta, \text{ we want to minimize “scale factor” } t \text{ under the constraint} \]

\[ W \cdot [\text{Risk}^C_\ell (T | H_1, ..., H_L)]_{\ell=1}^L \leq \alpha + t \beta \]

This problem can be safely approximated by the optimization problem

\[ \min_{\{\delta_{\ell \ell'}\}, t} \left\{ t : W \cdot \left[ \sum_{\ell' : (\ell, \ell') \not\in C} \Gamma(\delta_{\ell \ell'}) \right]_{\ell=1}^L \leq \alpha + t \beta \right\} \quad (\#) \]

\[ \text{Note: Assuming } \gamma(\cdot) \text{ nonincreasing on } \mathbb{R}_+ \text{ (as is the case, e.g., for Gaussian mixtures), function } \Gamma(\delta) = \int_\delta^\infty \gamma(s) \, ds \text{ is convex on } \mathbb{R}_+ \]

\[ \Rightarrow (\#) \text{ is an explicit Convex Programming problem!} \]
\[ \ell, \ell' : (\ell, \ell') \notin C \]
\[ \Rightarrow \quad \text{Opt}_{\ell \ell'} = \min_{u \in X_\ell, v \in X_{\ell'}} \frac{1}{2} \|u - v\|_2 > 0 = \text{Opt}_{\ell' \ell} \]
\[ \Rightarrow \quad u_*, v_*, \phi_{\ell \ell'}(\omega) = e_{\ell \ell'}^T \omega - c_{\ell \ell'} \equiv -\phi_{\ell \ell}(\omega) \quad e_{\ell \ell'} = \frac{u_* - v_*}{\|u_* - v_*\|_2}, \quad c_{\ell \ell'} = \frac{1}{2} e_{\ell \ell'}^T [u_* + v_*] \]
\[ \forall (u \in X_\ell, P \in \mathcal{P}_d^\gamma) : \quad \text{Prob}_{\xi \sim P} \{ \phi(u + \xi) < 0 \} \leq \Gamma(\text{Opt}_{\ell \ell'}) \]
\[ \forall (v \in X_{\ell'}, P \in \mathcal{P}_d^\gamma) : \quad \text{Prob}_{\xi \sim P} \{ \phi(v + \xi) \geq 0 \} \leq \Gamma(\text{Opt}_{\ell \ell'}) \]
\[ \Rightarrow \quad \Gamma(\delta) := \int_\delta^\infty \gamma(s) ds \]

\[ \star \textbf{Case of } K\text{-repeated observations, } K > 1. \text{ In the case of } \text{semi-stationary } K\text{-repeated observations } \omega^K = (\omega_1, ..., \omega_K), \text{ we act as follows:} \]

- For \((\ell, \ell') \notin C\), we build majority tests

\[ T_{\{\ell, \ell'\}}(\omega^K) = \begin{cases} \{\ell\}, & \text{Card}\{k \leq K : \phi_{\ell \ell'}(\omega_k) \geq 0\} \geq K/2 \\ \{\ell'\}, & \text{otherwise} \end{cases} \]

- Further, we use our general construction to assemble simple tests \(\{T_{\{\ell, \ell'\}} : (\ell, \ell') \notin C\}\) into test \(T_K\) deciding on \(H^K_1, ..., H^K_L\) via observation \(\omega^K\)

Note: By our results on majority tests, the associated with tests \(T_{\{\ell, \ell'\}}\) quantities \(\epsilon_{\ell \ell'}\) satisfy the relations

\[ \epsilon_{\ell \ell'} \leq \sum_{K/2 \leq k \leq K} \binom{K}{k} [\Gamma(\text{Opt}_{\ell \ell'})]^k [1 - \Gamma(\text{Opt}_{\ell \ell'})]^{K-k} \]

whence

\[ \text{Risk}^C_\ell(T_K | H_1, ..., H_L) \leq \sum_{\ell' : (\ell, \ell') \notin C} \sum_{K/2 \leq k \leq K} \binom{K'}{k} [\Gamma(\text{Opt}_{\ell \ell'})]^k [1 - \Gamma(\text{Opt}_{\ell \ell'})]^{K-k}. \]
Risk_\ell^C(\mathcal{T}_K|H_1,...,H_L) 
\leq \sum_{\ell':(\ell,\ell') \notin C} \sum_{K/2 \leq k \leq K} \binom{K}{k} [\Gamma(\text{Opt}_{\ell\ell'})]^k [1 - \Gamma(\text{Opt}_{\ell\ell'})]^{K-k}.

**Note:** By Standing Assumption, Opt_{\ell\ell'} > 0 when (\ell, \ell') \notin C
\Rightarrow \Gamma(\text{Opt}_{\ell\ell'}) < 1/2
\Rightarrow Risks Risk_\ell^C(\mathcal{T}_K|H_1,...,H_L) go to 0 exponentially fast as K \rightarrow \infty.
HYPOTHESIS TESTING, II

● Detector-Based Tests
  ● Detectors & Detector-Based Pairwise Tests
  ● Testing “up to Closeness”
  ● Simple Observation Schemes
    – Minimum Risk Detectors
    – Near-Optimal Tests
    – Sequential Hypothesis Testing
    – Measurement Design
Situation: Given two families $\mathcal{P}_1, \mathcal{P}_2$ of probability distributions on a given observation space $\Omega$ and an observation $\omega \sim P$ with $P$ known to belong to $\mathcal{P}_1 \cup \mathcal{P}_2$, we want to decide whether $P \in \mathcal{P}_1$ (hypothesis $H_1$) or $P \in \mathcal{P}_2$ (hypothesis $H_2$).

Detectors. A detector is a function $\phi : \Omega \to \mathbb{R}$. Risks of a detector $\phi$ w.r.t. $\mathcal{P}_1, \mathcal{P}_2$ are defined as

$$
\text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2] = \sup_{P \in \mathcal{P}_1} \int_{\Omega} e^{-\phi(\omega)} P(d\omega), \\
\text{Risk}_2[\phi|\mathcal{P}_1, \mathcal{P}_2] = \sup_{P \in \mathcal{P}_2} \int_{\Omega} e^{\phi(\omega)} P(d\omega)
$$

$\text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2] = \text{Risk}_2[-\phi|\mathcal{P}_2, \mathcal{P}_1]$

Simple test $T_\phi$ associated with detector $\phi$, given observation $\omega$,

- accepts $H_1$ when $\phi(\omega) \geq 0$,
- accepts $H_2$ when $\phi(\omega) < 0$.

Immediate observation:

$$
\text{Risk}_1(T_\phi|H_1, H_2) \leq \text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2] \\
\text{Risk}_2(T_\phi|H_1, H_2) \leq \text{Risk}_2[\phi|\mathcal{P}_1, \mathcal{P}_2]
$$

Reason: $\text{Prob}_{\omega \sim P} \{\omega : \psi(\omega) \geq 0\} \leq \int e^{\psi(\omega)} P(d\omega)$. 

3.1
\begin{align*}
\text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2] &= \sup_{P \in \mathcal{P}_1} \int_\Omega e^{-\phi(\omega)} P(d\omega), \\
\text{Risk}_2[\phi|\mathcal{P}_1, \mathcal{P}_2] &= \sup_{P \in \mathcal{P}_2} \int_\Omega e^{\phi(\omega)} P(d\omega),
\end{align*}

**Elementary Calculus of Detectors**

♣ Detectors admit simple “calculus:”

♣ **Renormalization:** $\phi(\cdot) \Rightarrow \phi_a(\cdot) = \phi(\cdot) - a$

\[ \Rightarrow \begin{cases} 
\text{Risk}_1[\phi_a|\mathcal{P}_1, \mathcal{P}_2] = e^a \text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2] \\
\text{Risk}_2[\phi_a|\mathcal{P}_1, \mathcal{P}_2] = e^{-a} \text{Risk}_2[\phi|\mathcal{P}_1, \mathcal{P}_2] 
\end{cases} \]

⇒ **What matters, is the product**

\[ [\text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2]]^2 := \text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2] \text{Risk}_2[\phi|\mathcal{P}_1, \mathcal{P}_2] \]

of partial risks of a detector. Shifting the detector by constant, we can distribute this product between factors as we want, e.g., always can make the detector balanced:

\[ \text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2] = \text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2] = \text{Risk}_2[\phi|\mathcal{P}_1, \mathcal{P}_2]. \]
Detectors are well-suited to passing to multiple observations. For $1 \leq k \leq K$, let

- $\mathcal{P}_{1,k}, \mathcal{P}_{2,k}$ be families of probability distributions on observation spaces $\Omega_k$,
- $\phi_k$ be detectors on $\Omega_k$.

Families $\{\mathcal{P}_{1,k}, \mathcal{P}_{2,k}\}_{k=1}^K$ give rise to families of product distributions on $\Omega^K = \Omega_1 \times \ldots \times \Omega_K$:

$$\mathcal{P}^K_\chi = \{P^K = P_1 \times \ldots \times P_K : P_k \in \mathcal{P}_{\chi,k}, 1 \leq k \leq K\}, \chi = 1, 2,$$

and detectors $\phi_1, \ldots, \phi_K$ give rise to detector $\phi^K$ on $\Omega^K$:

$$\phi^K(\omega_1, \ldots, \omega_K) = \sum_{k=1}^K \phi_k(\omega_k).$$

Observation: For $\chi = 1, 2$, we have

$$\text{Risk}_\chi[\phi^K|\mathcal{P}^K_1, \mathcal{P}^K_2] = \prod_{k=1}^K \text{Risk}_\chi[\phi_k|\mathcal{P}_{1,k}, \mathcal{P}_{2,k}].$$
\[ \phi^K(\omega_1, \ldots, \omega_K) = \sum_{k=1}^{K} \phi_k(\omega_k). \]

In the sequel, we refer to families \( \mathcal{P}^K_\chi \) as to direct products of families of distributions \( \mathcal{P}_{\chi,k} \) over \( 1 \leq k \leq K \):

\[
\mathcal{P}^K_\chi = \mathcal{P}^{\oplus,K}_\chi = \bigoplus_{k=1}^{K} \mathcal{P}_{\chi,k} := \left\{ P^K = P_1 \times \ldots \times P_K : P_k \in \mathcal{P}_{\chi,k}, 1 \leq k \leq K \right\}.
\]

We can define also quasi-direct products

\[
\mathcal{P}^{\otimes,K}_\chi = \bigotimes_{k=1}^{K} \mathcal{P}_{\chi,k}
\]

of the families \( \mathcal{P}_{\chi,k} \) over \( 1 \leq k \leq K \). By definition, \( \mathcal{P}^{\otimes,K}_\chi \) is comprised of all distributions \( P^K \) of random sequences \( \omega^K = (\omega, \ldots, \omega_K) \), \( \omega_k \in \Omega_k \), which can be generated as follows: in the nature there exists a random sequence \( \zeta^K = (\zeta_1, \ldots, \zeta_K) \) of “driving factors” such that for every \( k \leq K \), \( \omega_k \) is a deterministic function of \( \zeta^k = (\zeta_1, \ldots, \zeta_k) \), and the conditional, \( \zeta^{k-1} \) being fixed, distribution of \( \omega_k \) always belongs to \( \mathcal{P}_{\chi,k} \).

\[ \text{It is immediately seen that for } \chi = 1, 2 \text{ it holds} \]

\[ \text{Risk}_\chi[\phi^K|\mathcal{P}^{\oplus,K}_1, \mathcal{P}^{\oplus,K}_2] = \prod_{k=1}^{K} \text{Risk}_\chi[\phi_k|\mathcal{P}_{1,k}, \mathcal{P}_{2,k}]. \]
From pairwise detectors to detectors for unions. Assume that we are given an observation space $\Omega$ along with
- $R$ families $\mathcal{R}_r$, $r = 1, \ldots, R$ of “red” probability distributions on $\Omega$,
- $B$ families $\mathcal{B}_b$, $b = 1, \ldots, B$ of “brown” probability distributions on $\Omega$,
- detectors $\phi_{rb}(\cdot)$, $1 \leq r \leq R$, $1 \leq b \leq B$.

Let us aggregate the red and the brown families as follows

$$\mathcal{R} = \bigcup_{r=1}^{R} \mathcal{R}_r, \mathcal{B} = \bigcup_{b=1}^{B} \mathcal{B}_b$$

and assemble detectors $\phi_{rb}$ into a single detector

$$\phi(\omega) = \max_{r \leq R} \min_{b \leq B} \phi_{rb}(\omega).$$

Observation: We have

$$\text{Risk}_1[\phi|\mathcal{R}, \mathcal{B}] \leq \max_{r \leq R} \sum_{b \leq B} \text{Risk}_1[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b],$$
$$\text{Risk}_2[\phi|\mathcal{R}, \mathcal{B}] \leq \max_{b \leq B} \sum_{r \leq R} \text{Risk}_2[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b].$$

Indeed,

$$P \in \mathcal{R}_r \Rightarrow \int e^{-\max_{r \leq R} \min_{b \leq B} \phi_{rb}(\omega)} P(d\omega) = \int e^{\min_{r \leq R} \max_{b \leq B} [-\phi_{rb}(\omega)]} P(d\omega) \leq \int e^{\max_{b \leq B} [-\phi_{rb}(\omega)]} P(d\omega) \leq \sum_{b \leq B} \int e^{-\phi_{rb}(\omega)} P(d\omega) \leq \sum_{b \leq B} \text{Risk}_1[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b];$$

$$P \in \mathcal{B}_b \Rightarrow \int e^{\max_{r \leq R} \min_{b \leq B} \phi_{rb}(\omega)} P(d\omega) \leq \int e^{\max_{b \leq B} \phi_{rb}(\omega)} P(d\omega) \leq \sum_{r \leq R} \int e^{\phi_{rb}(\omega)} P(d\omega) \leq \sum_{r \leq R} \text{Risk}_2[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b];$$

$$\Rightarrow \text{Risk}_2[\phi|\mathcal{R}, \mathcal{B}] \leq \max_{b \leq B} \sum_{r \leq R} \text{Risk}_2[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b].$$
\textbf{Refinement:} W.l.o.g. we can assume that the detectors $\phi_{rb}$ are balanced:

$$
\epsilon_{rb} := \text{Risk}[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b] = \text{Risk}_1[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b] = \text{Risk}_2[\phi_{rb}|\mathcal{R}_r, \mathcal{B}_b].
$$

Consider matrices

$$
E = \begin{bmatrix}
\epsilon_{1,1} & \cdots & \epsilon_{1,B} \\
\vdots & \ddots & \vdots \\
\epsilon_{R,1} & \cdots & \epsilon_{R,B}
\end{bmatrix},
F = \begin{bmatrix}
\epsilon_{1,1} & \cdots & \epsilon_{1,B} \\
\vdots & \ddots & \vdots \\
\epsilon_{R,1} & \cdots & \epsilon_{R,B}
\end{bmatrix}
$$

The maximal eigenvalue $\theta$ of $F$ is the spectral norm $\|E\|_{2,2}$ of $E$, and the leading eigenvector $[g; f]$ can be selected to be positive \textit{(Perron-Frobenius Theorem)}.

\textbf{Note:} $\theta g = Ef$ & $\theta f = E^T g$

Let us pass from the detectors $\phi_{rb}$ to shifted detectors $\psi_{rb} = \phi_{rb} - \ln(f_b/g_r)$ and assemble the shifted detectors into the detector

$$
\psi(\omega) = \max_{r \leq R} \min_{b \leq B} \psi_{rb}(\omega)
$$

By previous observation

\begin{align*}
\text{Risk}_1(\psi|\mathcal{R}, \mathcal{B}) &\leq \max_r \sum_b \text{Risk}_1(\psi_{rb}|\mathcal{R}_r, \mathcal{B}_b) = \max_r \sum_b \epsilon_{rb}(f_b/g_r) \\
&= \max_r [(Ef)_r/g_r] = \|E\|_{2,2}
\end{align*}

\begin{align*}
\text{Risk}_2(\psi|\mathcal{R}, \mathcal{B}) &\leq \max_b \sum_r \text{Risk}_2(\psi_{rb}|\mathcal{R}_r, \mathcal{B}_b) = \max_b \sum_r \epsilon_{rb}(g_r/f_b) \\
&= \max_b [(E^T g)_b/f_b] = \|E\|_{2,2}
\end{align*}

\Rightarrow \textit{Partial risks of detector } \psi \textit{ on aggregated families } \mathcal{R}, \mathcal{B} \textit{ is } \leq \|E\|_{2,2}$.
Detector-Based Tests "Up to Closeness"

♦ Situation: We are given

- $L$ families of probability distributions $\mathcal{P}_\ell$, $\ell = 1, \ldots, L$, on observation space $\Omega$, giving rise to $L$ hypotheses $H_\ell$, on the distribution $P$ of random observation $\omega$

in $\Omega$:

$$H_\ell : P \in \mathcal{P}_\ell, \ 1 \leq \ell \leq L;$$

- closeness relation $C$;
- system of balanced detectors

$$\left\{ \phi_{\ell \ell'} : \ell < \ell', (\ell, \ell') \not\in C \right\}$$

along with upper bounds $\epsilon_{\ell \ell'}$ on detectors’ risks:

$$\forall (\ell, \ell' : \ell < \ell', (\ell, \ell') \not\in C) : \left\{ \begin{array}{l} \int_\Omega e^{-\phi_{\ell \ell'}(\omega)} P(d\omega) \leq \epsilon_{\ell \ell'} \ \forall P \in \mathcal{P}_\ell \\ \int_\Omega e^{\phi_{\ell \ell'}(\omega)} P(d\omega) \leq \epsilon_{\ell \ell'} \ \forall P \in \mathcal{P}_{\ell'} \end{array} \right.$$  

- Our goal is to build single-observation test deciding on hypotheses $H_1, \ldots, H_L$ up to closeness $C$.  

3.7
Construction: Let us set

\[ \phi_{\ell\ell'}(\omega) = \begin{cases} -\phi_{\ell\ell}(\omega), & \ell > \ell', (\ell, \ell') \not\in C \\ 0, & (\ell, \ell') \not\in C \end{cases} \]

\[ \epsilon_{\ell\ell'} = \begin{cases} \epsilon_{\ell\ell}, & \ell > \ell', (\ell, \ell') \not\in C \\ 1, & (\ell, \ell') \not\in C \end{cases} \]

thus ensuring that

\[ \phi_{\ell\ell'}(\cdot) \equiv -\phi_{\ell\ell}(\cdot), \quad \epsilon_{\ell\ell'} = \epsilon_{\ell\ell}, \quad 1 \leq \ell, \ell' \leq L \]

\[ \int_\Omega e^{-\phi_{\ell\ell'}(\omega)} P(d\omega) \leq \epsilon_{\ell\ell'} \forall (P \in \mathcal{P}_\ell, \ 1 \leq \ell, \ell' \leq L) \]

Given shifts \( a_{\ell\ell'} = -a_{\ell\ell} \), we specify test \( \mathcal{T} \) as follows:

Given observation \( \omega \), \( \mathcal{T} \) accepts all hypotheses \( H_\ell \) such that

\[ \phi_{\ell\ell'}(\omega) > a_{\ell\ell'} \forall (\ell' : (\ell, \ell') \not\in C) \]

and rejects all other hypotheses.

Proposition. The \( C \)-risk of \( \mathcal{T} \) can be upper-bounded as

\[ \text{Risk}^C(\mathcal{T}|H_1, \ldots, H_L) \leq \max_{\ell \leq L} \sum_{\ell' : (\ell, \ell') \not\in C} \epsilon_{\ell\ell'} e^{a_{\ell\ell'}} \]
Optimal shifts: Consider the symmetric nonnegative matrix

$$E = \left[ \epsilon_{\ell\ell'} \chi_{\ell\ell'} \right]_{\ell,\ell'}^{L} = 1, \quad \chi_{\ell\ell'} = \begin{cases} 1, & (\ell, \ell') \notin \mathcal{C} \\ 0, & (\ell, \ell') \in \mathcal{C} \end{cases}$$

and let $\theta = \|E\|_{2,2}$ be the spectral norm of $E$, or, which is the same under the circumstances, the largest eigenvalue of $E$. By Perron-Frobenius Theorem, for every $\theta' > \theta$ there exists a positive vector $f$ such that

$$Ef \leq \theta' f;$$

the same holds true when $\theta = \theta'$, provided the leading eigenvector of $E$ (which always can selected to be nonnegative) is positive.

Fact: With $\alpha_{\ell\ell'} = \ln(f_{\ell'}/f_{\ell})$, the risk bound from Proposition reads

$$\text{Risk}^{\mathcal{C}}(\mathcal{T}|H_{1}, ..., H_{L}) \leq \theta'.$$

Thus, assembling the detectors $\phi_{\ell\ell'}$ appropriately, one can get a test with $\mathcal{C}$-risk arbitrarily close to $\|E\|_{2,2}$. 
Utilizing repeated observations. Assuming $K$-repeated observations allowed, we can apply the above construction to

- $K$-repeated observation $\omega^K = (\omega_1, \ldots, \omega_K)$ in the role of $\omega$,
- quasi-direct degrees $\mathcal{P}_\ell^{\otimes, K} = \bigotimes_{k=1}^K \mathcal{P}_\ell$ in the role of families $\mathcal{P}_\ell$, and respective hypotheses $H_\ell^{\otimes, K}$ in the role of hypotheses $H_\ell$, $\ell = 1, \ldots, L$,
- detectors $\phi_{\ell\ell'}^K(\omega^K) = \sum_{k=1}^K \phi_{\ell\ell'}(\omega_k)$ in the role of detectors $\phi_{\ell\ell'}$, which allows to replace $\epsilon_{\ell\ell'}$ with $\epsilon_{K_{\ell\ell'}}$.

As a result, we get $K$-observation test $\mathcal{T}^K$ such that

$$\text{Risk}_C(\mathcal{T}^K|H_1^{\otimes, K}, \ldots, H_L^{\otimes, K}) \leq \theta'_K$$

where $\theta'_K$ can be made arbitrarily close (under favorable circumstances, even equal) to the quantity

$$\left\| \left[ \epsilon_{\ell\ell'}^{K} \chi_{\ell\ell'} \right]_{\ell\ell'}^{K} \right\|_{2,2} = \chi_{\ell\ell'} = \begin{cases} 1, & (\ell, \ell') \not\in C \\ 0, & (\ell, \ell') \in C \end{cases}$$

In particular, in the case when $\epsilon_{\ell\ell'} < 1$ whenever $(\ell, \ell') \not\in C$, we can ensure that the $C$-risk of $\mathcal{T}^K$ converges to 0 exponentially fast as $K \to \infty$. 

3.10
“Universality” of detector-based tests. Let $\mathcal{P}_\chi$, $\chi = 1, 2$, be two families of probability distributions on observation space $\Omega$, and let $H_\chi$, $\chi = 1, 2$, be associate hypotheses on the distribution of an observation.

Assume that there exists a simple deterministic or randomized test $T$ deciding on $H_1, H_2$ with risk $\leq \epsilon \in (0, 1/2)$. Then there exists a detector $\phi$ with

$$\text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2] \leq \epsilon_+ := 2\sqrt{\epsilon(1-\epsilon)} < 1.$$  

Indeed, let $T$ be deterministic, let $\Omega_{1\chi} = \{\omega \in \Omega : T(\omega) = \{\chi}\}$, $\chi = 1, 2$, and let

$$\phi(\omega) = \begin{cases} \frac{1}{2} \ln \left(\frac{1-\epsilon}{\epsilon}\right), & \omega \in \Omega_1 \\ \frac{1}{2} \ln (\epsilon/1-\epsilon), & \omega \in \Omega_2 \end{cases}$$

Then

$$\int e^{-\phi(\omega)} P(d\omega) = \sqrt{\epsilon/[1-\epsilon]}(1-\epsilon') + \sqrt{[1-\epsilon]/\epsilon\epsilon'} \leq 0$$

$$\leq \sqrt{\epsilon/[1-\epsilon]} + \left[\sqrt{[1-\epsilon]/\epsilon} - \sqrt{\epsilon/[1-\epsilon]}\right] \epsilon' = 2\sqrt{\epsilon(1-\epsilon)}$$

$$\int e^{\phi(\omega)} P(d\omega) = \sqrt{\epsilon/[1-\epsilon]}(1-\epsilon') + \sqrt{[1-\epsilon]/\epsilon\epsilon'} = 2\sqrt{\epsilon(1-\epsilon)}$$

$$\Rightarrow \text{Risk}_\chi(\phi|\mathcal{P}_1, \mathcal{P}_2) \leq 2\sqrt{\epsilon(1-\epsilon)}.$$
Now let \( T \) be randomized. Setting \( P_\chi^+ = \{ P \times \text{Uniform}[0, 1] : P \in P_\chi \} \), \( \chi = 1, 2, \Omega^+ = \Omega \times [0, 1] \), by above there exists a bounded detector \( \phi_+ : \Omega^+ \to \mathbb{R} \) such that

\[
\forall (P \in \mathcal{P}_1) : \int_{\Omega} \left[ \int_0^1 e^{-\phi_+(\omega, s)} \, ds \right] P(d\omega) \, ds \leq \epsilon_+ = 2\sqrt{\epsilon [1 - \epsilon]}, \\
\forall (P \in \mathcal{P}_2) : \int_{\Omega} \left[ \int_0^1 e^{\phi_+(\omega, s)} \, ds \right] P(d\omega) \leq \epsilon_+,
\]

whence, setting \( \phi(\omega) = \int_0^1 \phi(\omega, s) \, ds \) and applying Jensen’s Inequality,

\[
\forall (P \in \mathcal{P}_1) : \int_{\Omega} e^{-\phi(\omega)} P(d\omega) \leq \epsilon_+, \\
\forall (P \in \mathcal{P}_2) : \int_{\Omega} e^{\phi(\omega)} P(d\omega) \leq \epsilon_+.
\]

\[\blacklozenge\] Risk \( 2\sqrt{\epsilon [1 - \epsilon]} \) of the detector-based test induced by simple test \( T \) is “much worse” than the risk \( \epsilon \) of \( T \).

**However:** When repeated observations are allowed, we can compensate for risk deterioration \( \epsilon \mapsto 2\sqrt{\epsilon [1 - \epsilon]} \) by passing in the detector-based test from a single observation to a moderate number of them.
\[
\inf_{\phi} \left\{ \text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2] = \min \left\{ \epsilon : \begin{array}{c}
\int_{\Omega} e^{-\phi(\omega)} P(d\omega) \
\int_{\Omega} e^{\phi(\omega)} P(d\omega)
\end{array} \leq \begin{array}{c}
\epsilon \forall (P \in \mathcal{P}_1) \
\epsilon \forall (P \in \mathcal{P}_2)
\end{array} \right\} \right\}
\]

**Note:**
- The optimization problem specifying risk has constraints *convex* in \((\phi, \epsilon)\)
- When passing from families \(\mathcal{P}_\chi, \chi = 1, 2\), to their convex hulls, the risk of a detector remains intact.

♣ **Bottom line:** *It would be nice to be able to solve (1), thus arriving at the lowest risk detector-based tests.*

**But:** (1) is an optimization problem with *infinite-dimensional* decision “vector” and *infinitely many* constraints.

⇒ (1) in general is intractable.

**Simple observation schemes:** A series of special cases where (1) is efficiently solvable via Convex Optimization.
Simple Observation Schemes

♣ Simple Observation Scheme is a collection

\[ \mathcal{O} = ((\Omega, \Pi), \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F}), \]

where

- \((\Omega, \Pi)\) is a (complete separable metric) observation space \(\Omega\) with (\(\sigma\)-finite \(\sigma\)-additive) reference measure \(\Pi\), \(\text{supp} P = \Omega\);
- \(\{p_\mu(\cdot) : \mu \in \mathcal{M}\}\) is a parametric family of probability densities, taken w.r.t. \(\Pi\), on \(\Omega\), and
  - \(\mathcal{M}\) is a relatively open convex set in some \(\mathbb{R}^n\)
  - \(p_\mu(\omega)\): positive and continuous in \(\mu \in \mathcal{M}, \omega \in \Omega\)
- \(\mathcal{F}\) is a finite-dimensional space of continuous functions on \(\Omega\) containing constants and such that
  \[ \ln(p_\mu(\cdot)/p_\nu(\cdot)) \in \mathcal{F} \; \forall \mu, \nu \in \mathcal{M} \]
- For \(\phi \in \mathcal{F}\), the function
  \[ \mu \mapsto \ln \left( \int_{\Omega} e^{\phi(\omega)} p_\mu(\omega) P(d\omega) \right) \]
  is finite and concave in \(\mu \in \mathcal{M}\).
Example 1: Gaussian o.s., where $\Omega, \Pi$ is $\mathbb{R}^d$ with Lebesgue measure, $(\Omega = \mathbb{R}^d, d\omega), \{p_\mu(\cdot) = \mathcal{N}(\mu, I_d) : \mu \in \mathbb{R}^d\}$, \[ \mathcal{F} = \{ \text{affine functions on } \Omega \} \]
\[
\Rightarrow \left\{ \begin{aligned}
\ln(p_\mu(\cdot)/p_\nu(\cdot)) & \in \mathcal{F}, \\
\ln \left( \int_{\Omega} e^{a^T \omega + b} p_\mu(\omega) \Pi(d\omega) \right) & = a^T \mu + b + \frac{a^T a}{2} \text{ is concave in } \mu.
\end{aligned} \right.
\]
• Gaussian o.s. is the standard observation model in Signal Processing.

Example 2: Poisson o.s., where $(\Omega, \Pi)$ is the nonnegative part $\mathbb{Z}_d^+$ of integer lattice in $\mathbb{R}^d$ equipped with counting measure, \[ \{p_\mu(\omega) = \prod_{i=1}^{d} \frac{\omega_i^{\mu_i} e^{-\mu_i}}{\mu_i!} : \mu \in \mathcal{M} := \mathbb{R}^d_++\} \]
is the family of distributions of random vectors with independent across $i$ Poisson entries $\omega_i \sim \text{Poisson}(\mu_i)$, and \[ \mathcal{F} = \{ \text{affine functions on } \Omega \} \]
\[
\Rightarrow \left\{ \begin{aligned}
\ln(p_\mu(\cdot)/p_\nu(\cdot)) & \in \mathcal{F}, \\
\ln \left( \int_{\Omega} e^{a^T \omega + b} p_\mu(\omega) \Pi(d\omega) \right) & = \sum_i (e^{a_i} - 1) \mu_i \text{ is concave in } \mu.
\end{aligned} \right.
\]
Poisson o.s. arises in Poisson Imaging, including
• Positron Emission Tomography,
• Large binocular Telescope,
• Nanoscale Fluorescent Microscopy.
Example 3: Discrete o.s., where \((\Omega, \Pi)\) is finite set \(\{1, \ldots, d\}\) with counting measure,
\[
\{p_\mu(\omega) = \mu_\omega, \mu \in \mathcal{M} = \{\mu > 0 : \sum_{\omega=1}^{d} \mu_\omega = 1\}\}
\]
is the set of non-vanishing probability distributions on \(\Omega\), and
\[
\mathcal{F} = \{\text{all functions on } \Omega\}
\]
\[
\Rightarrow \left\{ \ln\left(\frac{p_\mu(\cdot)}{p_\nu(\cdot)}\right) \in \mathcal{F}, \right. 
\]
\[
\int_{\Omega} e^{\phi(\omega)} p_\mu(\omega) \Pi(d\omega) = \ln \sum_{\omega \in \Omega} e^{\phi(\omega)} \mu_\omega \text{ is concave in } \mu.
\]

Example 4: Direct product of simple o.s.’s. Simple o.s.’s
\[
\mathcal{O}_k = \left((\Omega_k, \Pi_k), \{p_{\mu_k, k}(\cdot) : \mu_k \in \mathcal{M}_k\}, \mathcal{F}_k\right), \quad 1 \leq k \leq K
\]
give rise to their direct product \(\otimes_{k=1}^{K} \mathcal{O}_k\) defined as the o.s.
\[
\left((\Omega^K, \Pi^K), \{p_{\mu^K}(\cdot) : \mu^K \in \mathcal{M}_K\}, \mathcal{F}_K\right),
\]
where
- \(\Omega^K = \Omega_1 \times \ldots \times \Omega_K, \Pi^K = \Pi_1 \times \ldots \times \Pi_K\)
- \(\mathcal{M}^K = \mathcal{M}_1 \times \ldots \times \mathcal{M}_K, p_{(\mu_1, \ldots, \mu_K)}(\omega_1, \ldots, \omega_K) = \prod_{k=1}^{K} p_{\mu_k, k}(\omega_k)\)
- \(\mathcal{F}_K = \{\phi(\underbrace{\omega_1, \ldots, \omega_K}_{\omega_K}) = \sum_{k=1}^{K} \phi_k(\omega_k) : \phi_k \in \mathcal{F}_k, 1 \leq k \leq K\}\)

Fact: Direct product of simple o.s.’s is a simple o.s.
Example 5: Degree of a simple o.s. When all \( K \) o.s.’s in direct product \( O^K = \bigotimes_{k=1}^K O_k \) are identical to each other:

\[
O_k = O := ((\Omega, \Pi), \{ p_\mu(\cdot) : \mu \in \mathcal{M} \}, \mathcal{F}), \quad 1 \leq k \leq K
\]

we can “restrict \( O^K \) to its diagonal,” arriving at \( K \)-th degree \( O^{(K)} \) of \( O \):

\[
O^{(K)} = \left( (\Omega^K, \Pi^K), \{ p^{(K)}_\mu(\cdot) : \mu \in \mathcal{M} \}, \mathcal{F}^{(K)} \right),
\]

\[
p^{(K)}_\mu(\omega_1, ..., \omega_K) = \prod_{k=1}^K p_\mu(\omega_k), \quad \mathcal{F}^{(K)} = \{ \phi^{(K)}(\omega^K) = \sum_{k=1}^K \phi(\omega_k) : \phi \in \mathcal{F} \}
\]

Fact: Degree of a simple o.s. is a simple o.s.
The function is continuous on its domain, convex in $\phi \in \mathcal{F}$, concave in $(\mu, \nu) \in M_1 \times M_2$, and possesses a saddle point ($\min$ in $\phi$, $\max$ in $(\mu, \nu)$): 

$$\exists (\phi^*_\bullet \in \mathcal{F}, (\mu^*_\bullet, \nu^*_\bullet) \in M_1 \times M_2): \Phi(\phi; \mu^*_\bullet, \nu^*_\bullet) \geq \Phi(\phi^*_\bullet; \mu^*_\bullet, \nu^*_\bullet) \geq \Phi(\phi^*_\bullet; \mu, \nu) \ \forall (\phi \in \mathcal{F}, (\mu, \nu) \in M_1 \times M_2)$$

- The component $\phi^*_\bullet$ of a saddle point $(\phi^*_\bullet, (\mu^*_\bullet, \nu^*_\bullet))$ of $\Phi$ is an optimal solution to (!), and 

$$\varepsilon_*(\mathcal{P}_1, \mathcal{P}_2) = \exp\{\Phi(\phi^*_\bullet; \mu^*_\bullet, \nu^*_\bullet)\}.$$
• A saddle point \((\phi^*, (\mu^*, \nu^*))\) can be found as follows. We solve the optimization problem

\[
\text{SadVal} = \max_{\mu \in M_1, \nu \in M_2} \ln \left( \int_{\Omega} \sqrt{p_{\mu}(\omega)p_{\nu}(\omega)} \Pi(d\omega) \right);
\]

which is a solvable convex optimization problem, and take an optimal solution to the problem as \((\mu^*, \nu^*)\). We then set

\[
\phi^*(\omega) = \frac{1}{2} \ln \left( \frac{p_{\mu^*}(\omega)}{p_{\nu^*}(\omega)} \right),
\]

thus getting an optimal detector \(\phi^* \in \mathcal{F}\). For this detector and the associated simple test \(T_{\phi^*}\),

\[
\text{Risk}(T_{\phi^*}|H_1, H_2) \leq \text{Risk}[\phi^*|\mathcal{P}_1, \mathcal{P}_2] = \text{Risk}_1[\phi^*|\mathcal{P}_1, \mathcal{P}_2] = \text{Risk}_2[\phi^*|\mathcal{P}_1, \mathcal{P}_2] = \varepsilon^*(\mathcal{P}_1, \mathcal{P}_2) = e^{\text{SadVal}} = \int_{\Omega} \sqrt{p_{\mu^*}(\omega)p_{\nu^*}(\omega)} \Pi(d\omega).
\]
Gaussian o.s. \( \mathcal{P}_\chi = \{ \mathcal{N}(\mu, I_d) : \mu \in M_\chi \}, \chi = 1, 2 \):

- Problem \( \max_{\mu \in M_1, \nu \in M_2} \ln \left( \int \sqrt{p_\mu(\omega)p_\nu(\omega)} \prod(d\omega) \right) \) reads

\[
\max_{\mu \in M_1, \nu \in M_2} \left[ -\frac{1}{8} \| \mu - \nu \|_2^2 \right]
\]

- The optimal balanced detector and its risk are given by

\[
\phi^*_\chi(\omega) = \frac{1}{2} [\mu^* - \nu^*] \omega - c,
\]
\[
(\mu^*, \nu^*) \in \text{Argmin}_{\mu \in M_1, \nu \in M_2} \| \mu - \nu \|_2^2
\]
\[
c = \frac{1}{4} [\mu^* - \nu^*]^T [\mu^* + \nu^*]
\]
\[
\varepsilon^*_\chi(\mathcal{P}_1, \mathcal{P}_2) = \exp \left\{ -\frac{\| \mu^* - \nu^* \|_2^2}{8} \right\}
\]

Note: We are in the “signal plus noise” model of observations with noise \( \sim \mathcal{N}(0, I_d) \). The test \( T^*_\phi \) is nothing but the pairwise Euclidean separation test associated with \( X_\chi = M_\chi, \chi = 1, 2 \).
Poison o.s. $\mathcal{P}_\chi = \{ \bigotimes_{i=1}^d \text{Poisson}(\mu_i) : \mu = [\mu_1; \ldots; \mu_d] \in M_\chi \}$, $\chi = 1, 2$:

- Problem $\max_{\mu \in M_1, \nu \in M_2} \ln \left( \int \sqrt{p_\mu(\omega)p_\nu(\omega)} \prod (d\omega) \right)$ reads

$$
\max_{\mu \in M_1, \nu \in M_2} \left[ -\frac{1}{2} \sum_{i=1}^d (\sqrt{\mu_i} - \sqrt{\nu_i})^2 \right]
\sum_i \left[ \sqrt{\mu_i \nu_i} - \frac{1}{2} \mu_i - \frac{1}{2} \nu_i \right]
$$

- The optimal balanced detector and its risk are given by

$$
\phi^*(\omega) = \frac{1}{2} \sum_{i=1}^d \left[ \ln(\mu_i^*/\nu_i^*) \omega_i + \nu_i^* - \mu_i^* \right],
(\mu^*, \nu^*) \in \text{Argmax} \sum_i \left[ \sqrt{\mu_i \nu_i} - \frac{1}{2} \mu_i - \frac{1}{2} \nu_i \right]
$$

$$
\varepsilon^*(\mathcal{P}_1, \mathcal{P}_2) = \exp \left\{ -\frac{1}{2} \sum_i (\sqrt{\mu_i^*} - \sqrt{\nu_i^*})^2 \right\}
$$

3.21
Discrete o.s.

\[ \mathcal{P}_\chi = \{ \mu \in M_\chi \}, M_\chi \subset \Delta^o_d = \{ \mu \in \mathbb{R}^d_+ : \sum_\omega \mu_\omega = 1, \mu > 0 \}, \]

\[ \chi = 1, 2 \]

- Problem \( \max_{\mu \in M_1, \nu \in M_2} \ln \left( \int \sqrt{p_\mu(\omega)p_\nu(\omega)} \prod(d\omega) \right) \) reads

\[
\max_{\mu \in M_1, \nu \in M_2} \sum_\omega \sqrt{\mu_\omega \nu_\omega}
\]

- The optimal balanced detector and its risk are given by

\[
\phi_*(\omega) = \frac{1}{2} \ln(\mu_\omega^*/\nu_\omega^*), \omega \in \Omega = \{1, \ldots, d\}
\]

\[
(\mu^*, \nu^*) \in \text{Argmin} \sum_\omega \sqrt{\mu_\omega \nu_\omega}
\]

\[
\varepsilon_*(\mathcal{P}_1, \mathcal{P}_2) = \sum_\omega \sqrt{\mu_\omega^* \nu_\omega^*}
\]
**Direct product of simple o.s.’s.** Let

\[ O_k = \big((\Omega_k, \pi_k), \{p_{\mu_k,k}(\cdot) : \mu_k \in \mathcal{M}_k\}, \mathcal{F}_k\big), \ 1 \leq k \leq K, \]

be simple o.s.’s, and \( M_{\chi,k} \subset \mathcal{M}_k, \chi = 1, 2, \) be nonempty convex compact sets. Consider the simple o.s.

\[ ((\Omega^K, \pi^K), \{\pi_{\mu^K} : \mu^K \in \mathcal{M}^K\}, \mathcal{F}^K) = \bigotimes_{k=1}^{K} O_k \]

along with two compact convex sets

\[ M_{\chi} = M_{\chi,1} \times \ldots \times M_{\chi,K}, \chi = 1, 2. \]

**Question:** What is the problem responsible for the smallest risk detector for the families of distributions \( \mathcal{P}^K_\chi, \mathcal{P}^K_2 \) associated in \( O^K \) with the sets \( M_{\chi}, \chi = 1, 2 \)?

**Answer:** This is the separable problem

\[
\max_{\mu^K \in \mathcal{M}_1, \nu^K \in \mathcal{M}_2} \sum_{k=1}^{K} \ln \left( \int_{\Omega^K} \sqrt{p_{\mu_k,k}(\omega^K)p_{\nu_k,k}(\omega^K)\prod_{k}(d\omega^K)} \right)
\]

\Rightarrow \text{Minimum risk balanced detector for } \mathcal{P}^K_1, \mathcal{P}^K_2 \text{ can be chosen as}

\[
\phi^K_*(\omega_1, \ldots, \omega_K) = \sum_{k=1}^{K} \phi_{*,k}(\omega_k), \\
\phi_{*,k}(\omega_k) = \frac{1}{2} \ln \left( \frac{p_{\mu_k^*,k}(\omega_k)}{p_{\nu_k^*,k}(\omega_k)} \right)
\]

\[
\left(\mu_k^*, \nu_k^*\right) \in \operatorname{Argmax} \ln \left( \int_{\Omega_k} \sqrt{p_{\mu_k,k}(\omega_k)p_{\nu_k,k}(\omega_k)\prod_k(d\omega_k)} \right)
\]

and

\[
\varepsilon_*(\mathcal{P}^K_1, \mathcal{P}^K_2) = \prod_{k=1}^{K} \varepsilon_*(\mathcal{P}_{1,k}, \mathcal{P}_{2,k}),
\]

where \( \mathcal{P}_{\chi,k} \) are the families of distributions associated in \( O_k \) with \( M_{\chi,k}, \chi = 1, 2. \)

3.23
Remark: The families of distributions $\mathcal{P}_K^\chi$ are direct products of the families $\mathcal{P}_\chi^k$ over $k = 1, \ldots, K$. From Detector Calculus, extending $\mathcal{P}_K^\chi$ to families $\mathcal{P}_\chi^{\otimes, K}$ of quasi-direct products of families $\mathcal{P}_\chi^k$, $k = 1, \ldots, K$, we still have

$$\text{Risk}[\phi_K^* | \mathcal{P}_1^{\otimes, K}, \mathcal{P}_2^{\otimes, K}] \leq \prod_{k=1}^{K} \varepsilon^*(\mathcal{P}_1^k, \mathcal{P}_2^k),$$

whence also $\varepsilon_K = \varepsilon^*(\mathcal{P}_1^K, \mathcal{P}_2^K) \leq \varepsilon^*(\mathcal{P}_1^{\otimes, K}, \mathcal{P}_2^{\otimes, K}) \leq \varepsilon_K$

$$\Rightarrow \varepsilon^*(\mathcal{P}_1^{\otimes, K}, \mathcal{P}_2^{\otimes, K}) = \prod_{k=1}^{K} \varepsilon^*(\mathcal{P}_1^k, \mathcal{P}_2^k).$$
Degree of a simple o.s. Let
\[ \mathcal{O} = ((\Omega, \Pi), \{p_\mu(\cdot) : \mu \in \mathcal{M}\}, \mathcal{F}) \]
be a simple o.s., and \( M_\chi \subset \mathcal{M}, \chi = 1, 2 \), be nonempty convex compact sets. Consider the \( K \)-th degree of \( \mathcal{O} \), that is, the simple o.s.
\[ \mathcal{O}^{(K)} = \left( (\Omega^K, \Pi^K), \{p_\mu^{(K)}(\omega_1, ..., \omega_K) = \prod_{k=1}^K p_\mu(\omega_k) : \mu \in \mathcal{M}\}, \mathcal{F}^{(K)} \right). \]

Question: What is the problem responsible for the smallest risk detector for the families of distributions \( \mathcal{P}_\chi^{(K)} \) associated in \( \mathcal{O}^{(K)} \) with the sets \( M_\chi, \chi = 1, 2 \)?

Answer: This is the separable problem
\[
\max_{\mu \in M_1, \nu \in M_2} \ln \left( \int_{\Omega^K} \sqrt{p_\mu^{(K)}(\omega^K)p_\nu^{(K)}(\omega^K)} \Pi^K(d\omega^K) \right)
\]

Minimum risk balanced detector for \( \mathcal{P}_1^{(K)}, \mathcal{P}_2^{(K)} \) can be chosen as
\[
\phi^*_K(\omega_1, ..., \omega_K) = \sum_{k=1}^K \phi^*_k(\omega_k), \\
\phi^*_k(\omega_k) = \frac{1}{2} \ln \left( \frac{p_\mu(\omega_k)}{p_\nu(\omega_k)} \right)
\]
\[
\left( \mu^*, \nu^* \right) \in \text{Argmax} \ln \left( \int_{\Omega} \sqrt{p_\mu(\omega)p_\nu(\omega)} \Pi(d\omega) \right)
\]

and
\[
\varepsilon^*_\chi(\mathcal{P}_1^{(K)}, \mathcal{P}_2^{(K)}) = [\varepsilon^*_\chi(\mathcal{P}_1, \mathcal{P}_2)]^K,
\]
where \( \mathcal{P}_\chi \) are the families of distributions associated in \( \mathcal{O} \) with \( M_\chi, \chi = 1, 2 \).
Remark: The families of distributions $\mathcal{P}_\chi^{(K)}$ are direct products $\mathcal{P}_\chi^{\oplus,K}$ of the families $\mathcal{P}_\chi$. From Detector Calculus, extending $\mathcal{P}_\chi^{(K)}$ to families $\mathcal{P}_\chi^{\otimes,K}$ of quasi-direct degrees of families $\mathcal{P}_\chi$, we still have

$$\text{Risk}[\phi_*^{(K)}|\mathcal{P}_1^{\oplus,K}, \mathcal{P}_2^{\oplus,K}] \leq \left[\varepsilon_*(\mathcal{P}_1, \mathcal{P}_2)\right]^K, =: \varepsilon_K,$$

whence also $\varepsilon_K = \varepsilon_*(\mathcal{P}_1^{(K)}, \mathcal{P}_2^{(K)}) \leq \varepsilon_*(\mathcal{P}_1^{\otimes,K}, \mathcal{P}_2^{\otimes,K}) \leq \varepsilon_K$

$$\Rightarrow \varepsilon_*(\mathcal{P}_1^{\otimes,K}, \mathcal{P}_2^{\otimes,K}) = \left[\varepsilon_*(\mathcal{P}_1, \mathcal{P}_2)\right]^K.$$
Proposition A. Let

$$\mathcal{O} = ((\Omega, \Pi), \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F})$$

be a simple o.s., and $$\mathcal{M}_\chi \subset \mathcal{M}, \chi = 1, 2,$$ be nonempty convex compact sets, giving rise to families of distributions

$$\mathcal{P}_\chi = \{P : P \text{ has density } p_\mu(\cdot) \text{ w.r.t. } \Pi \text{ with } \mu \in \mathcal{M}_\chi\}, \chi = 1, 2,$$

hypotheses

$$H_\chi : P \in \mathcal{P}_\chi, \chi = 1, 2,$$

on the distribution of a random observation $$\omega \in \Omega,$$ and minimum risk detector $$\phi_*$$ for $$\mathcal{P}_1, \mathcal{P}_2.$$

Assume that in the nature there exists a simple single-observation test, deterministic or randomized, $$T$$ with

$$\text{Risk}(T|H_1, H_2) \leq \epsilon < 1/2.$$  

Then the risk of the simple test $$T_{\phi_*}$$ accepting $$H_1$$ when $$\phi_*(\omega) \geq 0$$ and accepting $$H_2$$ otherwise “is comparable” to $$\epsilon$$:

$$\text{Risk}(T_{\phi_*}|H_1, H_2) \leq \epsilon_+ := 2\sqrt{\epsilon(1-\epsilon)} < 1.$$
Proof. From what we called “universality” of detector-based tests, there exists a detector $\phi$ with $\text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2] \leq \epsilon_+$, and $\text{Risk}[\phi^*|\mathcal{P}_1, \mathcal{P}_2]$ can be only less than $\text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2]$. $\Box$
Proposition B. Let $\mathcal{O} = ((\Omega, \Pi), \{p_\mu: \mu \in M\}, \mathcal{F})$ be a simple o.s., and $M_\chi \subset M$, $\chi = 1, 2$, be nonempty convex compact sets, giving rise to families of distributions

$\mathcal{P}_\chi = \{P: P \text{ has density } p_\mu(\cdot) \text{ w.r.t. } \Pi \text{ with } \mu \in M_\chi\}$, $\chi = 1, 2$

their diagonal direct degrees

$\mathcal{P}^{\circ,K}_\chi = \{P \times \ldots \times P: P \in \mathcal{P}_\chi\}$, $\chi = 1, 2$, $K = 1, 2, \ldots$

hypotheses $H^K_\chi: P \in \mathcal{P}^{\circ,K}_\chi$, $\chi = 1, 2$, $K = 1, 2, \ldots$ on the distribution of a random $K$-repeated observation $\omega^K = (\omega_1, \ldots, \omega_K) \in \Omega^K$, and minimum risk detector $\phi_*$ for $\mathcal{P}_1, \mathcal{P}_2$.

Assume that in the nature there positive integer $K_*$ and a simple $K_*$-observation test, deterministic or randomized, $T_{K_*}$ capable to decide on the hypotheses $H^K_\chi$, $\chi = 1, 2$, with risk $\leq \epsilon < 1/2$. Then the test $T_{\phi_*,K}$ deciding on $H^K_\chi$, $\chi = 1, 2$, by accepting $H^K_1$ whenever $\phi(K)(\omega^K) := \sum_{k=1}^K \phi_*(\omega_k) \geq 0$ and accepting $H^K_2$ otherwise, satisfies

$$\text{Risk}(T_{\phi_*}, |H^K_1, H^K_2) \leq \epsilon \ \forall K \geq \hat{K}_* = \frac{2}{1 - \frac{\ln(4(1-\epsilon))}{\ln(1/\epsilon)}} K_*.$$

Moreover, this risk bound remains true when the hypotheses $H^K_\chi$ are extended to $H^{\otimes,K}_\chi$ stating that the distribution $P$ of $\omega^K$ belongs to the quasi direct $K$-th degree of $\mathcal{P}_\chi$, $\chi = 1, 2$.

Note that $\hat{K}_*/K_* \rightarrow 2$ as $\epsilon \rightarrow +0$. 

3.29
Proof. As we know, $K_*$-th degree $\mathcal{O}^{(K)}$ of $\mathcal{O}$ is simple o.s. along with $\mathcal{O}$, and $\phi^{(K)}_*$ is the minimum risk detector for the families $\mathcal{P}^{\ominus,K_*}_\chi$, $\chi = 1, 2$, the risk of this detector being $[\varepsilon_*(\mathcal{P}_1, \mathcal{P}_2)]^K$. By Proposition A as applied to $\mathcal{O}^{(K)}$ in the role of $\mathcal{O}$, we have

$$[\varepsilon_*(\mathcal{P}_1, \mathcal{P}_2)]^{K_*} \leq 2\sqrt{\epsilon(1 - \epsilon)} \Rightarrow \varepsilon_*(\mathcal{P}_1, \mathcal{P}_2) \leq [2\sqrt{\epsilon(1 - \epsilon)}]^{1/K_*} < 1.$$ 

By Detector Calculus, it follows that for $K = 1, 2, \ldots$ it holds

$$\text{Risk}[\phi^{(K)}_* | \mathcal{P}_1^{\otimes,K}, \mathcal{P}_2^{\otimes,K}] \leq [2\sqrt{\epsilon(1 - \epsilon)}]^{K/K_*}$$

and the right hand side is $\leq \epsilon$ whenever $K \geq \hat{K}_*$.  \qed
Near-Optimality of Detector-Based Up to Closeness Testing in Simple Observation Schemes

♠ Situation: We are given a simple o.s.

\[ \mathcal{O} = (\Omega, \Pi), \{ p_\mu : \mu \in \mathcal{M} \}, \mathcal{F} \]

and a collection of nonempty convex compact subsets \( M_\ell, 1 \leq \ell \leq L \) giving rise to

- Families \( \mathcal{P}_\ell = \{ P : P \text{ admits density } \pi_\mu, \mu \in M_\ell \text{ w.r.t. } \Pi \}, \ell = 1, \ldots, L \), along with quasi-direct degrees \( \mathcal{P}_\ell^{\otimes K} = \bigotimes_{k=1}^K \mathcal{P}_\ell \) of \( \mathcal{P}_\ell \) and hypotheses \( H_\ell^{\otimes K} : P^K \in \mathcal{P}_\ell^{\otimes K} \) on the distribution of \( K \)-repeated observation \( \omega^K = (\omega_1, \ldots, \omega_K) \),
- minimum-risk balanced single-observation detectors \( \phi_{\ell \ell'}(\omega) \) for \( \mathcal{P}_\ell, \mathcal{P}_{\ell'} \) along with their risks \( \varepsilon_\star(\mathcal{P}_\ell, \mathcal{P}_{\ell'}) \), \( 1 \leq \ell < \ell' \leq L \), and \( K \)-repeated versions

\[ \phi_{\ell \ell'}^K(\omega^K) = \sum_{k=1}^K \phi_{\ell \ell'}(\omega_k) \]

of \( \phi_{\ell \ell'} \) such that

\[ \text{Risk}[\pi_{\ell \ell'}^{(K)} | H_\ell^{\otimes K}, H_{\ell'}^{\otimes K}] \leq [\varepsilon_\star(\mathcal{P}_\ell, \mathcal{P}_{\ell'})]^K. \]

♠ Assume that in addition to the above data, we are given a closeness relation \( C \) on \( \{1, \ldots, L\} \). Applying Calculus of Detectors, for every positive integer \( K \), setting

\[ \theta_K = \left\| \varepsilon_\star^K(\mathcal{P}_\ell, \mathcal{P}_{\ell'}) \cdot \left\{ \begin{array}{ll} 1, & (\ell, \ell') \notin C \\ 0, & (\ell, \ell') \in C \end{array} \right\}_{\ell, \ell'=1}^L \right\|_{2,2} \]

we can assemble the outlined data, in a computationally efficient fashion, into a \( K \)-observation test \( \mathcal{T}^K \) deciding on \( H_\ell^{\otimes K}, 1 \leq \ell \leq L \), with \( C \)-risk upper-bounded as follows:

\[ \text{Risk}^C(\mathcal{T}^K | H_1^{\otimes K}, \ldots, H_L^{\otimes K}) \leq \varkappa \theta_K \]

(\( \varkappa > 1 \) can be selected to be as close to 1 as we want).

3.31
Proposition. In the just described situation, assume that for some $\epsilon < 1/2$ and $K_*$ in the nature there exists a $K_*$-observation test $\mathcal{T}$, deterministic or randomized, deciding on the hypotheses

$$H_{\ell}^{\otimes, K_*} : \omega^{K_*} = (\omega_1, ..., \omega_{K_*})$$

is an i.i.d. sample drawn from a $P \in \mathcal{P}_{\ell}$, $\ell = 1, ..., L$, with $C$-risk $\leq \epsilon$. Then the test $\mathcal{T}^K$ with

$$K \geq 2 \left\lceil \frac{1 + \ln(\kappa L) / \ln(1/\epsilon)}{1 - \ln(4(1 - \epsilon)) / \ln(1/\epsilon)} \right\rceil K_*$$

\rightarrow 1 \text{ as } \epsilon \rightarrow 0$$

decides on $H_{\ell}^{\otimes, K_*}$, $\ell = 1, ..., L$, with $C$-risk $\leq \epsilon$ as well.

Proof. Let us fix $\ell, \ell'$ such that $(\ell, \ell') \not\in C$, and let us convert $\mathcal{T}$ into a simple $K_*$-observation test $\mathcal{T}'$ deciding on $H_{\ell}^{\otimes, K_*}, H_{\ell'}^{\otimes, K_*}$ as follows: whenever $\ell \in \mathcal{T}(\omega^K)$, $\mathcal{T}'$ accepts $H_{\ell}^{\otimes, K_*}$ and rejects $H_{\ell'}^{\otimes, K_*}$, otherwise the test accepts $H_{\ell}^{\otimes, K_*}$ and rejects $H_{\ell'}^{\otimes, K_*}$. It is immediately seen that

$$\text{Risk}(\mathcal{T}'|H_{\ell}^{\otimes, K_*}, H_{\ell'}^{\otimes, K_*}) \leq \epsilon.$$ 

Indeed, let $P^{K_*} = P \times ... \times P$ be the distribution of $\omega^{K_*}$. Whenever $P^{K_*}$ obeys $H_{\ell}^{\otimes, K_*}$, $\mathcal{T}$ must accept the hypothesis with $P^{K_*}$-probability $\geq 1 - \epsilon$, whence

$$\text{Risk}_1(\mathcal{T}'|H_{\ell}^{\otimes, K_*}, H_{\ell'}^{\otimes, K_*}) \leq \epsilon.$$ 

If $P^{K_*}$ obeys $H_{\ell'}^{\otimes, K_*}$, the $P^{K_*}$-probability of the event “$\mathcal{T}$ accepts $H_{\ell}^{\otimes, K_*}$ and rejects $H_{\ell'}^{\otimes, K_*}$” is $\leq \epsilon$, since $H_{\ell}^{\otimes, K_*}, H_{\ell'}^{\otimes, K_*}$ are not $C$-close to each other.

$\Rightarrow$ $P^{K_*}$-probability to reject $H_{\ell}^{\otimes, K_*}$ is at least $1 - \epsilon$

$\Rightarrow$ $\text{Risk}_2(\mathcal{T}'|H_{\ell}^{\otimes, K_*}, H_{\ell'}^{\otimes, K_*}) \leq \epsilon.$
$H^\oplus, K, H^\oplus, K'$ can be decided upon by a simple test with risk $\leq \epsilon$.

- $H^\oplus, K$, $H^\oplus, K'$ can be decided upon with risk $\leq \epsilon < 1/2$

  $\Rightarrow \varepsilon^\ast(P^\oplus, K, P^\oplus, K') \leq 2\sqrt{\epsilon(1 - \epsilon)} < 1$ (Calculus of Detectors)

  $\Rightarrow \varepsilon^\ast(P, P') \leq \left[2\sqrt{\epsilon(1 - \epsilon)}\right]^{1/K^\ast} < 1$ (since $\mathcal{O}$ is a simple o.s.)

  $\Rightarrow \theta_K \leq \left[2\sqrt{\epsilon(1 - \epsilon)}\right]^{K/K^\ast} L$

  $\Rightarrow \text{Risk}^C(T^K|H^\otimes_1, K, \ldots, H^\otimes_L, K) \leq \varkappa \theta_K \leq \epsilon$ when

  $$K/K^\ast \geq 2^{\frac{1 + \ln(\varkappa L)/\ln(1/\epsilon) - \ln(4(1 - \epsilon))}{\ln(1/\epsilon)}}.$$
How it Works: Illustration I
Selecting the Best in a Family of Estimates

Problem:

- We are given a simple o.s. \( \mathcal{O} = ((\Omega, \Pi), \{p_\mu : \mu \in \mathcal{M}\}, \mathcal{F}) \) and have access to stationary \( K \)-repeated observations

\[ \omega_k \sim p_{A(x^*)}(\cdot), \quad k = 1, \ldots, K, \]

of unknown signal \( x_* \) known to belong to a given convex compact set \( X \subset \mathbb{R}^n \).

\( [x \mapsto A(x)]: \) affine mapping such that \( A(X) \subset \mathcal{M} \).

- We are given \( M \) candidate estimates \( x_i \in \mathbb{R}^n \), \( 1 \leq i \leq M \), of \( x_* \), a norm \( \| \cdot \| \) on \( \mathbb{R}^n \), and a reliability tolerance \( \epsilon \in (0, 1) \)

- Ideal Goal: Use observations \( \omega_1, \ldots, \omega_K \) to identify \((1 - \epsilon)\)-reliably the \( \| \cdot \| \)-closest to \( x_* \) point among \( x_1, \ldots, x_M \).

- Actual Goal:

Given \( \alpha \geq 1, \beta \geq 0 \) and a grid \( \Gamma = \{r_0 > r_1 > \ldots > r_N > 0\} \), use observations \( \omega_1, \ldots, \omega_K \) to identify \((1 - \epsilon)\)-reliably a point \( x_i(\omega_K) \) such that

\[
\|x_* - x_i(\omega_K)\| \leq \alpha \rho(x_*) + \beta
\]

\[
\rho(x) := \min \{r : r \in \Gamma, \quad r \geq \min_i \|x - x_i\|\}
\]

\( \rho(x) \) is grid approximation of \( \min_i \|x - x_i\| \)
A Solution:

- Introduce $M(N + 1)$ hypotheses

$$H_{ij} : \omega_k \sim p_{A(x)}(\cdot) \text{ for some } x \in X_{ij} := \{x \in X : \|x - x_i\| \leq r_j\}.$$

- Define closeness $C = C_{\alpha,\beta} : ij C\text{-close to } i'j' \iff \|x_i - x_{i'}\| \leq \bar{\alpha}(r_j + r_{j'}) + \beta$ $[\bar{\alpha} = \frac{\alpha - 1}{2}]$

- Build minimum risk pairwise tests $\phi_{i,j,i',j'}$, along with their risks $\epsilon_{i,j,i',j'}$, for hypotheses $H_{i,j}$.
  - $\epsilon_{i,j,i',j'} = 1 \text{ for some } (i,j, i', j') \notin C \Rightarrow \text{ reject } (\alpha, \beta)$
  - $\epsilon_{i,j,i',j'} < 1 \text{ for all } (i,j, i', j') \notin C \Rightarrow \text{ find the smallest } K = K_\epsilon(\alpha, \beta) \text{ such that the } K\text{-observation detector-based test } T^K \text{ decides on } \{H_{i,j} : i \leq M, 0 \leq j \leq N\} \text{ with } C\text{-risk} \leq \epsilon$.

  $\Rightarrow$ applying $T^K$ to observation $\omega^K$, select among the accepted hypotheses $H_{i,j}$ the one, $H_{i^*(\omega^K), j^*(\omega^K)}$ with the largest $j$. $i(\omega^K)$ is built.
Fact: In the situation in question, whenever \((\alpha, \beta)\) is not rejected, the resulting inference \(\omega^K \mapsto i(\omega^K)\) meets the design specifications:

\[
(x^* \in X, \omega_k \sim p_{A(x^*)}(\cdot) \text{ are independent across } k \leq K) \Rightarrow \Pr\{\|x - x_i(\omega^K)\| \leq \alpha \rho(x^*) + r\} \geq 1 - \epsilon.
\]

Fact: In the situation in question, assume that for some \(\epsilon \in (0, 1/2), a, b \geq 0\) and positive integer \(K\) in the nature there exists an inference \(\omega^K \mapsto i^*(\omega^K)\) such that

\[
\forall x^* \in X : \Pr\{\|x^* - x_i^*(\omega^K)\| \leq a \rho(x^*) + b\} \geq 1 - \epsilon.
\]

Then the pair \((\alpha = 2a + 1, \beta = 2b)\) is not rejected by the above construction, and

\[
K_\epsilon(\alpha, \beta) \leq \text{Ceil} \left( \frac{2 \ln(M(N + 1))/\ln(1/\epsilon) - K}{1 - \ln(4(1 - \epsilon))/\ln(1/\epsilon)} \right).
\]
Numerical illustration: Given noisy observation

\[ \omega = Ax + \sigma \xi, \; \xi \sim \mathcal{N}(0, I_n) \]

of the “discretized primitive” \( Ax \) of a signal \( x = [x^1; \ldots; x^n] \in \mathbb{R}^n \):

\[ [Ax]_j = \frac{1}{n} \sum_{s=1}^{j} x^s, \; 1 \leq j \leq n, \]

for \( i = 1, \ldots, \kappa \) we have built Least Squares polynomial, of order \( i - 1 \), approximations \( x_i \) of \( x \):

\[ x_i = \arg\min_{x \in \mathcal{P}_i} \| Ax - \omega \|_2^2 \]

\[ \mathcal{P}_i = \{ x = [x^1; \ldots; x^n] : x^s \text{ is polynomial, of degree } \leq i - 1, \text{ in } s \} \]

and now want to use \( K \) additional observations to identify the nearly closest to \( x_* \), in the norm

\[ \| u \| = \frac{1}{n} \sum_{i=1}^{n} |u_i| \]

on \( \mathbb{R}^n \), among the points \( x_i, \; 1 \leq i \leq \kappa \).
Numerical experiment \([\epsilon = 0.01, n = 128, \sigma = 0.01, 
\kappa = 5, \alpha = 3, \beta = 0.05]\)

\[
\begin{array}{c|c|c|c|c|c}
   \hline
   i & 1 & 2 & 3 & 4 & 5 \\
   \hline
   \|x - x_i\| & 0.5348 & 0.33947 & 0.23342 & 0.16313 & 0.16885 \\
   \hline
\end{array}
\]

Top: \(x_\ast\) and \(x_{i}\). Bottom: the primitive of \(x_\ast\)

- Computation yielded \(K = 3\). But
  - with \(K = 3\), in sample of 1000 simulations, \textit{not a single case of wrong identification of the exactly closest to} \(x_\ast\) \textit{point was observed}: in every single simulation, we got \(i(\omega^3) = 4\), in spite of the theoretical guarantee as poor as poor as

  \[
  \|x_\ast - x_{i(\omega^3)}\| \leq 3\rho(x_\ast) + 0.05
  \]

  — the same was true when \(K = 3\) was replaced with \(K = 1\);

  — replacing \(K = 3\) with \(K = 1\) \textit{and increasing} \(\sigma\) \textit{from} 0.01 \textit{to} 0.05, the procedure started to make imperfect conclusions. However, the exactly closest to \(x_\ast\) point \(x_4\) was identified correctly in as many as 961 of 1000 simulations, and the empirical mean \(E\{\|x_\ast - x_{i(\omega^1)}\| - \rho(x_\ast)\}\) was as small as 0.0024.

3.38
Problem: An unknown signal $x$ known to belong to a given convex compact set $X \subset \mathbb{R}^n$ is observed according to

$$\omega = Ax + \sigma \xi, \; \xi \sim \mathcal{N}(0, I_d)$$

Our goal is to recover the value at $x$ of a linear-fractional functional $F(z) = f^Tz/e^Tz$, with $e^Tz > 0$, $z \in X$.

Illustration: We are given noisy measurements of voltages $V_i$ at some nodes $i$ and currents $I_{ij}$ in some arcs $(i, j)$ of an electric circuit, and want to recover the resistance of a particular arc $(\hat{i}, \hat{j})$:

$$r_{\hat{i}\hat{j}} = \frac{V_{\hat{j}} - V_{\hat{i}}}{I_{\hat{i}\hat{j}}}$$
A circuit with 8 nodes and 11 arcs

\[ x = [\text{voltages at nodes}; \text{currents in arcs}] \]

\[ Ax = [\text{observable voltages}; \text{observable currents}] \]

- Currents are measured in blue arcs only
- Voltages are measured in magenta nodes only
- We want to recover resistance of red arc

\[ X : \begin{cases} 
\text{conservation of current, except for nodes } \#1,8 \\
\text{zero voltage at node } \#1, \text{ nonnegative currents} \\
\text{current in red arc at least 1, total of currents at most 33} \\
\text{Ohm Law, resistances of arcs between 1 and 10} 
\end{cases} \]
**Strategy:** Given $L$,

- split the range $\Delta = [\min_{x \in X} F(x), \max_{x \in X} F(x)]$ into $L$ consecutive bins $\Delta_\ell$ of length $\delta_L = \text{length}(\Delta)/L$,
- define the convex compact sets

$$X_\ell = \{x \in X : F(x) \in \Delta_\ell\}, \ M_\ell = \{Ax : x \in X_\ell\}, \ 1 \leq \ell \leq L$$

- decide on $L$ hypotheses $H_\ell : P = \mathcal{N}(\mu, \sigma^2 I), \mu \in M_\ell$ on the distribution $P$ of observation $\omega = Ax + \sigma \xi$ up to closeness $C$ “$H_\ell$ is close to $H_{\ell'}$ if and only if $|\ell - \ell'| \leq 1$”
- estimate $F(x)$ by the center of masses of all accepted bins.

**Fact:** For the resulting test $\mathcal{T}$, with probability $\geq 1 - \text{Risk}^C(\mathcal{T}|H_1, ..., H_L)$ the estimation error does not exceed $\delta_L$. 

3.40
**Implementation and results:** Given target risk $\epsilon$ and $L$, we selected the largest $\sigma$ for which $\text{Risk}^C(T|H_1,\ldots,H_L) \leq \epsilon$.

- This is what we get in our Illustration for $\epsilon = 0.01$: $\Delta = [1, 10]$

<table>
<thead>
<tr>
<th>$L$</th>
<th>8</th>
<th>16</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_L$</td>
<td>$9/8 \approx 1.13$</td>
<td>$9/16 \approx 0.56$</td>
<td>$9/32 \approx 0.28$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.024</td>
<td>0.010</td>
<td>0.005</td>
</tr>
<tr>
<td>$\sigma_{opt}/\sigma \leq$</td>
<td>1.31</td>
<td>1.31</td>
<td>1.33</td>
</tr>
</tbody>
</table>

- $\sigma_{opt}$ – the largest $\sigma$ for which “in the nature” there exists a test deciding on $H_1,\ldots,H_L$ with $C$-risk $\leq 0.01$

- **Red data:** Risks $\epsilon_{\ell\ell'}$ of pairwise tests are bounded via risks of optimal detectors, $C$-risk of $T$ is bounded by

$$\left\| \left[ \epsilon_{\ell\ell'} \cdot \chi(\ell,\ell') \notin C \right]_{\ell,\ell'=1}^L \right\|_{2,2};$$

- **Brown data:** Risks $\epsilon_{\ell\ell'}$ of pairwise tests are bounded via error function, $C$-risk of $T$ is bounded by

$$\max_{\ell} \sum_{\ell':(\ell,\ell') \notin C} \epsilon_{\ell\ell'}.$$
Motivating example: Opinion Polls. Population-wide elections with $L$ candidates are to be held.

Preferences of a voter are represented by $L$-dimensional basic orth $[0; ..., 0; 1; 0; ...; 0]$ with 1 in position $\ell$ meaning voting for candidate $\neq \ell$.

Equivalently: Preference $\omega$ of a voter is a vertex in the $L$-dimensional probabilistic simplex

$$\Delta_L = \{p \in \mathbb{R}^L : p \geq 0, \sum_{\ell} p_{\ell} = 1\}.$$

The average $\mu = [\mu_1; ...; \mu_L]$ of preferences of all voters “encodes” election’s outcome: $\mu_\ell$ is the fraction of voters supporting $\ell$-th candidate, and the winner corresponds to the largest entry in $\mu$ (assumed to be uniquely defined).

Note: $\mu$ is a probabilistic vector: $\mu \in \Delta_L$. We think of $\mu$ as of a probability distribution on the $L$-element set $\Omega = \text{Ext}(\Delta_L)$ of vertices of $\Delta_L$.

Our goal is to design opinion poll – to select $K$ voters at random from the uniform over voters population distribution and to observe their preferences, in order to predict, with reliability $1 - \epsilon$, election’s outcome.

Poll’s model is drawing stationary $K$-repeated observation $\omega^K = (\omega_1, ..., \omega_K), \omega_k \in \Omega$, from distribution $\mu$. 

3.42
♠ Assume once for ever that the elections never end with “near tie,” that is, the fraction of votes for the winner is at least by a known margin $\delta$ larger than the fraction of votes for every no-winner, and introduce $L$ hypotheses on the distribution $\mu$ from which $\omega_1, \ldots, \omega_K$ are drawn:

$$H_\ell : \mu \in \mathcal{P}_\ell = \{\mu \in \Delta_L : \mu_\ell \geq \mu_{\ell'} + \delta, \forall \ell' \neq \ell\}, \ell = 1, \ldots, L$$

Our goal is to specify $K$ in a way which allows to decide on $H_1, \ldots, H_L$ via stationary $K$-repeated observations with risk $\leq \epsilon$.

♠ We are in the case of Discrete o.s., and can use our machinery to build a near-optimal $K$-observation test deciding on $H_1, \ldots, H_L$ up to trivial closeness $C$ “$H_\ell$ is close to $H_{\ell'}$ iff $\ell = \ell'$” and then select the smallest $K$ for which the $C$-risk of this test is $\leq \epsilon$.

♠ **Illustration** $L = 2$: In this case $\Omega$ is two-point set of basic orths in $\mathbb{R}^2$, the minimum risk single-observation detector is

$$\phi^*(\omega) = \frac{1}{2} \ln \left( \frac{1 + \delta}{1 - \delta} \right) [\omega^1 - \omega^2] : \Omega \to \mathbb{R}$$

and $\text{Risk}[\phi^*|\mathcal{P}_1, \mathcal{P}_2] = 1 - \delta^2$

$$\Rightarrow K = \text{Ceil} \left( \frac{\ln(1/\epsilon)}{\ln(1/(1-\delta^2))} \right) \approx \frac{1}{\delta^2} \ln(1/\epsilon).$$

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>0.3162</th>
<th>0.1000</th>
<th>0.0316</th>
<th>0.0100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K, K$</td>
<td>51, 88</td>
<td>534, 917</td>
<td>5379, 9206</td>
<td>53820, 92064</td>
</tr>
</tbody>
</table>

$K$: lower bound on optimal poll size.
### USA Presidential Elections-2016:

<table>
<thead>
<tr>
<th>State</th>
<th>Actual margin</th>
<th>Poll size, lower bound</th>
<th>Poll size, upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wisconsin</td>
<td>0.0041</td>
<td>24,576</td>
<td>88,663</td>
</tr>
<tr>
<td>Pennsylvania</td>
<td>0.0038</td>
<td>28,978</td>
<td>104,545</td>
</tr>
<tr>
<td>Michigan</td>
<td>0.0012</td>
<td>281,958</td>
<td>1,017,227</td>
</tr>
</tbody>
</table>

Confidence level 95%
$K \geq \frac{1}{\delta^2} \ln(1/\epsilon)$

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>0.3162</th>
<th>0.1000</th>
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</tr>
</tbody>
</table>

*K: lower bound on optimal poll size*

**Bad news:** Required size of opinion poll grows rapidly as “winning margin” decreases.

♠ **Question:** *Can we do better?*

♠ **Partial remedy:** Let us pass to sequential tests, where we attempt to make conclusion before all $K$ respondents required by the worst-case-oriented analysis are interviewed.

**Hope:** If elections are about to be “landslide” (i.e., in unknown to us actual distribution $\mu^*$ of voters’ preferences the winner beats all other candidates by margin $\delta^* \gg \delta$), the winner hopefully can be identified after a relatively small number of interviews.
Strategy. We select a number $S$ of attempts and associate with attempt $s$ number $K(s)$ of observations, $K(1) < \ldots < K(S)$.

$s$-th attempt to make inference is made when $K(s)$ observations are collected. When it happens, we apply to the collected so far observation $\omega^{K(s)} = (\omega_1, \ldots, \omega_{K(s)})$ a test $T_s$ which, depending on $\omega^{K(s)}$,

- either accepts exactly one of the hypotheses $H_1, \ldots, H_L$, in which case we terminate,
- or claims that information collected so far does not allow to make an inference, in which case we pass to collecting more observations (when $s < S$) or terminate (when $s = S$).

Specifications: We want the overall procedure to be

- conclusive: an inference should be made in one of the $S$ attempts (thus, when attempt $S$ is reached, making inference becomes a must);
- reliable: whenever the true distribution $\mu_*$ underlying observations obeys one of our $L$ hypotheses, the $\mu_*$-probability for this hypothesis to be eventually accepted should be $\geq 1 - \epsilon$, where $\epsilon \in (0, 1)$ is a given in advance risk bound.
An implementation:

- We select somehow the number of attempts $S$ and set $\delta_s = \delta^s / S$ so that $\delta_1 > \delta_2 > \ldots > \delta_S = \delta$. Besides this, we split risk bound $\epsilon$ into $S$ parts $\epsilon_s$:
  
  $$\epsilon_s > 0, \ s \leq S \ & \ \sum_{s=1}^{S} \epsilon_s = \epsilon;$$

- For $s < S$, we define $2L$ hypotheses

  $$H_{2\ell-1}^s = H_{\ell} : \mu \in P_{2\ell-1}^s = P_{\ell} := \{ \mu \in \Delta_L : \mu_{\ell} \geq \delta_S + \max_{\ell' \neq \ell} \mu_{\ell'} \}$$

  “weak hypothesis”

  $$H_{2\ell}^s = \{ \mu \in P_{2\ell}^s := \{ \mu \in \Delta_L : \mu_{\ell} \geq \delta_s + \max_{\ell' \neq \ell} \mu_{\ell'} \} \subset P_{\ell}$$

  “strong hypothesis”

  $1 \leq \ell \leq L$, and assign $H_{2\ell-1}^s$ and $H_{2\ell}^s$ with color $\ell$, $1 \leq \ell \leq L$.

- For $s = S$ we introduce $L$ hypotheses $H_{\ell}^S = H_{\ell}$, $\ell = 1, \ldots, L$, with $H_{\ell}^S$ assigned color $\ell$.

- For $s < S$, we introduce closeness relation $C_s$ on the collection of hypotheses $H_1^s, \ldots, H_{2L}^s$ as follows:
  - the only hypotheses close to a strong hypothesis $H_{2\ell}^s$ are the hypotheses $H_{2\ell}^s$ and $H_{2\ell-1}^s$ of the same color;
  - the only hypotheses close to a weak hypothesis $H_{2\ell-1}^s$ are all weak hypotheses and the strong hypothesis $H_{2\ell}$ of the same color as $H_{2\ell-1}$.

- For $s = S$, the $C_s$-closeness is trivial: $H_{\ell}^S \equiv H_{\ell}$ is $C_S$-close to $H_{\ell'}^S \equiv H_{\ell'}$ if and only if $\ell = \ell'$.
3-candidate hypotheses in probabilistic simplex $\Delta_3$

- **[weak green]** $M_1$: dark green + light green: candidate A wins with margin $\geq \delta_s$
- **[strong green]** $M_1^s$: dark green: candidate A wins with margin $\geq \delta_s > \delta_S$
- **[weak red]** $M_2$: dark red + pink: candidate B wins with margin $\geq \delta_s$
- **[strong red]** $M_2^s$: dark red: candidate B wins with margin $\geq \delta_s > \delta_S$
- **[weak blue]** $M_3$: dark blue + light blue: candidate C wins with margin $\geq \delta_s$
- **[strong blue]** $M_3^s$: dark blue: candidate C wins with margin $\geq \delta_s > \delta_S$

- $H_{s2\ell-1}^s: \mu \in M_\ell$ [weak hypothesis]
  
  weak hypothesis $H_{s2\ell-1}^s$ is $C_s$-close to itself, to all other weak hypotheses and to strong hypothesis $H_{2\ell}^s$ of the same color as $H_{2\ell-1}^s$

- $H_{2\ell}^s: \mu \in M_\ell^s$ [strong hypothesis]
  
  strong hypothesis $H_{2\ell}^s$ is $S$-close only to itself and to weak hypothesis $H_{2\ell-1}^s$ of the same color as $H_{2\ell-1}^s$
• **Note:** We are in the case of stationary repeated observations in Discrete o.s., and the hypotheses \( H^s_j \) are of the form “i.i.d. observations \( \omega_1, \omega_2, ... \) are drawn from distribution \( \mu \in M^s_j \) with nonempty closed convex sets \( M^s_j \subset \Delta_L \), and sets \( M^s_j, M^s_{j'} \) with \((j, j') \notin C_s\) do not intersect

\[ \Rightarrow \] the risks of the minimum-risk pairwise detectors for \( P^s_j, P^s_{j'}, (j, j') \notin C_s \), are < 1

\[ \Rightarrow \] we can efficiently find out the smallest \( K = K(s) \) for which our machinery produces a test \( T = T_s \) deciding, via stationary \( K(s) \)-repeated observations, on the hypotheses \( \{H^s_j\}_j \) with \( C_s \)-risk \( \leq \epsilon_s \).

• It is easily seen that \( K(1) < K(2) < ... < K(S - 1) \). In addition, discarding all attempts \( s < S \) with \( K(s) < K(S) \) and renumbering the remaining attempts, we may assume w.l.o.g. that \( K(1) < K(2) < ... < K(S) \).

♠ **Our inference routine** works as follows: we observe \( \omega_k, k = 1, 2, ..., K(S) \) (i.e., carry interviews with one by one randomly selected voters), and perform \( s \)-th attempt to make conclusion when \( K(s) \) observations are acquired (\( K(s) \) interviews are completed).

At \( s \)-th attempt, we apply the test \( T_s \) to observation \( \omega^{K(s)} \). If the test does accept some of the hypotheses \( H^s_j \) and all accepted hypotheses have the same color \( \ell \), we accept \( \ell \)-th of our original hypotheses \( H_1, ..., H_L \) (i.e., predict that \( \ell \)-th candidate will be the winner) and terminate, otherwise we proceed to next observations (i.e., next interviews) (when \( s < S \)) or claim the winner to be, say, the first candidate and terminate (when \( s = S \)).

3.49
Facts:

- The risk of the outlined sequential hypothesis testing procedure is $\leq \epsilon$: whenever the distribution $\mu_*$ underlying observations obeys hypothesis $H_\ell$ for some $\ell \leq L$, the $\mu_*$-probability of the event “$H_\ell$ is the only accepted hypothesis” is at least $1 - \epsilon$.

- The worst-case volume of observations $K(S)$ is within logarithmic factor from the minimal number of observations allowing to decide on the hypotheses $H_1, \ldots, H_L$ with risk $\leq \epsilon$.

- Whenever the distribution $\mu_*$ underlying observations obeys strong hypothesis $H_{s\ell}^s$ for some $\ell$ and $s$ (“distribution $\mu_*$ of voters’ preferences corresponds to winning margin at least $\delta_s$”), the conclusion, with $\mu_*$-probability $\geq 1 - \epsilon$, will be made in course of the first $s$ attempts (i.e., in course of the first $K(s)$ interviews).

Informally: In landslide elections, the winner will be predicted reliably after a small number of interviews.
How it Works: 2-Candidate Elections

♠ Setup:
• # of candidates $L = 2$
• # $\delta_S = 10^{-s/4}$
• range of # of attempts $S$: $1 \leq S \leq 8$

♠ Numerical Results:

<table>
<thead>
<tr>
<th>$S$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = \delta_S$</td>
<td>0.5623</td>
<td>0.3162</td>
<td>0.1000</td>
<td>0.0562</td>
<td>0.0316</td>
<td>0.0100</td>
</tr>
<tr>
<td>$K$</td>
<td>25</td>
<td>88</td>
<td>287</td>
<td>917</td>
<td>9206</td>
<td>92098</td>
</tr>
<tr>
<td>$K(S)$</td>
<td>25</td>
<td>152</td>
<td>1594</td>
<td>5056</td>
<td>16005</td>
<td>160118</td>
</tr>
</tbody>
</table>

Volume $K$ of non-sequential test, number of attempts $S$ and worst-case volume $K(S)$ of sequential test as functions of winning margin $\delta = \delta_S$. Risk $\epsilon$ is set to 0.01.

Note: Worst-case volume of sequential test is essentially worse than the volume of non-sequential test.

But: When drawing the true distribution $\mu_*$ of voters’ preferences at random from the uniform distribution on the set of $\mu$’s with winning margin $\geq 0.01$, the typical size of observations used by Sequential test with $S = 8$ prior to termination is $\ll K(S)$:

Empirical Volume of Sequential test

<table>
<thead>
<tr>
<th>median</th>
<th>mean</th>
<th>60%</th>
<th>65%</th>
<th>75%</th>
<th>80%</th>
<th>85%</th>
<th>90%</th>
<th>95%</th>
<th>100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>177</td>
<td>9182</td>
<td>177</td>
<td>397</td>
<td>617</td>
<td>1223</td>
<td>1829</td>
<td>8766</td>
<td>87911</td>
<td>160118</td>
</tr>
</tbody>
</table>

Column "X%": empirical X%-quantile of test’s volume. Data over 1,000 experiments. Empirical risk: 0.01
Observation: In our Hypothesis Testing setup, observation scheme is our “environment” and is completely out of our control. However, there are situations where the observation scheme is under our partial control.

Example: Opinion Poll revisited. In our original Opinion Poll problem, a particular voter was represented by basic orth $\omega = [0; \ldots; 0; 1; 0; \ldots; 0] \in \mathbb{R}^L$, with entry 1 in position $\ell$ meaning that the voter prefers candidate $\ell$ to all other candidates. Our goal was to predict the winner by observing preferences of respondents selected at random from uniform distribution on voters’ population.

However: Imagine we can split voters in $I$ non-intersecting groups (say, according to age, education, gender, income, occupation,...) in such a way that we have certain a priori knowledge of the distribution of preferences within the groups. In this situation, our poll can be organized as follows:

- We assign the groups with nonnegative weights $q_i$ summing up to 1
- To organize an interview, we first select at random one of the groups, with probability $q_i$ to select group $i$, and then select a respondent from $i$-th group at random, from uniform distribution on the group.
• We assign the groups with nonnegative weights $q_i$ summing up to 1
• To organize an interview, we first select at random one of the groups, with probability $q_i$ to select group $i$, and then select a respondent from $i$-th group at random, from uniform distribution on the group.

**Note:** When $q_i$ is equal to the fraction $\theta_i$ of group $i$ in the entire population, the above policy reduces to the initial one. It can make sense, however, to use $q_i$ different from $\theta_i$, with $q_i \ll \theta_i$ if a priori information about preferences of voters from $i$-th group is rich, and $q_i \gg \theta_i$ if this information is poor. Hopefully, this will allow us to make more reliable predictions with the same total number of interviews.
The model of outlined situation is as follows:

- We characterize distribution of preferences within group \( i \) by vector \( \mu^i \in \Delta_L \). for \( 1 \leq \ell \leq L \), \( \ell \)-th entry in \( \mu^i \) is the fraction of voters in group \( i \) voting for candidate \( \ell \);

Note: The population-wide distribution of voters’ preferences is \( \mu = \sum_{i=1}^{I} \theta_i \mu^i \).

- A priori information on distribution of preferences of voters from group \( i \) is modeled as the inclusion \( \mu^i \in M^i \), for some known subset \( M^i \subset \Delta_L \) which we assume to be nonempty convex compact set.

- Output of particular interview is pair \((i,j)\), where \( i \in \{1,\ldots,I\} \) is selected at random according to probability distribution \( q \), and \( j \) is the candidate preferred by respondent selected from group \( i \) at random, according to uniform distribution on the group.

\( \Rightarrow \) Our observation (outcome from an interview) becomes

\[
\omega := (i, \ell) \in \Omega = \{1, \ldots, I\} \times \{1, \ldots, L\},
\]

\[
\text{Prob}\{\omega = (i, j)\} = p(i,j) := q_i \mu^i_j.
\]

The hypotheses to be decided upon are

\[
H_\ell[q] : p \in \mathcal{P}_\ell[q] := \left\{ \{p_{ij} = q_i \mu^i_j\}_{1 \leq i, j \leq L} : \begin{bmatrix} \sum_{i} \theta_i \mu^i \end{bmatrix}_\ell \geq \delta + \begin{bmatrix} \sum_{i} \theta_i \mu^i \end{bmatrix}_{\ell'} \quad \forall (\ell' \neq \ell) \right\}
\]

\( H_\ell[q], \ell = 1, \ldots, L, \) states that the “signal” \( \vec{\mu} = [\mu^1; \ldots; \mu^I] \) underlying distribution \( p \) of observations \( \omega \) induces population-wide distribution \( \sum_i \theta_i \mu^i \) of votes resulting in electing candidate \( \ell \) with winning margin \( \geq \delta \).
\[ H_\ell[q] : p \in \mathcal{P}_\ell[q] := \left\{ p_{ij} = q_i \mu_i^j \right\}_{1 \leq i \leq I, 1 \leq j \leq L} : \left\{ \begin{array}{l}
\mu^i \in M^i \forall i, \\
\left[ \sum_i \theta_i \mu^i \right]_{\ell} \geq \delta + \left[ \sum_i \theta_i \mu^i \right]_{\ell'} \forall (\ell' \neq \ell)
\end{array} \right\} \]

\[\star\text{ Note:}\] Hypotheses \( H_\ell[q] \) are of the form

\[ H_\ell[q] = \left\{ p = A[q] \bar{\mu} : \bar{\mu} : = [\mu^1; \ldots; \mu^L] \in \mathcal{M}^\ell \right\}, \]

where \( \mathcal{M}^\ell, \ell = 1, \ldots, L, \) are nonempty nonintersecting convex compact subsets in \( \Delta_L \times \ldots \times \Delta_L \)

\[\star\text{ Note:}\] Opinion Poll with \( K \) interviews corresponds to stationary \( K \)-repeated observation in Discrete o.s. with \( IL \)-element observation space \( \Omega \)

\[ \Rightarrow \text{ Given } K, \text{ we can use our machinery to design a near-optimal detector-based test } T_K \text{ deciding via stationary } K \text{-repeated observation (i.e., via the outcomes of } K \text{ interviews) on hypotheses } H_\ell[q], \ell = 1, \ldots, L \text{ up to trivial closeness } \]

\[ \text{“} H_\ell[q] \text{ is close to } H_{\ell'}[q] \text{ iff } \ell = \ell' \text{”} \]

This test will predict the winner with reliability \( 1 - \text{Risk}(T_K|H_1[q], \ldots, H_L[q]) \).
\[ H_\ell[q] = \{ p = A[q]\vec{\mu} : \vec{\mu} := [\mu^1; \ldots; \mu^L] \in M^\ell \}, \]
\[ [A[q]\vec{\mu}]_{ij} = q_i \mu^i_j, \]

\[ \text{By our theory, setting } \chi_{\ell\ell'} = \begin{cases} 0, & \ell = \ell' \\ 1, & \ell \neq \ell' \end{cases}, \text{ we have} \]

\[ \text{Risk}(T_K|H_1[q], \ldots, H_L[q]) \leq \epsilon_K[q] := \left\| \left[ \epsilon_{\ell\ell'}^K[q] \chi_{\ell\ell'} \right]_{\ell,\ell'=1}^L \right\|_{2,2}, \]
\[ \epsilon_{\ell\ell'}[q] = \max_{\vec{\mu} \in M^\ell, \vec{\nu} \in M^\ell'} \sum_{i,j} \sqrt{[A[q]\vec{\mu}]_{ij} [A[q]\vec{\nu}]_{ij}} \]
\[ = \max_{\vec{\mu} \in M^\ell, \vec{\nu} \in M^\ell'} \sum_{i=1}^I q_i \left[ \sum_{j=1}^L \sqrt{\mu^i_j \nu^i_j} \right] \Phi(q; \vec{\mu}, \vec{\nu}) \]

**Note:** \( \Phi(q; \vec{\mu}, \vec{\nu}) \) is linear in \( q \).

\[ \text{Let us carry out Measurement Design – optimization of } \epsilon_K[q] \text{ in } q. \]

\[ \text{Main observation: } \epsilon_K[q] = \Gamma(\Psi(q)), \text{ where} \]
- \( \Gamma(Q) = \left\| [(Q_{\ell\ell'})^K \chi_{\ell\ell'}]_{\ell,\ell'=1}^L \right\|_{2,2} \) is efficiently computable convex and entrywise nondecreasing function on the space of nonnegative \( L \times L \) matrices
- \( \Psi(q) \) is matrix-valued function with efficiently computable convex in \( q \) and nonnegative entries

\[ \Psi_{\ell\ell'}(q) = \max_{\vec{\mu} \in M^\ell, \vec{\nu} \in M^\ell'} \Phi(q; \vec{\mu}, \vec{\nu}) \]
\[ \Rightarrow \text{Optimal selection of } q_i \text{'s reduces to solving explicit convex problem} \]
\[ \min_q \left\{ \Gamma(\Psi(q)) : q = [q_1; \ldots; q_I] \geq 0, \sum_{i=1}^I q_i = 1 \right\} \]
How it Works: Measurement Design in Election Polls

♠ Setup:
• Opinion Poll problem with $L$ candidates and winning margin $\delta = 0.05$
• Reliability $\epsilon = 0.01$
• A priori information on voters’ preferences in groups:

$$M^i = \{\mu_i^i \in \Delta_L : p^i_{\ell} - u_i \leq \mu_i^i \leq p^i_{\ell} + u_i, \ell \leq L\}$$

- $p^i$: randomly selected probabilistic vector
- $u_i$: uncertainty level

♠ Sample of results:

<table>
<thead>
<tr>
<th>$L$</th>
<th>$I$</th>
<th>Group sizes $\theta$</th>
<th>Uncertainty levels $u$</th>
<th>$K_{\text{ini}}$</th>
<th>$q_{\text{opt}}$</th>
<th>$K_{\text{opt}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>$\theta = [0.50; 0.50]$</td>
<td>$u = [0.03; 1.00]$</td>
<td>1212</td>
<td>$[0.44; 0.56]$</td>
<td>1194</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$[0.50; 0.50]$</td>
<td>$[0.02; 1.00]$</td>
<td>2699</td>
<td>$[0.00; 1.00]$</td>
<td>1948</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>$[0.33; 0.33; 0.33]$</td>
<td>$[0.02; 0.03; 1.00]$</td>
<td>3177</td>
<td>$[0.00; 0.46; 0.54]$</td>
<td>2726</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>$[0.25; 0.25; 0.25; 0.25]$</td>
<td>$[0.02; 0.03; 1.00]$</td>
<td>2556</td>
<td>$[0.00; 0.13; 0.32; 0.55]$</td>
<td>2086</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>$[0.25; 0.25; 0.25; 0.25]$</td>
<td>$[1.00; 1.00; 1.00; 1.00]$</td>
<td>4788</td>
<td>$[0.25; 0.25; 0.25; 0.25]$</td>
<td>4788</td>
</tr>
</tbody>
</table>

Effect of measurement design. $K_{\text{ini}}$ and $K_{\text{opt}}$ are the poll sizes required for 0.99-reliable prediction of the winner when $q_i = \theta_i$ and $q = q_{\text{opt}}$, respectively.

Note: Uncertainty $= 1.00 \iff$ No a priori information
In numerous situations, we do have partial control of observation scheme and thus can look for optimal Measurement Design.

**However:** the situations where optimal Measurement Design can be found efficiently, like in design of Election Polls, are rare.

Additional examples of these rare situations are *Poisson o.s. and Gaussian o.s. with time control.*
Poisson o.s. with time control. Typical models where Poisson o.s. arises are as follows:

- “in the nature” there exists a “signal” $x$ known to belong to some convex compact set $\subset \mathbb{R}^n$

For example, in Positron Emission Tomography, $x$ is (discretized) density of radioactive tracer administered to patient

- We observe random vector $\omega \in \mathbb{R}^m$ with independent entries $\omega_i \sim \text{Poisson}(a_i^T x)$, and want to make inferences on $x$. For example, in PET, tracer disintegrates, and every disintegration act results in pair of gamma-quants flying in opposite directions along a randomly oriented line passing through disintegration point. This line is registered when two detector cells are (nearly) simultaneously hit:

The data acquired in PET study are the numbers $\omega_i$ of lines registered in bins (pairs of detector cells) $i = 1, \ldots, m$ over a time horizon $T$, and

$$\omega_i \sim \text{Poisson}(T \sum_{j=1}^n p_{ij} x_j)$$

$[p_{ij}$: probability for line emanated from voxel $j = 1, \ldots, n$ to cross pair $i = 1, \ldots, m$ of detector cells

$\Rightarrow A = T \begin{bmatrix} p_{ij} \end{bmatrix}_{i \leq m, j \leq n}$
\[ \omega = \{ \omega_i \sim \text{Poisson}([Ax]_i) \}_{i \leq m} \]

- In some situations, the sensing matrix \( A \) can be partially controlled:

\[
A = A[q] := \text{Diag}\{q\}A_*
\]

- \( A_* \): given \( m \times n \) matrix; \( q \in Q \): vector of control parameters.

For example, in a whole body PET scan the position of the patient w.r.t. the scanner is updated several times to cover the entire body.

The data acquired in position \( \iota \) form subvector \( \omega^{\iota} \) in the entire observation \( \omega = [\omega^1; \ldots; \omega^I] \):

\[
\omega^{\iota}_i \sim \text{Poisson}([t_\iota A^{\iota}x]_i), \ 1 \leq i \leq \bar{m} = m/I
\]

\[
[A^{\iota} : \text{given matrices}; \ t_\iota : \text{duration of study in position } \iota]
\]

implying that \( \omega = \text{Diag}\{q\}A_* \) with properly selected \( A_* \) and \( q \) of the form

\[
q = [t_1; \ldots; t_1; \ldots; t_I; \ldots; t_I]_{\bar{m}}\bar{m}
\]

- Let our goal be to decide, up to a given closeness \( C \), on \( L \) hypotheses on the distribution of Poisson observation \( \omega \):

\[
H^q_\ell : \omega \sim \text{Poisson}([A[q]x]_1) \times \ldots \times \text{Poisson}([A[q]x]_m) \& x \in X_\ell
\]

\( X_\ell \): given convex compact sets, \( 1 \leq \ell \leq L \).
\( H^q_\ell : \omega_i \sim \text{Poisson}([A[q]x]_i) \) are independent across \( i \leq m \) and \( x \in X_\ell \)

\[
A = A[q] := \text{Diag} \{ q \} A_*,
\]

• \( A_* \): given \( m \times n \) matrix; • \( q \in Q \): control parameters.

♩ By our theory, the (upper bound on the) \( C \)-risk of near-optimal test deciding on \( H^q_\ell, \ell = 1, \ldots, L \), is

\[
\epsilon(q) = \left\| \left[ \exp \{ \text{Opt}_{\ell\ell'}(q) \} \chi_{\ell\ell'} \right]_{\ell,\ell' = 1}^L \right\|_{2,2}
\]

\[
\begin{align*}
\chi_{\ell\ell'} &= \begin{cases} 
0, & (\ell, \ell') \in C \\
1, & (\ell, \ell') \notin C
\end{cases} \\
\text{Opt}_{\ell\ell'}(q) &= \max_{u \in X_\ell, v \in X_{\ell'}} -\frac{1}{2} \sum_{i=1}^m \left( \sqrt{[A[q]u]_i} - \sqrt{[A[q]v]_i} \right)^2
\end{align*}
\]

• Similarly to the Opinion Polls, \( \epsilon(q) = \Gamma(\Psi(q)) \), where

• \( \Gamma(Q) = \left\| \left[ \exp \{ Q_{\ell\ell'} \} \chi_{\ell\ell'} \right]_{\ell,\ell' = 1}^L \right\|_{2,2} \) is a convex entrywise nondecreasing function of \( L \times L \) matrix \( Q \)

• \( [\Psi(q)]_{\ell\ell'} = \exp \left\{ \max_{u \in X_\ell, v \in X_{\ell'}} \sum_{i=1}^m q_i \left( \sqrt{[A_*u]_i[A_*v]_i} - \frac{1}{2} [A_*u]_i - \frac{1}{2} [A_*v]_i \right) \right\} \)

Note: similarly to Opinion Polls, \( [\Psi(q)]_{\ell\ell'} \) is efficiently computable and convex in \( q \)

⇒ Assuming the set \( Q \subset \mathbb{R}_+^m \) of allowed controls \( q \) convex, optimizing \( \epsilon(q) \) over \( q \in Q \) is an explicitly given convex optimization problem.
An efficiently solvable Measurement Design problem in Gaussian o.s.

\[ \omega = A[q]x + \xi, \quad \xi \sim \mathcal{N}(0, I_m) \]

[• A[q] partially controlled sensing matrix; • q ∈ Q: control parameters.]

is the one where

\[ A[q] = \text{Diag}\{\sqrt{q_1}, \ldots, \sqrt{q_m}\} \& Q \subset \mathbb{R}_+^m \text{ is a convex compact set} \]

In this case, minimizing Q-risk of test deciding up to closeness C on L hypotheses

\[ H^q_\ell : \omega \sim \mathcal{N}(A[q]x, I_m), \quad x \in X_\ell, \quad 1 \leq \ell \leq L \]

associated with nonempty convex compact sets \( X_\ell \) reduces to solving convex problem

\[
\min_{q \in Q} \Gamma(\Psi(q))
\]

where

\[
\Gamma(Q) = \| [\exp\{Q_{\ell\ell'}/8\}X_\ell X_{\ell'}]_{\ell,\ell' \leq L} \|_{2,2}
\]

is convex entrywise nondecreasing function of \( L \times L \) matrix \( Q \), and

\[
[\Psi(q)]_{\ell\ell'} = \max_{u \in X_\ell, v \in X_{\ell'}} \left[ -\|A[q](u - v)\|_2^2 \right]
\]

\[
= -\min_{u \in X_\ell, v \in X_{\ell'}} (u - v)^T A^*_T \text{Diag}\{q\} A_*(u - v)
\]

is efficiently computable convex function of \( q \in Q \).
Illustration. In some applications, “the physics” beyond Gaussian o.s. \( \omega = Ax + \xi \) is as follows. There are \( \ell \) sensors measuring analogous vector-valued continuous time signal \( x \) (nearly constant on the observation horizon) in the presence of noise. The output of sensor \( \#i \) is

\[
\omega_i = \frac{1}{|\Delta_i|} \int_{\Delta_i} [a_{i,*}^T x + B_i(t)] dt
\]

- \( \Delta_i \) : continuous time interval
- \( B_i(t) \) : "Brownian motion:" \( \frac{1}{|\Delta|} \int_{\Delta} B_i(t) dt \sim \mathcal{N}(0, |\Delta|^{-1}) \), \( \int_{\Delta} B_i(t) dt, \int_{\Delta'} B_i(t) dt \) are independent when \( \Delta \cap \Delta' = \emptyset \)
- Brownian motions \( B_i(t) \) are independent across \( i \)

- When all sensors work in parallel for unit time, we arrive at the standard Gaussian o.s. \( \omega = A_{*}x + \xi, \xi \sim \mathcal{N}(0, I_m) \).
- When sensors work on consecutive segments \( \Delta_1, ..., \Delta_m \) of durations \( q_i = |\Delta_i| \), we arrive at

\[
\omega_i = a_{i,*}^T x + q_i^{-1/2} \xi_i, \xi_i \sim \mathcal{N}(0, 1) \text{ are independent across } i
\]

Rescaling observations according to \( \omega_i \mapsto \sqrt{q_i} \omega_i \), we arrive at the desired partially controlled observation scheme

\[
\omega = \text{Diag}\{\sqrt{q_1}, ..., \sqrt{q_m}\} A_{*}x + \xi, \xi \sim \mathcal{N}(0, I_m)
\]

A natural selection of \( Q \) is, e.g.,

\[
Q = \{ q \geq 0 : \sum_i q_i = n \}
\]

(setting “time budget” to the same value as in the case of consecutive observations of duration 1 each).
HYPOTHESIS TESTING, III

Beyond simple observation schemes
Observation: A “common denominator” of minimum risk detectors for simple o.s.’s is their affinity in observations:

- the optimal detectors in Gaussian and Poisson o.s.’s are affine “as they are”
- encoding observation space $\Omega = \{1, \ldots, d\}$ of Discrete o.s. by vertices $e_i, i = 1, \ldots, d$, of the standard simplex $\Delta_d = \{x \in \mathbb{R}^d : x \geq 0, \sum_j x_j = 1\}$, every function on $\Omega$ becomes affine
  $\Rightarrow$ we can treat optimal detector in Discrete o.s. as affine function on $\mathbb{R}^d$.
- operations with optimal detectors induced by taking direct products of basic simple o.s.’s or passing to repeated observations preserves affinity.

Fact: “Reasonable” (perhaps, non-optimal) affine detectors can be found, in a computationally efficient way, in many important situations which are beyond simple o.s.’s.
Setup

Given an observation space $\Omega = \mathbb{R}^d$, consider a triple $\mathcal{H}, \mathcal{M}, \Phi$, where

- $\mathcal{H}$ is a nonempty closed convex set in $\Omega$ symmetric w.r.t. the origin,
- $\mathcal{M}$ is a compact convex set in some $\mathbb{R}^n$,
- $\Phi(h; \mu) : \mathcal{H} \times \mathcal{M} \rightarrow \mathbb{R}$ is a continuous function convex in $h \in \mathcal{H}$ and concave in $\mu \in \mathcal{M}$.

$\mathcal{H}, \mathcal{M}, \Phi$ specify a family $S[\mathcal{H}, \mathcal{M}, \Phi]$ of probability distributions on $\Omega$. A probability distribution $P$ belongs to the family iff there exists $\mu \in \mathcal{M}$ such that

$$\ln \left( \int_{\Omega} e^{h^T \omega} P(d\omega) \right) \leq \Phi(h; \mu) \quad \forall h \in \mathcal{H} \quad (\star)$$

We refer to $\mu$ ensuring $(\star)$ as to parameter of distribution $P$.
- **Warning**: A distribution $P$ may have many different parameters!
- We refer to triple $\mathcal{H}, \mathcal{M}, \Phi$ satisfying the above requirements as to regular data, and to $S[\mathcal{H}, \mathcal{M}, \Phi]$ – as to the simple family of distributions induced by these data.
Example 1: Gaussian and sub-Gaussian distributions. When $\mathcal{M} = \{(u, \Theta)\} \subset \mathbb{R}^d \times \text{int} \mathbb{S}_+^d$ is a convex compact set such that $\Theta \succ 0$ for all $(u, \Theta) \in \mathcal{M}$, $\mathcal{H} = \mathbb{R}^d$ and $\Phi(h; u, \Theta) = h^T y + \frac{1}{2} h^T \Theta h$, $S = S[\mathcal{H}, \mathcal{M}, \Phi]$ contains all probability distributions $P$ which are sub-Gaussian with parameters $(u, \Theta)$, meaning that
\[
\ln \left( \int_{\Omega} e^{h^T \omega} P(d\omega) \right) \leq h^T u + \frac{1}{2} h^T \Theta h \quad \forall h, \quad (1)
\]
and, in addition, the “parameter” $(u, \Theta)$ belongs to $\mathcal{M}$.

Note: Whenever $P$ is sub-Gaussian with parameters $(u, \Theta)$, $u$ is the expectation of $P$.

Note: $\mathcal{N}(u, \Theta) \in S$ whenever $(u, \Theta) \in \mathcal{M}$; for $P = \mathcal{N}(u, \Theta)$, (1) is an identity.
Example 2: Poisson distributions. When $\mathcal{M} \subset \mathbb{R}^d_+$ is a convex compact set, $\mathcal{H} = \mathbb{R}^d$ and

$$\Phi(h; \mu) = \sum_{i=1}^{d} \mu_i (e^{h_i} - 1),$$

$S = S[\mathcal{H}, \mathcal{M}, \Phi]$ contains distributions of all $d$-dimensional random vectors $\omega_i$ with independent across $i$ entries

$$\omega_i \sim \text{Poisson}(\mu_i)$$

such that $\mu = [\mu_1; \ldots; \mu_d] \in \mathcal{M}$.
Example 3: Discrete distributions. When

$$\mathcal{M} = \{\mu \in \mathbb{R}^d : \mu \geq 0, \sum_j \mu_j = 1\}$$

is the probabilistic simplex in $\mathbb{R}^d$, $\mathcal{H} = \mathbb{R}^d$ and

$$\Phi(h; \mu) = \ln \left( \sum_{i=1}^d \mu_i e^{h_i} \right),$$

$S = S[\mathcal{H}, \mathcal{M}, \Phi]$ contains all discrete distributions supported on the vertices of the probabilistic simplex.
Example 4: Distributions with bounded support. Let $X \subset \mathbb{R}^d$ be a nonempty convex compact set with support function $\phi_X(\cdot)$:

$$\phi_x(y) = \max_{x \in X} y^T x : \mathbb{R}^d \to \mathbb{R}^d.$$  

When $\mathcal{M} = X$, $\mathcal{H} = \mathbb{R}^d$ and

$$\Phi(h; \mu) = h^T \mu + \frac{1}{8}[\phi_X(h) + \phi_X(-h)]^2, \quad (2)$$

$S = S[\mathcal{H}, \mathcal{M}, \Phi]$ contains all probability distributions supported on $X$, and for such a distribution $P$, $\mu = \int_X \omega P(d\omega)$ is a parameter of $P$.

**Note:** When $G$, $0 \in G$, is a convex compact set, the conclusion in Example 4 remains valid when function (2) is replaced with the smaller function

$$\Phi(h; \mu) = \min_{g \in G} \left[ \mu^T (h - g) + \frac{1}{8}[\phi_X(h - g) + \phi_X(g - h)]^2 + \phi_X(g) \right].$$
♠ Fact: Simple families of probability distributions admit “calculus:”

♠ [summation] For \(1 \leq \ell \leq L\), let \(\lambda_\ell\) be reals, and let \(\mathcal{H}_\ell, \mathcal{M}_\ell, \Phi_\ell\) be regular data with common observation space: \(\mathcal{H}_\ell \subset \Omega = \mathbb{R}^d\). Setting

\[
\mathcal{H} = \{h \in \mathbb{R}^d : \lambda_\ell h \in \mathcal{H}_\ell, 1 \leq \ell \leq L\}, \mathcal{M} = \mathcal{M}_1 \times ... \times \mathcal{M}_L, \\
\Phi(h; \mu_1, ..., \mu_L) = \sum_{\ell=1}^{L} \Phi_\ell(\lambda_\ell h; \mu_\ell),
\]

we get regular data with the following property:

Whenever random vectors \(\xi_\ell \sim P_\ell \in S[\mathcal{H}_\ell, \mathcal{M}_\ell, \Phi_\ell], 1 \leq \ell \leq L\), are independent across \(\ell\), the distribution \(P\) of the random vector \(\xi = \sum_{\ell=1}^{L} \lambda_\ell \xi_\ell\) belongs to \(S[\mathcal{H}, \mathcal{M}, \Phi]\). Denoting by \(\mu_\ell\) parameters of \(P_\ell\), \(\mu = [\mu_1; ...; \mu_L]\) can be taken as parameter of \(P\).

♠ [direct product] For \(1 \leq \ell \leq L\), let \(\mathcal{H}_\ell, \mathcal{M}_\ell, \Phi_\ell\) be regular data with observation spaces \(\Omega_\ell = \mathbb{R}^{d_\ell}\). Setting

\[
\mathcal{H} = \mathcal{H}_1 \times ... \times \mathcal{H}_L \subset \Omega = \mathbb{R}^{d_1+...+d_L}, \\
\mathcal{M} = \mathcal{M}_1 \times ... \times \mathcal{M}_L, \\
\Phi(h_1, ..., h_L; \mu_1, ..., \mu_L) = \sum_{\ell=1}^{L} \Phi_\ell(h_\ell; \mu_\ell),
\]

we get regular data with the following property:

Whenever \(P_\ell \in S[\mathcal{H}_\ell, \mathcal{M}_\ell, \Phi_\ell], 1 \leq \ell \leq L\), the direct product distribution \(P = P_1 \times ... \times P_L\) belongs to \(S[\mathcal{H}, \mathcal{M}, \Phi]\). Denoting by \(\mu_\ell\) parameters of \(P_\ell\), \(\mu = [\mu_1; ...; \mu_L]\) can be taken as parameter of \(P\).
[marginal distribution] Let $\mathcal{H}, \mathcal{M}, \Phi$ be regular data with observation space $\mathbb{R}^d$, and let $\omega \mapsto A\omega + a : \mathbb{R}^d \mapsto \Omega = \mathbb{R}^\delta$. Setting

$$\bar{\mathcal{H}} = \{ h \in \mathbb{R}^\delta : A\bar{T} h \in \mathcal{H} \}, \quad \bar{\Phi}(h; \mu) = h^T a + \Phi(A\bar{T} h; \mu),$$

we get regular data $\bar{\mathcal{H}}, \mathcal{M}, \bar{\Phi}$ with the following property:

Whenever $\xi \sim P \in S[\mathcal{H}, \mathcal{M}, \Phi]$, the distribution $\bar{P}$ of the random variable $\omega = A\xi + a$ belongs to the simple family $S[\bar{\mathcal{H}}, \mathcal{M}, \bar{\Phi}]$, and parameter of $P$ is a parameter of $\bar{P}$ as well.
Main observation: When deciding on simple families of distributions, affine tests and their risks can be efficiently computed via Convex Programming:

Theorem. Let $\mathcal{H}_\chi, \mathcal{M}_\chi, \Phi_\chi, \chi = 1, 2$, be two collections of regular data with compact $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{H}_1 = \mathcal{H}_2 =: \mathcal{H}$, and let

$$
\Psi(h) = \max_{\mu_1 \in \mathcal{M}_1, \mu_2 \in \mathcal{M}_2} \frac{1}{2} \left[ \Phi_1(-h; \mu_1) + \Phi_2(h, \mu_2) \right] : \mathcal{H} \to \mathbb{R}
$$

Then $\Psi$ is efficiently computable continuous convex function, and for every $h \in \mathcal{H}$, setting

$$
\phi(\omega) = h^T \omega + \frac{1}{2} \left[ \max_{\mu_1 \in \mathcal{M}_1} \Phi_1(-h; \mu_1) - \max_{\mu_2 \in \mathcal{M}_2} \Phi_2(h; \mu_2) \right],
$$

one has

$$
\text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2] \leq \exp\{\Psi(h)\} \quad [\mathcal{P}_\chi = \mathcal{S}[\mathcal{H}, \mathcal{M}_\chi, \Phi_\chi]]
$$

In particular, if convex-concave function $\Phi(h; \mu_1, \mu_2)$ possesses a saddle point $h_*, (\mu_1^*, \mu_2^*)$ on $\mathcal{H} \times (\mathcal{M}_1 \times \mathcal{M}_2)$, the affine detector

$$
\phi_*(\omega) = h_*^T \omega + \frac{1}{2} \left[ \Phi_1(-h_*; \mu_1^*) - \Phi_2(h_*; \mu_2^*) \right]
$$

admits risk bound

$$
\text{Risk}[\phi|\mathcal{P}_1, \mathcal{P}_2] \leq \exp\{\Phi(h_*; \mu_1^*, \mu_2)\}$$
Indeed, let $h \in \mathcal{H}$. Setting $\mu_1^* \in \text{Argmax}_{\mu_1 \in \mathcal{M}_1} \Phi_1(-h; \mu_1)$, $\mu_2^* \in \text{Argmax}_{\mu_2 \in \mathcal{M}_2} \Phi_2(h; \mu_2)$, we have

$P \in \mathcal{P}_1 := \mathcal{S} \left[ \mathcal{H}, \mathcal{M}_1, \Phi_1 \right] \Rightarrow$

$\exists \mu_1 \in \mathcal{M}_1 : \mathbb{E}_{\omega \sim P} \left\{ e^{-h^T \omega} P(d\omega) \right\} \leq e^{\Phi_1(-h; \mu_1)} \Rightarrow$

$e^{\mathbb{E}_{\omega \sim P} \left\{ e^{-\phi(\omega)} P(d\omega) \right\}} \leq e^{\Phi_1(-h; \mu_1^*) - \kappa} = e^{\Psi(h)} \Rightarrow$

$$\text{Risk}_1[\phi|\mathcal{P}_1, \mathcal{P}_2] \leq e^{\Psi(h)}.$$  

Similarly,

$P \in \mathcal{P}_2 := \mathcal{S} \left[ \mathcal{H}, \mathcal{M}_2, \Phi_2 \right] \Rightarrow$

$\exists \mu_2 \in \mathcal{M}_2 : \mathbb{E}_{\omega \sim P} \left\{ e^{h^T \omega} P(d\omega) \right\} \leq e^{\Phi_2(h; \mu_2)} \Rightarrow$

$e^{\mathbb{E}_{\omega \sim P} \left\{ e^{\phi(\omega)} P(d\omega) \right\}} \leq e^{\Phi_2(h; \mu_2^*) + \kappa} = e^{\Psi(h)} \Rightarrow$

$$\text{Risk}_2[\phi|\mathcal{P}_1, \mathcal{P}_2] \leq e^{\Psi(h)}.$$
Numerical Illustration. Given observation

\[ \omega = Ax + \sigma A \text{Diag} \{\sqrt{x_1}, \ldots, \sqrt{x_n}\} \xi \quad [\xi \sim \mathcal{N}(0, I_n)] \]

of an unknown signal \( x \) known to belong to a given convex compact set \( M \subset \mathbb{R}^n_{++} \), we want to decide on two hypotheses \( H_\chi : x \in X_\chi, \chi = 1, 2 \) with risk 0.01.
\( X_\chi \): convex compact subsets of \( X \).

Novelty: Noise intensity depends on the signal!

- Introducing regular data \( \mathcal{H}_\chi = \mathbb{R}^n, \mathcal{M}_\chi = X_\chi \),

\[ \Phi_\chi(h, \mu) = h^T A \mu + \frac{\sigma^2}{2} h^T [A \text{Diag}\{\mu\} A^T] h \quad [\chi = 1, 2] \]

distribution of observations under \( H_\chi \) belongs to \( \mathcal{S}[\mathcal{H}, \mathcal{M}_\chi, \Phi_\chi] \).

- An affine detector for families \( \mathcal{P}_\chi \) of distributions obeying \( H_\chi \), \( \chi = 1, 2 \), is given by the saddle point of the function

\[ \Phi(h; \mu_1, \mu_2) = \frac{1}{2} \left[ h^T [\mu_2 - \mu_1] + \frac{\sigma^2}{2} h^T A \text{Diag}\{\mu_1 + \mu_2\} A^T h \right] \]

Data: \( n = 16, \sigma = 0.1 \), target risk 0.01,

- \( A = U \text{Diag}\{0.01^{(i-1)/15}, i \leq 16\} V \) with random orthogonal \( U, V \),

- \( X_1 = \left\{ x \in \mathbb{R}^{16} : 0.001 \leq x_1 \leq \delta, \quad 0.001 \leq x_i \leq 1, i \geq 2 \right\} \)

- \( X_2 = \left\{ x \in \mathbb{R}^{16} : 2\delta \leq x_1 \leq 1, \quad 0.001 \leq x_i \leq 1, i \geq 2 \right\} \)

Results:

\( \delta = 0.1 \Rightarrow \text{Risk}[\phi_*|\mathcal{P}_1, \mathcal{P}_2] = 0.4346 \Rightarrow 6\text{-repeated observation} \)

\( \delta = 0.01 \Rightarrow \text{Risk}[\phi_*|\mathcal{P}_1, \mathcal{P}_2] = 0.9201 \Rightarrow 56\text{-repeated observation} \)

- Safe “Gaussian o.s. approximation” of the above observation scheme requires 37-repeated observations to handle \( \delta = 0.1 \) and 3685-repeated observation to handle \( \delta = 0.01 \).
♣ **Sub-Gaussian case.** For \( \chi = 1, 2 \), let \( U_{\chi} \subset \Omega = \mathbb{R}^d \) and \( \mathcal{V}_{\chi} \subset \text{int} \mathcal{S}_d^+ \) be convex compact sets. Setting

\[
\mathcal{M}_{\chi} = U_{\chi} \times \mathcal{V}_{\chi}, \quad \Phi(h; u, \Theta) = h^T u + \frac{1}{2} h^T \Theta h : \mathcal{H} \times \mathcal{M}_{\chi} \to \mathbb{R},
\]

the regular data \( \mathcal{H} = \mathbb{R}^d, \mathcal{M}_{\chi}, \Phi \) specify the families

\[
\mathcal{P}_{\chi} = S[\mathbb{R}^d, U_{\chi} \times \mathcal{V}_{\chi}, \Phi]
\]

of sub-Gaussian distributions with parameters from \( U_{\chi} \times \mathcal{V}_{\chi} \).

♠ **Saddle point problem** responsible for design of affine detector for \( \mathcal{P}_1, \mathcal{P}_2 \) reads

\[
\text{SadVal} = \min_{h \in \mathbb{R}^d} \max_{u_1 \in U_1, u_2 \in U_2, \Theta_1 \in \mathcal{V}_1, \Theta_2 \in \mathcal{V}_2} \frac{1}{2} \left[ h^T (u_2 - u_1) + \frac{1}{2} h^T [\Theta_1 + \Theta_2] h \right]
\]

- Saddle point \((h^*; (u_1^*, u_2^*, \Theta_1^*, \Theta_2^*))\) does exist and satisfies

\[
h^* = [\Theta_1^* + \Theta_2^*]^{-1} [u_1^* - u_2^*], \quad \text{SadVal} = -\frac{1}{4} [u_1^* - u_2^*] [\Theta_1^* + \Theta_2^*]^{-1} [u_1^* - u_2^*] = -\frac{1}{4} h^T_* [u_1^* - u_2^*]
\]

- The associated affine detector and its risk are

\[
\phi^*_*(\omega) = h^T_* [\omega - \frac{1}{2} [u_1^* + u_2^*]] = [u_1^* - u_2^*]^T [\Theta_1^* + \Theta_2^*]^{-1} [\omega - \frac{1}{2} [u_1^* + u_2^*]]
\]

\[
\text{Risk}(\phi^*_*|\mathcal{P}_1, \mathcal{P}_2) \leq \exp\{\text{SadVal}\} = \exp\left\{-\frac{1}{4} [u_1^* - u_2^*] [\Theta_1^* + \Theta_2^*]^{-1} [u_1^* - u_2^*]\right\}
\]

♥ **Note:** In the *symmetric case* \( \mathcal{V}_1 = \mathcal{V}_2 \) \((h^*; (u_1^*, u_2^*, \Theta_1^*, \Theta_2^*))\) can be selected to have \( \Theta_1^* = \Theta_2^* =: \Theta^* \). In this case, the affine detector we end up with is the minimum risk detector for \( \mathcal{P}_1, \mathcal{P}_2 \).
What is “affine?” Quadratic Lifting

♣ We have developed a technique for building reasonable **affine** detectors for simple families of distributions.

**But:** Given observation $\zeta \sim P$, we can subject it to **nonlinear** transformation $\zeta \mapsto \omega = \psi(\zeta)$, e.g., to **quadratic lifting**

$$\zeta \mapsto \omega = (\zeta, \zeta \zeta^T)$$

and treat as our observation $\omega$ rather than the “true” observation $\zeta$. **Affine in $\omega$ detectors are nonlinear in $\zeta$**.

**Example:** Detectors affine in the quadratic lifting $\omega = (\zeta, \zeta \zeta^T)$ of $\zeta$ are exactly the **quadratic** functions of $\zeta$.

♠ We can try to apply our machinery for building affine detectors to nonlinear transformations of true observations, thus arriving at nonlinear detectors.

- **Bottleneck:** To apply the outlined strategy to a pair $\mathcal{P}_1, \mathcal{P}_2$ of families of distributions of interest, we need to cover the families $\mathcal{P}_\chi^+$ of distributions of $\omega = \psi(\zeta)$ induced by distributions $P \in \mathcal{P}_\chi$ of $\zeta$, $\chi = 1, 2$, by simple families of distributions.

- **What is ahead:** Simple “coverings” of quadratic lifts of (sub)Gaussian distributions.
Situation: Given are:

- a compact nonempty set $U \subset \mathbb{R}^n$
- an affine mapping $u \mapsto A(u) = A[u; 1] : \mathbb{R}^n \to \mathbb{R}^d$
- a convex compact set $V \subset \text{int } S^d_+$.

The above data specify families of probability distributions of random observations

$$\omega = (\zeta, \zeta^T), \quad \zeta = A(u) + \xi \in \mathbb{R}^d,$$

specifically,

- the family $\mathcal{G}$ of all distributions of $\omega$ induced by deterministic $u \in U$ and Gaussian noise $\xi \sim \mathcal{N}(0, \Theta \in V)$

- the family $\mathcal{SG}$ of all distributions of $\omega$ induced by deterministic $u \in U$ and sub-Gaussian, with parameters $(0, \Theta \in V)$ noise $\xi$

Goal: To cover $\mathcal{G}$ ($\mathcal{SG}$) by a simple family of distributions.
Gaussian case

Proposition. Given the above data $U, A(u) = A[u; 1], V$, let us select

- $\gamma \in (0, 1)$
- a computationally tractable convex compact set

\[ Z \subset Z^+ = \{ Z \in S^{n+1} : Z \succeq 0, Z_{n+1,n+1} = 1 \} \]

such that $[u; 1][u; 1]^T \in Z \ \forall u \in U$

- A matrix $\Theta_* \in S^d$ and $\delta \in [0, 2]$ such that

\[ \forall (\Theta \in \mathcal{V}) : \Theta \preceq \Theta_* & \|\Theta^{1/2}\Theta_*^{-1/2} - I_d\| \leq \delta \]

($\|\cdot\|$ is the spectral norm).

Let us set

\[ B = \begin{bmatrix} A \\ 0, \ldots, 0, 1 \end{bmatrix} \in \mathbb{R}^{(d+1) \times (n+1)}, \ M = \mathcal{V} \times Z \]

\[ \mathcal{H} = \{(h, H) \in \mathbb{R}^d \times S^d : -\gamma \Theta_*^{-1} \preceq H \preceq \gamma \Theta_*^{-1} \} \]

\[ \Phi_{A,\mathcal{Z}}(h, H; \Theta, Z) = -\frac{1}{2} \ln \text{Det}(I - \Theta_*^{1/2}H\Theta_*^{1/2}) + \frac{1}{2} \text{Tr}([\Theta - \Theta_*]H) + \frac{\delta(2+\delta)\|\Theta_*^{1/2}H\Theta_*^{1/2}\|_F^2}{2(1-\|\Theta_*^{1/2}H\Theta_*^{1/2}\|_F)} \]

\[ + \frac{1}{2} \text{Tr} \left( ZB^T \left[ \begin{array}{c|c} H \\ h^T \end{array} \right] h + [H, h]^T [\Theta_*^{-1} - H]^{-1} [H, h] \right) \]

\[ \mathcal{H} \times M \to \mathbb{R} \]

Then \( \mathcal{H}, M, \Phi_{A,\mathcal{Z}} \) is efficiently computable regular data, and

\[ \mathcal{G} \subset S[\mathcal{H}, M, \Phi_{A,\mathcal{Z}}]. \]
Proposition. Given the above data \( U, A(u) = A[u; 1], V \), let us select

- \( \gamma, \gamma^+ \in (0, 1) \) with \( \gamma < \gamma^+ \)
- a computationally tractable convex compact set

\[ \mathcal{Z} \subset \mathcal{Z}^+ = \{ Z \in S^{n+1} : Z \succeq 0, Z_{n+1,n+1} = 1 \} \]

such that \([u; 1][u; 1]^T \in \mathcal{Z} \ \forall u \in U\)
- A matrix \( \Theta_* \in S^d \) and \( \delta \in [0, 2] \) such that

\[ \forall (\Theta \in \mathcal{V}) : \Theta \preceq \Theta_* \ \& \ |\Theta^{1/2}\Theta_*^{-1/2} - I_d| \leq \delta \]

Let us set

\[ B = \begin{bmatrix} A \\ 0, \ldots, 0, 1 \end{bmatrix} \in \mathbb{R}^{(d+1) \times (n+1)} \]

\[ \mathcal{H} = \{ (h, H) \in \mathbb{R}^d \times S^d : -\gamma \Theta_*^{-1} \preceq H \preceq \gamma \Theta_*^{-1} \} \]

\[ \mathcal{H}^+ = \{ (h, H, G) \in \mathbb{R}^d \times S^d \times S^d : -\gamma^+ \Theta_*^{-1} \preceq H \preceq G \preceq \gamma^+ \Theta_*^{-1}, 0 \preceq G \} \]

\[ \mathcal{M} = \mathcal{Z} \]

\[ \Phi_{A,Z}(h, H; Z) = \min_{G:(h,H,G)\in\mathcal{H}^+} \left\{ -\frac{1}{2} \ln \text{Det}(I - \Theta_*^{1/2}G\Theta_*^{1/2}) \right\} : \mathcal{H} \times \mathcal{M} \to \mathbb{R} \]

Then \( \mathcal{H}, \mathcal{M}, \Phi_{A,Z} \) is efficiently computable regular data, and

\[ SG \subset S[\mathcal{H}, \mathcal{M}, \Phi_{A,Z}] \].
How to specify $Z$. To apply the above construction, one should specify a computationally tractable convex compact set

$$Z \subset Z^+ = \{Z \in S^{n+1} : Z \succeq 0, Z_{n+1,n+1} = 1\}$$

the smaller the better, such that $u \in U \rightarrow [u; 1][u; 1]^T \in Z$

The ideal selection is

$$Z = Z[U] = \text{Conv}\{[u; 1][u; 1]^T : u \in U\}$$

However: $Z[U]$ usually is computationally intractable.

Important exception:

$$Q \succ 0, U = \{u : u^T Qu \leq 1\} \Rightarrow Z[U] = \{Z \in Z^+ : \sum_{i,j=1}^n Z_{ij}Q_{ij} \leq 1\}$$

“Simple” case: When $U$ is given by quadratic inequalities:

$$U = \{u \in \mathbb{R}^n : [u; 1]^T Q_s [u; 1] \leq q_s, 1 \leq s \leq S\}$$

we can set

$$Z = \{Z \in S^{n+1} : Z \succeq 0, Z_{n+1,n+1} = 1, \text{Tr}(Q_s Z) \leq q_s, 1 \leq s \leq S\}.$$
Quadratic Lifting – Does it Pay?

♣ **Situation:** Let for \( \chi = 2, 1 \) be given

- convex compact sets \( U_\chi \subset \mathbb{R}^{n_\chi} \)
- affine mappings \( u_\chi \mapsto \mathcal{A}_\chi(u_\chi) : \mathbb{R}^{n_\chi} \to \mathbb{R}^d \)
- convex compact sets \( \mathcal{V}_\chi \subset \text{int} \ S_d^+ \).

These data define families \( \mathcal{G}_\chi \) of Gaussian distributions:

\[
\mathcal{G}_\chi = \{N(\mathcal{A}_\chi(u_\chi), \Theta_\chi) : u_\chi \in U_\chi, \Theta_\chi \in \mathcal{V}_\chi\}
\]

♠ Our machinery offers two types of detectors for \( \mathcal{G}_1, \mathcal{G}_2 \):

♠ **Affine detector** \( \phi_{\text{aff}} \) yielded by the solution to the saddle point problem

\[
\text{SadVal}_{\text{aff}} = \min_{h \in \mathbb{R}^d} \max_{u_1 \in U_1, u_2 \in U_2, \Theta_1 \in \mathcal{V}_1, \Theta_2 \in \mathcal{V}_2} \frac{1}{2} \left[ h^T [\mathcal{A}_2(u_2) - \mathcal{A}_1(u_1)] + \frac{1}{2} h^T [\Theta_1 + \Theta_2] h \right]
\]

with \( \text{Risk}(\phi_{\text{aff}}|\mathcal{G}_1, \mathcal{G}_2) \leq \exp\{\text{SadVal}_{\text{aff}}\} \)

♠ **Quadratic detector** \( \phi_{\text{lift}} \) yielded by the solution to the saddle point problem

\[
\text{SadVal}_{\text{lift}} = \min_{(h,H) \in \mathcal{H}} \max_{\Theta_1 \in \mathcal{V}_1, \Theta_2 \in \mathcal{V}_2} \frac{1}{2} \left[ \Phi_{\mathcal{A}_1, \mathcal{Z}_1}(-h, -H; \Theta_1) + \Phi_{\mathcal{A}_2, \mathcal{Z}_2}(h, H; \Theta_2) \right]
\]

with \( \text{Risk}(\phi_{\text{lift}}|\mathcal{G}_1, \mathcal{G}_2) \leq \exp\{\text{SadVal}_{\text{lift}}\} \)

♠ **Fact:** Assume that the sets \( \mathcal{V}_\chi \) contain \( \succeq \)-largest elements. Then with proper selection of the “design parameters” \( \mathcal{Z}_\chi, \Theta_*(\chi) \) participating in the construction of \( \Phi_{\mathcal{A}_\chi, \mathcal{Z}_\chi} \), \( \chi = 1, 2 \), passing from affine to quadratic detectors helps:

\[
\text{SadVal}_{\text{lift}} \leq \text{SadVal}_{\text{aff}}
\]
Numerical illustration:

- $U_1 = U_1^\rho = \{u \in \mathbb{R}^{12} : u_i \geq \rho, 1 \leq i \leq 12\}$,
  
- $U_2 = U_2^\rho = -U_1^\rho$, $A_1 = A_2 \in \mathbb{R}^{8 \times 13}$;

- $V_\chi = \{\Theta^*(\chi) = \sigma_\chi^2 I_8\}$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>unrestricted $H$ and $h$</th>
<th>$H = 0$</th>
<th>$h = 0$</th>
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</tr>
<tr>
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<td>1</td>
<td>4</td>
<td>0.41</td>
<td>1.00</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Risk of quadratic detector $\phi(\zeta) = h^T \zeta + \frac{1}{2} \zeta^T H \zeta + \kappa$

We see that

- when deciding on families of Gaussian distributions with common covariance matrix and expectations varying in associated with the families convex sets, passing from affine to quadratic detectors does not help.

- in general, both affine and purely quadratic components in a quadratic detector are useful.

- when deciding on families of Gaussian distributions in the case where distributions from different families can have close expectations, affine detectors are useless, while the quadratic ones are not.
Illustration: Simple Change Point Detection

Frames from a noisy “movie”
When the picture starts to change?
♣ Model: We observe one by one vectors (“vectorized” 2D images)

\[ \omega_t = x_t + \xi_t, \]

- \( x_t \): deterministic image
- \( \xi_t \sim \mathcal{N}(0, \sigma^2 I_d) \): independent across observation noises.

**Note:** We know a range \([\sigma, \bar{\sigma}]\) of \(\sigma\), but perhaps do not know \(\sigma\) exactly.

- We know that \(x_1 = x_2\) and want to check whether \(x_1 = \ldots = x_K\) (“no change”) or there is a change.

♣ Goal: Given an upper bound \(\epsilon > 0\) on the probability of false alarm, we want to design a sequential change detection routine capable to detect change, if any.
Approach:

- Pass from observations $\omega_t$, $1 \leq t \leq K$, to observations

  $\zeta_t = \omega_t - \omega_1 = \underbrace{x_t - x_1}_{y_t} + \underbrace{\xi_t - \xi_1}_{\eta_t}$, $2 \leq t \leq K$

- Test hypothesis $H_0 : y_2 = \ldots = y_K = 0$ vs. alternative

  $\bigcup_{k=2}^{K} H_k^\rho$, $H_k^\rho : y_2 = \ldots = y_{k-1} = 0, \|y_k\|_2 \geq \rho$

via our machinery for testing

magenta hypothesis $H_0$

vs.

brown hypotheses $H_2^\rho, \ldots, H_K^\rho$

via quadratic liftings $\zeta_t \zeta_t^T$ of observations $\zeta_t$ up to closeness

$C$: all brown hypotheses are close to each other and are not close to the magenta hypothesis

- Find the smallest $\rho$ for which the $C$-risk of the resulting inference is $\leq \epsilon$, and utilize this inference in change point detection.
How It Works

♠ Setup: \( \dim y = 256^2 = 65536, \sigma = 10, \sigma^2/\sigma^2 = 2, \)
\( K = 9, \epsilon = 0.01 \)

♠ Inference: At time \( t = 2, ..., K \), compute

\[
\phi_*(\zeta_t) = -2.7138 \frac{\|\zeta_t\|_2^2}{10^5} + 366.9548.
\]

\( \phi_*(\zeta_t) < 0 \Rightarrow \) conclude that the change took place and terminate
\( \phi_*(\zeta_t) \geq 0 \Rightarrow \) conclude that there was no change so far and proceed to the next image, if any

♠ Note:

• When magenta hypothesis \( H_0 \) holds true, the probability not to claim change on time horizon \( 2, ..., K \) is at least 0.99.
• When a brown hypothesis \( H^0_k \) holds true, the change at time \( \leq K \) is detected with probability at least 0.99, provided \( \rho \geq \rho_* = 2716.6 \) (average per pixel energy in \( y_k \) at least by 12% larger than \( \sigma^2 \))
• No test can 0.99-reliably decide via \( \zeta_1, ..., \zeta_k \) on \( H^0_k \) vs. \( H_0 \), provided \( \rho/\rho_* < 0.965 \).
• In the movie, the change takes place at time 3 and is detected at time 4.
ESTIMATING SIGNALS IN GAUSSIAN O.S.

- Problem of interest
- Developing tools, Optimization
  - Conic Programming
  - Conic Duality
- Developing tools, Statistics
  - Gauss-Markov Theorem
- Optimizing linear estimates
- Near-optimality of linear estimates
- Byproduct on Semidefinite Relaxation
Situation: “In the nature” there exists a signal $x$ known to belong to a given convex compact set $\mathcal{X} \subset \mathbb{R}^n$. We observe corrupted by noise affine image of the signal:

$$\omega = Ax + \sigma \xi \in \Omega = \mathbb{R}^m$$

- $A$: given $m \times n$ sensing matrix
- $\xi$: random observation noise

Goal: To recover the image $Bx$ of $x$ under a given linear mapping

- $B$: given $k \times n$ matrix.

Risk of a candidate estimate $\hat{x}(\cdot) : \Omega \to \mathbb{R}^k$ is defined as

$$\text{Risk}[\hat{x}|\mathcal{X}] = \sup_{x \in \mathcal{X}} \mathbb{E}_\xi \left\{ \| Bx - \hat{x}(Ax + \sigma \xi) \|_2^2 \right\}$$

$\Rightarrow$ Risk$^2$ is the worst-case, over $x \in \mathcal{X}$, expected $\| \cdot \|_2^2$ recovery error.

Agenda: Under appropriate assumptions on $\mathcal{X}$, we shall show that

- One can build, in a computationally efficient fashion, (nearly) the best, in terms of risk, estimate in the family of linear estimates

$$\hat{x}(\omega) = \hat{x}_H(\omega) = H^T \omega \quad [H \in \mathbb{R}^{m \times k}]$$

- The resulting linear estimate is nearly optimal among all estimates, linear and nonlinear alike.
Why linear estimates?

♠ As it was announced, a “nearly optimal” linear estimate can be built in a computationally efficient fashion.

♠ In contrast,
  • Exactly minimax optimal estimate is unknown even in the simplest case when the observation is
    \[ \omega = x + \eta \]
    with \( \eta \sim \mathcal{N}(0, \sigma^2) \) and \( x \in \mathcal{X} = [-1, 1] \)
  • “Standard” Maximum Likelihood estimate can be disastrously bad even in the simple case
    \[ \omega = x + \eta, \quad \eta \sim \mathcal{N}(0, \sigma^2 I_n), \quad \mathcal{X} = \{ x \in \mathbb{R}^n : \|x\|_2 \leq 1 \}, \quad Bx = x_1 \]

In this case, natural implementation of ML estimate is

Build signal \( \tilde{x} \) most likely yielding the observation:

\[ \omega \mapsto \tilde{x} = \underset{\|u\|_2 \leq 1}{\arg\min} \|\omega - u\|_2 \]

and take \( \tilde{x}_1 \) as the estimate of \( Bx = x_1 \).

For \( \sigma \) small and fixed and \( n \) large, with overwhelming probability \( \tilde{x} = \omega/\|\omega\|_2 \approx \omega/\sqrt{n\sigma^2} \), implying that \( |\tilde{x}_1| \leq \frac{O(1)}{\sigma \sqrt{n}} \), and the risk of the ML estimate is \( O(1) \), as compared to the minimax optimal risk \( O(\sigma) \).
Developing Tools, Optimization
“Structure-Revealing” Representation of Convex Problem: Conic Programming

When passing from a Linear Programming program

$$\min_x \left\{ c^T x : Ax - b \geq 0 \right\} \quad (*)$$

to a nonlinear convex one, the traditional wisdom is to replace linear inequality constraints

$$a_i^T x - b_i \geq 0$$

with nonlinear ones:

$$g_i(x) \geq 0 \quad [g_i \text{ are concave}]$$

There exists, however, another way to introduce nonlinearity, namely, to replace the coordinate-wise vector inequality

$$y \geq z \iff y - z \in \mathbb{R}^m_+ = \{ u \in \mathbb{R}^m : u_i \geq 0 \ \forall i \} \quad [y, z \in \mathbb{R}^m]$$

with another vector inequality

$$y \geq_K z \iff y - z \in \mathbb{K} \quad [y, z \in \mathbb{R}^m]$$

where $\mathbb{K}$ is a regular cone (i.e., closed, pointed and convex cone with a nonempty interior) in $\mathbb{R}^m$. 

5.3
\[ y \geq_K z \iff y - z \in K \quad [y, z \in \mathbb{R}^m] \]

**K**: closed, pointed and convex cone in \( \mathbb{R}^m \) with a nonempty interior.

Requirements on \( K \) ensure that \( \geq_K \) obeys the usual rules for inequalities:

- \( \geq_K \) is a *partial order*:
  
  \[ x \geq_K x \quad \forall x \quad [\text{reflexivity}] \]
  \[ (x \geq_K y \& y \geq_K x) \Rightarrow x = y \quad [\text{antisymmetry}] \]
  \[ (x \geq_K y, y \geq_K z) \Rightarrow x \geq_K z \quad [\text{transitivity}] \]

- \( \geq_K \) is compatible with linear operations: the validity of \( \geq_K \) inequality is preserved when we multiply both sides by the same nonnegative real and add to it another valid \( \geq_K \)-inequality;

- in a sequence of \( \geq_K \)-inequalities, one can pass to limits:
  
  \[ \{a_i \geq_K b_i, i = 1, 2, \ldots \& a_i \to a \& b_i \to b\} \Rightarrow a \geq_K b \]

- one can define the strict version \( >_K \) of \( \geq_K \):

  \[ a >_K b \iff a - b \in \text{int } K. \]

Arithmetics of \( >_K \) and \( \geq_K \) inequalities is completely similar to the arithmetics of the usual coordinate-wise \( \geq \) and \( > \).
\* \* LP problem:

\[
\min_x \{ c^T x : Ax - b \geq 0 \} \iff \min_x \{ c^T x : Ax - b \in \mathbb{R}^m_+ \}
\]

\* \* General Conic problem:

\[
\min_x \{ c^T x : Ax - b \geq_K 0 \} \iff \min_x \{ c^T x : Ax - b \in K \}
\]

- \( (A, b) \) – data of conic problem
- \( K \) - structure of conic problem

\* Note: Every convex problem admits equivalent conic reformulation

\* Note: With conic formulation, convexity is “built in”; with the standard MP formulation convexity should be kept in mind as an additional property.

\* (??) A general convex cone has no more structure than a general convex function. Why conic reformulation is “structure-revealing”?

\* (!!) As a matter of fact, just 3 types of cones allow to represent an extremely wide spectrum (“essentially all”) of convex problems!
\[
\min_x \{ c^T x : Ax - b \geq_K 0 \} \iff \min_x \{ c^T x : Ax - b \in K \}
\]

♣ Three Magic Families of cones:

- **\(\mathcal{LP}\): Nonnegative orthants** \(\mathbb{R}_+^m\) – direct products of \(m\) nonnegative rays \(\mathbb{R}_+ = \{ s \in \mathbb{R} : s \geq 0 \}\) giving rise to Linear Programming programs

  \[
  \min_s \left\{ c^T x : a_\ell^T x - b_\ell \geq 0, \ 1 \leq \ell \leq q \right\}.
  \]

- **\(\mathcal{CQP}\): Direct products of Lorentz cones**

  \(\mathbb{L}_+^p = \{ u \in \mathbb{R}^p : u_p \geq \left( \sum_{i=1}^{p-1} u_i^2 \right)^{1/2} \}\) giving rise to Conic Quadratic programs

  \[
  \min_x \left\{ c^T x : \|Ax - b_\ell\|_2 \leq c_\ell^T x - d_\ell, \ 1 \leq \ell \leq q \right\}.
  \]

- **\(\mathcal{SDP}\): Direct products of Semidefinite cones**

  \(\mathbb{S}_+^p = \{ M \in \mathbb{S}^p : M \succeq 0 \}\) giving rise to Semidefinite programs

  \[
  \min_x \left\{ c^T x : \frac{\lambda_{\min}(A_\ell(x))}{\lambda_{\min}(S)} \geq 0, \ 1 \leq \ell \leq q \right\}.
  \]

where \(\mathbb{S}^p\) is the space of \(p \times p\) real symmetric matrices, \(A_\ell(x) \in \mathbb{S}^p\) are affine in \(x\) and \(\lambda_{\min}(S)\) is the minimal eigenvalue of \(S \in \mathbb{S}^p\).

- **Note:** Constraint stating that a symmetric matrix affinely depending on decision variables is \(\succeq 0\) is called **LMI** – Linear Matrix Inequality.
What can be reduced to $LP/CQP/SDP$?

Calculus of Conic programs

Let $\mathcal{K}$ be a family of regular cones closed w.r.t. taking direct products.

Definition: A $\mathcal{K}$-representation of a set $X \subset \mathbb{R}^n$ is a representation

$$X = \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m : Ax + Bu - b \in \mathcal{K}\} \quad (*)$$

where $\mathcal{K} \in \mathcal{K}$.

$X$ is called $\mathcal{K}$-representable, if $X$ admits a $\mathcal{K}$-r.

Note: Minimizing a linear objective $c^T x$ over a $\mathcal{K}$-representable set $X$ reduces to a conic program on a cone from $\mathcal{K}$.

Indeed, given $(*)$, problem $\min_{x \in X} c^T x$ is equivalent to

$$\text{Opt} = \min_{x,u} \left\{c^T x : Ax + Bu - b \in \mathcal{K}\right\}$$

Definition: A $\mathcal{K}$-representation of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a $\mathcal{K}$-representation of the epigraph of $f$:

$\text{Epi}\{f\} := \{(x, t) : t \geq f(x)\}$

$$= \{x, t : \exists v : Px + pt + Qv - q \in \mathcal{K}\}, \quad K \in \mathcal{K}$$

$f$ is called $\mathcal{K}$-representable, if $f$ admits a $\mathcal{K}$-r.
Note:

- A level set of a $\mathcal{K}$-r. function is $\mathcal{K}$-r.: 
  \[
  \text{Epi}\{f\} := \{(x, t) : t \geq f(x)\} =\{x, t : \exists v : Px + pt + Qu - q \in \mathcal{K}\}
  \Rightarrow \{x : f(x) \leq c\} =\{x : \exists v : Px + Qu - [q - cp] \in \mathcal{K}\}
  \]

- Minimization of a $\mathcal{K}$-r. function $f$ over a $\mathcal{K}$-r. set $X$ reduces to a conic program on a cone from $\mathcal{K}$:

\[
  \begin{align*}
  x \in X & \iff \exists u : Ax + Bu - b \in \mathcal{K}_X \\
  t \geq f(x) & \iff \exists v : Px + pt + Qv - q \in \mathcal{K}_f
  \end{align*}
\]

\[
\min_{x \in X} f(x) \iff \min_{t, x, u, v} \left\{ t : [Ax + Bu - b; Px + pt + Qv - q] \in \mathcal{K}_X \times \mathcal{K}_f \right\} \in \mathcal{C}
\]

5.8
Investigating “expressive abilities” of generic Magic conic problems reduces to answering the question

What are $\mathcal{LP}/\mathcal{CQP}/\mathcal{SDP}$-r. functions/sets?

“Built-in” restriction is Convexity: A $\mathcal{K}$-representable set/function must be convex.

Good news: Convexity, essentially, is the only restriction: for all practical purposes, all convex sets/functions arising in applications are $\mathcal{SDP}$-r. Quite rich families of convex functions/sets are $\mathcal{LP}/\mathcal{CQP}$-r.

Note: Nonnegative orthants are direct products of (1-dimensional) Lorentz cones, and Lorentz cones are intersections of semidefinite cones and properly selected linear subspaces $\Rightarrow \mathcal{LP} \subset \mathcal{CQP} \subset \mathcal{SDP}$.
Let $\mathcal{K}$ be a family of regular cones closed w.r.t. taking direct products and passing from a cone $K$ to its dual cone

$$K_* = \{ \lambda : \langle \lambda, \xi \rangle \geq 0 \ \forall \xi \in K \}$$

Note: $K_*$ is regular cone provided $K$ is so, and

$$(K_*)_* = K$$

Fact: $\mathcal{K}$-representable sets/functions admit fully algorithmic calculus: all basic convexity-preserving operations with functions/sets, as applied to $\mathcal{K}$-r. operands, produce $\mathcal{K}$-r. results, and the resulting $\mathcal{K}$-r.’s are readily given by $\mathcal{K}$-r.’s of the operands.

“Calculus rules” are independent of what $\mathcal{K}$ is.

⇒ Starting with “raw materials” (characteristic for $\mathcal{K}$ elementary $\mathcal{K}$-r. sets/functions) and applying calculus rules, we can recognize $\mathcal{K}$-representability and get explicit $\mathcal{K}$-r.’s of sets/functions of interest.
Basics of “calculus of $\mathcal{K}$-representability”:

[Sets:] If $X_1, \ldots, X_k$ are $\mathcal{K}$-r. sets, so are their
- intersections,
- direct products,
- images under affine mappings,
- inverse images under affine mappings.

[Functions:] If $f_1, \ldots, f_k$ are $\mathcal{K}$-r. functions, so are their
- linear combinations with nonnegative coefficients,
- superpositions with affine mappings.

Moreover, if $F, f_1, \ldots, f_k$ are $\mathcal{K}$-r. functions, so is the superposition $F(f_1(x), \ldots, f_k(x))$ provided that $F$ is monotonically nondecreasing in its arguments.

More advanced convexity-preserving operations preserve $\mathcal{K}$-representability under (pretty mild!) regularity conditions. This includes

- for sets: taking conic hulls and convex hulls of (finite) unions and passing from a set to its recessive cone, or polar, or support function
- for functions: partial minimization, projective transformation, and taking Fenchel dual.

Note: Calculus rules are simple and algorithmic

$\Rightarrow$ Calculus can be run on a compiler [used in cvx].
\[
\begin{align*}
\text{Illustration} & \\
\min c^T x + d^T y & \\
y \geq 0, \quad Ax + By \leq b & \\
2y_1^2 - y_2 - y_3^3 + 3y_2^2 - \frac{1}{3}y_4^3 & \leq e^T x + 4y_1^\frac{1}{3}y_2^\frac{2}{3} + 5y_3^\frac{1}{3}y_4^\frac{2}{3} & \\
\begin{bmatrix}
x_1 - x_2 & x_3 + x_2 \\
x_3 + x_2 & x_2 - x_4 & x_5 - 6 \\
x_5 - 6 & x_6 + x_7 & -x_8 & x_5
\end{bmatrix} & \succeq 0 & \\
\begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 \\
x_2 & x_6 & x_7 & x_8 & x_9 \\
x_3 & x_7 & x_{10} & x_{11} & x_{12} \\
x_4 & x_8 & x_{11} & x_{13} & x_{14} \\
x_5 & x_9 & x_{12} & x_{14} & x_{15}
\end{bmatrix} & \succeq 0 & \\
\det \left( \begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 \\
x_2 & x_6 & x_7 & x_8 & x_9 \\
x_3 & x_7 & x_{10} & x_{11} & x_{12} \\
x_4 & x_8 & x_{11} & x_{13} & x_{14} \\
x_5 & x_9 & x_{12} & x_{14} & x_{15}
\end{bmatrix} \right) & \geq 1 & \\
\text{Sum of 2 largest singular values of } & \begin{bmatrix}
x_1 & x_2 & x_3 \\
x_4 & x_5 & x_6 \\
x_7 & x_8 & x_9 \\
x_{10} & x_{11} & x_{12} \\
x_{13} & x_{14} & x_{15}
\end{bmatrix} & \text{is } \leq 6 & \\
1 - \sum_{i=1}^{6} [x_i - x_{i+1}]s^i & \leq 0, \quad \frac{3}{2} \leq s \leq 6 & \\
\sum_{i=1}^{4} x_{2i}\cos(i\phi) - \sum_{i=1}^{4} x_i\sin(i\phi) & \leq 1, \quad \frac{\pi}{3} \leq \phi \leq \frac{\pi}{2} & \\
\end{align*}
\]

- the blue part of the problem is in \(\mathcal{LP}\)
- the blue-magenta part of the problem is in \(\mathcal{CQP}\) and can be approximated, in a polynomial time fashion, by \(\mathcal{LP}\)
- the entire problem is in \(\mathcal{SDP}\)

and the reductions to \(\mathcal{LP}/\mathcal{CQP}/\mathcal{SDP}\) are “fully algorithmic.”
Conic Duality

Conic Programming admits nice Duality Theory completely similar to LP Duality.

**Primal problem:**

\[
\min_x \left\{ c^T x : \begin{cases} \quad Ax - b \geq_K 0 \\ \quad Rx = r \end{cases} \right\}
\]

\[\iff\]

[passing to primal slack \( \xi = Ax - b \)]

\[
\min_\xi \left\{ e^T \xi : \xi \in [\mathcal{L} - b] \cap K \right\} \quad (P)
\]

\[
e : A^Te + R^Tf = c \text{ for some } f
\]

\[
\mathcal{L} = \{Au : Ru = 0\}
\]

**Dual problem:**

\[
\max_{y,z} \left\{ b^T y : A^Ty + R^Tz = c, \ y \geq_{K_*} 0 \right\}
\]

\[\iff\]

\[
\max_y \left\{ b^T y : y \in K_*, \exists z : A^Ty + R^Tz = c \right\}
\]

\[
\max_y \left\{ b^T y : y \in [\mathcal{L}^\perp + e] \cap K_* \right\} \quad (D)
\]

\[
[K_* : \text{cone dual to } K]
\]

Thus,

- the dual problem is conic along with primal
- the duality is completely symmetric

Note: Cones from Magic Families are self-dual, so that the dual of a Linear/Conic Quadratic/Semidefinite program is of exactly the same type.

5.13
Derivation of the Dual Problem

♣ Primal problem:
\[
\text{Opt}(P) = \min_x \left\{ c^T x : A_i x - b_i \in K^i, \; i \leq m \right\} \quad (P)
\]

♠ Goal: find a systematic way to bound \( \text{Opt}(P) \) from below.

♠ Simple observation: When \( y_i \in K^i_* \), the scalar inequality 
\[
y_i^T A_i x \geq y_i^T b_i
\]
is a consequence of the constraint \( A_i x - b_i \in K^i \). If \( z \) is a vector of the same dimension as \( r \), the scalar inequality 
\[
z^T R x \geq z^T r
\]
is a consequence of the constraint \( R x = r \).

⇒ Whenever \( y_i \in K^i_* \) for all \( i \) and \( z \) is a vector of the same dimension as \( r \), the scalar linear inequality 
\[
\left[ \sum_i A_i^T y_i + R^T z \right]^T x \geq \sum_i b_i^T y_i + r^T z
\]
is a consequence of the constraints in \( (P) \)

⇒ Whenever \( y_i \in K^i_* \) for all \( i \) and \( z \) is a vector of the same dimension as \( r \) such that 
\[
\sum_i A_i^T y_i + R^T z = c,
\]
the quantity \( \sum_i b_i^T y_i + r^T z \) is a lower bound on \( \text{Opt}(P) \).

• The Dual problem

\[
\text{Opt}(D) = \max_{y_i,z} \left\{ \sum_i b_i^T y_i + r^T z : y_i \in K^i_*, \; i \leq m \right\} \quad (D)
\]
is just the problem of maximizing this lower bound on \( \text{Opt}(P) \).
Definition: A conic problem

\[
\min_x \left\{ c^T x : \begin{array}{l}
A_i x - b_i \in K^i, \ i \leq m \\
Ax \leq b \\
Rx = r
\end{array} \right\} \tag{C'}
\]

is called strictly feasible, if there exists a feasible solution \( \bar{x} \) where all conic and \( \leq \) constraints are satisfied strictly:

\[
A_i \bar{x} - b_i \in \text{int} K^i \ \forall i \ \& \ A\bar{x} < b,
\]

and is called essentially strictly feasible, if there exists a feasible solution \( \bar{x} \) where all non-polyhedral constraints are satisfied strictly:

\[
A_i \bar{x} - b_i \in \text{int} K^i \ \forall i.
\]
Conic Programming Duality Theorem. Consider a conic problem

\[ \text{Opt}(P) = \min_x \left\{ c^T x : A_i x - b_i \in K^i, i \leq m \right\} \]  

along with its dual

\[ \text{Opt}(D) = \max_{y, z} \left\{ \sum_i b_i^T y_i + r^T z : \sum_i A_i^T y_i + R^T z = c \right\} \]

Then:

♠ [Symmetry] Duality is symmetric: the dual problem is conic, and its dual is (equivalent to) the primal problem;

♠ [Weak duality] One has \( \text{Opt}(D) \leq \text{Opt}(P) \);

♠ [Strong duality] Let one of the problems be essentially strictly feasible and bounded. Then the other problem is solvable, and

\[ \text{Opt}(D) = \text{Opt}(P) \]

In particular, if both problems are essentially strictly feasible, both are solvable with equal optimal values.
\[
\min_x \left\{ c^T x : \begin{array}{l}
A_i x - b_i \in K^i, i \leq m \\
R x = r
\end{array} \right\} \quad (P)
\]

\[
\max_{y_i, z} \left\{ \sum_i b_i^T y_i + r^T z : \begin{array}{l}
y_i \in K^i, i \leq m \\
\sum_i A_i^T y_i + R^T z = c
\end{array} \right\} \quad (D)
\]

**Conic Programming Optimality Conditions:**

Let both \((P)\) and \((D)\) be essentially strictly feasible. Then a pair \((x, \{\{y_i\}, z\})\) of primal and dual feasible solutions is comprised of optimal solutions to the respective problems if and only if

- **[Zero Duality Gap]**
  \[
  \text{DualityGap}(x, \{\{y_i\}, z\}) := c^T x - [\sum_i b_i^T y_i + r^T z] = 0
  \]
  Indeed,
  \[
  \text{DualityGap}(x, \{\{y_i\}, z\}) = \left[ c^T x - \text{Opt}(P) \right] + \left[ \text{Opt}(D) - [\sum_i b_i^T y_i + r^T z] \right] \\
  \geq 0 \quad \geq 0
  \]
  and if and only if

- **[Complementary Slackness]**
  \[
  [A_i x - b_i]^T y_i = 0, i \leq m
  \]
  Indeed,
  \[
  \sum_i [A_i x - b_i]^T y_i \\
  \geq 0
  \]
  \[
  = [\sum_i A_i^T y_i] x - \sum_i b_i^T y_i = [c - R^T z]^T x - \sum_i b_i^T y_i \\
  = c^T x - \sum_i b_i^T y_i + r^T z = \text{DualityGap}(x, \{\{y_i\}, z\})
  \]
Conic Duality, same as the LP one, is

- *fully algorithmic*: to write down the dual, given the primal, is a purely mechanical process
- *fully symmetric*: the dual problem “remembers” the primal one

*Cf. Lagrange Duality:*

\[
\begin{align*}
\min_x \{ f(x) : g_i(x) \leq 0, \ i = 1, \ldots, m \} \quad (P) \\
\downarrow \\
\max_{y \geq 0} L(y) \quad (D)
\end{align*}
\]

\[
L(y) = \min_x \left\{ f(x) + \sum_i y_i g_i(x) \right\}
\]

- Dual “exists in the nature”, but is given implicitly; its objective, typically, is not available in a closed form
- Duality is asymmetric: given \( L(\cdot) \), we, typically, cannot recover \( f \) and \( g_i \)...
Lemma: Symmetric block matrix $\begin{bmatrix} P & S^T \\ S & R \end{bmatrix}$ with $R \succ 0$ is positive semidefinite if and only if the matrix $P - S^T R^{-1} S$ is so.

Proof:

$\begin{bmatrix} P & S^T \\ S & R \end{bmatrix} \succeq 0$ iff

$$0 \leq \min_{u,v} [u^T Pu + 2u^T S^T v + v^T Rv]$$

$$= \min_u \left[ \min_v [u^T Pu + 2u^T S^T v + v^T Rv] \right]$$

achieved when $v = -R^{-1} Su$

$$= \min_u u^T \left[ P - S^T R^{-1} S \right] u.$$
An alternative to the minimax risks defined as the worst, over the signals of interest, performance of a statistical inference, is average performance, with the average taken over some prior probability distribution on the signals.

In the problem of \( \| \cdot \|_2 \)-recovering \( Bx \) via noisy observation

\[
\omega = Ax + \sigma \xi, \; \xi \sim P
\]

this alternative reads as follows:

(!) Given a probability distribution \( \Pi \) of signal \( x \in \mathbb{R}^n \), find an estimate \( \hat{x}(\cdot) \) which minimizes

\[
\text{Risk}^2(\hat{x}|\Pi) := \int \Pi \left\{ \int \mathbb{R}^m \| Bx - \hat{x}(Ax + \sigma \xi) \|_2^2 P(d\xi) \right\} \Pi(dx)
\]

– the average, over the distribution \( \Pi \) of signals \( x \), of expected \( \| \cdot \|_2^2 \) recovery error of \( Bx \) via observation \( Ax + \sigma \xi \).

Let \( Q \) be the joint distribution of \( (x, \omega = Ax + \sigma \xi) \) on \( \mathbb{R}^n_x \times \mathbb{R}^m_\omega \) induced by \( \Pi, \; P \) :

- the marginal distribution of \( x \) induced by \( Q \) is \( \Pi \);
- the conditional, given \( x \), distribution of \( \omega \) is the distribution of random vector \( Ax + \sigma \xi \) induced by the distribution \( P \) of \( \xi \).
Given a probability distribution $\Pi$ of signal $x \in \mathbb{R}^n$, find an estimate $\hat{x}(\cdot)$ which minimizes

$$\text{Risk}^2(\hat{x}|\Pi) := \int \left\{ \int_{\mathbb{R}^m} \| Bx - \hat{x}(Ax + \sigma\xi) \|^2_2 P(d\xi) \right\} \Pi(dx)$$

the average, over the distribution $\Pi$ of signals $x$, of expected $\| \cdot \|^2_2$ recovery error of $Bx$ via observation $Ax + \sigma\xi$.

Joint distribution of $(x, \omega = Ax + \sigma\xi)$ induced by distributions $\Pi$ and $p$ of independent r.v.’s $x$, $\xi$ gives rise to

- the conditional, $\omega$ given, distribution $R_\omega$ of $x$,
- marginal distribution $W$ of $\omega$.

We have

$$\text{Risk}^2(\hat{x}|\Pi) = \int_{\mathbb{R}^n \times \mathbb{R}^m} \| Bx - \hat{x}(\omega) \|^2_2 Q(dx, d\omega)$$

$$= \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} \| Bx - \hat{x}(\omega) \|^2_2 R_\omega(dx) \right\} W(d\omega)$$

Evident Fact: Assuming that a probability distribution $S$ on $\mathbb{R}^k$ possesses finite second moments, one has

$$\min_{c \in \mathbb{R}^k} \int_{\mathbb{R}^n} \| Bx - c \|^2_2 S(dx) = \int_{\mathbb{R}^n} \| Bx - c_* \|^2_2 S(dx),$$

$$c_* = \int_{\mathbb{R}^n} Bx S(dx).$$

An optimal, in terms of $\text{Risk}^2(\hat{x}|\Pi)$, estimate $\hat{x}(\cdot)$ of $Bx$ via $\omega = Ax + \sigma\xi$ is given by

$$\hat{x}_*(\omega) = \int_{\mathbb{R}^n} Bx R_\omega(dx).$$

5.21
**Corollary [Gauss-Markov Theorem]:** Let \( x \in \mathbb{R}^n \) and \( \xi \in \mathbb{R}^m \) be independent zero mean Gaussian random vectors. Assuming \( \sigma > 0 \) and the covariance matrix of \( \xi \) to be positive definite, an optimal solution \( \hat{x}(\cdot) \) to the risk minimization problem

\[
\min_{\hat{x}(\cdot)} \mathbb{E}_{x,\xi} \left\{ \| Bx - \hat{x}(Ax + \sigma \xi) \|_2^2 \right\}
\]

exists and can be selected as a linear function of \( \omega = Ax + \sigma \xi \).

Indeed, by the above, we can take

\[
\hat{x}_*(\omega) = \int_{\mathbb{R}^n} Bx R_\omega(dx),
\]

\( R_\omega \): the conditional, \( \omega \) given, distribution of \( x \) induced by the distribution of \( \zeta = (x, \omega = Ax + \sigma \xi) \) with independent zero mean Gaussian \( x \) and \( \xi \). Assuming w.l.o.g. that the covariance matrix of \( x \) is positive definite, random variable \( \zeta = (x, \omega) \) clearly is zero mean Gaussian with positive definite covariance matrix

\[
\Rightarrow \text{the conditional, } \omega \text{ given, density of } x \text{ is }
\]

\[
r_\omega(x) = c \exp\left\{ \omega^T S x - \frac{1}{2} x^T D x \right\} \quad [D > 0]
\]

\[
\Rightarrow \hat{x}_*(\omega) = c \int_{\mathbb{R}^n} Bx \exp\left\{ \omega^T S x - \frac{1}{2} x^T D x \right\} dx \text{ is a linear function of } \omega.
\]
Under the premise of Gauss-Markov theorem, with $x \sim \mathcal{N}(0, Q)$ and $\xi \sim \mathcal{N}(0, I_m)$, direct computation shows that

$$\hat{x}_*(\omega) = \left[\sigma^2 I_m + AQAT\right]^{-1} AQBT \omega$$

$$\text{Risk}^2(\hat{x}_*|\mathcal{N}(0, Q)) = \text{Tr}(BQBT - BQAT\left[\sigma^2 I_m + AQAT\right]^{-1} AQBT)$$
Optimizing Linear Estimates

♣ Situation: “In the nature” there exists a signal $x$ known to belong to a given convex compact set $\mathcal{X} \subset \mathbb{R}^n$. We observe corrupted by noise affine image of the signal:

$$\omega = Ax + \sigma \xi \in \Omega = \mathbb{R}^m$$

- $A$: given $m \times n$ sensing matrix
- $\xi$: random noise

♠ Goal: To recover the image $Bx$ of $x$
- $B$: given $\nu \times n$ matrix.

♠ Risk of a candidate estimate $\hat{x}(\cdot) : \Omega \to \mathbb{R}^\nu$ is

$$\text{Risk}[\hat{x}|\mathcal{X}] = \sup_{x \in \mathcal{X}} \sqrt{\mathbb{E}_\xi \left\{ \|Bx - \hat{x}(Ax + \sigma \xi)\|_2^2 \right\}}$$

♣ Assumption on noise: $\xi$ is zero mean with unit covariance matrix.
⇒ The risk of a linear estimate $\hat{x}_H(\omega) = H^T \omega$ is given by

$$\text{Risk}^2[\hat{x}_H|\mathcal{X}] = \max_{x \in \mathcal{X}} \mathbb{E}_\xi \left\{ \|[B - H^T A]x - \sigma H^T \xi\|_2^2 \right\}$$

$$= \max_{x \in \mathcal{X}} \left\{ \|[B - H^T A]x\|_2^2 + \sigma^2 \mathbb{E}_\xi \{\text{Tr}(H^T \xi \xi^T H)\} \right\}$$

$$= \sigma^2 \text{Tr}(H^T H) + \max_{x \in \mathcal{X}} \text{Tr}([B - H^T A]xx^T [B^T - A^T H]) \underbrace{\Psi(H)}_{\text{Ψ}(H)}$$

♥ Note: $\Psi$ is convex ⇒ building the minimum risk linear estimate reduces to solving convex minimization problem

$$\text{Opt} = \min_H \left[ \Psi(H) + \sigma^2 \text{Tr}(H^T H) \right]. \quad (\star)$$

But: Convex function $\Psi$ is given implicitly and can be difficult to compute, making $(\star)$ difficult as well.
\[
\text{Opt} = \min_H \left[ \sigma^2 \text{Tr}(H^T H) + \psi(H) \right] \\
\psi(H) = \max_{x \in \mathcal{X}} \text{Tr}([-B - H^T A] x x^T [B^T - A^T H]) 
\]

\(\heartsuit\textbf{Fact:}\) Basically, the only cases when (*) is known to be easy are those when

- \(\mathcal{X}\) is given as a convex hull of finite set of moderate cardinality
- \(\mathcal{X}\) is an ellipsoid.

\(\mathcal{X}\) is a box \(\Rightarrow\) computing \(\psi\) is NP-hard...

\(\spadesuit\textbf{When }\psi\textbf{ is difficult to compute, we can to replace }\psi\textbf{ in the design problem (\ast) with an efficiently computable convex upper bound }\psi^+(H)\).

We are about to consider a family of sets \(\mathcal{X} - \text{ellitopes} -\) for which reasonably tight bounds \(\psi^+\) of desired type are available.
A basic ellitope is a set $\mathcal{Y} \subset \mathbb{R}^N$ given as

$$\mathcal{Y} = \{ y \in \mathbb{R}^N : \exists t \in \mathcal{T} : y^T S_k y \leq t_k, \; k \leq K \}$$

where

- $S_k \succeq 0$ are positive semidefinite matrices with $\sum_k S_k \succ 0$
- $\mathcal{T}$ is a convex compact subset of $K$-dimensional nonnegative orthant $\mathbb{R}_+^K$ such that
  - $\mathcal{T}$ contains some positive vectors
  - $\mathcal{T}$ is monotone: if $0 \leq t' \leq t$ and $t \in \mathcal{T}$, then $t' \in \mathcal{T}$ as well.

An ellitope $\mathcal{X}$ is linear image of a basic ellitope:

$$\mathcal{X} = \{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^N, t \in \mathcal{T} : x = Fy, \; y^T S_k y \leq t_k, \; k \leq K \}$$

- $F$ is a given $n \times N$ matrix,

Note: Every ellitope is a symmetric w.r.t. the origin convex compact set.
**Examples of basic ellitopes:**

A. Ellipsoid centered at the origin
   \[(K = 1, \mathcal{T} = [0; 1])\]

B. (Bounded) intersection of \( K \) ellipsoids/elliptic cylinders centered at the origin
   \[\mathcal{T} = \{t \in \mathbb{R}^K : 0 \leq t_k \leq 1, k \leq K\}\]

C. Box \( \{x \in \mathbb{R}^n : -1 \leq x_i \leq 1\} \)
   \[\mathcal{T} = \{t \in \mathbb{R}^n : 0 \leq t_k \leq 1, k \leq K = n\}, x^T S_k x = x_k^2\]

D. \( \ell_p \)-ball \( \mathcal{X} = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\} \) with \( p \geq 2 \)
   \[\mathcal{T} = \{t \in \mathbb{R}^n_+ : \|t\|_{p/2} \leq 1\}, x^T S_k x = x_k^2, k \leq K = n\]

♠ **Ellitopes admit fully algorithmic calculus:** if \( \mathcal{X}_i, 1 \leq i \leq I \), are ellitopes, so are their
   - intersection \( \bigcap_i \mathcal{X}_i \)
   - direct product \( \mathcal{X}_1 \times \ldots \times \mathcal{X}_I \)
   - arithmetic sum \( \mathcal{X}_1 + \ldots + \mathcal{X}_I \)
   - linear images \( \{Ax : x \in \mathcal{X}_i\} \)
   - inverse linear images \( \{y : Ay \in \mathcal{X}_i\} \) under linear embedding \( A \)
Observation: Let
\[
\mathcal{X} = \{x : \exists (t \in \mathcal{T}, y) : x = F y, y^T S_k y \leq t_k, k \leq K\} \quad (\ast)
\]
be an ellitope. Given a quadratic form \( x^T W x \), \( W \in S^n \), we have
\[
\max_{x \in \mathcal{X}} x^T W x \leq \min_{\lambda} \left\{ \phi_{\mathcal{T}}(\lambda) : \lambda \geq 0, \sum_{k=1}^{K} \lambda_k S_k \succeq F^T W F \right\}
\]
\[
\phi_{\mathcal{T}}(\lambda) = \max_{t \in \mathcal{T}} t^T \lambda : \text{ support function of } \mathcal{T}
\]
Indeed, we have
\[
\lambda \geq 0, F^T W F \preceq \sum_k \lambda_k S_k, x \in \mathcal{X} \Rightarrow \\
\exists (t \in \mathcal{T}, y) : y^T S_k y \leq t_k \forall k \leq K, x = F y \Rightarrow \\
\exists (t \in \mathcal{T}, y) : x^T W x = y^T F^T W F y \leq \sum_k \lambda_k y^T S_k y \leq \sum_k \lambda_k t_k \leq \phi_{\mathcal{T}}(\lambda) \Rightarrow \\
x^T W x \leq \phi_{\mathcal{T}}(\lambda).
\]
\[ \mathcal{X} = \{ x : \exists (t \in \mathcal{T}, y) : x = Fy, y^T S_k y \leq t_k, k \leq K \} \quad (\ast) \]

\[ \blacklozenge \textbf{Corollary: Let } \mathcal{X} \text{ be the ellitope } (\ast). \text{ Then the function} \]
\[ \psi(H) = \max_{x \in \mathcal{X}} \text{Tr}((B - H^T A)xx^T(B^T - A^T H)) \]
\[ = \max_{x \in \mathcal{X}} x^T[(B^T - A^T H)(B - H^T A)]x \]

\[ \text{can be upper-bounded as} \]
\[ \psi(H) \leq \overline{\psi}(H) \]
\[ := \min_{\lambda} \left\{ \phi_T(\lambda) : \lambda \geq 0, F^T [B^T - A^T H][B - H^T A]F \leq \sum_k \lambda_k S_k \right\} \]
\[ \text{[Schur Complement Lemma]} \]
\[ = \min_{\lambda} \left\{ \phi_T(\lambda) : \lambda \geq 0, \left[ \frac{\sum_k \lambda_k S_k}{[B - H^T A]F} \right] F^T [B^T - A^T H] I_\nu \succeq 0 \right\} \]

\[ \text{The function } \overline{\psi}(H) \text{ is real-valued and convex, and is efficiently computable whenever } \phi_T \text{ is so, that is, whenever } \mathcal{T} \text{ is computationally tractable.} \]
**Bottom line:** Given matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{\nu \times n}$ and an ellitope

$$\mathcal{X} = \{ x : \exists (t \in \mathcal{T}, y) : x = Fy, y^T S_k y \leq t_k, k \leq K \} \quad (*)$$

contained in $\mathbb{R}^n$, consider the convex optimization problem

$$\text{Opt} = \min_{H, \lambda} \left\{ \phi_T(\lambda) + \sigma^2 \text{Tr}(H^T H) : \lambda \geq 0, \left[ \sum_k \lambda_k S_k \right] F^T [B^T - A^T H] \leq 0 \right\}$$

Assuming the noise $\xi$ in observation $\omega = Ax + \sigma \xi$ zero mean with unit covariance matrix, the risk of the linear estimator $\hat{x}_{H^*}(.).$ induced by the optimal solution $H^*$ to the problem (this solution clearly exists provided $\sigma > 0$) satisfies the risk bound

$$\text{Risk}[\hat{x}_{H^*}|\mathcal{X}] \leq \sqrt{\text{Opt}}.$$ 

**What is ahead:** We are about to prove that in the case of $\xi \sim \mathcal{N}(0, I_m)$, Opt is “nearly” the same as the ideal minimax risk

$$\text{Risk}_{\text{Opt}} = \inf_{\hat{x}(.)} \text{Risk}[\hat{x}|\mathcal{X}],$$

where $\inf$ is taken w.r.t. all, not necessarily linear, estimates $\hat{x}(\cdot)$.
How It Works: Inverse Heat Equation

♣ **Situation:** Square plate is heated at time 0 and is rest to cool; the temperature at the plate’s boundary is all the time is kept 0.
Given given noisy measurements, taken along \( m \) points, of plate’s temperature at time \( t_1 \), we want to recover distribution of temperature at time a given time \( t_0 \), \( 0 < t_0 < t_1 \).

♠ **The model:** The temperature field \( u(t;p,q) \) evolves according to *Heat Equation*

\[
\frac{\partial}{\partial t} u(t;p,q) = \left[ \frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2} \right] u(t;p,q), \quad t \geq 0, \quad (p,q) \in S
\]

- \( t \): time
- \( S = \{(p,q), -1 \leq p, q \leq 1\} \): the plate

with boundary conditions \( u(t;p,q) \big|_{(p,q) \in \partial S} \equiv 0 \).

♥ It is convenient to represent \( u(t;p,q) \) by its expansion

\[
\phi_k(s) = \begin{cases} 
\cos(\omega_{2i-1}s), & \omega_{2i-1} = (i - 1/2)\pi \\
\sin(\omega_{2i}s), & \omega_{2i} = i\pi
\end{cases} \\
\phi_k(s) = \begin{cases} 
\cos(\omega_{2i-1}s), & k = 2i - 1 \\
\sin(\omega_{2i}s), & k = 2i
\end{cases} \\
\]

\( \bigstar \) Note: \( \phi_k(s) \) are harmonic oscillations vanishing at \( s = \pm 1 \).
\[
\phi_k(s) = \begin{cases} 
\cos(\omega_{2i-1}s), & \omega_{2i-1} = (i-1/2)\pi \quad k = 2i - 1 \\
\sin(\omega_{2i}s), & \omega_{2i} = i\pi \quad k = 2i
\end{cases}
\]

Note:

- \{\phi_{k\ell}(p, q) = \phi_k(p)\phi_\ell(q)\}_{k, \ell} form an orthonormal basis in \(L_2(S)\)
- \(\phi_{k\ell}(\cdot)\) meet the boundary conditions \(\phi_{k\ell}(p, q) \big|_{(p, q) \in \partial S} = 0\)
- in terms of the coefficients \(x_{k\ell}(t)\), the Heat Equation becomes

\[
\frac{d}{dt}x_{k\ell}(t) = -[\omega_k^2 + \omega_\ell^2]x_{k\ell}(t) \Rightarrow \\
x_{k\ell}(t) = e^{-[\omega_k^2 + \omega_\ell^2]t}x_{k\ell}(0).
\]

We select integer discretization parameter \(N\) and

- restrict (*) to terms with \(1 \leq k, \ell \leq 2N - 1\)
- discretize the spatial variable \((p, q)\) to reside in the grid

\[
G_N = \{P_{ij} = (p_i, p_j) = \left(\frac{i}{N} - 1, \frac{j}{N} - 1\right), 1 \leq i, j \leq 2N - 1\}
\]

Note: Restricting functions \(\phi_{k\ell}(\cdot, \cdot)\), \(1 \leq k, \ell \leq 2N - 1\) on grid \(G_N\), we get orthogonal basis in \(\mathbb{R}^{(2N-1) \times (2N-1)}\).
We arrive at the model as follows:

- The signal $x$ underlying observation is

$$x = \{x_{k\ell} := x_{k\ell}(t_0), 1 \leq k, \ell \leq 2N - 1\} \in \mathbb{R}^{(2N-1) \times (2N-1)}$$

- The observation is

$$\omega = A(x) + \sigma \xi \in \mathbb{R}^m, \xi \sim \mathcal{N}(0, I_m)$$

$$[A(x)]_\nu = \sum_{k,\ell=1}^{2N-1} x_{k\ell} e^{-[\omega_k^2 + \omega_\ell^2](t_1 - t_0)} \phi_k(p_{i(\nu)}) \phi_\ell(p_{j(\nu)}) x_{k\ell}$$

- $(p_{i(\nu)}, p_{j(\nu)}) \in S, 1 \leq \nu \leq m$: measurement points

- We want to recover the restriction $B(x)$ of $u(t_0; p, q)$ to some grid, say, square grid

$$G_K = \{(r_i = \frac{i}{K} - 1, r_j = \frac{j}{K} - 1), 1 \leq i, j \leq 2K - 1\} \subset S,$$

resulting in

$$[B(x)]_{ij} = \sum_{k,\ell=1}^{2N-1} \phi_k(r_i) \phi_\ell(r_j) x_{k\ell}$$

- We assume that the initial distribution of temperatures $[u(0; p_i, p_j)]_{i,j=1}^{2N-1}$ satisfies $\|u\|_2 \leq R$, for some given $R$, implying that $x$ resides in the ellipsope, namely, the ellipsoid

$$\mathcal{X} = \left\{ \{x_{k\ell}\} \in \mathbb{R}^{(2N-1) \times (2N-1)} : \sum_{k,\ell} \left[ e^{[\omega_k^2 + \omega_\ell^2]t_0} x_{k\ell} \right]^2 \leq R^2 \right\}$$
\[ u(t; p_i, p_j) = \sum_{k, \ell} e^{-[\omega_k^2 + \omega_\ell^2][t - t_0]} \phi_k(p_i) \phi_\ell(p_j) x_{k\ell} \]

\[ [A(x)]_\nu = \sum_{k, \ell=1}^{2N-1} x_{k\ell} e^{-[\omega_k^2 + \omega_\ell^2][t_1 - t_0]} \phi_k(p_i(\nu)) \phi_\ell(p_j(\nu)) x_{k\ell} \]

\red{Bad news:} Contributions of high frequency (with large \( \omega_k^2 + \omega_\ell^2 \)) signal components \( x_{k\ell} \) to \( A(x) \) decrease exponentially fast with high decay rate as \( t_1 - t_0 \) grows

⇒ High frequency components \( x_{k\ell} \) are impossible to recover from observations at time \( t_1 \), unless \( t_1 \) is very small.

\[ x = \left\{ \{x_{k\ell}\} : \sum_{k, \ell} \left[ e^{[\omega_k^2 + \omega_\ell^2]t_0} x_{k\ell} \right]^2 \leq R^2 \right\} \]

\[ [B(x)]_{ij} = \sum_{k, \ell=1}^{2N-1} \phi_k(r_i) \phi_\ell(r_j) x_{k\ell} \]

\red{Good news:} High frequency components \( x_{k\ell} \) of \( x \in \mathcal{X} \) are very small, provided \( t_0 \) is not too small

⇒ There is no necessity to recover well high frequency components of signal from observations!
Numerical results $N = 32$, $m = 125$, $K = 6$, $t_0 = 0.01$, $t_1 = 0.02$, $\sigma = 0.001$, $R = 15$

Minimax risk of optimal linear estimate: 0.1707

63 x 63 grid $G_{63}$ and $m = 125$ measurement points

Sample results

- left: $b = B(x)$
- center: sample optimal linear recovery $\hat{b} = H^T \omega$ of $b = B(x)$
- right: naive recovery $\tilde{b} = B(\tilde{x})$
  $\tilde{x}$: Least Squares solution to $A(x) = \omega$
Near-Optimality of Linear Estimates

\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^N, t \in \mathcal{T} : x = Fy, y^T S_k y \leq t_k \ \forall k \leq K \} \]

\[ \blacklozenge \textbf{Simple observation:} \text{ When recovering } Bx, \ x \in \mathcal{X}, \text{ from observation } \omega = Ax + \sigma \xi, \text{ we lose nothing when assuming that the signal is} \]

\[ y \in \mathcal{Y} = \{ y : \exists t \in \mathcal{T} : y^T S_k y \leq t_k \ \forall k \leq K \} \]

the observation is \( \omega = [AF]y + \sigma \xi \), and the entity to be recovered is \([BF]y\). With this transformation, families of all estimates, all linear estimates and their risks remain intact

\[ \Rightarrow \text{We lose nothing when assuming that } F \text{ is the identity:} \]

\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k \ \forall k \leq K \} \]
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in T : x^T S_k x \leq t_k \ \forall k \leq K \} \]

⇒ We can build linear estimate satisfying
\[ \text{Risk}^2[\hat{x}|\mathcal{X}] \leq \text{Opt} = \min_{H, \lambda} \left\{ \phi_T(\lambda) + \sigma^2 \text{Tr}(H^T H) : \right. \]
\[ \left. \lambda \geq 0, \left[ \begin{array}{c} \sum_k \lambda_k S_k \\ B - H^T A \\ I_\nu \end{array} \right] \preceq 0 \right\} \]

♠ Our course of actions is as follows.
A. Imagine for a moment that we can specify Gaussian prior \( \mathcal{N}(0, Q) \) for the signal \( x \) to be supported on \( \mathcal{X} \). Then by Gauss-Markov Theorem the quantity
\[ \psi(Q) = \text{Tr}(BQB^T - BQA^T [\sigma^2 I_m + AQ A^T]^{-1} AQB^T) \]
would be a lower bound on \( \text{Risk}^2_{\text{Opt}} \).

B. Of course, for a nonzero \( Q \), \( \mathcal{N}(0, Q) \) cannot be supported on bounded set \( \mathcal{X} \). We, however, can select \( Q \) to enforce the “overwhelming part” of the probability mass of \( \mathcal{N}(0, Q) \) to sit in \( \mathcal{X} \), thus making “slightly reduced” \( \psi(Q) \) a lower bound on \( \text{Risk}^2_{\text{Opt}} \).

C. We have \( \mathbb{E}_{\eta \sim \mathcal{N}(0, Q)} \{ \eta^T S \eta \} = \text{Tr}(SQ) \)
⇒ Selecting \( Q \succeq 0 \) according to
\[ \exists t \in T : \text{Tr}(QS_k) \leq t_k, k \leq K \]

we ensure that \( \eta \sim \mathcal{N}(0, Q) \) “at average” sits in \( \mathcal{X} \)
⇒ We may hope that imposing on \( Q \succeq 0 \) restriction
\[ \exists t \in T : \text{Tr}(QS_k) \leq \rho t_k, k \leq K, \quad [\rho > 0] \]
we enforce \( \eta \sim \mathcal{N}(0, Q) \) to take values in \( \mathcal{X} \) with probability controlled by \( \rho \) and approaching 1 as \( \rho \to +0 \).
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k \ \forall k \leq K \} \]

⇒ We can build linear estimate satisfying

\[
\text{Risk}^2[\hat{x} | \mathcal{X}] \leq \text{Opt} = \min_{H, \lambda} \left\{ \phi_T(\lambda) + \sigma^2 \text{Tr}(H^T H) : \lambda \geq 0, \left[ \begin{array}{c} \frac{\sum_k \lambda_k S_k}{B - H^T A} \\ \frac{B^T - A^T H}{I^\nu} \end{array} \right] \succeq 0 \right\}
\]

D. The above considerations give rise to parametric optimization problem

\[ \text{Opt}_\ast(\rho) = \max_{Q \succeq 0} \left\{ \psi(Q) : \exists t \in \mathcal{T} : \text{Tr}(Q S_k) \leq \rho t_k \ \forall k \leq K \right\} \]

We may hope that for small \( \rho \) “slightly corrected” \( \text{Opt}_\ast(\rho) \) is a lower bound on \( \text{Risk}^2_{\text{Opt}} \).

♠ Miracle: It turns out by Conic Duality that \( \text{Opt}_\ast(\rho) \geq \rho \text{Opt} \)

⇒ The outlined lower bounds on \( \text{Risk}^2_{\text{Opt}} \) can be expressed via \( \text{Opt} \), which ultimately results in

\[ \sqrt{\text{Opt}} \leq O \left( \sqrt{\ln(1/\text{Risk}_{\text{Opt}})} \text{Risk}_{\text{Opt}} \right) \]

provided \( \text{Risk}_{\text{Opt}} \) is small

⇒ Risk of efficiently computable linear estimate is just by logarithmic factor worse that the “ideal” risk \( \text{Risk}_{\text{Opt}} \).

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 [~, Justifying Miracle:

\[
\text{Opt}_\ast(1) = \max_{Q,t} \left\{ \text{Tr}(BQB^T - BQA^T[\sigma^2 I_m + AQA^T]^{-1} AQB^T) : t \in T, Q \succeq 0, \text{Tr}(QS_k) \leq t_k \forall k \right\}
\]

♥ Step 1: passing to linear objective

\[
\text{Opt}_\ast(1) = \max_{Q,t} \left\{ \text{Tr}(BQB^T - BQA^T[\sigma^2 I_m + AQA^T]^{-1} AQB^T) : t \in T, Q \succeq 0, \text{Tr}(QS_k) \leq t_k \forall k \right\}
\]

\[
= \max_{Q,t,G} \left\{ \text{Tr}(BQB^T) - \text{Tr}(G) : \begin{array}{c}
G - BQA^T[\sigma^2 I_m + AQA^T]^{-1} AQB^T \succeq 0 \\
t \in T, Q \succeq 0, \text{Tr}(QS_k) \leq t_k \forall k
\end{array} \right\}
\]

[Schur Complement Lemma]

\[
= \max_{Q,t,G} \left\{ \text{Tr}(BQB^T) - \text{Tr}(G) : \begin{bmatrix}
G & BQA^T \\
AQB^T & \sigma^2 I_m + AQA^T
\end{bmatrix} \succeq 0 \\
t \in T, Q \succeq 0, \text{Tr}(QS_k) \leq t_k \forall k
\right\}
\]
$$\text{Opt}_*(1) = \max_{Q,t,G} \left\{ \text{Tr}(BQB^T) - \text{Tr}(G) : \begin{bmatrix} G & BQA^T \\ AQB^T & \sigma^2 I_m + AQA^T \end{bmatrix} \succeq 0 \right\}$$

$$t \in \mathcal{T}, Q \succeq 0, \text{Tr}(QS_k) \leq t_k \forall k \leq K$$

♥ **Step 2:** Conic representation of constraint $t \in \mathcal{T}$.

- Let $\mathcal{T}^+ = \{[t; 1] \in \mathbb{R}^{K+1} : t \in \mathcal{T}\}$, and let $\mathcal{T} \in \mathbb{R}^{K+1}$ be the set of nonnegative multiples of vectors from $\mathcal{T}^+$.

- $\mathcal{T}$ is a regular cone (since $\mathcal{T}$ is a convex compact set with a nonempty interior)

- $\mathcal{T} = \{t : [t; 1] \in \mathcal{T}\}$

- The cone $\mathcal{T}_*$ dual to $\mathcal{T}$ is $\{[y; s] \in \mathbb{R}^{k+1} : s \geq \phi_\mathcal{T}(-g)\}$

Indeed, $\{[y; s] \in \mathcal{T}_*\} \Leftrightarrow \{y^T t + s \tau \geq 0 \forall [t; \tau] \in \mathcal{T}\}$

$\Leftrightarrow \{y^T t + s \geq 0 \forall t : [t; 1] \in \mathcal{T}\} \Leftrightarrow \{s \geq -y^T t \forall t \in \mathcal{T}\}$

$\Leftrightarrow \{s \geq \max_{t \in \mathcal{T}} [-y]^T t\}$

$\Leftrightarrow \{s \geq \phi_\mathcal{T}(-y)\}$

$\Rightarrow$ We have

$$\text{Opt}_*(1) = \max_{Q,t,G} \left\{ \text{Tr}(BQB^T) - \text{Tr}(G) : \begin{bmatrix} G & BQA^T \\ AQB^T & \sigma^2 I_m + AQA^T \end{bmatrix} \succeq 0 \right\}$$

($\ast$)

**Note:** Problem ($\ast$) clearly is strictly feasible and solvable.

5.40
\[ \text{Opt}_*(1) = \max_{Q,t,G} \left\{ \text{Tr}(BQB^T) - \text{Tr}(G) : \begin{bmatrix} G & BQA^T \\ AQB^T & \sigma^2 I_m + AQA^T \end{bmatrix} \succeq 0, \begin{bmatrix} U \\ V \\ W \end{bmatrix} \succeq 0, [t; 1] \in T, Q \succeq 0, t_k - \text{Tr}(QS_k) \geq 0 \forall k \leq K, [g; s] \in T_*, P \succeq 0, \lambda_k \geq 0 \right\} \]

red: Lagrange multipliers

\[ \text{Step 3: passing from } (*) \text{ to the dual problem.} \]

- Taking inner products of the constraints of \((*)\) with Lagrange multipliers and summing up the results, we get the consequence of the constraints of \((*)\):

\[
\begin{align*}
\text{Tr}(UG) + 2\text{Tr}(V^TAQB^T) + \text{Tr}(W[\sigma^2 I + AQA^T]) \\
g^T t + s + \text{Tr}(PQ) + \sum_k \lambda_k [t_k - \text{Tr}(QS_k)] \geq 0
\end{align*}
\]

\[\downarrow\]

\[
\begin{align*}
-\text{Tr}(UG) - \text{Tr} \left( \left[ \sum_k \lambda_k S_k - P - A^TWA - B^TV^TA - A^TVB \right] Q \right) \\
- [g + \lambda]^T t \leq s + \sigma^2 \text{Tr}(W)
\end{align*}
\]

\[
\begin{bmatrix} U \\ V \\ W \end{bmatrix} \succeq 0, s \geq \phi_T(-g), P \succeq 0, \lambda \geq 0
\]
\[
-\text{Tr}(UG) - \text{Tr} \left( \left[ \sum_k \lambda_k S_k - P - A^T W A - B^T V^T A - A^T V B \right] Q \right) \\
- \left[ g + \lambda \right]^T t \leq s + \sigma^2 \text{Tr}(W)
\]

\[
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix} \succeq 0, s \geq \phi_T(-g), P \succeq 0, \lambda \geq 0
\]

To get the dual problem, we impose on Lagrange multiplies the requirement that the body of the aggregated – the red – constraint, as a function of \(G, Q, t\), is identically equal to the objective \(\text{Tr}(BQ B^T) - \text{Tr}(G)\) of (\(*\)), and minimize under this restriction (and the red constraints on the multipliers) the right hand side of the aggregated constraint. By Conic Duality, we get

\[
\text{Opt}_*(1) = \min_{U,V,W,g,s,\lambda,P} \left\{ s + \sigma^2 \text{Tr}(W) : \\
U = I_\nu, g = -\lambda \\
\sum_k \lambda_k S_k - P - A^T W A - B^T V^T A - A^T V B = B^T B \\
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix} \succeq 0, s \geq \phi_T(-g), P \succeq 0, \lambda \geq 0
\right\}
\]

\[
= \min_{V,W,\lambda} \left\{ \phi_T(\lambda) + \sigma^2 \text{Tr}(W) : \\
B^T B + A^T W A + B^T V^T A + A^T V B \leq \sum_k \lambda_k S_k \\
\begin{bmatrix}
I_\nu \\
V \\
W
\end{bmatrix} \succeq 0, \lambda \geq 0
\right\}
\]

\[
\leftarrow W \succeq V V^T
\]

\[
= \min_{V,\lambda} \left\{ \phi_T(\lambda) + \sigma^2 \text{Tr}(V V^T) : \lambda \geq 0, \\
B^T B + A^T V V^T A + B^T V^T A + A^T V B \leq \sum_k \lambda_k S_k
\right\}
\]

\[
= \min_{V,\lambda} \left\{ \phi_T(\lambda) + \sigma^2 \text{Tr}(V V^T) : \lambda \geq 0, \\
\frac{\sum_k \lambda_k S_k}{B + V^T A} B^T + A^T V \\
\frac{I_\nu}{I_\nu}
\right\} \succeq 0
\]

\[
= \min_{H[:=-V],\lambda} \left\{ \phi_T(\lambda) + \sigma^2 \text{Tr}(H^T H) : \lambda \geq 0, \\
\frac{\sum_k \lambda_k S_k}{B - H^T A} B^T + A^T H \\
\frac{I_\nu}{I_\nu}
\right\} \succeq 0
\]

\[
= \text{Opt}
\]
$\mathcal{X} = \{x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k \ \forall k \leq K\}$

⇒ **We can build linear estimate satisfying**

\[
\text{Risk}^2[\hat{x}|\mathcal{X}] \leq \text{Opt} = \min_{H,\lambda} \left\{ \phi_T(\lambda) + \sigma^2 \text{Tr}(H^T H) : \lambda \geq 0, \left[ \frac{1}{B - H^T A} \frac{B^T - A^T H}{I_\nu} \right] \succeq 0 \right\} = \text{Opt}_*(1)
\]

where

\[
\text{Opt}_*(\rho) = \max_{Q,t} \{ \psi(Q) : Q \succeq 0, t \in \mathcal{T}, \text{Tr}(QS_k) \leq \rho t_k \ \forall k \}\]

\[
\left[ \sqrt{\psi(Q)} : \text{optimal Bayesian risk of recovering } Bx \text{ when } x \sim \mathcal{N}(0, Q) \right] \tag{\ast_\rho}
\]

**♠ Easy fact:** When $0 < \rho \leq 1$, one has

\[
\text{Opt}_*(\rho) \geq \rho \text{Opt}_*(1) = \rho \text{Opt}.
\]

Indeed, from formula for $\psi(Q)$ it follows immediately that

$\psi(\rho Q) \geq \rho \psi(Q), \ 0 \leq \rho \leq 1$.

**♠ Fact:** If $Q_\rho$ stems from optimal solution to (\ast_\rho), then the probability for $\xi \sim \mathcal{N}(0, Q_\rho)$ to be outside of $\mathcal{X}$ approaches zero extremely fast as $\rho$ decreases:

\[
\pi(\rho) := \text{Prob}_{\xi \sim \mathcal{N}(0, Q_\rho)}\{\xi \not\in \mathcal{X}\} \leq K \exp\{-\frac{1-\rho+\rho \ln(\rho)}{2\rho}\}
\]

<table>
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<th>$\rho$</th>
<th>0.10</th>
<th>0.05</th>
<th>0.04</th>
<th>0.03</th>
<th>0.02</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
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<td>1.0e0</td>
<td>3.4e–2</td>
<td>3.1e–3</td>
<td>5.5e–5</td>
<td>1.6e–8</td>
<td>3.2e–18</td>
</tr>
</tbody>
</table>

\[ K = 100 \]

⇒ **For reasonably small $\rho$, everything is as if $\mathcal{N}(0, Q_\rho)$ were supported on $\mathcal{X}$, so that, say, $0.99 \text{Opt}_*(\rho) \geq 0.99 \rho \text{Opt}$ is a lower bound on $\text{Risk}^2_{\text{Opt}}$.**
With Miracle at our disposal, implementing the already outlined strategy for bounding $\text{Risk}_{\text{Opt}}$ from below, we arrive at

**Theorem.** Let us associate with ellitope

$$\mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k, \forall k \leq K \}$$

the convex compact set

$$\mathcal{Q} = \{ Q \in \mathbb{S}^n : Q \succeq 0, \exists t \in \mathcal{T} : \text{Tr}(Q S_k) \leq t_k, k \leq K \},$$

and the quantity

$$M^* = \max_{Q \in \mathcal{Q}} \sqrt{\text{Tr}(BJB^T)}.$$

Then the linear estimate $\hat{x}_{H^*}(\omega) = H^T \omega$ of $Bx$, $x \in \mathcal{X}$, via observation $\omega = Ax + \sigma \xi$, $\xi \sim \mathcal{N}(0, I_m)$, given by the optimal solution to the convex optimization problem

$$\text{Opt} = \min_{H, \lambda} \left\{ \phi_T(\lambda) + \sigma^2 \text{Tr}(H^T H) : \lambda \geq 0, \begin{bmatrix} \sum_k \lambda_k S_k & B^T - A^T H \\ B - H^T A & I \end{bmatrix} \succeq 0 \right\}$$

satisfies the risk bound

$$\text{Risk}[\hat{x}_{H^*} | \mathcal{X}] \leq \sqrt{\text{Opt}} \leq 4 \sqrt{\ln \left( \frac{6M^2 \sqrt{K}}{\text{Risk}_{\text{Opt}}^2[\mathcal{X}]} \right)} \text{Risk}_{\text{Opt}}[\mathcal{X}],$$

where

$$\text{Risk}_{\text{Opt}}[\mathcal{X}] = \inf_{\hat{x}(\cdot)} \sup_{x \in \mathcal{X}} \sqrt{\mathbb{E}_{\xi \sim \mathcal{N}(0, I_m)} \left\{ \| Bx - \hat{x}(Ax + \sigma \xi) \|^2 \right\}}$$

$\inf$ being taken with respect to all, linear and nonlinear alike, estimates $\hat{x}(\cdot)$, is the optimal minimax risk.
\( x \in \mathcal{X}, \omega = Ax + \sigma \xi \quad \bar{x}(\omega) \approx Bx \)

\( \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in T : x^T S_k x \leq t_k \ \forall k \leq K \} \)

♣ **Extension:** **Relative risk.** When “very large” signals are allowed, it might make sense to switch from the usual risk to its relative version—“\( S \)-risk” defined as follows:

\[
\text{Risk}_{S}[\bar{x}|\mathcal{X}] = \min \left\{ \sqrt{\tau} : \right.
\]

\[
E_\xi \left\{ \| Bx - \bar{x}(Ax + \sigma \xi) \|_2^2 \right\} \leq \tau [1 + x^T S x] \forall x \in \mathcal{X}
\]

\( S : \) fixed positive semidefinite “risk calibrating” matrix

**Note:** Setting \( S = 0 \) recovers the usual “plain” risk.

♣ **Our results on design of near-optimal, in terms of plain risk, linear estimates extend to the case of \( S \)-risk:**

**A. Design** of near optimal linear estimate \( \tilde{x}_{H^*}(\omega) = H^T \omega \) is given by an optimal solution \((H^*, \tau^*, \lambda^*)\) to the convex optimization problem

\[
\text{Opt} = \min_{H, \tau, \lambda} \left\{ \tau : \begin{bmatrix} \sum_k \lambda_k S_k + \tau S & B^T - A^T H \\ B - H^T A & I_\nu \end{bmatrix} \geq 0, \right. \\
\sigma^2 \text{Tr}(H^T H) + \phi_T(\lambda) \leq \tau, \ \lambda \geq 0 \}
\]

For the resulting estimate, it holds

\[
\text{Risk}_{S}[\tilde{x}_{H^*}|\mathcal{X}] \leq \sqrt{\text{Opt}},
\]

provided \( \xi \) is zero mean with unit covariance matrix.
B. Near-optimality properties of the estimate $\hat{x}_{H^*}$ remain the same as in the case of plain risk:

When $\xi \sim \mathcal{N}(0, I_m)$, one has

$$\text{RiskS}[\hat{x}_{H^*} | \mathcal{X}] \leq 4 \sqrt{\ln \left( \frac{6M^2_* \sqrt{K}}{\text{RiskS}_{\text{Opt}}^2[\mathcal{X}]} \right)} \text{RiskS}_{\text{Opt}}[\mathcal{X}],$$

where

$$M_* = \max_Q \left\{ \sqrt{\text{Tr}(BQB^T)} : Q \succeq 0, \exists t \in T : \text{Tr}(QS_k) \leq t_k \forall k \leq K \right\}$$

and

$$\text{RiskS}_{\text{Opt}}[\mathcal{X}] = \inf_{\hat{x}(\cdot)} \text{RiskS}[\hat{x} | \mathcal{X}].$$
\( x \in \mathcal{X}, \omega = Ax + \sigma \xi \Rightarrow \hat{x}(\omega) \approx Bx \)

\* In the case \( \mathcal{X} = \mathbb{R}^n \), the best linear estimate is yielded by optimal solution to the convex problem

\[
\text{Opt} = \min_{H, \tau} \left\{ \tau : \left[ \begin{array}{c} \tau S \\ B - HTA \\ \sigma^2 \text{Tr}(HTH) \end{array} \right] \geq 0, \right. \\
\left. \sigma^2 \text{Tr}(HTH) \leq \tau \right\} \quad (\ast)
\]

A feasible solution \( \tau, H \) to (\ast) gives rise to linear estimate \( \hat{x}_H(\omega) = HT\omega \) such that

\[
\text{RiskS}[\hat{x}_H|\mathbb{R}^n] \leq \sqrt{\tau},
\]

provided \( \xi \) is zero mean and with unit covariance matrix.

\* Proposition. Assume that \( B \neq 0 \) and (\ast) is feasible. Then the problem is solvable, and its optimal solution \( \text{Opt}, H_* \) gives rise to linear estimate \( \hat{x}_{H_*}(\omega) = H_*^T\omega \) with \( S \)-risk \( \sqrt{\text{Opt}} \).

When \( \xi \sim \mathcal{N}(0, I_m) \), this estimate is minimax optimal:

\[
\text{RiskS}[\hat{x}_{H_*}|\mathbb{R}^n] = \sqrt{\text{Opt}} = \text{RiskS}_{\text{Opt}}[\mathbb{R}^n].
\]
Byproduct on Semidefinite Relaxation

♠ Theorem Let $C$ be a symmetric $n \times n$ matrix and $\mathcal{X}$ be an ellitope:

$$
\mathcal{X} = \{ x \in \mathbb{R}^n : \exists (t \in T, y) : x = Fy, y^T S_k y \leq t_k \ \forall k \leq K \}.
$$

Then the efficiently computable quantity

$$
\text{Opt} = \min_{\lambda} \left\{ \phi_T(\lambda) : \lambda \geq 0, F^T C F \preceq \sum_k \lambda_k S_k \right\}
$$

$$
\phi_T(\lambda) = \max_{t \in T} \lambda^T t
$$

is a tight upper bound on

$$
\text{Opt}_* = \max_{x \in \mathcal{X}} x^T C x :
$$

namely,

$$
\text{Opt}_* \leq \text{Opt} \leq 4 \ln(5K) \text{Opt}_*.
$$

Note: $\text{Opt}_*$ is difficult to compute within 4% accuracy when $\mathcal{X}$ is as simple as the unit box in $\mathbb{R}^n$. 

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Let \( \mathcal{X} \) be given by quadratic inequalities:

\[
\mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in T : x^T S_k x \leq t_k, k \leq K \} \neq \emptyset
\]

[\( T \) : nonempty convex compact set]

We have

\[
\text{Opt}_* := \max_{x \in \mathcal{X}} x^T C x \\
\leq \text{Opt} := \min_{\lambda} \{ \phi_T(\lambda) : \lambda \geq 0, C \preceq \sum_k \lambda_k S_k \}
\leq \Theta \cdot \text{Opt}_*
\]

What can be said about tightness factor \( \Theta \)?

**Facts:**

**A.** Assuming \( K = 1 \) and Slater condition: \( \bar{x}^T S_1 \bar{x} < t \) for some \( \bar{x} \) and some \( t \in T \), one can set \( \Theta = 1 \).

[famous \( \mathcal{S} \)-Lemma]

**B.** Assuming that \( x^T S_k x = x_k^2, k \leq K = \dim x, T = [0; 1]^K \), and \( C \) is Laplacian of a graph (i.e., off-diagonal entries in \( C \) are nonpositive and all row sums are zero), one can set \( \Theta = 1.1382... \)

[MAXCUT Theorem of Goemans and Williamson, 1995]

**Note:** Laplacian of a graph always is \( \succeq 0 \)

**C.** Assuming that \( C \succeq 0 \) and all matrices \( S_k \) are diagonal, one can set \( \Theta = \frac{\pi}{2} = 1.5708... \)

[\( \frac{\pi}{2} \) Theorem, Nesterov, 1998]
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k, k \leq K \} \neq \emptyset \]

[\mathcal{T} : \text{nonempty convex compact set}]

\[ \Rightarrow \]

\[ \text{Opt}_* := \max_{x \in \mathcal{X}} x^T C x \leq \text{Opt} := \min_{\lambda} \{ \phi_{\mathcal{T}}(\lambda) : \lambda \geq 0, C \preceq \sum_k \lambda_k S_k \} \leq \Theta \cdot \text{Opt}_* \]

\textbf{D. Assuming } \mathcal{X} \text{ is an ellitope (i.e., } S_k \succeq 0, \sum_k S_k \succ 0 \text{ and } \mathcal{T} \text{ contains a positive vector), one can set } \Theta = 4 \ln(5K) \]

\textbf{Note:} In the case of D, \( \Theta \) indeed can be as large as \( O(\ln(K)) \)
A byproduct of Theorem is the following useful fact:

**Theorem [upper-bounding of operator norms]** Let \( \| \cdot \|_x \) be a norm on \( \mathbb{R}^N \) such that the unit ball \( X \) of the norm is an ellitope:

\[
X := \{ x : \| x \|_x \leq 1 \} = \{ x : \exists (t \in \mathcal{T}, y) : x = Py, y^T S_k y \leq t_k, k \leq K \}
\]

Let, further, \( \| \cdot \| \) be a norm on \( \mathbb{R}^M \) such that the unit ball \( B^* \) of the norm \( \| \cdot \|_\ast \) conjugate to \( \| \cdot \| \) is an ellitope:

\[
B^* := \{ u \in \mathbb{R}^m : u^T v \leq 1 \forall (v, \| v \| \leq 1) \} = \{ u : \exists (r \in \mathcal{R}, z) : u = Qz, z^T R_\ell z \leq r_\ell, \ell \leq L \}
\]

Then the efficiently computable quantity

\[
\text{Opt}(C) = \min_{\lambda, \mu} \left\{ \phi_\mathcal{T}(\lambda) + \phi_\mathcal{R}(\mu) : \begin{array}{c}
\lambda \geq 0, \mu \geq 0 \\
\left[ \begin{array}{c}
\sum_\ell \mu_\ell R_\ell \\
\frac{1}{2} P^T C^T Q
\end{array} \right] \begin{array}{c}
\sum_\ell \mu_\ell R_\ell \\
\frac{1}{2} P^T C^T Q
\end{array} = 0 \\
\sum_k \lambda_k S_k \end{array} \right. \\
\left. C \in \mathbb{R}^{M \times N} \right\}
\]

is a convex in \( C \) upper bound on the operator norm

\[
\| C \|_{\| \cdot \|_x \to \| \cdot \|} = \max_x \{ \| Cx \| : \| x \|_x \leq 1 \}
\]

of the mapping \( x \mapsto Cx \), and this bound is reasonably tight:

\[
\| C \|_{\| \cdot \|_x \to \| \cdot \|} \leq \text{Opt}(C) \leq 4 \ln(5(K + L)) \| C \|_{\| \cdot \|_x \to \| \cdot \|}.
\]
Indeed, the operator norm in question is the maximum of a quadratic form over an ellitope:

\[
\|C\|_{\|\cdot\| \rightarrow \|\cdot\|} = \max_{x \in X, u \in B_*} [x; u]^T \begin{bmatrix} \quad C \quad \\ C^T \end{bmatrix} [x; u] = \max_{y; z \in W} [y; z]^T \begin{bmatrix} \quad Q^T C P \quad \\ P^T C^T Q \end{bmatrix} [y; z]
\]

where \(W\) is the basic ellitope given by

\[
W = \left\{ [y; z] : \exists [t; r] \in \mathcal{T} \times \mathcal{R} : \begin{array}{l}
y^T S_k y \leq t_k, k \leq K \\
z^T R_\ell z \leq r_\ell, \ell \leq L
\end{array} \right\}.
\]
The above results on tightness of semidefinite relaxation, we speak about tightness of the Semidefinite Relaxation upper bound on the maximum of a quadratic form over an ellitope:

$$\text{Opt}_* = \max_{x,t} \left\{ x^T C x : x^T S_k x \leq t_k, k \leq K, t \in T \right\}$$  (∗)

(perhaps with additional restrictions on $C$ and the ellitope).

**Fact:** Semidefinite relaxation admits an alternative description as follows:

*Let us associate with (∗) another optimization problem where instead of deterministic candidate solutions $(x, t)$ we are looking for random solutions $(\xi, \tau)$ satisfying the constraints at average:*

$$\text{Opt}^+ = \max_{\xi,\tau} \left\{ \mathbb{E}\{\xi^T C \xi\} : \mathbb{E}\{\xi^T S_k \xi\} \leq \mathbb{E}\{\tau_k\} \right\}$$  (!)

**Immediate observation:** Property of a random solution $(\xi, \tau)$ to be feasible for (!) depends solely on the matrix $Q = \mathbb{E}\{\xi \xi^T\}$ and the vector $t = \mathbb{E}\{\tau\}$, so that

$$\text{Opt}^+ = \max_{Q,t} \left\{ \text{Tr}(CQ) : \begin{array}{l} \text{Tr}(S_k Q) \leq t_k \\ Q \succeq 0, t \in T \end{array} \right\}$$  (#)
\[ \text{Opt}_\star = \max_{x, t} \left\{ x^T C x : \exists (t \in T) : x^T S_k x \leq t_k \right\} \quad (\ast) \]
\[ \text{Opt}^+ = \max_{Q, t} \left\{ \text{Tr}(CQ) : \text{Tr}(S_k Q) \leq t_k, Q \succeq 0, [t; 1] \in T \right\} \quad (\#) \]

\[ \text{Note:} \; (\#) \text{ is strictly feasible and bounded, and the problem} \]
\[ \text{Opt} = \min_{\lambda} \left\{ \phi_T(\lambda) : \lambda \geq 0, C \preceq \sum_k \lambda_k S_k \right\} \]

\[ \text{specifying Semidefinite relaxation upper bound on Opt is is nothing but the conic dual to (\#)} \]
\[ \Rightarrow \text{Opt}^+ = \text{Opt}. \]

• (\#) suggests the following recipe for quantifying the conservaism of the upper bound Opt on Opt\_\star:
  — Find an optimal solution \( Q_\star, t_\star \) to (\#) and treat \( Q_\star \succeq 0 \) as the covariance matrix of random vector \( \xi \) (many options!)
  — Random solutions \( (\xi, t_\star) \) satisfy \( (\ast) \) “at average.” Try to “correct” them to get feasible solutions to \( (\ast) \) and look how “costly” this correction is in terms of the objective.

For example, in Goemans-Williamson MAXCUT and in Nesterov’s \( \pi/2 \) Theorems, where \( x^T C x \) is maximized over the unit box \( \{ \| x \|_\infty \leq 1 \} \), one selects \( \xi \sim N(0, Q_\star) \) and “corrects” \( \xi \) according to \( \xi \mapsto \text{sign}[\xi] \).
\[ \text{Opt}_* = \max_{x,t} \left\{ x^T C x : \exists (t \in \mathcal{T}) : x^T S_k x \leq t_k \right\} \quad (\ast) \]

\[ \text{Opt}^+ = \max_{Q,t} \left\{ \text{Tr}(CQ) : \text{Tr}(S_k Q) \leq t_k, Q \succeq 0, [t; 1] \in \mathcal{T} \right\} \quad (\#) \]

\[ \text{♠ This is how the above recipe works in the general ellitopic case:} \]

A. Let \((Q_*, t^*)\) be an optimal solution to \((\#)\). Set

\[ \overline{C} := Q_*^{1/2} C Q_*^{1/2} = U D U^T \]

\((U\) is orthogonal, \(D\) is diagonal). 

B. Let \(\xi = Q_*^{1/2} U \zeta\) with Rademacher random \(\zeta\) (\(\zeta_i\) take values \(\pm 1\) with probability \(1/2\) and are independent across \(i\)), so that \(E\{\xi \xi^T\} = Q_*\). Note that

\[ \xi^T C \xi = \zeta^T U^T [Q_*^{1/2} C Q_*^{1/2}] U \zeta = \zeta^T D \zeta \]

\[ \equiv \text{Tr}(D) = \text{Tr}(Q_*^{1/2} C Q_*^{1/2}) \equiv \text{Tr}(C Q_*) \]

\[ = \text{Opt}, \]

\[ E\{\xi^T S_k \xi\} = E\{\zeta^T U^T Q_*^{1/2} S_k Q_*^{1/2} U \zeta\} \]

\[ = \text{Tr}(U^T Q_*^{1/2} S_k Q_*^{1/2} U) \]

\[ = \text{Tr}(Q_*^{1/2} S_k Q_*^{1/2}) = \text{Tr}(S_k Q_*) \]

\[ \leq t^*_k, k \leq K \]
\[ \xi^T C \xi \equiv \text{Opt} \quad (a) \]
\[ \mathbb{E}\{\xi^T S_k \xi\} \leq t^*_k, k \leq K \quad (b) \]

**C.** Since \( S_k \succeq 0 \) and \( \xi \) is "light-tail" (it comes from Rademacher random vector), simple bounds on probabilities of large deviations combine with (b) to imply that

\[ \forall (\gamma \geq 0, k \leq K) : \]
\[ \text{Prob}\{\xi : \xi^T S_k \xi > \gamma t^*_k\} \leq O(1) \exp\{-O(1)\gamma\} \]

\[ \Rightarrow \] with \( \gamma_* = O(1) \ln(K + 1) \), there exists a realization \( \hat{\xi} \) of \( \xi \) such that
\[ \hat{\xi}^T S_k \hat{\xi} \leq \gamma_* t^*_k, k \leq K \]
\[ \Rightarrow (\xi^* = \hat{\xi}/\sqrt{\gamma_*}, t^*) \) is feasible for

\[ \text{Opt}_{x,t} = \max_{x,t} \left\{ x^T C x : \exists (t \in T) : x^T S_k x \leq t_k \right\} \quad (*) \]

\[ \Rightarrow \text{Opt}_{x,t} \geq \hat{\xi}^T C \hat{\xi} / \gamma_* = \text{Opt} / \gamma_* \text{ (look at (a)!)} \]
“Simple bounds on probabilities of large deviations” stem from the following

**Mini-Lemma:** Let $P$ be positive semidefinite $N \times N$ matrix with trace $\leq 1$ and $\zeta$ be $N$-dimensional Rademacher random vector. Then

$$E\left\{ \exp \left\{ \frac{\zeta^T P \zeta}{3} \right\} \right\} \leq \sqrt{3}.$$  

**Mini-Lemma $\Rightarrow$ bounds:** We have

$$\xi^T S_k \xi = \zeta^T U^T Q_*^{1/2} S_k Q_*^{1/2} U \zeta$$

and $\text{Tr}(P_k) = \text{Tr}(Q_*^{1/2} S_k Q_*^{1/2})/t_*^k = \text{Tr}(S_k Q_*)/t_*^k \leq 1$

$\Rightarrow$ [Mini-Lemma] $E\left\{ \exp \left\{ \frac{\zeta^T P_k \zeta}{3} \right\} \right\} \leq \sqrt{3}$

$\Rightarrow$ Prob$\{\zeta^T P_k \zeta > 3 \rho\} \leq \sqrt{3}e^{-\rho}$

$\Rightarrow$ Prob$\{\xi^T S_k \xi > \gamma t_*^k\} = \text{Prob}\{\zeta^T P_k \zeta > \gamma\} \leq \sqrt{3}e^{-\gamma/3}$. 
**Proof of Mini-Lemma:** Let \( P = \sum_i \sigma_i f_i f_i^T \) be the eigenvalue decomposition of \( P \), so that \( f_i^T f_i = 1 \), \( \sigma_i \geq 0 \), and \( \sum_i \sigma_i \leq 1 \). The function

\[
f(\sigma_1, \ldots, \sigma_N) = \mathbb{E} \left\{ e^{\frac{1}{3} \sum_i \sigma_i \zeta^T f_i f_i^T \zeta} \right\}
\]

is convex on the simplex \( \{ \sigma \geq 0, \sum_i \sigma_i \leq 1 \} \) and thus attains its maximum over the simplex at a vertex, implying that for some \( h = f_i, h^T h = 1 \), it holds

\[
\mathbb{E} \{ e^{\frac{1}{3} \zeta^T P \zeta} \} \leq \mathbb{E} \{ e^{\frac{1}{3} (h^T \zeta)^2} \}.
\]

Let \( \xi \sim \mathcal{N}(0, 1) \) be independent of \( \zeta \). We have

\[
\mathbb{E}_\xi \left\{ \exp \left\{ \frac{1}{3} (h^T \zeta)^2 \right\} \right\} = \mathbb{E}_\zeta \left\{ \mathbb{E}_\xi \left\{ \exp \left\{ \sqrt{2/3} h^T \zeta \right\} \right\} \right\}
\]

\[
= \mathbb{E}_\zeta \left\{ \prod_{s=1}^N \mathbb{E}_\zeta \left\{ \exp \left\{ \sqrt{2/3} \xi h_s \zeta_s \right\} \right\} \right\}
\]

\[
= \mathbb{E}_\zeta \left\{ \exp \left\{ \sqrt{2/3} \xi h_s \zeta_s \right\} \right\} \leq \mathbb{E}_\zeta \left\{ \prod_{s=1}^N \exp \left\{ \xi^2 h_s^2 / 3 \right\} \right\}
\]

\[
= \mathbb{E}_\zeta \left\{ \exp \left\{ \xi^2 / 3 \right\} \right\} = \sqrt{3}
\]

\[ \square \]
So far, we have considered a problem of recovering $Bx$ from observation

$$\omega = Ax + \eta$$

where

- $x$ is unknown signal known to belong to a given basic ellitope

$$\mathcal{X} = \{x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k, k \leq K\}$$

- $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{\nu \times n}$ are given matrices
- $\eta \sim \mathcal{N}(0, \sigma^2 I_m)$ is observation noise
- (squared) risk of a candidate estimate is the worst-case, over $x \in \mathcal{X}$, expected squared $\| \cdot \|_2$-norm of recovery error:

$$\text{Risk}^2[\hat{x} | \mathcal{X}] = \sup_{x \in \mathcal{X}} \mathbb{E} \left\{ \| Bx - \hat{x}(Ax + \eta) \|_2^2 \right\}.$$
We are about to extend our results to the situation where
• Noise $\eta$ not necessary is zero mean Guassian; we allow
the distribution $P$ of noise to be unknown in advance and to
depend on signal $x$.

Assumption: The covariance matrix of $P$ admits a known upper bound:

$$\text{Cov}[P] := \mathbb{E}_{\eta \sim P} \{\eta \eta^T\} \preceq Q$$

for a given $Q > 0$

• We measure recovering error in a given norm $\|\cdot\|$, not
necessarily the Euclidean one, and define risk of a candidate
estimate $\hat{x}(\cdot)$ as

$$\text{Risk}_{Q,\|\cdot\|}[\hat{x}|X] = \sup_{x \in X} \sup_{P:\text{Cov}[P] \preceq Q} \mathbb{E}_{\eta \sim P} \{\|Bx - \hat{x}(Ax + \eta)\|\}$$

Assumption: The unit ball $B_*$ of the norm conjugate to $\|\cdot\|$ is an ellitope:

$$\|u\| = \max_{h \in B_*} h^T u,$$

$$B_* = \{h : \exists (y \in \mathbb{R}^M, r \in \mathcal{R}) : h = Fy, \ y^T R_\ell y \leq r_\ell \ \forall \ell \leq L\}$$
Building Linear Estimate

\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k \ \forall k \leq K \} \]

\[ \omega = Ax + \eta \implies \hat{x}_H(\omega) = H^T \omega \approx Bx \]

\[ \text{We have} \]

\[
\text{Risk}_{Q, \| \cdot \|}[\hat{x}_H | \mathcal{X}]
= \sup_{x \in \mathcal{X}} \sup_{P: \text{Cov}[P] \preceq Q} \mathbb{E}_{\eta \sim Q} \left\{ \| Bx - H^T [Ax + \eta] \| \right\}
\leq \sup_{x \in \mathcal{X}} \sup_{P: \text{Cov}[P] \preceq Q} \mathbb{E}_{\eta \sim Q} \left\{ \| [B - H^T Ax] \| + \| H^T \eta \| \right\}
\leq \Phi(H) + \Psi_Q[H],
\]

\[
\Phi(H) = \max_{x \in \mathcal{X}} \| [B - H^T A]x \|
\]

\[
\Psi_Q(H) = \sup_{P: \text{Cov}[P] \preceq Q} \mathbb{E}_{\eta \sim P} \left\{ \| H^T \eta \| \right\}
\]
Next,

\[ B_\ast = \{ u = My : y \in \mathcal{Y} \}, \]
\[ \mathcal{Y} = \{ y : \exists r \in \mathcal{R} : y^T R_\ell y \leq r_\ell \ \forall \ell \leq L \} \]

whence

\[ \Phi(H) := \max_{x \in \mathcal{X}} \|[B - H^T A]x\| \]
\[ = \max_{[u; x] \in B_\ast \times \mathcal{X}} [u; x]^T \left[ \frac{1}{2}[B^T - A^T H] \left[ \frac{1}{2}[B - H^T A] \right] \right] [u; x] \]
\[ = \max_{[y; x] \in \mathcal{Y} \times \mathcal{X}} [y; x]^T \left[ \frac{1}{2}[B^T - A^T H] F y \left[ \frac{1}{2}F^T [B - H^T A] \right] \right] [y; x] \]

[semidefinite relaxation; note that \( \mathcal{Y} \times \mathcal{X} \) is an ellitope]

\[ \leq \overline{\Phi}(H) := \min_{\lambda, \mu} \left\{ \phi_T(\lambda) + \phi_R(\mu) : \lambda \geq 0, \mu \geq 0, \right. \]
\[ \left. \left[ \frac{\sum_\ell \mu_\ell R_\ell}{\frac{1}{2}[A^T H - B^T] F} \frac{1}{2}F^T [H^T A - B] \right] \succeq 0 \right\} \]
\[ \phi_T(\lambda) = \max_{t \in T} \lambda^T t, \phi_R(\mu) = \max_{r \in \mathcal{R}} \mu^T r \]
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T S_k x \leq t_k, k \leq K \} \]
\[ B_* = \{ u = F y : y \in \mathcal{Y} \}, \]
\[ \mathcal{Y} = \{ y : \exists r \in \mathcal{R} : y^T R \ell y \leq r_\ell \ \forall \ell \leq L \} \]
\[ \omega = Ax + \eta \Rightarrow \hat{x}_H(\omega) = H^T \omega \approx Bx \]
\[ \text{Risk}_{Q,\|\cdot\|}[\hat{x}_H|\mathcal{X}] \leq \overline{\Phi}(H) + \psi_Q(H), \]
\[ \overline{\Phi}(H) = \min_{\lambda, \mu} \left\{ \phi_T(\lambda) + \phi_R(\mu) : \lambda \geq 0, \mu \geq 0, \right\} \]
\[ \psi_Q(H) = \sup_{P : \text{Cov}[P] \preceq Q} \mathbb{E}_{\eta \sim P} \left\{ \| H^T \eta \| \right\} \]

\[ \textbf{Lemma: One has} \]
\[ \psi_Q(H) \leq \overline{\psi}_Q(H) : = \min_{\Theta, \kappa} \left\{ \text{Tr}(Q \Theta) + \phi_R(\kappa) : \kappa \geq 0, \right\} \]
\[ \left[ \frac{\sum \kappa_\ell R_\ell}{\frac{1}{2} H F} \left| \frac{1}{2} F^T H^T \overline{\Theta} \right| \right] \succeq 0 \]
**Lemma:** One has

$$\psi_Q(H) \leq \overline{\psi}_Q(H) := \min_{\Theta, \kappa} \left\{ \operatorname{Tr}(Q \Theta) + \phi_R(\kappa) : \kappa \geq 0, \right\}$$

Indeed, let $(\kappa, \Theta)$ be feasible for the right hand side problem, and let $\operatorname{Cov}[P] \preceq Q$

$$\|H^T \xi\| = \max_{u \in B_*} [-u^T H^T \xi] = \max_{y \in Y} [-y^T F^T H^T \xi]$$

$$\leq \max_{y \in Y} \left[ y^T [\sum_\ell \kappa_\ell R_\ell] y + \xi^T \Theta \xi \right] \quad \text{[by (*)]}$$

$$= \max_{r \in \mathcal{R}, y} \left\{ y^T [\sum_\ell \kappa_\ell R_\ell] y + \xi^T \Theta \xi : y^T R_\ell y \leq r_\ell, \ell \leq L \right\}$$

$$\leq \max_{r \in \mathcal{R}} \left\{ \sum_\ell \kappa_\ell r_\ell + \xi^T \Theta \xi \right\}$$

$$\leq \phi_R(\kappa) + \xi^T \Theta \xi = \phi_R(\kappa) + \operatorname{Tr}(\Theta[\xi \xi^T]).$$

Taking expectation in $\xi$, the conclusion of Lemma follows. \(\square\)

**Illustration:** When $\| \cdot \| = \| \cdot \|_p$, $p \in [1, 2]$, Lemma states that whenever $\operatorname{Cov}[P] \preceq Q$, one has

$$\mathbb{E}_{\eta \sim P} \left\{ \|H^T \eta\|_p \right\} \leq \left\| \left[ \|\text{Col}_1[Q^{1/2} H]\|_2 ; \ldots ; \|\text{Col}_\nu[Q^{1/2} H]\|_2 \right] \right\|_p$$

5.64
\textbf{Summary:} Consider convex optimization problem

\[ \text{Opt} = \min_{H, \lambda, \mu, \kappa, \Theta} \left\{ \phi_T(\lambda) + \phi_R(\mu) + \phi_R(\kappa) + \text{Tr}(Q\Theta) : \begin{array}{l}
\lambda \geq 0, \mu \geq 0, \kappa \geq 0, \\
\frac{1}{2}\left[ A^T H - B^T \right] F \left[ \frac{1}{2} F^T \left[ H^T A - B \right] \right] \geq 0, \\
\frac{1}{2} H F \left[ \frac{1}{2} F^T H^T \right] \geq 0 \end{array} \right\} \]

The problem is solvable, and the $H$-component $H_*$ of its optimal solution yields linear estimate

\[ \hat{x}_{H_*}(\omega) = H^T \omega \]

such that

\[ \text{Risk}_{Q, \| \cdot \|}[\hat{x}_{H_*} | \mathcal{X}] \leq \text{Opt}. \]
Fact: *In the case of zero mean Gaussian observation noise, the estimate $\hat{x}_{H_*}$ is near-optimal:*

♠ **Theorem:** We have

$$\text{Risk}_{Q,\|\cdot\|}[\hat{x}_{H_*}, \mathcal{X}] \leq \text{Opt} \leq C \sqrt{\ln(2L) \ln \left( \frac{2M_*^2 K}{\text{RiskOpt}_{Q,\|\cdot\|}} \right)} \text{RiskOpt}_{Q,\|\cdot\|},$$

where

- $C$ is a positive absolute constant,
- $\text{RiskOpt}_{Q,\|\cdot\|} = \inf_{\hat{x}()} \left[ \sup_{x \in \mathcal{X}} \mathbb{E}_{\eta \sim \mathcal{N}(0, Q)} \left\{ \| Bx - \hat{x}(Ax + \eta) \| \right\} \right]$ the infimum being taken over all estimates, is the minimax optimal $\| \cdot \|$-risk corresponding to $\mathcal{N}(0, Q)$ observation noise,

- $K, L$ are the “sizes” of ellitopes $\mathcal{X}$ and $\mathcal{B}_* := F\mathcal{Y}$:
  
  $\mathcal{X} = \{ x : \exists t \in T : x^T S_k x \leq t_k, k \leq K \},$
  
  $\mathcal{Y} = \{ y : \exists r \in \mathcal{R} : y^T R_\ell y \leq r_\ell, \ell \leq L \},$

- $M_*^2 = \max_W \left\{ \mathbb{E}_{\eta \sim \mathcal{N}(0, I_n)} \| BW^{1/2} \eta \|^2 : W \in Q \right\}$, where

$Q := \{ W : W \succeq 0, \exists t \in T : \text{Tr}(S_k W) \leq t_k, k \leq K \}$

5.66
One of the key facts underlying near-optimality is important by its own right:

**Fact:** When the unit ball $B_*$ of the norm $\| \cdot \|_*$ conjugate to $\| \cdot \|$ is an ellitope:

$$B_* = \{ u : \exists r \in \mathbb{R}, y : u = F y, y^T R_\ell y \leq r_\ell, \ell \leq L \}$$

and $\eta \sim \mathcal{N}(0, Q)$, the upper bound

$$\mathbb{E}_{\eta}\{\|H^T \eta\|\} \leq \Psi_Q(H) := \min_{\Theta, \kappa} \left\{ \begin{array}{c} \text{Tr}(Q \Theta) + \phi_R(\kappa) : \kappa \geq 0, \\
\frac{1}{2} R_{\ell} F^T H^T \Theta \end{array} \right\} \succeq 0$$

is tight:

$$\mathbb{E}_{\eta}\{\|H^T \eta\|\} \leq \Psi_Q(H) \leq O(1) \sqrt{\ln(2L)} \mathbb{E}_{\eta}\{\|H^T \eta\|\}. $$
Recovery under uncertain-but-bounded noise

♣ So far, we have considered recovering $Bx, x \in \mathcal{X}$, from observation

$$\omega = Ax + \eta$$

affected by random noise $\eta$. We are about to consider the case when $\eta$ is “uncertain-but-bounded:” all we know is that

$$\eta \in \mathcal{H}$$

with a given convex and compact set $\mathcal{H}$.

♠ In the case in question, natural definition of risk of a candidate estimate $\hat{x}(\cdot)$ is

$$\text{Risk}_{\mathcal{H}, \|\cdot\|}[\hat{x}(\cdot)|\mathcal{X}] = \sup_{x \in \mathcal{X}, \eta \in \mathcal{H}} \|Bx - \hat{x}(Ax + \eta)\|.$$  

♠ Observation: Signal recovery under uncertain-but-bounded noise reduces to the situation where there is no observation noise at all.

Indeed, let us treat as the signal underlying observation the pair $z = [x; \eta] \in Z := X \times \mathcal{H}$ and replace $A$ with $\bar{A} = [A, 0]$ and $B$ with $\bar{B} = [B, 0]$, so that

$$\omega = \bar{A}[x; \eta] \quad \& \quad Bx = \bar{B}[x; \eta],$$

thus reducing signal recovery to recovering $\bar{B}z, z \in Z$, from noiseless observation $\bar{A}z$.  

5.68
Let us focus on the problem of recovering the image \( Bx \in \mathbb{R}^\nu \) of unknown signal \( x \in \mathbb{R}^n \) known to belong to signal set \( \mathcal{X} \subset \mathbb{R}^n \) via observation

\[
\omega = Ax \in \mathbb{R}^m.
\]

Given norm \( \| \cdot \| \) on \( \mathbb{R}^\nu \), we quantify the performance of an estimate \( \hat{x}(\cdot) : \mathbb{R}^m \to \mathbb{R}^\nu \) by its \( \| \cdot \| \)-risk

\[
\text{Risk}_{\| \cdot \|}[\hat{x}|\mathcal{X}] = \sup_{x \in \mathcal{X}} \| Bx - \hat{x}(Ax) \|.
\]

**Observation:** Assuming that \( \mathcal{X} \) is computationally tractable convex compact set and \( \| \cdot \| \) is computationally tractable, it is easy to build an efficiently computable optimal within factor 2 nonlinear estimate:

Given \( \omega \), let us solve the convex feasibility problem

Find \( y \in \mathcal{Y}[\omega] := \{ y \in \mathcal{X} : Ay = \omega \} \).

and take, as \( \hat{x}(\omega) \), the vector \( By \), where \( y \) is (any) solution to the feasibility problem.

**Note:** When \( \omega \) stems from a signal \( x \in \mathcal{X} \), the set \( \mathcal{Y}[\omega] \) contains \( x \)

\[\Rightarrow \hat{x}(\cdot) \text{ is well defined}\]
\[ x \in \mathcal{X}, \omega = Ax \Rightarrow \hat{x}(\omega) = By \]
\[ [y \in \mathcal{Y}[\omega] = \{ y \in \mathcal{X} : Ay = \omega \}] \]

\begin{align*}
\blacklozenge \text{Performance analysis: Let} \\
\mathcal{R} &= \max_{y, z} \left\{ \frac{1}{2} \| B[y - z] \| : y, z \in \mathcal{X}, A[y - z] = 0 \right\} \\
&= \frac{1}{2} \| B[y_* - z_*] \| \quad [y_*, z_* \in \mathcal{X}, A[y_* - z_*] = 0] \\
\end{align*}

\textbf{Claim A: For every estimate } \tilde{x}(\cdot) \text{ it holds } \text{Risk}_{\| \cdot \| \| [\tilde{x} | \mathcal{X}] \geq \mathcal{R}.} \]

\text{Indeed, the observation } \omega = Ay_* = Az_* \text{ stems from both } y_* \text{ and } z_*, \text{ whence the } \| \cdot \| -\text{risk of every estimate is at least } \frac{1}{2} \| y_* - z_* \| = \mathcal{R}. \\

\textbf{Claim B: One has } \text{Risk}_{\| \cdot \| \| [\hat{x} | \mathcal{X}] \leq 2\mathcal{R}.} \]

\text{Indeed, let } \omega = Ax \text{ with } x \in \mathcal{X}, \text{ and let } \hat{x}(\omega) = B\hat{y} \text{ with } \hat{y} \in \mathcal{Y}[\omega]. \text{ Then both } x, \hat{y} \text{ belong to } \mathcal{Y}[\omega] \Rightarrow \frac{1}{2} \| B[x - \hat{y}] \| \leq \mathcal{R}. \]
We have built optimal, within factor 2, estimate. How to upper-bound its $\| \cdot \|$-risk?

**Observation:** Let $\mathcal{X}$ and the unit ball $\mathcal{B}_*$ of the norm $\| \cdot \|_*$ conjugate to $\| \cdot \|$ be ellitopes:

$$
\mathcal{X} = \{x = Py : y \in \mathcal{Y} := \{y : \exists t \in T : y^T S_k y \leq t_k, k \leq K\}\}
$$
$$
\mathcal{B}_* = \{u = Qv : v \in \mathcal{V} := \{v : \exists r \in \mathcal{R} : v^T R_\ell v \leq r_\ell, \ell \leq L\}\}
$$

Then the $\| \cdot \|$-risk of the optimal, within factor 2, efficiently computable nonlinear estimate $\hat{x}(\cdot)$ can be tightly lower- and upper-bounded as follows.

- Assuming $\text{Ker} A \cap \mathcal{X} = \{0\}$ (otherwise the risk is zero), the set $\mathcal{X}_A = \{x \in \mathcal{X} : Ax = 0\}$ is an ellitope:

$$
\mathcal{X}_A = \{x = Fw, w \in \mathcal{W} := \{w : \exists t \in T : w^T T_k w \leq t_k, k \leq K\}\}
$$

Indeed, setting $E = \{y : APy = 0\}$, the set

$$
\mathcal{Y}_A = \{y \in E : \exists t \in T : y^T S_k y \leq t_k, k \leq K\}
$$

is a basic ellitope in some $\mathbb{R}^{N'} \Rightarrow \mathcal{X}_A = \{Py : y \in \mathcal{Y}_A\}$ is an ellitope.

$$
\mathcal{R} := \max_{x',x'' \in \mathcal{X}} \{\frac{1}{2} \|B[x' - x'']\| : Ax'[x' - x''] = 0\}
$$
$$
= \max_{x \in \mathcal{X}_A} \|Bx\| = \max_{w \in \mathcal{W}} \|BFw\|
$$
$$
= \|BF\|_{\|\cdot\|_w \to \|\cdot\|} [\|\cdot\|_w : \text{norm with the unit ball } \mathcal{W}]
$$

$$
\Rightarrow \mathcal{R} \leq \text{Opt} \leq 4 \ln(5[K + L]) \mathcal{R},
$$

where the efficiently computable quantity $\text{Opt}$ is given by

$$
\text{Opt} = \min_{\lambda, \mu} \left\{ \phi_T(\lambda) + \phi_R(\mu) : \begin{array}{c}
\lambda \geq 0, \mu \geq 0
\end{array} \right\}
$$

$$
= \min_{\lambda, \mu} \left\{ \lambda \geq 0, \mu \geq 0 : \begin{array}{c}
\frac{1}{2} \sum_{\ell} \sum_{k} \lambda_k T_k \left[ \frac{\lambda T_k}{2} B F \right] \leq 0
\end{array} \right\}.
$$

$$
\Rightarrow \text{The optimal } \| \cdot \|\text{-risk is } \geq \mathcal{R} \geq \frac{\text{Opt}}{4 \ln(5[K + L])}, \text{ and } \text{Risk}_{\| \cdot \|}[\hat{x}, \mathcal{X}] \leq 2 \mathcal{R} \leq 2 \text{Opt}.
$$
In fact, under mild assumptions a linear estimate is near-optimal:

**Theorem.** Consider the problem of recovering $Bx$ in $\| \cdot \|$, $x \in \mathcal{X}$, via observation $\omega = Ax$. Let the signal set $\mathcal{X}$ be an ellitope, and the unit ball $\mathcal{B}_*$ of the norm conjugate to $\| \cdot \|$ be a basic ellitope:

\[
\mathcal{X} = \{ x = Py : y \in \mathcal{Y} := \{ y : \exists t \in \mathcal{T} : y^T S_k y \leq t_k, k \leq K \} \}
\]
\[
\mathcal{B}_* = \{ u : \exists r \in \mathcal{R} : u^T R_{\ell} u \leq r_{\ell}, \ell \leq L \}
\]

Then the linear estimate $\hat{x}(\omega) = H_*^T \omega$ yielded by the $H$-component of optimal solution to the efficiently solvable optimization problem

\[
\text{Opt} = \min_{\lambda, \mu, H} \left\{ \phi_T(\lambda) + \phi_R(\mu) : \lambda \geq 0, \mu \geq 0,
\left[ \frac{\sum_\ell \mu_\ell R_\ell}{\frac{1}{2} P^T (B^T - A^T H)} \right] \frac{1}{2} [B - H^T A] P \right\} \succeq 0 \}
\]

is near-optimal:

\[
\text{Risk}_{\| \cdot \|}[\hat{x}_{H^*} | \mathcal{X}] \leq \text{Opt} \leq 4 \ln(5[K + L]) \text{Risk}^*_{\| \cdot \|} [\mathcal{X}],
\]

where

\[
\text{Risk}^*_{\| \cdot \|} [\mathcal{X}] = \inf_{\hat{x}(\cdot)} \text{Risk}_{\| \cdot \|}[\hat{x} | \mathcal{X}],
\]

$\inf$ being taken over all estimates, linear and nonlinear alike, is the minimax optimal risk.
Sketch of the proof:

A. The $\| \cdot \|_\cdot\|\cdot\|\cdot\|\cdot\|$-risk of a linear estimate $\hat{x}_H(\omega) = H^T \omega$ is

$$\text{Risk}_{\| \cdot \|}(\hat{x}_H|\mathcal{X}) = \max_{x \in \mathcal{X}} \| [B-H^T A]x \| = \max_{y \in \mathcal{Y}} \| [B-H^T A]Py \|$$

which is nothing but

$$\| [B-H^T A]P \|_{\| \cdot \|_{\cdot \rightarrow \| \cdot \|}},$$

where $\| \cdot \|_{\cdot \rightarrow \| \cdot \|$ is the norm with the unit ball $\mathcal{Y}$. By Theorem on upper-bounding operator norms, we have

$$\text{Risk}_{\| \cdot \|}(\hat{x}_H|\mathcal{X}) \leq \min_{\lambda,\mu} \left\{ \phi_T(\lambda) + \phi_R(\mu) : \begin{array}{l} \lambda \geq 0, \mu \geq 0 \\ \frac{1}{2} P^T [B^T - A^T H] \left[ \frac{1}{2} [B - H^T A]P \right] \frac{1}{\sum_k \lambda_k S_k} \right\} \succeq 0 \right\}$$

$$\Rightarrow \text{Risk}_{\| \cdot \|}(\hat{x}_H|\mathcal{X}) \leq \text{Opt}.$$

B. As we have seen, the quantity

$$\mathcal{R} = \max_x \{ \| Bx \| : Ax = 0, x \in \mathcal{X} \}$$

is a lower bound on the minimal optimal $\| \cdot \|-risk \text{ Risk}^*_{\| \cdot \|}(\mathcal{X})$, and $\mathcal{R}$ can be tightly (within factor $4 \ln ([K + L])$ upper-bounded by the optimal value in an explicit conic problem. On the closest inspection, heavily utilizing conic duality, the optimal value in question turns out to be equal to $\text{Opt}$

$$\Rightarrow \text{The (upper bound on the) } \| \cdot \|\cdot\|-risk \text{ of the linear estimate } \hat{x}_{H*} \text{ is within the factor } 4 \ln (5[K + L]) \text{ of the lower bound } \mathcal{R} \text{ on the minimax optimal } \| \cdot \|\cdot\|-risk.$$
From Ellitopes to Spectratopes

♫ Fact: All our results extend from ellitopes – sets of the form
\[
\{ y \in \mathbb{R}^N : \exists t \in \mathcal{T}, z : y = Pz, z^T S_k z \leq t_k, k \leq K \}
\]
\[
\mathcal{T} \subset \mathbb{R}_+^K : \text{monotone convex compact intersecting int } \mathbb{R}_+^K
\]
which played the roles of signal sets, ranges of bounded noise, and unit balls of the norms conjugate to the norms \( \| \cdot \| \) in which the recovering error is measured, to a wider family – spectratopes

basic spectratope:
\[
\mathcal{Y} = \{ y \in \mathbb{R}^N : \exists t \in \mathcal{T}, S_k^2[y] \leq t_k I_{d_k}, k \leq K \}
\]
\[
S_k[y] = \sum_j y_j S_k^{kj}, S_k^{kj} \in S^d_k, y \neq 0 \Rightarrow \sum_k S_k^2[y] \neq 0
\]
\( \mathcal{T} \) as in \((E)\)

spectratope:
\[
\mathcal{Z} = \{ z = Py : y \in \mathcal{Y} := \{ y : \exists t \in \mathcal{T} : S_k^2[y] \leq t_k I_{d_k}, k \leq K \} \}
\]
\[ \mathcal{T} \text{ and } S_k[\cdot] \text{ as in basic spectratope} \]

With this extension, we get, e.g., access to
• matrix boxes \( \mathcal{X} = \{ x \in \mathbb{R}^{p \times q} : \| x \|_{2,2} \leq 1 \} \) or their symmetric versions \( \mathcal{X} = \{ x \in S_+^p : -I \leq x \leq I \} \) as signal sets
• nuclear norm \( \| u \|_{\text{nuc}} \) (sum of singular values of a matrix) as the norm quantifying recovery error

♫ Modifications of the results when passing from ellitopes to spectratopes are as follows:
A. The “size” \( K \) of an ellitope \((E)\) (logs of these sizes participate in our tightness factors) in the case of spectratope \((S)\) becomes \( D = \sum_k d_k \)
B. SDP relaxation bound for the quantity

\[
\text{Opt}_* = \max_y \{ y^T B y : \exists t \in T, z : y = P z, S^2_k[z] \preceq t_k I_{d_k}, k \leq K \}
\]

\[
= \max_z \left\{ z^T \hat{B} z : t \in T, S^2_k[z] \preceq t_k I_{d_k}, k \leq K \right\}, \hat{B} = P^T B P
\]

is as follows:

We associate with \( S_k[z] = \sum_j z_j S^{kj} \), \( S^{kj} \in S^{dk} \), two linear mappings:

\[
Q \mapsto S_k[Q] : S^{\dim z} \to S^{dk} : S_k[Q] = \sum_{i,j} \frac{1}{2} Q_{ij} [S^{ki} S^{kj} + S^{kj} S^{ki}]
\]

\[
\Lambda \mapsto S^*_k[\Lambda] : S^{dk} \to S^{\dim z} : \quad \left[ S^*_k[\Lambda] \right]_{ij} = \frac{1}{2} \text{Tr}(\Lambda [S^{ki} S^{kj} + S^{kj} S^{ki}])
\]

Note:

- \( S^2_k[z] = S_k[zz^T] \)
- the mappings \( S_k \) and \( S^*_k \) are conjugates of each other w.r.t. the Frobenius inner product:

\[
\text{Tr}(S_k[Q] \Lambda) = \text{Tr}(QS^*_k[\Lambda]) \quad \forall (Q \in S^{\dim z}, \Lambda \in S^{dk})
\]

Selecting \( \Lambda_k \succeq 0, k \leq K \), such that \( \sum_k S^*_k[\Lambda_K] \succeq \hat{B} \), for

\[
z \in Z = \{ z : \exists t \in T : S^2_k[z] \preceq t_k I_{d_k}, k \leq K \}
\]

we have \( \exists t \in T : S^2_k[z] \preceq t_k I_{d_k}, k \leq K \) \implies

\[
z^T \hat{B} z \leq z^T \left[ \sum_k S^*_k[\Lambda_k] \right] z = \sum_k z^T S^*_k[\Lambda_k] z = \sum_k \text{Tr}(S^*_k[\Lambda_k][zz^T]) = \sum_k \text{Tr}(\Lambda_k S^2_k[z]) \leq \sum_k t_k \text{Tr}(\Lambda_k) \leq \phi_T(\lambda[\Lambda]), \quad \phi_T(\lambda) = \max_{t \in T} t^T \lambda, \lambda[\Lambda] = [\text{Tr}(\Lambda_1); ..., \text{Tr}(\Lambda_K)]
\]

\[
\implies \text{Opt}_* \leq \text{Opt} := \min_{\Lambda = \{ \Lambda_k, k \leq K \}} \left\{ \phi_T(\lambda[\Lambda]) : \Lambda_k \succeq 0, k \leq K, \hat{B} \preceq \sum_k S^*_k[\Lambda_k] \right\}
\]
\section*{Theorem. SDP relaxation bound}

\[ \text{Opt} := \min_{\Lambda = \{\Lambda_k, k \leq K\}} \left\{ \phi_T(\lambda[\Lambda]) : \Lambda_k \succeq 0, k \leq K, \hat{B} \leq \sum_k S_k^*[\Lambda_k] \right\} \]

on the quantity

\[ \text{Opt}_* = \max_y \left\{ y^T B y : \exists t \in T, z : y = P z, S_k^2[z] \leq t_k I_{d_k}, k \leq K \right\} = \max_{z,t} \left\{ z^T \hat{B} z : t \in T, S_k^2[z] \leq t_k I_{d_k}, k \leq K \right\} \]

is tight:

\[ \text{Opt}_* \leq \text{Opt} \leq 2 \ln(2 \sum_k d_k) \text{Opt}_*. \]

\textbf{Note:} The role of elementary Mini-Lemma in the spectratopic case is played by the following fundamental matrix concentration result:

\textbf{Noncommutative Khintchine Inequality} [Lust-Picard 1986, Pisier 1998, Buchholz 2001] \textit{Let } A_i \in S^d, 1 \leq i \leq N, \text{ be deterministic matrices such that}

\[ \sum_i A_i^2 \leq I_d, \]

and let \( \zeta \) be \( N \)-dimensional \( N(0, I_N) \) or Rademacher random vector. Then for all \( s \geq 0 \) it holds

\[ \text{Prob} \left\{ \| \sum_i \zeta_i A_i \|_{2,2} \geq s \right\} \leq 2d \exp\{-s^2/2\}. \]