

Assignment 1: From the Definition of Convexity to Helley Theorem

**Exercise 1** Mark in the following list the sets which are convex:

1.  $\{x \in \mathbf{R}^2 : x_1 + i^2 x_2 \leq 1, i = 1, \dots, 10\}$
2.  $\{x \in \mathbf{R}^2 : x_1^2 + 2ix_1x_2 + i^2x_2^2 \leq 1, i = 1, \dots, 10\}$
3.  $\{x \in \mathbf{R}^2 : x_1^2 + ix_1x_2 + i^2x_2^2 \leq 1, i = 1, \dots, 10\}$
4.  $\{x \in \mathbf{R}^2 : x_1^2 + 5x_1x_2 + 4x_2^2 \leq 1\}$
5.  $\{x \in \mathbf{R}^{10} : x_1^2 + 2x_2^2 + 3x_3^2 + \dots + 10x_{10}^2 \leq 2004x_1 - 2003x_2 + 2002x_3 - \dots + 1996x_9 - 1995x_{10}\}$
6.  $\{x \in \mathbf{R}^2 : \exp\{x_1\} \leq x_2\}$
7.  $\{x \in \mathbf{R}^2 : \exp\{x_1\} \geq x_2\}$
8.  $\{x \in \mathbf{R}^n : \sum_{i=1}^n x_i^2 = 1\}$
9.  $\{x \in \mathbf{R}^n : \sum_{i=1}^n x_i^2 \leq 1\}$
10.  $\{x \in \mathbf{R}^n : \sum_{i=1}^n x_i^2 \geq 1\}$
11.  $\{x \in \mathbf{R}^n : \max_{i=1, \dots, n} x_i \leq 1\}$
12.  $\{x \in \mathbf{R}^n : \max_{i=1, \dots, n} x_i \geq 1\}$
13.  $\{x \in \mathbf{R}^n : \max_{i=1, \dots, n} x_i = 1\}$
14.  $\{x \in \mathbf{R}^n : \min_{i=1, \dots, n} x_i \leq 1\}$
15.  $\{x \in \mathbf{R}^n : \min_{i=1, \dots, n} x_i \geq 1\}$
16.  $\{x \in \mathbf{R}^n : \min_{i=1, \dots, n} x_i = 1\}$

**Exercise 2** Which ones of the following three statements are true?

1. The convex hull of a closed set in  $\mathbf{R}^n$  is closed
2. The convex hull of a closed convex set in  $\mathbf{R}^n$  is closed
3. The convex hull of a closed and bounded set in  $\mathbf{R}^n$  is closed and bounded

For true statements, present proofs; for wrong, give counterexamples.

Hint: Recall that a bounded and closed subset of  $\mathbf{R}^n$  is compact and that there exists Caratheodory Theorem.

The next two exercises are optional.

**Exercise 3** A cake contains  $300 \text{ g}^1$  of raisins (you may think of every one of them as of a 3D ball of positive radius). John and Jill are about to divide the cake according to the following rules:

- first, Jill chooses a point  $a$  in the cake;
- second, John makes a cut through  $a$ , that is, chooses a 2D plane  $\Pi$  passing through  $a$  and takes the part of the cake on one side of the plane (both  $\Pi$  and the side are up to John, with the only restriction that the plane should pass through  $a$ ); all the rest goes to Jill.

1. Prove that it may happen that Jill cannot guarantee herself 76 g of the raisins
2. Prove that Jill always can choose  $a$  in a way which guarantees her at least 74 g of the raisins
3. Consider  $n$ -dimensional version of the problem, where the raisins are  $n$ -dimensional balls, the cake is a domain in  $\mathbf{R}^n$ , and “a cut” taken by John is defined as the part of the cake contained in the half-space

$$\{x \in \mathbf{R}^n : e^T(x - a) \geq 0\},$$

where  $e \neq 0$  is the vector (“inner normal to the cutting hyperplane”) chosen by John. Prove that for every  $\epsilon > 0$ , Jill can guarantee to herself at least  $\frac{300}{n+1} - \epsilon$  g of raisins, but in general cannot guarantee to herself  $\frac{300}{n+1} + \epsilon$  g.

**Exercise 4** Prove the following Kirchner’s Theorem:

Assume that  $X = \{x_1, \dots, x_k\}$  and  $Y = \{y_1, \dots, y_m\}$  are finite sets in  $\mathbf{R}^n$ , with  $k + m \geq n + 2$ , and all the points  $x_1, \dots, x_k, y_1, \dots, y_m$  are distinct. Assume that for any subset  $S \subset X \cup Y$  comprised of  $n + 2$  points the convex hulls of the sets  $X \cap S$  and  $Y \cap S$  do not intersect. Then the convex hulls of  $X$  and  $Y$  also do not intersect.

Hint: Assume, on contrary, that the convex hulls of  $X$  and  $Y$  intersect, so that

$$\sum_{i=1}^k \lambda_i x_i = \sum_{j=1}^m \mu_j y_j$$

for certain nonnegative  $\lambda_i$ ,  $\sum_i \lambda_i = 1$ , and certain nonnegative  $\mu_j$ ,  $\sum_j \mu_j = 1$ , and look at the expression of this type with the minimum possible total number of nonzero coefficients  $\lambda_i, \mu_j$ .

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<sup>1)</sup>grams

## Assignment 2: General Theorem on Alternative

General Theorem on Alternative has several particular cases (in fact equivalent to the GTA). The goal of the subsequent exercises is to become acquainted with these particular cases.

**Exercise 5** Derive from GTA the following

**Gordan's Theorem on Alternative** Let  $A$  be an  $m \times n$  matrix. One of the inequality systems

$$(I) \quad Ax < 0, x \in \mathbf{R}^n,$$

$$(II) \quad A^T y = 0, 0 \neq y \geq 0, y \in \mathbf{R}^m,$$

has a solution if and only if the other one has no solutions.

**Exercise 6** Derive from GTA the following

**Inhomogeneous Farkas Lemma** A linear inequality

$$a^T x \leq p \tag{1}$$

is a consequence of a solvable system of linear inequalities

$$Nx \leq q \tag{2}$$

if and only if it is a "linear consequence" of the system and the trivial inequality

$$0^T x \leq 1,$$

i.e., if it can be obtained by taking weighted sum, with nonnegative coefficients, of the inequalities from the system and this trivial inequality.

Algebraically: (1) is a consequence of solvable system (2) if and only if

$$a = N^T \nu$$

for some nonnegative vector  $\nu$  such that

$$\nu^T q \leq p.$$

**Exercise 7** Derive from GTA the following

**Motzkin's Theorem on Alternative** The system

$$Sx < 0, Nx \leq 0$$

has no solutions if and only if the system

$$S^T \sigma + N^T \nu = 0, \sigma \geq 0, \nu \geq 0, \sigma \neq 0$$

has a solution.

**Exercise 8** Which of the following systems of linear inequalities with 2 unknowns have, and which have no solutions (for the systems which are solvable, point out a solution; for the unsolvable systems, explain why they are so):

- $\begin{cases} x + y \geq 2 \\ 2x - y \geq 1 \\ -5x + y \geq -5 \end{cases}$
- $\begin{cases} x + y \geq 2 \\ 2x - y \geq 1 \\ -5x + y \geq -4 \end{cases}$
- $\begin{cases} x + y \geq 2 \\ 2x - y \geq 1 \\ -5x + y \geq -3.5 \end{cases}$

**Exercise 9** Consider the linear inequality

$$x + y \leq 2$$

and the system of linear inequalities

$$\begin{cases} x \leq 1 \\ -x \leq -100 \end{cases}$$

Our inequality clearly is a consequence of the system – it is satisfied at every solution to it (simply because there are no solutions to the system at all). According to the Inhomogeneous Farkas Lemma, the inequality should be a linear consequence of the system and the trivial inequality  $0 \leq 1$ , i.e., there should exist nonnegative  $\nu_1, \nu_2$  such that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \nu_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \nu_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \nu_1 - 1000\nu_2 \leq 2,$$

which clearly is not the case. What is the reason for the observed “contradiction”?

Assignment 3: Separation and Extreme Points

**Exercise 10** Mark in the below list the cases when the linear form  $f^T x$  separates the sets  $S$  and  $T$

- $S = \{0\} \subset \mathbf{R}, T = \{0\} \subset \mathbf{R}, f^T x = x$
- $S = \{0\} \subset \mathbf{R}, T = [0, 1] \subset \mathbf{R}, f^T x = x$
- $S = \{0\} \subset \mathbf{R}, T = [-1, 1] \subset \mathbf{R}, f^T x = x$
- $S = \{x \in \mathbf{R}^3 : x_1 = x_2 = x_3\}, T = \{x \in \mathbf{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}\}, f^T x = x_1 - x_2$
- $S = \{x \in \mathbf{R}^3 : x_1 = x_2 = x_3\}, T = \{x \in \mathbf{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}\}, f^T x = x_3 - x_2$
- $S = \{x \in \mathbf{R}^3 : -1 \leq x_1 \leq 1\}, T = \{x \in \mathbf{R}^3 : x_1^2 \geq 4\}, f^T x = x_1$
- $S = \{x \in \mathbf{R}^2 : x_2 \geq x_1^2, x_1 \geq 0\}, T = \{x \in \mathbf{R}^2 : x_2 = 0\}, f^T x = -x_2$

**Exercise 11** Consider the set

$$M = \left\{ x \in \mathbf{R}^{2004} : \begin{array}{l} x_1 + x_2 + \dots + x_{2004} \geq 1 \\ x_1 + 2x_2 + 3x_3 + \dots + 2004x_{2004} \geq 10 \\ x_1 + 2^2x_2 + 3^2x_3 + \dots + 2004^2x_{2004} \geq 10^2 \\ \dots\dots\dots \\ x_1 + 2^{2002}x_2 + 3^{2002}x_3 + \dots + 2004^{2002}x_{2004} \geq 10^{2002} \end{array} \right\}$$

Is it possible to separate this set from the set  $\{x_1 = x_2 = \dots = x_{2004} \leq 0\}$ ? If yes, what could be a separating plane?

**Exercise 12** Let  $M$  be a convex set in  $\mathbf{R}^n$  and  $x$  be an extreme point of  $M$ . Prove that if

$$x = \sum_{i=1}^m \lambda_i x_i$$

is a representation of  $x$  as a convex combination of points  $x_i \in M$  with positive weights  $\lambda_i$ , then  $x = x_1 = \dots = x_m$ .

**Exercise 13** Let  $M$  be a closed convex set in  $\mathbf{R}^n$  and  $\bar{x}$  be a point of  $M$ . Prove that if there exists a linear form  $a^T x$  such that  $\bar{x}$  is the unique maximizer of the form on  $M$ , then  $\bar{x}$  is an extreme point of  $M$ .

**Exercise 14** Find all extreme points of the set

$$\{x \in \mathbf{R}^2 \mid -x_1 + 2x_2 \leq 8, 2x_1 + x_2 \leq 9, 3x_1 - x_2 \leq 6, x_1, x_2 \geq 0\}.$$

**Exercise 15** Mark with "y" those of the below statements which are true:

- If  $M$  is a nonempty convex set in  $\mathbf{R}^n$  which does not contain lines, then  $M$  has an extreme point
- If  $M$  is a convex set in  $\mathbf{R}^n$  which has an extreme point, then  $M$  does not contain lines

- If  $M$  is a nonempty closed convex set in  $\mathbf{R}^n$  which does not contain lines, then  $M$  has an extreme point
- If  $M$  is a closed convex set in  $\mathbf{R}^n$  which has an extreme point, then  $M$  does not contain lines
- If  $M$  is a nonempty bounded convex set in  $\mathbf{R}^n$ , then  $M$  is the convex hull of  $\text{Ext}(M)$
- If  $M$  is a nonempty closed and bounded convex set in  $\mathbf{R}^n$ , then  $M$  is the convex hull of  $\text{Ext}(M)$
- If  $M$  is a nonempty closed convex set in  $\mathbf{R}^n$  which is equal to the convex hull of  $\text{Ext}(M)$ , then  $M$  is bounded

**Exercise 16** A  $n \times n$  matrix  $\pi$  is called *double stochastic*, if all its entries are nonnegative, and the sums of entries in every row and every column are equal to 1, as it is the case with the unit matrix or, more generally, with a permutation matrix – the one which has exactly one nonzero entry (equal to 1) in every column and every row, e.g.,

$$\pi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Double stochastic matrices of a given order  $n$  clearly form a nonempty bounded convex polyhedral set  $\mathcal{D}$  in  $\mathbf{R}^{n \times n}$ . What are the extreme points of the set? The answer is given by the following

**Theorem** (Birkhoff) *The extreme points of the polytope  $\mathcal{D}$  of  $n \times n$  double stochastic matrices are exactly the permutation matrices of order  $n$ .*

*Try to prove the Theorem.*

*Hint: Using the algebraic characterization of extreme points of a polyhedral set, verify that in an extreme point, there is a column with just one nonzero entry.*

The Birkhoff Theorem is the source of a number of important inequalities; some of these inequalities will be the subject of optional exercises in the coming hometasks.

## Assignment 4: Convex Functions

**Exercise 17** Mark by "c" those of the following functions which are convex on the indicated domains:

- $f(x) \equiv 1$  on  $\mathbf{R}$
- $f(x) = x$  on  $\mathbf{R}$
- $f(x) = |x|$  on  $\mathbf{R}$
- $f(x) = -|x|$  on  $\mathbf{R}$
- $f(x) = -|x|$  on  $\mathbf{R}_+ = \{x \geq 0\}$
- $\exp\{x\}$  on  $\mathbf{R}$
- $\exp\{x^2\}$  on  $\mathbf{R}$
- $\exp\{-x^2\}$  on  $\mathbf{R}$
- $\exp\{-x^2\}$  on  $\{x \mid x \geq 100\}$

**Exercise 18** Prove that the following functions are convex on the indicated domains:

- $\frac{x^2}{y}$  on  $\{(x, y) \in \mathbf{R}^2 \mid y > 0\}$
- $\ln(\exp\{x\} + \exp\{y\})$  on the 2D plane.

**Exercise 19** A function  $f$  defined on a convex set  $Q$  is called log-convex on  $Q$ , if it takes real positive values on  $Q$  and the function  $\ln f$  is convex on  $Q$ . Prove that

- a log-convex on  $Q$  function is convex on  $Q$
- the sum (more generally, linear combination with positive coefficients) of two log-convex functions on  $Q$  also is log-convex on the set.

Hint: use the result of the previous Exercise + your knowledge on operations preserving convexity

**Exercise 20** For  $n$ -dimensional vector  $x$ , let  $\hat{x} = (\hat{x}^1, \dots, \hat{x}^n)^T$  be the vector obtained from  $x$  by rearranging the coordinates in the non-ascending order. E.g., with  $x = (2, 1, 3, 1)^T$ ,  $\hat{x} = (3, 2, 1, 1)^T$ . Let us fix  $k$ ,  $1 \leq k \leq n$ .

- Is the function  $\hat{x}^k$  ( $k$ -th largest entry in  $x$ ) a convex function of  $x$ ?
- Is the function  $s_k(x) = \hat{x}^1 + \dots + \hat{x}^k$  (the sum of  $k$  largest entries in  $x$ ) convex?

**Exercise 21** Consider a Linear Programming program

$$\min_x \{c^T x : Ax \leq b\}$$

with  $m \times n$  matrix  $A$ , and let  $x^*$  be an optimal solution to the problem. It means that  $x^*$  is a minimizer of differentiable convex function  $f(x) = c^T x$  on convex set  $Q = \{x \mid Ax \leq b\}$  and therefore, according to Proposition 2.5.1,  $\nabla f(x^*)$  should belong to the normal cone of  $A$  at  $x^*$  – this is the necessary and sufficient condition for optimality of  $x^*$ . What does this condition mean in terms of the data  $A, b, c$ ?

Optional exercises:

**Exercise 22** Let  $f(x)$  be a convex symmetric function of  $x \in \mathbf{R}^n$  (symmetry means that  $f$  remains unchanged when permuting the coordinates of the argument, as it is the case with  $\sum_i x_i$ , or  $\max_i x_i$ ). Prove that if  $\pi$  is a double stochastic  $n \times n$  matrix (see Optional exercise from Assignment 3), then

$$f(\pi x) \leq f(x) \quad \forall x$$

**Exercise 23** Let  $f(x)$  be a convex symmetric function on  $\mathbf{R}^n$ . For a symmetric  $n \times n$  matrix  $X$ , let  $\lambda_1(X) \geq \lambda_2(X) \geq \dots \geq \lambda_n(X)$  be the eigenvalues of  $X$  taken with their multiplicities and arranged in the nondecreasing order. Prove

1. For every orthogonal  $n \times n$  matrix  $U$  and symmetric  $n \times n$  matrix  $X$ ,

$$f(\text{diag}(UXU^T)) \leq f(\lambda(X))$$

where  $\text{diag}(A)$  stands for the diagonal of square matrix;

2. The function

$$F(X) = f(\lambda(X))$$

of symmetric  $n \times n$  matrix  $X$  is convex.

**Exercise 24** For a  $10 \times 10$  symmetric matrix  $X$ , what is larger – the sum of two largest diagonal entries or the sum of two largest eigenvalues?



Assignment 5: Optimality Conditions in Convex Programming

**Exercise 25** Find the minimizer of a linear function

$$f(x) = c^T x$$

on the set

$$V_p = \{x \in \mathbf{R}^n \mid \sum_{i=1}^n |x_i|^p \leq 1\};$$

here  $p, 1 < p < \infty$ , is a parameter. What happens with the solution when the parameter becomes 0.5?

**Exercise 26** Let  $a_1, \dots, a_n > 0, \alpha, \beta > 0$ . Solve the optimization problem

$$\min_x \left\{ \sum_{i=1}^n \frac{a_i}{x_i^\alpha} : x > 0, \sum_i x_i^\beta \leq 1 \right\}$$

**Exercise 27** Consider the optimization problem

$$\max_{x,t} \left\{ \xi^T x + \tau t + \ln(t^2 - x^T x) : (t, x) \in X = \{t > \sqrt{x^T x}\} \right\}$$

where  $\xi \in \mathbf{R}^n, \tau \in \mathbf{R}$  are parameters. Is the problem convex<sup>2)</sup>? What is the domain in space of parameters where the problem is solvable? What is the optimal value? Is it convex in the parameters?

**Exercise 28** Consider the optimization problem

$$\max_{x,y} \{ax + by + \ln(\ln y - x) + \ln(y) : (x, y) \in X = \{y > \exp\{x\}\}\},$$

where  $a, b \in \mathbf{R}$  are parameters. Is the problem convex? What is the domain in space of parameters where the problem is solvable? What is the optimal value? Is it convex in the parameters?

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<sup>2)</sup>A maximization problem with objective  $f(\cdot)$  and certain constraints and domain is called convex if the equivalent minimization problem with the objective  $(-f)$  and the original constraints and domain is convex.

Assignment 6: Optimality Conditions

**Exercise 29** Consider the problem of minimizing the linear form

$$f(x) = x_2 + 0.1x_1$$

on the 2D plane over the triangle with the vertices  $(1, 0)$ ,  $(0, 1)$ ,  $(0, 1/2)$  (draw the picture!).

- 1) Verify that the problem has unique optimal solution  $x_* = (1, 0)$
- 2) Verify that the problem can be posed as the LP program

$$\min_x \{x_2 + 0.1x_1 : x_1 + x_2 \leq 1, x_1 + 2x_2 \geq 1, x_1, x_2 \geq 0\}.$$

Prove that in this formulation of the problem the KKT necessary optimality condition is satisfied at  $x_*$ .

What are the active at  $x_*$  constraints? What are the corresponding Lagrange multipliers?

- 3) Verify that the problem can be posed as a nonlinear program with inequality constraints

$$\min_x \{x_2 + 0.1x_1 : x_1 \geq 0, x_2 \geq 0, (x_1 + x_2 - 1)(x_1 + 2x_2 - 1) \leq 0\}.$$

Is the KKT Optimality condition satisfied at  $x_*$ ?

**Exercise 30** Consider the following elementary problem:

$$\min_x \{f(x_1, x_2) = x_1^2 - x_2 : h(x) \equiv x_2 = 0\}$$

with the evident unique optimal solution  $(0, 0)$ . Is the KKT condition satisfied at this solution?

Rewrite the problem equivalently as

$$\min_x \{f(x_1, x_2) = x_1^2 - x_2 : h(x) \equiv x_2^2 = 0\}.$$

What about the KKT condition in this equivalent problem?

**Exercise 31** Consider an inequality constrained optimization problem

$$\min_x \{f(x) : g_i(x) \leq 0, i = 1, \dots, m\}.$$

Assume that  $x_*$  is locally optimal solution,  $f, g_i$  are continuously differentiable in a neighbourhood of  $x_*$  and the constraints  $g_i$  are concave in this neighbourhood. Prove that  $x_*$  is locally optimal solution to the linearized problem

$$\min_x \left\{ f(x_*) + (x - x_*)^T \nabla f(x_*) : (x - x_*)^T \nabla g_j(x_*) = 0, j \in J(x_*) = \{j : g_j(x_*) = 0\} \right\}.$$

Is  $x_*$  a KKT point of the problem?

**Exercise 32** Let  $a_1, \dots, a_n$  be positive reals, and let  $0 < s < r$  be two reals. Find maximum and minimum of the function

$$\sum_{i=1}^n a_i |x_i|^r$$

on the surface

$$\sum_{i=1}^n |x_i|^s = 1.$$

Optional exercises:

**Exercise 33** Recall S-Lemma:

If  $A, B$  are two symmetric  $n \times n$  matrices such that

$$\bar{x}^T B \bar{x} > 0$$

for certain  $\bar{x}$ , then the implication

$$x^T B x \geq 0 \Rightarrow x^T A x \geq 0$$

holds true iff there exists  $\lambda \geq 0$  such that  $A - \lambda B \succeq 0$

( $P \succeq 0$  means that  $P$  is symmetric positive semidefinite:  $P = P^T$  and  $x^T P x \geq 0$  for all  $x \in \mathbf{R}^n$ ). The proof given in Lecture on optimality conditions (see Transparencies) was incomplete – it was taken for granted that certain optimization problem has an optimal solution. Fill the gap in the proof or find an alternative proof.

**Exercise 34** Let  $A = A^T$  be a symmetric  $n \times n$  matrix, and let  $B, C$  be  $m \times n$  matrices,  $C \neq 0$ . Then all matrices

$$A + B^T \Delta C + C^T \Delta^T B$$

corresponding to  $m \times m$  matrices  $\Delta$  with  $\|\Delta\| \leq 1$ <sup>3)</sup> are positive semidefinite iff there exists  $\lambda \geq 0$  such that the matrix

$$\left[ \begin{array}{c|c} A - \lambda C^T C & -B^T \\ \hline -B & \lambda I_m \end{array} \right]$$

is positive semidefinite.

This fact has important applications in Control and Optimization under uncertainty.

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<sup>3)</sup>here  $\|\Delta\|$  is the standard matrix norm:  $\|\Delta\| = \max_u \{\|\Delta u\|_2 : \|u\|_2 \leq 1\}$ . The statement remains true when replacing the matrix norm with the Frobenius norm  $\|\Delta\|_{Fro} = \sqrt{\sum_{i,j} \Delta_{ij}^2}$ .